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REGULAR OPERATORS ON HILBERT C^*-MODULES

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ABSTRACT. A regular operator T on a Hilbert C^*-module is defined just like a closed operator on a Hilbert space, with the extra condition that the range of $(I + T^*T)$ is dense. Semiregular operators are a slightly larger class of operators that may not have this property. It is shown that, like in the case of regular operators, one can, without any loss in generality, restrict oneself to semiregular operators on C^*-algebras. We then prove that for abelian C^*-algebras as well as for subalgebras of the algebra of compact operators, any closed semiregular operator is automatically regular. We also determine how a regular operator and its extensions (and restrictions) are related. Finally, using these results, we give a criterion for a semiregular operator on a liminal C^*-algebra to have a regular extension.

KEYWORDS: *Hilbert C^*-modules, unbounded operators, C^*-algebras.*

MSC (2000): 46H25, 47C15.

1. INTRODUCTION

Hilbert C^*-modules were first studied by Kaplansky ([4]) for abelian C^*-algebras, and later for more general C^*-algebras by Rieffel ([9]) and Paschke ([7]). Kasparov developed the theory further, and used them to get deep and far-reaching results ([5]) in KK-theory. As the name suggests, Hilbert C^*-modules are very similar to Hilbert spaces, with C^*-algebra elements playing the role of scalars. With the development of quantum groups and noncommutative geometry, the study of Hilbert C^*-modules has assumed further importance. In noncommutative geometry, for example, the role of vector bundles on (noncommutative) spaces is played by Hilbert C^*-modules. For a locally compact quantum group G, the C^*-algebra

$\mathcal{A} = C_0(G)$ of "continuous vanishing-at-infinity functions" on G is a Hilbert C^*-module over itself. Analogues of bounded operators in the Hilbert C^*-module context are *adjointable operators*. For example, for any C^*-algebra considered as a Hilbert C^*-module over itself, adjointable operators are the elements of its multiplier algebra. In various contexts that the Hilbert C^*-modules arise, one also needs to study "unbounded adjointable operators", or what are now known as *regular operators*. These were first introduced by Baaj and Julg in [1], where they gave a nice construction of Kasparov bimodules in KK-theory using regular operators. Later they were rediscovered by Woronowicz ([11]) while investigating noncompact quantum groups. He considered C^*-algebras rather than general Hilbert C^*-modules (we shall see in Section 3 that there is no loss in generality in doing this), and called them elements affiliated to the C^*-algebra. "Coordinate functions" on locally compact noncompact quantum groups are examples of such objects. Representations of locally compact noncompact groups (quantum as well as classical) are examples of regular operators on more general Hilbert C^*-modules. Lance gave a brief indication in his book ([6]) about the possible role Hilbert modules might play in studying representations of quantum groups.

Let us quickly recall the definition of a regular operator. Let $\mathcal{A}$ be a C^*-algebra. An operator T from a Hilbert $\mathcal{A}$-module E to another Hilbert $\mathcal{A}$-module F is said to be *regular* if

(a) T is closed and densely defined,
(b) its adjoint T^* is also densely defined, and
(c) range of $I + T^*T$ is dense in F.

Note that if we set $\mathcal{A} = \mathbb{C}$, i.e. if we take E and F to be Hilbert spaces, then this is exactly the definition of a closed operator, except that in that case, both the second and the third condition follow from the first one. In the Hilbert C^*-module context, one needs to add these extra conditions in order to get a reasonably good theory. But when one deals with specific unbounded operators on concrete Hilbert C^*-modules, it is usually extremely difficult to verify the last condition, though the first two conditions are relatively easy to check. So it would be interesting to find other more easily manageable conditions that are equivalent to the last condition above. In [11], Woronowicz gave a criterion based on the graph of an operator for it to be regular, and to this date, this remains the only attempt in this direction.

In the present paper, we will consider a somewhat larger class of operators that we call semiregular operators, which are, roughly speaking, operators satisfying the first two conditions above. We then investigate the following two problems, namely, under what conditions are they regular, and when do they admit regular extensions. We will assume elements of C^*-algebra theory as can be found for example in Dixmier ([2]). For an account on the C^*-module theory required, we refer the reader to Lance ([6]).

NOTATIONS. $\mathcal{H}$ will denote a complex separable Hilbert space. $\mathcal{B}_0(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ will respectively stand for the space of compact operators on $\mathcal{H}$ and the space of all bounded operators on it. $\mathcal{A}$ denotes a C^*-algebra, usually nonunital, and $M(\mathcal{A})$ is its multiplier algebra. The algebra $\mathcal{A}$ will always be assumed to be separable. π, with or without sub- (or super-) scripts will usually denote representations of the C^*-algebra under consideration, and $\mathcal{H}_\pi$ will be the Hilbert space on which the representation acts. E and F will denote Hilbert C^*-modules, generally over a C^*-algebra $\mathcal{A}$. $\langle E, E\rangle$ will denote the linear span of $\{\langle x, y\rangle : x, y \in E\}$ in $\mathcal{A}$. For a Hilbert C^*-module E, $\mathcal{K}(E)$ and $\mathcal{L}(E)$ will denote respectively the space of all "compact" operators on E and the space of all adjointable operators on E. Operators on Hilbert modules or on Hilbert spaces will be denoted by $S, T, \mathbf{s}, \mathbf{t}$ etc. For an operator T, $\mathrm{G}(T)$ will denote its graph and D_T or $\mathrm{D}(T)$ its domain. For a topological space X, $C_0(X)$ (respectively $C_\mathrm{c}(X)$) will denote the algebra of continuous functions on X vanishing at infinity (resp. with compact support).

Before we end this section, let us state here a Stone-Weierstrass type theorem for C^*-algebras that will be very useful in studying regular operators on Hilbert C^*-modules.

THEOREM 1.1. *Let $\mathcal{A}$ be a separable C^*-algebra, $\widehat{\mathcal{A}}$ being its spectrum. Let J be a right ideal in $\mathcal{A}$ such that $\pi(J)$ is dense in $\pi(\mathcal{A})$ for all $\pi \in \widehat{\mathcal{A}}$. Then J is dense in $\mathcal{A}$.*

Normally, in Stone-Weierstrass type results, the subspace J is assumed to be a $*$-subalgebra, and the proof goes roughly like this: if J is not dense in $\mathcal{A}$, one constructs a nonzero state on $\mathcal{A}$ that vanishes on J. The corresponding GNS representation must also vanish on J. Using separability, one can now get a point $\pi \in \widehat{\mathcal{A}}$ that vanishes on J. Since this is not the case, one reaches a contradiction. Thus the key step in the proof is the construction of the state, which is made possible by the condition that J is closed under involution. In our case, J is not necessarily $*$-closed; but as Lemma 2.9.4 in [2] tells us, the condition that it is a right ideal is strong enough to guarantee that such a construction is still possible.

2. SEMIREGULAR OPERATORS

Let us start with the following definition.

DEFINITION 2.1. Let E and F be Hilbert $\mathcal{A}$-modules. An operator $T : E \to F$ is called *semiregular* if

(i) D_T is a dense right submodule in E (i.e. $\mathrm{D}_T\mathcal{A} \subseteq \mathrm{D}_T$);
(ii) T is closable;
(iii) T^* is densely defined.

Observe that from (iii), it follows that T is $\mathcal{A}$-linear. Any regular operator is of course semiregular. But there are also many semiregular operators that are not regular, as the following example illustrates.

Let $\mathcal{A} = C[0,1]$, and $E = C[0,1] \otimes L_2(0,1)$. Let

$$\begin{aligned} D &= \{f \in L_2(0,1) : f \text{ absolutely continuous, } f' \in L_2(0,1)\},\\ D_0 &= \{f \in L_2(0,1) : f \text{ absolutely continuous, } f' \in L_2(0,1), f(0) = f(1)\},\\ D_{00} &= \{f \in L_2(0,1) : f \text{ absolutely continuous, } f' \in L_2(0,1), f(0) = f(1) = 0\}. \end{aligned}$$

Define T on D by $Tf = \mathrm{i}f'$. Let $T_0 = T|D_0$. Now define an operator $\mathbf{t}$ on E as follows:

$$\mathrm{D}(\mathbf{t}) = \{f \in E : f_0 \in D_{00},\ f_\pi \in D_0 \text{ for } 0 < \pi \leqslant 1,\ \pi \mapsto f'_\pi \text{ continuous}\},$$

$$(\mathbf{t}f)(\pi) = \mathrm{i}f'_\pi.$$

PROPOSITION 2.2. *The operator* $\mathbf{t}$ *defined above is a closed semiregular non-regular operator.*

Proof. Let us first of all show that

$$\mathrm{D}(\mathbf{t})_\pi = \begin{cases} D_{00} & \text{if } \pi = 0,\\ D_0 & \text{if } 0 < \pi \leqslant 1. \end{cases}$$

From the definition of $\mathrm{D}(\mathbf{t})$, it is clear that $\mathrm{D}(\mathbf{t})_0 \subseteq D_{00}$ and $\mathrm{D}(\mathbf{t})_\pi \subseteq D_0$ for $0 < \pi \leqslant 1$. To show the reverse inclusions, take $f \in D_{00}$. Define $g(\pi, x) = f(x)$. Check that $g \in \mathrm{D}(\mathbf{t})$. Therefore $\mathrm{D}(\mathbf{t})_0 = D_{00}$. Next, fix some $\pi_0 \in (0,1]$ and take $f \in D_0$. This time, take $g(\pi, x) = \frac{\pi}{\pi_0} f(x)$. Then $g_{\pi_0} = f$ and $g \in \mathrm{D}(\mathbf{t})$, so that $\mathrm{D}(\mathbf{t})_{\pi_0} = D_0$.

It is easy to check that for $f, g \in \mathrm{D}(\mathbf{t})$, $\langle \mathbf{t}f, g\rangle = \langle f, \mathbf{t}g\rangle$. Therefore $\mathbf{t} \subseteq \mathbf{t}^*$. From this and from the fact that $C[0,1] \otimes_{\mathrm{alg}} D_{00} \subseteq \mathrm{D}(\mathbf{t})$, it is clear that $\mathbf{t}$ is semiregular. To show that it is closed, take $f_n \in \mathrm{D}(\mathbf{t})$ such that f_n converges to f

and $\mathbf{t}f_n$ converges to $\widehat{f}$. Then $(f_n)_\pi \in \mathrm{D}(\mathbf{t})_\pi$, $f_\pi = \lim(f_n)_\pi$ and $\widehat{f}_\pi = \lim(\mathbf{t}f_n)_\pi$. Since $(f_n)_\pi \in \mathrm{D}(\mathbf{t})_\pi$, $(\mathbf{t}f_n)_\pi = \mathrm{i}(f_n)'_\pi$. Therefore by closedness of $\mathbf{t}_\pi$, $f_\pi \in \mathrm{D}(\mathbf{t})_\pi$, $\mathrm{i}f'_\pi = \widehat{f}_\pi$. $\pi \mapsto \mathrm{i}f'_\pi = \widehat{f}_\pi$ is continuous and hence $f \in \mathrm{D}(\mathbf{t})$, $\widehat{f} = \mathbf{t}f$.

Finally, if $\mathbf{t}$ is regular, then $\{(I+\mathbf{t}^*\mathbf{t})f : f \in \mathrm{D}(\mathbf{t}^*\mathbf{t})\}$ is dense in E, so that for any π, $\{(I+\mathbf{t}^*\mathbf{t})f(\pi) : f \in \mathrm{D}(\mathbf{t}^*\mathbf{t})\}$ is dense in $E_\pi = L_2(0,1)$. Now

$$\{(I+\mathbf{t}^*\mathbf{t})f(0) : f \in \mathrm{D}(\mathbf{t}^*\mathbf{t})\} \subseteq \{f - f'' : f \in D_{00},\ f' \in D_0\}.$$

Notice that the right hand side is equal to $\mathrm{ran}\,(I + T_0T^*)$. So its orthogonal complement is given by $\ker(I + TT_0^*) = \ker(I + TT_0) = \{f : f \in D_0,\ f' \in D,\ f = f''\}$. Hence the function $f : x \mapsto \exp(x) + \exp(1-x) \in \ker(I + TT_0)$. Therefore $\{(I+\mathbf{t}^*\mathbf{t})f(0) : f \in \mathrm{D}(\mathbf{t}^*\mathbf{t})\}$ can not be dense in $L_2(0,1)$, which means, $\mathbf{t}$ can not be regular. ∎

This operator $\mathbf{t}$ shares many features with regular operators. Here are two of them.

PROPOSITION 2.3. *The operator* $\mathbf{t}$ *satisfies the following:*
(i) $\mathbf{t}^*$ *is regular;*
(ii) $\mathrm{D}(\mathbf{t}^*\mathbf{t})$ *is a core for* $\mathbf{t}$.

Proof. (i) We will first show that $\mathrm{D}(\mathbf{t}^*)_\pi = D_0$ for all π. Take any $f \in \mathrm{D}(\mathbf{t}^*)$. Then for any $g \in \mathrm{D}(\mathbf{t})$, $\langle \mathbf{t}g,\ f\rangle = \langle g,\ \mathbf{t}^*f\rangle$. So $\langle \mathbf{t}_\pi g_\pi,\ f_\pi\rangle = \langle g_\pi,\ (\mathbf{t}^*f)_\pi\rangle$. This means $f_\pi \in \mathrm{D}(\mathbf{t}_\pi{}^*)$ and $\mathbf{t}_\pi{}^* f_\pi = (\mathbf{t}^*f)_\pi$. In our context, $f_0 \in D$, $f_\pi \in D_0$ for $0 < \pi \leqslant 1$. Therefore $\mathrm{D}(\mathbf{t}^*)_0 \subseteq D$, $\mathrm{D}(\mathbf{t}^*)_\pi \subseteq D_0$ for $0 < \pi \leqslant 1$. We have already observed that $\mathbf{t} \subseteq \mathbf{t}^*$. Hence $\mathrm{D}(\mathbf{t}^*)_\pi = D_0$ for $0 < \pi \leqslant 1$. Now choose an $f \in \mathrm{D}(\mathbf{t}^*)$. Then for any $\pi \in (0,1]$,

$$\langle \mathbb{1}, \mathbf{t}^*f\rangle(\pi) = \mathrm{i}(f_\pi(1) - f_\pi(0)) = 0,$$
$$\langle \mathbb{1}, \mathbf{t}^*f\rangle(0) = \mathrm{i}(f_0(1) - f_0(0)).$$

By continuity, $f_0(1) = f_0(0)$, i.e. $\mathrm{D}(\mathbf{t}^*)_0 \subseteq D_0$. To show the reverse inclusion, take $f \in D_0$. Define $\widetilde{f}(\pi,x) = f(x)$, $h(\pi,x) = \mathrm{i}f'(x)$. Then for any $g \in \mathrm{D}(\mathbf{t})$, $\langle \mathbf{t}g, \widetilde{f}\rangle = \langle g, h\rangle(\pi)$. Therefore $\widetilde{f} \in \mathrm{D}(\mathbf{t}^*)$. Since $\widetilde{f}_0 = f$, we get $D_0 \subseteq \mathrm{D}(\mathbf{t}^*)_0$.

Let us denote $\mathbf{t}^*$ by $\mathbf{s}$ in this paragraph. From what we have seen, $\mathbf{s}_\pi{}^* = \mathbf{s}_\pi$ for all π. Hence $\mathrm{D}(\mathbf{s}_\pi{}^*\mathbf{s}_\pi) = \{f \in D_0 : f' \in D_0\}$. Clearly $\mathrm{D}(\mathbf{s}^*\mathbf{s})_\pi \subseteq \mathrm{D}(\mathbf{s}_\pi{}^*\mathbf{s}_\pi)$. Take any $f \in \mathrm{D}(\mathbf{s}_\pi{}^*\mathbf{s}_\pi)$. Define $\widetilde{f}(\pi,x) = f(x)$, $h(\pi,x) = -f''(x)$. Then for any $g \in \mathrm{D}(\mathbf{s})$, $\langle \mathbf{s}g, \mathbf{s}\widetilde{f}\rangle = \langle g, h\rangle$. Therefore $\widetilde{f} \in \mathrm{D}(\mathbf{s}^*\mathbf{s})$ and $\mathbf{s}^*\mathbf{s}\widetilde{f} = h$. Since $\widetilde{f}_\pi = f$, we have $\mathrm{D}(\mathbf{s}^*\mathbf{s})_\pi = \mathrm{D}(\mathbf{s}_\pi{}^*\mathbf{s}_\pi)$. This implies $\{(I+\mathbf{s}^*\mathbf{s})g(\pi) : g \in \mathrm{D}(\mathbf{s}^*\mathbf{s})\} = \{(I+\mathbf{s}_\pi{}^*\mathbf{s}_\pi)f : f \in \mathrm{D}(\mathbf{s}_\pi{}^*\mathbf{s}_\pi)\}$. Each $\mathbf{s}_\pi$ is closed, so the right hand side is dense in $L_2(0,1)$. Consequently,$\{(I+\mathbf{s}^*\mathbf{s})g : g \in \mathrm{D}(\mathbf{s}^*\mathbf{s})\}$ is dense in E (essentially by Theorem 1.1), which means $\mathbf{s}$ is regular.

(ii) It is easy to see that

$$\mathrm{D}(\mathbf{t}^*\mathbf{t})_\pi \subseteq \begin{cases} \{f \in D_0 : f' \in D_0\} & \text{if } 0 < \pi \leqslant 1, \\ \{f \in D_{00} : f' \in D_0\} & \text{if } \pi = 0. \end{cases}$$

For any $f \in D_{00}$ such that $f' \in D_0$, the function $g(\pi, x) = f(x)$ can easily be seen to be in $\mathrm{D}(\mathbf{t}^*\mathbf{t})$. Therefore in the second case above, we actually have equality. To show that equality holds in the second case as well, choose an $f \in D_0$ for which $f' \in D_0$ and define $g(\pi, x) = \frac{\pi}{\pi_0} f(x)$. One can then verify that $g \in \mathrm{D}(\mathbf{t}^*\mathbf{t})$. So we now have

$$\mathrm{D}(\mathbf{t}^*\mathbf{t})_\pi = \begin{cases} \{f \in D_0 : f' \in D_0\} & \text{if } 0 < \pi \leqslant 1, \\ \{f \in D_{00} : f' \in D_0\} & \text{if } \pi = 0. \end{cases}$$

We now prove that $\mathrm{D}(\mathbf{t}^*\mathbf{t})_\pi$ is a core for $\mathbf{t}_\pi$ for all π. Notice that for $\pi > 0$, $\mathrm{D}(\mathbf{t}^*\mathbf{t})_\pi = \mathrm{D}({T_0}^2)$, and $\mathbf{t}_\pi = T_0$. So the above assertion holds in this case. Suppose $\pi = 0$. Take an $f \in D_{00} = \mathrm{D}(\mathbf{t}_0) = \mathrm{D}(T|D_{00})$. Choose $g_n \in D_{00}$ such that

$${g_n}'(0) = -f'(0), \quad {g_n}'(1) = -f'(1), \quad \|g_n\| \to 0, \quad \|{g_n}'\| \to 0,$$

then $f + g_n \in D_{00}$, $(f + g_n)' \in D_{00} \subseteq D_0$, and

$$\|f + g_n - f\| = \|g_n\| \to 0, \quad \|(f + g_n)' - f'\| = \|{g_n}'\| \to 0.$$

Thus $\mathrm{D}(\mathbf{t}^*\mathbf{t})_0$ is a core for $\mathbf{t}_0$.

We are now ready to show that $\mathrm{D}(\mathbf{t}^*\mathbf{t})$ is a core for $\mathbf{t}$. Take an $f \in \mathrm{D}(\mathbf{t})$. Let $\varepsilon > 0$ be any given number. Choose a partition $0 = \pi_0 < \pi_1 < \cdots < \pi_n = 1$ of $[0, 1]$ such that

$$\|f_\pi - f_{\pi'}\| < \varepsilon, \quad \|f'_\pi - {f_{\pi'}}'\| < \varepsilon,$$

whenever π and π' belong to the same subinterval. Choose $h_i \in \mathrm{D}(\mathbf{t}^*\mathbf{t})_{\pi_i}$ satisfying

$$\|f_{\pi_i} - h_i\| < \varepsilon, \quad \|{f_{\pi_i}}' - {h_i}'\| < \varepsilon.$$

Define g as follows:

$$g(\pi, x) = \frac{\pi_{i+1} - \pi}{\pi_{i+1} - \pi_i} h_i(x) + \frac{\pi - \pi_i}{\pi_{i+1} - \pi_i} h_{i+1}(x), \quad \pi_i \leqslant \pi \leqslant \pi_{i+1}.$$

Then $g_{\pi_i} = h_i$, $g'_\pi \in D_0$ for all π, $g_\pi \in D_0$ for all $\pi > 0$, $g_0 \in D_{00}$ and the maps $\pi \mapsto g'_\pi$ and $\pi \mapsto g''_\pi$ are continuous. From these, one can now easily check that $g \in \mathrm{D}(\mathbf{t}^*\mathbf{t})$, and

$$\|g_\pi - f_\pi\| < 5\varepsilon, \quad \|g'_\pi - f'_\pi\| < 5\varepsilon.$$

Hence we have $\|g - f\| = \sup_\pi \|g_\pi - f_\pi\| < 5\varepsilon$ and $\|\mathbf{t}g - \mathbf{t}f\| = \sup_\pi \|g'_\pi - f'_\pi\| < 5\varepsilon$. Thus $\mathrm{D}(\mathbf{t}^*\mathbf{t})$ is a core for $\mathbf{t}$. ∎

REMARKS 2.4. (i) The example constructed above is very similar in spirit to an example of a nonregular selfadjoint operator first constructed by Hilsum ([3]).

(ii) Propositions 2.2 and 2.3 together imply that for a closed semiregular operator, regularity of its adjoint does not ensure regularity of the original operator, i.e. Corollary 9.6 in [6] is false.

3. SEMIREGULAR OPERATORS ON C^*-ALGEBRAS

Let $\mathcal{A}$ be a C^*-algebra and E be a Hilbert $\mathcal{A}$-module. Any regular operator on E is uniquely determined by its z-transform (or the bounded transform) which is an element z of $\mathcal{L}(E)$ satisfying the following two conditions:

$$\|z\| \leqslant 1, \quad (1 - z^*z)^{1/2}E \text{ dense in } E. \tag{3.1}$$

The space $\mathcal{K}(E)$ of compact operators on E is a C^*-algebra, and hence is a Hilbert C^*-module over itself. Any regular operator on this C^*-algebra is uniquely determined by its z-transform w in $\mathcal{L}(\mathcal{K}(E))$ that obeys the following:

$$\|w\| \leqslant 1, \quad (1 - w^*w)^{1/2}\mathcal{K}(E) \text{ dense in } \mathcal{K}(E). \tag{3.2}$$

Now, via the isomorphism $\mathcal{L}(E) \cong \mathcal{L}(\mathcal{K}(E))$, the set of elements in $\mathcal{L}(E)$ satisfying (3.1) can be identified with the set of elements in $\mathcal{L}(\mathcal{K}(E))$ satisfying (3.2), which means, regular operators on E can be identified with regular operators on $\mathcal{K}(E)$. One can therefore deal with regular operators on C^*-algebras without any loss of generality. In this section, we will show that the same is true for semiregular operators as well. The main trouble in this case is that we do not have z-transforms at our disposal any more.

Let us denote by $\mathrm{R}(E)$ the space of all regular operators on a Hilbert C^*-module E, and by $\mathrm{SR}(E)$ the space of all semiregular operators on E. Define two maps $\varphi_1 : \mathrm{SR}(E) \to \mathrm{SR}(\mathcal{K}(E))$ and $\varphi_2 : \mathrm{SR}(\mathcal{K}(E)) \to \mathrm{SR}(E)$ as follows: let $S \in \mathrm{SR}(\mathcal{K}(E))$, $T \in \mathrm{SR}(E)$. Let

$$\begin{aligned}
\mathrm{D}(\varphi_1(T)) &:= \mathrm{span}\,\{|x\rangle\langle y| : x \in \mathrm{D}_T,\, y \in E\},\\
\varphi_1(T)|x\rangle\langle y| &:= |Tx\rangle\langle y|,\; |x\rangle\langle y| \in \mathrm{D}(\varphi_1(T));\\
\mathrm{D}(\varphi_2(S)) &:= \mathrm{span}\,\{ax : a \in \mathrm{D}_S,\, x \in E\},\\
\varphi_2(S)(ax) &:= (Sa)x,\; ax \in \mathrm{D}(\varphi_2(S)).
\end{aligned}$$

From the semiregularity of S and T, it follows that $\varphi_1(T)$ and $\varphi_2(S)$ are well defined and semiregular. Let us now list some properties of these two maps φ_1 and φ_2.

LEMMA 3.1. *Let* $S, S_1, S_2 \in \mathrm{SR}(\mathcal{K}(E))$ *and* $T, T_1, T_2 \in \mathrm{SR}(E)$. *Then we have*

$$T_1 \subseteq T_2 \Rightarrow \varphi_1(T_1) \subseteq \varphi_1(T_2),$$
$$S_1 \subseteq S_2 \Rightarrow \varphi_2(S_1) \subseteq \varphi_2(S_2).$$

Proof. Straightforward. ∎

LEMMA 3.2. *Let* $S \in \mathrm{SR}(\mathcal{K}(E))$, $T \in \mathrm{SR}(E)$. *Then*

$$\varphi_1(T) \subseteq \varphi_1(\overline{T}) \subseteq \overline{\varphi_1(T)},$$
$$\varphi_2(S) \subseteq \varphi_2(\overline{S}) \subseteq \overline{\varphi_2(S)}.$$

Proof. The first inclusion in both cases follows from the previous lemma. For the second inclusion, take $\sum |x_i\rangle\langle y_i| \in \mathrm{D}(\varphi_1(\overline{T}))$, where the x_i's come from $\mathrm{D}_{\overline{T}}$. There exist $x_i^{(n)} \in \mathrm{D}_T$ such that $x_i = \lim_n x_i^{(n)}$ and $\overline{T}x_i = \lim_n T x_i^{(n)}$. Therefore $\sum |x_i\rangle\langle y_i| = \lim_n \sum |x_i^{(n)}\rangle\langle y_i|$, and $\varphi_1(\overline{T})(\sum |x_i\rangle\langle y_i|) = \sum |\overline{T}x_i\rangle\langle y_i| = \lim_n \varphi_1(T)(\sum |x_i^{(n)}\rangle\langle y_i|)$. So $\big(\sum |x_i\rangle\langle y_i|, \varphi_1(\overline{T})(\sum |x_i\rangle\langle y_i|)\big) \in \mathrm{G}(\overline{\varphi_1(T)})$. Thus $\mathrm{G}(\varphi_1(\overline{T})) \subseteq \mathrm{G}(\overline{\varphi_1(T)})$, i.e. $\varphi_1(\overline{T}) \subseteq \overline{\varphi_1(T)}$.

Next, take $(ax, \varphi_2(\overline{S})(ax)) \in \mathrm{G}(\varphi_2(\overline{S}))$. Then $a \in \mathrm{D}_{\overline{S}}$, so that there exist $a_n \in \mathrm{D}_S$ such that $a = \lim_n a_n$ and $\overline{S}a = \lim_n Sa_n$. Therefore $ax = \lim_n a_n x$, and $(\overline{S}a)x = \lim_n (Sa_n)x$. Since $a_n x \in \mathrm{D}_{\varphi_2(S)}$ and $\varphi_2(S)(a_n x) = (Sa_n)x$, it follows that $(ax, (\overline{S}a)x) \in \mathrm{G}(\overline{\varphi_2(S)})$. Therefore $\varphi_2(\overline{S}) \subseteq \overline{\varphi_2(S)}$. ∎

LEMMA 3.3. *Let* $S \in \mathrm{SR}(\mathcal{K}(E))$, $T \in \mathrm{SR}(E)$. *Then:*

(i) $\varphi_1(\varphi_2(S)) \subseteq S$, $\varphi_1(\overline{\varphi_2(S)}) \subseteq \overline{S}$;
(ii) $\varphi_2(\varphi_1(T)) \subseteq T$, $\varphi_2(\overline{\varphi_1(T)}) \subseteq \overline{T}$.

Proof. The first inclusion in both cases is trivial. The second inclusion follows from the first and the foregoing lemma. ∎

LEMMA 3.4. *Let* $S \in \mathrm{SR}(\mathcal{K}(E))$, $T \in \mathrm{SR}(E)$. *Let E be separable. Then:*

(i) $\mathrm{D}(\varphi_1(\varphi_2(S)))$ *is a core for* $\overline{S}$;
(ii) $\mathrm{D}(\varphi_2(\varphi_1(T)))$ *is a core for* $\overline{T}$.

Proof. By definition,

$$\begin{aligned}\mathrm{D}(\varphi_1(\varphi_2(S))) &= \operatorname{span}\{|y\rangle\langle z| : y \in \mathrm{D}(\varphi_2(S)), z \in E\}\\ &= \operatorname{span}\{a|x\rangle\langle z| : a \in \mathrm{D}_S,\ x, z \in E\}.\end{aligned}$$

Take $(a, \overline{S}a) \in \mathrm{G}(\overline{S})$. There exist $a_n \in \mathrm{D}_S$ such that $a = \lim_n a_n$, $\overline{S}a = \lim_n Sa_n$. Since E is separable, so is $\mathcal{K}(E)$. Hence by Proposition 1.7.2 in [2], there is an approximate identity $\{p_n\} \in \operatorname{span}\{|x\rangle\langle z| : x, z \in E\}$ such that $\|bp_n - b\|$ converges to zero for all $b \in \mathcal{K}(E)$, and $\|p_n\| \leqslant 1$. Hence $\|a_n p_n - a\| \leqslant \|a_n - a\| \, \|p_n\| + \|ap_n - a\|$, which implies $a = \lim\limits_n a_n p_n$. Therefore

$$\begin{aligned}\|S(a_n p_n) - \overline{S}a\| &\leqslant \|(Sa_n)p_n - (\overline{S}a)p_n\| + \|(\overline{S}a)p_n - \overline{S}a\| \\ &\leqslant \|Sa_n - \overline{S}a\| + \|(\overline{S}a)p_n - \overline{S}a\|,\end{aligned}$$

and consequently, $\overline{S}a = \lim\limits_n S(a_n p_n)$. Since $a_n p_n \in \mathrm{D}(\varphi_1(\varphi_2(S)))$, $\mathrm{D}(\varphi_1(\varphi_2(S)))$ is a core for $\overline{S}$.

Next, take $(x, \overline{T}x) \in \mathrm{G}(\overline{T})$. There exist $\{x_n\} \in \mathrm{D}_T$ such that $x = \lim\limits_n x_n$, and $\overline{T} = \lim\limits_n T x_n$. Now $\operatorname{span}\{\langle y, z\rangle : y, z \in E\}$ is a dense two-sided ideal in $\overline{\langle E, E\rangle}$. Again, by Proposition 1.7.2 in [2], it admits an approximate identity $\{\xi_n\}$ of $\overline{\langle E, E\rangle}$ with $\|\xi_n\| \leqslant 1$. Check that $x = \lim\limits_n x_n \xi_n$. Since

$$\begin{aligned}\mathrm{D}(\varphi_2(\varphi_1(T))) &= \operatorname{span}\{az : a \in \mathrm{D}(\varphi_1(T)), z \in E\} \\ &= \operatorname{span}\{|x\rangle\langle y|z : x \in \mathrm{D}_T,\ y, z \in E\} \\ &= \operatorname{span}\{x\langle y, z\rangle : x \in \mathrm{D}_T,\ y, z \in E\},\end{aligned}$$

$x_n \xi_n$ is in $\mathrm{D}(\varphi_2(\varphi_1(T)))$ for all n. Therefore $\overline{T}x = \lim\limits_n T x_n = \lim\limits_n (Tx_n)\xi_n = \lim\limits_n T(x_n \xi_n)$. Thus $\mathrm{D}(\varphi_2(\varphi_1(T)))$ is a core for $\overline{T}$. ∎

Let us call two semiregular operators T_1 and T_2 equivalent if their closures are equal. In such a case we will write $T_1 \sim T_2$. Clearly this is an equivalence relation.

LEMMA 3.5. *Let $S, S_1, S_2 \in \mathrm{SR}(\mathcal{K}(E))$ and $T, T_1, T_2 \in \mathrm{SR}(E)$. Then one has*

(i) $\varphi_1\varphi_2(S) \sim S$, $\varphi_2\varphi_1(T) \sim T$;
(ii) $T_1 \sim T_2 \Leftrightarrow \varphi_1(T_1) \sim \varphi_1(T_2)$;
(iii) $S_1 \sim S_2 \Leftrightarrow \varphi_2(S_1) \sim \varphi_2(S_2)$.

Proof. (i) is a consequence of Lemma 3.3 and Lemma 3.4.

For (ii), assume $T_1 \sim T_2$, i.e. $\overline{T}_1 = \overline{T}_2$. Then $\varphi_1(\overline{T}_1) = \varphi_1(\overline{T}_2)$. By Lemma 3.2, $\varphi_1(T_1) \subseteq \varphi_1(\overline{T_1}) \subseteq \overline{\varphi_1(T_1)}$. Therefore $\overline{\varphi_1(T_1)} = \overline{\varphi_1(\overline{T}_1)}$. Similarly $\overline{\varphi_1(T_2)} = \overline{\varphi_1(\overline{T}_2)}$. Hence $\overline{\varphi_1(T_1)} = \overline{\varphi_1(T_2)}$. Conversely, if $\overline{\varphi_1(T_1)} = \overline{\varphi_1(T_2)}$, one has $\varphi_2(\overline{\varphi_1(T_1)}) = \varphi_2(\overline{\varphi_1(T_2)})$. By Lemma 3.2

$$\varphi_2\varphi_1(T_i) \subseteq \varphi_2(\overline{\varphi_1(T_i)}) \subseteq \overline{\varphi_2\varphi_1(T_i)}, \quad i = 1, 2.$$

Hence by (i), $\overline{T}_i = \overline{\varphi_2(\overline{\varphi_1(T_i)})}$, $i = 1, 2$, which now implies $\overline{T}_1 = \overline{T}_2$.

Proof of (iii) is exactly similar. ∎

If we denote by $\mathrm{sr}(E)$ (respectively $\mathrm{sr}(\mathcal{K}(E))$) the space of all semiregular operators on E (resp. $\mathcal{K}(E)$) modulo the above equivalence relation, then the above lemma tells us that the maps $\varphi_1 : \mathrm{sr}(E) \to \mathrm{sr}(\mathcal{K}(E))$ and $\varphi_2 : \mathrm{sr}(\mathcal{K}(E)) \to \mathrm{sr}(E)$ are one-one, onto and are inverses of each other. Therefore we can identify a semiregular operator T on E with its image $\varphi_1(T)$ on $\mathcal{K}(E)$. Also, from the definitions of φ_1 and φ_2 it follows easily that if T (respectively S) is regular on E (resp. $\mathcal{K}(E)$) with z-transform z_T (resp. z_S), then $\overline{\varphi_1(T)}$ (resp. $\overline{\varphi_2(S)}$) is regular on $\mathcal{K}(E)$ (resp. E) with the same z-transform. Thus the identification of semiregular operators that we are making is compatible with the identification of regular operators on the two spaces that we have already made in the beginning of this section using their z-transforms.

4. ABELIAN C^*-ALGEBRAS

In this section, we will prove that on C^*-algebras of the form $C_0(X)$, where X is a locally compact Hausdorff space, any semiregular operator is given by multiplication by a continuous function, thereby implying that it is regular.

PROPOSITION 4.1. *Let X be a locally compact Hausdorff space, and let T be a closed semiregular operator on $C_0(X)$. Then T is regular.*

Proof. The proof is quite elementary. The key observation in the proof is the fact that the Pedersen ideal of $C_0(X)$ is $C_{\mathrm{c}}(X)$, that is, any dense ideal in $C_0(X)$ contains $C_{\mathrm{c}}(X)$ (see 5.6.3, p. 176, [8]). Let Λ be the set $\{K \subseteq X : K \text{ compact}\}$ ordered by inclusion. For each $K \in \Lambda$, choose a $f_K \in C_{\mathrm{c}}(X)$ such that $0 \leqslant f_K(x) \leqslant 1$ for all x, and $f_K(x) = 1$ for all x in K. We have already observed that $C_{\mathrm{c}}(X) \subseteq \mathrm{D}_T$, so that each f_K is in the domain of T. Our claim now is that the net $\{Tf_K\}_{K\in\Lambda}$ converges pointwise to a continuous function f on X. First, let us show that for any $x \in X$, the net $\{Tf_K(x)\}_{K\in\Lambda}$ converges. Let $K_0 = \operatorname{supp} f_{\{x\}}$. Then for any $K \supseteq K_0$, $f_K f_{\{x\}} = f_{\{x\}}$. Therefore $Tf_{\{x\}} = (Tf_K)f_{\{x\}}$. Evaluating at the point x, we get $(Tf_K)(x) = (Tf_{\{x\}})(x)$ whenever $K_0 \subseteq K$. So $\{Tf_K(x)\}_{K\in\Lambda}$ converges. Define $f(x) := \lim_K Tf_K(x)$. Take any compact subset K of X. Let S_K be the support of f_K. Then for any $K_0 \supseteq S_K$, $Tf_{K_0}(x) = Tf_K(x)$ for all $x \in K$. Hence $f(x) = Tf_K(x)$ for all $x \in K$. Since $Tf_K \in C_0(X)$, f is continuous on K. This being true for any compact subset K of X, f is continuous on X.

Next observe that $Tg = fg$ for all $g \in C_{\mathrm{c}}(X)$. Indeed, if $K = \operatorname{supp} g$, then $g = f_K g$. Therefore $Tg = (Tf_K)g$. Since $Tf_K = f$ on K, and $g = 0$ outside K, we have $Tg = fg$. If we denote by T_f the operator $g \mapsto fg$ on $C_0(X)$ (with maximal domain), then T_f is regular, $T_f|C_{\mathrm{c}}(X) = T|C_{\mathrm{c}}(X)$. Let $D = (1 + |f|^2)^{1/2} C_{\mathrm{c}}(X)$.

It is easy to see that D is dense in $C_0(X)$. Therefore $C_{\mathrm{c}}(X) = (1+|f|^2)^{-1/2}D$ is a core for T_f. So $T_f = \overline{T_f|C_{\mathrm{c}}(X)} = \overline{T|C_{\mathrm{c}}(X)}$. Since T is closed, this implies

$$T_f \subseteq T, \tag{4.1}$$

and hence $T^* \subseteq {T_f}^*$. Since D_{T^*} is a dense ideal in $C_0(X)$, we get $C_{\mathrm{c}}(X) \subseteq \mathrm{D}_{T^*}$. So $T^*|C_{\mathrm{c}}(X) = {T_f}^*|C_{\mathrm{c}}(X)$. Now, ${T_f}^*$ is just multiplication by $\overline{f}$ and $C_{\mathrm{c}}(X)$ is a core for ${T_f}^*$. Therefore ${T_f}^* = \overline{{T_f}^*|C_{\mathrm{c}}(X)} = \overline{T^*|C_{\mathrm{c}}(X)} \subseteq T^*$. Since T_f is regular, this implies $T^{**} \subseteq {T_f}^{**} = T_f$, and consequently, $T \subseteq T_f$. This, along with (4.1), implies $T = T_f$. Thus T is regular. ∎

REMARK 4.2. Let $\mathcal{A}$ be a unital C^*-algebra. Observe that $C_0(X) \otimes \mathcal{A}$ can be identified with the space $C_0(X,\mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X that vanish at infinity, with its usual norm. Notice also that Tietze's extension theorem continues to hold for $\mathcal{A}$-valued continuous functions. Using this, it is not too difficult to show that if D is a dense right ideal in $C_0(X,\mathcal{A})$, then it must contain $C_{\mathrm{c}}(X,\mathcal{A})$, the space of all compactly supported $\mathcal{A}$-valued functions on X. Having proved this, notice now that the proof of Proposition 4.1 remains valid if one replaces $C_0(X)$ by $C_0(X,\mathcal{A})$ and $C_{\mathrm{c}}(X)$ by $C_{\mathrm{c}}(X,\mathcal{A})$. Thus Proposition 4.1 continues to hold for semiregular operators on C^*-algebras of the form $C_0(X) \otimes \mathcal{A}$ as well.

5. SUBALGEBRAS OF $\mathcal{B}_0(\mathcal{H})$

We will deal with non abelian C^*-algebras in this section. The simplest case of course is the algebra $\mathcal{B}_0(\mathcal{H})$ of compact operators on a Hilbert space. We have seen in Section 3 that semiregulars on $\mathcal{B}_0(\mathcal{H})$ are, up to taking closures, same as semiregular operators on $\mathcal{H}$. Therefore it is natural to expect that in this case, all semiregular operators are regular. The following proposition says that this is indeed the case.

PROPOSITION 5.1. *Let $\mathcal{H}$ be a complex separable Hilbert space. Then any closed semiregular operator on $\mathcal{B}_0(\mathcal{H})$ is regular.*

Proof. This is a straightforward consequence of the results in Section 3. Let T be a closed semiregular operator on $\mathcal{B}_0(\mathcal{H})$. Then $\varphi_2(T)$ is a semiregular operator on $\mathcal{H}$, which simply means that it is closable and densely defined. Therefore its closure $\overline{\varphi_2(T)}$ is a regular operator on $\mathcal{H}$. From the remarks following Lemma 3.5, $\varphi_1(\overline{\varphi_2(T)})$ is a regular operator on $\mathcal{B}_0(\mathcal{H})$. By Lemmas 3.3 and 3.5, we now obtain $\varphi_1\varphi_2(T) \subseteq \varphi_1\overline{\varphi_2(T)} \subseteq \overline{\varphi_1\varphi_2(T)} = \overline{T} = T$. Hence $T = \varphi_1(\overline{\varphi_2(T)})$. But since $\overline{\varphi_2(T)}$ is regular, so is $\varphi_1(\overline{\varphi_2(T)})$. Thus T is regular. ∎

Let us now consider the next simplest class of C^*-algebras, namely, the sub C^*-algebras of $\mathcal{B}_0(\mathcal{H})$.

Let $\mathcal{A}$ be a C^*-algebra. Denote by $\widehat{\mathcal{A}}$ its spectrum, i.e. the space of all irreducible representations of $\mathcal{A}$, equipped with its usual hull-kernel topology. Let T be a semiregular operator on $\mathcal{A}$, with domain $\mathrm{D}(T)$. Since $\mathrm{D}(T)$ is a dense right ideal in $\mathcal{A}$, for any $\pi \in \widehat{\mathcal{A}}$, $\mathrm{D}(T)_\pi := \{\pi(a) : a \in \mathrm{D}(T)\}$ is a dense right ideal in the C^*-algebra $\pi(\mathcal{A})$. Define an operator T_π on $\mathrm{D}(T)_\pi$ by the prescription

$$T_\pi \pi(a) := \pi(Ta), \quad a \in \mathrm{D}(T). \tag{5.1}$$

To see that this is well defined, notice that if $\pi(a) = \pi(b)$, then for any $c \in \mathrm{D}(T^*)$,

$$\langle \pi(Ta) - \pi(Tb), \pi(c)\rangle = \langle \pi(a) - \pi(b), \pi(T^*c)\rangle = 0.$$

Since $\mathrm{D}(T^*)$ is dense in $\mathcal{A}$, $\pi(Ta) = \pi(Tb)$. The equality

$$\langle \pi(Ta), \pi(c)\rangle = \langle \pi(a), \pi(T^*c)\rangle, \quad \forall\, a \in \mathrm{D}(T), c \in \mathrm{D}(T^*)$$

shows that T_π is closable and $(T^*)_\pi \subseteq (T_\pi)^*$, therby implying that ${T_\pi}^*$ is densely defined. Thus T_π is a semiregular operator on $\pi(\mathcal{A})$.

LEMMA 5.2. *If T is regular, then each $\overline{T_\pi}$ is regular.*

Proof. All we need to show is that $\operatorname{ran}(I + {T_\pi}^* T_\pi) = \pi(\mathcal{A})$. Take any $b \in \mathcal{A}$. By regularity of T, there is an $a \in \mathrm{D}(T^*T) \subseteq \mathrm{D}(T)$ such that $(I + T^*T)a = b$. But then

$$\begin{aligned}(I + {T_\pi}^* T_\pi)\pi(a) &= \pi(a) + {T_\pi}^* T_\pi \pi(a) = \pi(a) + {T_\pi}^* \pi(Ta)\\ &= \pi(a) + (T^*)_\pi \pi(Ta) = \pi(a) + \pi(T^*Ta)\\ &= \pi((I + T^*T)a) = \pi(b).\end{aligned}$$

Thus $\operatorname{ran}(I + {T_\pi}^* T_\pi) = \pi(\mathcal{A})$. ∎

LEMMA 5.3. *Let S and T be semiregular operators on $\mathcal{A}$. Then for each $\pi \in \widehat{\mathcal{A}}$,*

(i) *$S \subseteq T \Rightarrow S_\pi \subseteq T_\pi$;*
(ii) *$\mathrm{D}(S^*S)_\pi \subseteq \mathrm{D}({S^*}_\pi S_\pi) \subseteq \mathrm{D}({S_\pi}^* S_\pi)$;*
(iii) *if $\mathrm{D}(S^*S)$ is a core for S, then $\mathrm{D}(S^*S)_\pi$ is a core for S_π.*

Proof. Proof of the first two parts are trivial. For (iii), take $(\pi(a), \overline{S_\pi}\pi(a)) \in \mathrm{G}(\overline{S_\pi})$. Choose $a_0 \in \mathrm{D}(S)$ such that

$$\|(\pi(a_0), S_\pi \pi(a_0)) - (\pi(a), \overline{S_\pi}\pi(a))\| < \varepsilon.$$

Since $\mathrm{D}(S^*S)$ is a core for S, there is an $a_1 \in \mathrm{D}(S^*S)$ such that $\|(a_1, Sa_1) - (a_0, Sa_0)\| < \varepsilon$. But then $\|(\pi(a_1), \pi(Sa_1)) - (\pi(a_0), \pi(Sa_0))\| < \varepsilon$, so that

$$\|(\pi(a_1), S_\pi \pi(a_1)) - (\pi(a), \overline{S_\pi}\pi(a))\| < 2\varepsilon.$$

Thus $\mathrm{D}(S^*S)_\pi$ is a core for S_π. ∎

PROPOSITION 5.4. *Let S be a closed semiregular operator on $\mathcal{A}$. If each $\overline{S_\pi}$ is regular, and $\mathrm{D}(S^*S)_\pi$ is a core for ${S_\pi}^*S_\pi$ for all $\pi \in \widehat{\mathcal{A}}$, then S is regular.*

Proof. The given condition implies that any element of the form $(I + {S_\pi}^*S_\pi)\pi(a)$ can be approximated by an element of the form $(I + {S_\pi}^*S_\pi)\pi(b)$ where $b \in \mathrm{D}(S^*S)$, which, in turn, implies that $\pi\{(I + S^*S)b : b \in \mathrm{D}(S^*S)\}$ is dense in $\pi(\mathcal{A})$ for all $\pi \in \widehat{\mathcal{A}}$. By Theorem 1.1, $\operatorname{ran}(I + S^*S)$ is dense in $\mathcal{A}$. ∎

LEMMA 5.5. *Assume that $\mathcal{A}$ is separable or* GCR*, and $\widehat{\mathcal{A}}$ has discrete topology. Then for any $a \in \mathcal{A}$ and $\pi \in \widehat{\mathcal{A}}$, there is a unique element $a_{(\pi)} \in \mathcal{A}$ such that*

$$\pi'(a_{(\pi)}) = \begin{cases} 0 & \text{if } \pi' \neq \pi, \\ \pi(a) & \text{if } \pi' = \pi. \end{cases}$$

Proof. Let $J = \bigcap_{\pi' \neq \pi} \ker \pi'$. If $J \subseteq \ker \pi$, then π will belong to the closure of $\{\pi\}^{\mathrm{c}}$. But $\{\pi\}$ is open. Hence $J \nsubseteq \ker \pi$. Therefore $\pi|J$ is nonzero, and since J is an ideal, $\pi|J$ is actually irreducible. From the assumptions, it follows that $\mathcal{A}$ is liminal; therefore we get $\pi(J) = \mathcal{B}_0(\mathcal{H}_\pi) = \pi(\mathcal{A})$. So there is an element $a_{(\pi)} \in J$ such that $\pi(a_{(\pi)}) = \pi(a)$. Obviously $\pi'(a_{(\pi)}) = 0$ for all other π'. Uniqueness is now obvious. ∎

LEMMA 5.6. *Under the same assumptions as in the previous lemma, we have:*

(i) $(ab)_{(\pi)} = a \cdot b_{(\pi)}$;

(ii) $\pi(a) = \lim_n \pi(a^{(n)}) \Rightarrow a_{(\pi)} = \lim_n a^{(n)}_{(\pi)}$;

(iii) *if S is a closed semiregular operator on $\mathcal{A}$, then $a \in \mathrm{D}(S)$ implies $a_{(\pi)} \in \mathrm{D}(S)$ and $Sa_{(\pi)} = (Sa)_{(\pi)}$ for all $\pi \in \widehat{\mathcal{A}}$.*

Proof. (i) and (ii) are obvious. For (iii), choose an approximate identity $\{e^{(n)}\}$ in $\mathcal{A}$. It is easy to verify that $ae^{(n)}_{(\pi)}$ converges to $a_{(\pi)}$ and $S\big(ae^{(n)}_{(\pi)}\big)$ converges. The result follows by closedness of S. ∎

LEMMA 5.7. *Let $\mathcal{A}$ be as above, and let S be a closed semiregular operator on $\mathcal{A}$. Then:*

(i) S_π *is closed;*

(ii) ${S_\pi}^* = (S^*)_\pi$, *and*

(iii) $(S^*S)_\pi = {S_\pi}^*S_\pi$.

Proof. (i) Take $\pi(a) \in \mathrm{D}(\overline{S_\pi})$. There exist $a^{(n)} \in \mathrm{D}(S)$ such that $\pi(a^{(n)})$ converges to $\pi(a)$ and $S_\pi\pi(a^{(n)})$ converges. By the previous lemma, $a^{(n)}_{(\pi)}$ converges to $a_{(\pi)}$, and $(Sa^{(n)})_{(\pi)}$ converges. By closedness of S, we conclude that $a_{(\pi)} \in \mathrm{D}(S)$ and $Sa_{(\pi)} = \lim(Sa^{(n)})_{(\pi)}$. Therefore $\pi(a) = \pi(a_{(\pi)}) \in \mathrm{D}(S)_\pi = \mathrm{D}(S_\pi)$.

(ii) We only have to show the inclusion ${S_\pi}^* \subseteq (S^*)_\pi$. Take $\pi(a) \in \mathrm{D}({S_\pi}^*)$. For any $b \in \mathrm{D}(S)$ and $\pi' \in \widehat{\mathcal{A}}$, $\pi' \neq \pi$, we have $\pi'(\langle Sb, a_{(\pi)}\rangle) = \pi'(Sb)^*\pi'(a_{(\pi)}) = 0$, and

$$\pi(\langle Sb, a_{(\pi)}\rangle) = \pi(Sb)^*\pi(a_{(\pi)}) = \langle S_\pi\pi(b), \pi(a)\rangle = \langle \pi(b), {S_\pi}^*\pi(a)\rangle.$$

Since ${S_\pi}^*\pi(a) \in \pi(\mathcal{A})$, ${S_\pi}^*\pi(a) = \pi(c)$ for some $c \in \mathcal{A}$. Hence $\pi(\langle Sb, a_{(\pi)}\rangle) = \langle \pi(b), \pi(c)\rangle = \pi(\langle b, c\rangle)$. Therefore $\pi'(\langle Sb, a_{(\pi)}\rangle) = \pi'(\langle b, c_{(\pi)}\rangle)$ for all π', i.e. $\langle Sb, a_{(\pi)}\rangle = \langle b, c_{(\pi)}\rangle$. This implies $a_{(\pi)} \in \mathrm{D}(S^*)$ and consequently $\pi(a) = \pi(a_{(\pi)}) \in \mathrm{D}(S^*)_\pi$.

(iii) Again, the inclusion $\mathrm{D}(S^*S)_\pi \subseteq \mathrm{D}({S_\pi}^*S_\pi)$ is obvious. To show the reverse inclusion, take $\pi(a) \in \mathrm{D}({S_\pi}^*S_\pi)$. Then $\pi(a) \in \mathrm{D}(S_\pi) = \mathrm{D}(S)_\pi$, and $S_\pi\pi(a) \in \mathrm{D}({S_\pi}^*) = \mathrm{D}(S^*)_\pi$. This means $a_{(\pi)} \in \mathrm{D}(S)$, and $(Sa)_{(\pi)} = Sa_{(\pi)} \in \mathrm{D}(S^*)$, i.e. $a_{(\pi)} \in \mathrm{D}(S^*S)$. Hence $\pi(a) = \pi(a_{(\pi)}) \in \mathrm{D}(S^*S)_\pi$. ∎

As a consequence of the above results, we now obtain the following:

THEOREM 5.8. *Let $\mathcal{H}$ be a complex separable Hilbert space. Any closed semiregular operator on a C^*-subalgebra of $\mathcal{B}_0(\mathcal{H})$ is regular.*

Proof. Follows from Proposition 5.4 and Lemma 5.7. ∎

Let G be a compact quantum group ([10]), and let $\widehat{G}$ denote its Pontryagin dual. Then the C^*-algebra $C_0(\widehat{G})$ has discrete spectrum. Therefore in this context, we can rephrase the previous result as follows.

PROPOSITION 5.9. *Let G be a compact quantum group. Any closed semiregular operator on $C_0(\widehat{G})$ is regular.*

6. EXTENSIONS OF SEMIREGULAR OPERATORS

We will be concerned with more general classes of C^*-algebras in this section. As the example in Section 2 suggests, we can not possibly expect results like Theorem 5.8 to hold once we go beyond subalgebras of $\mathcal{B}_0(\mathcal{H})$. However in many cases, it is possible to get regular extensions of semiregular operators. But before we go to extensions of semiregular operators, let us find out how a regular operator is related to its extensions and restrictions.

PROPOSITION 6.1. *Let T be a regular operator on a C^*-algebra $\mathcal{A}$ with z-transform z. Let u be an isometry in $M(\mathcal{A})$ obeying the following condition:*

$$(u^*(I - z^*z)u)^{1/2} = (I - z^*z)^{1/2}u. \tag{6.1}$$

Then $z_S := zu$ is the z-transform of a regular restriction S of T.

Proof. Since z is the z-transform of a regular operator, $(I-z^*z)^{1/2}\mathcal{A}$ is dense in $\mathcal{A}$, and u is an isometry. So $u^*(I - z^*z)^{1/2}\mathcal{A}$ is also dense in $\mathcal{A}$. Now from the given conditions, we get

$$(I - z^*z)^{1/2}u = u^*(I - z^*z)^{1/2}.$$

Therefore $(I - z^*z)^{1/2}u\mathcal{A}$ is dense in $\mathcal{A}$. It follows then that $(u^*(I - z^*z)u)\mathcal{A} = (I-z_S{}^*z_S)\mathcal{A}$ is dense in $\mathcal{A}$. This means $(I-z_S{}^*z_S)^{1/2}\mathcal{A}$ is also dense in $\mathcal{A}$. Clearly $\|z_S\| \leqslant 1$. Hence there is a unique regular operator S whose z-transform is z_S.

To show that S is a restriction of T, it is enough to prove that $S = T$ on $(I - z_S{}^*z_S)\mathcal{A}$, since this is a core for S. We will prove that:

(i) $(I - z_S{}^*z_S)\mathcal{A} \subseteq (I - z^*z)^{1/2}\mathcal{A}$, and
(ii) if $(I - z_S{}^*z_S)a = (I - z^*z)^{1/2}b$, then $z_S(I - z_S{}^*z_S)^{1/2}a = zb$.

(i) is a direct consequence of (6.1). For (ii), assume that $(I - z_S{}^*z_S)a = (I - z^*z)^{1/2}b$. This means $(u^*(I - z^*z)u)a = (I - z^*z)^{1/2}b$, which, together with (6.1) and injectivity of the operator $(I-z^*z)^{-1/2}$, implies that $u(I-z^*z)^{1/2}ua = b$. Therefore $zb = zu(I - z^*z)^{1/2}ua = z_S(I - z^*z)^{1/2}ua = z_S(I - z_S{}^*z_S)^{1/2}a$. ∎

PROPOSITION 6.2. *Let T be a regular operator on $\mathcal{A}$ with z-transform z. A regular operator S with z-transform z_S is a restriction of T if and only if $z_S = zu$ for some isometry u in $M(\mathcal{A})$ obeying Equation* (6.1).

Proof. We have seen that if u is such an isometry, then $z_S := zu$ defines a regular restriction of T. Now conversely, suppose S is a regular restriction of T. Since $\mathrm{D}(S) \subseteq \mathrm{D}(T)$, we have $(I - z_S{}^*z_S)^{1/2}\mathcal{A} \subseteq (I - z^*z)^{1/2}\mathcal{A}$. Therefore the operator $w_{S,T} := (I - z^*z)^{-1/2}(I - z_S{}^*z_S)^{1/2}$ is everywhere defined on $\mathcal{A}$. $(I - z_S{}^*z_S)^{1/2}$ is bounded, $(I - z^*z)^{-1/2}$ is closed; so $w_{S,T}$ is also closed. Hence $w_{S,T}$ is bounded. Observe next that for any $a, b \in \mathcal{A}$,

$$\langle w_{S,T}a, (I - z^*z)^{1/2}b\rangle = \langle (I - z_S{}^*z_S)^{1/2}a, b\rangle = \langle a, (I - z_S{}^*z_S)^{1/2}b\rangle.$$

This implies $\mathrm{D}(T) \subseteq \mathrm{D}(w_{S,T}{}^*)$, i.e. $w_{S,T}{}^*$ is densely defined. Together with the boundedness of $w_{S,T}$, this means it is adjointable.

For any $a \in \mathcal{A}$, $(I - z_S{}^* z_S)^{1/2} a = (I - z^* z)^{1/2} w_{S,T} a \in \mathrm{D}(S) \subseteq \mathrm{D}(T)$. Since $S = T$ on $\mathrm{D}(S)$, we have $z_S a = z w_{S,T} a$ for all $a \in \mathcal{A}$. So $z_S = z w_{S,T}$. Next, take any $a, b \in \mathcal{A}$. Then

$$\begin{aligned}\langle w_{S,T}{}^*(I - z^* z) w_{S,T} a, b\rangle &= \langle (I - z^* z)^{1/2} w_{S,T} a, (I - z^* z)^{1/2} w_{S,T} b\rangle \\ &= \langle (I - z_S{}^* z_S) a, b\rangle.\end{aligned}$$

Therefore $w_{S,T}{}^*(I - z^* z) w_{S,T} = I - z_S{}^* z_S$. It now follows immediately that $w_{S,T}{}^* w_{S,T} = I$, and $\big(w_{S,T}{}^*(I - z^* z) w_{S,T}\big)^{1/2} = (I - z^* z)^{1/2} w_{S,T}$. ∎

PROPOSITION 6.3. *Let T be a regular operator on $\mathcal{A}$ with z-transform z. A regular operator S with z-transform z_S is an extension of T if and only if $z_S = uz$ for some coisometry u satisfying the following equation:*

$$(u(I - zz^*)u^*)^{1/2} = u(I - zz^*)^{1/2}. \tag{6.2}$$

Proof. Use the previous proposition and the fact that S is an extension of T if and only if S^* is a restriction of T^*. ∎

Let S be a closed semiregular operator on a C^*-algebra $\mathcal{A}$. For any $\pi \in \widehat{\mathcal{A}}$, if we define S_π by Equation (5.1), with S replacing T, then $\overline{S_\pi}$ is a closed semiregular operator on $\pi(\mathcal{A})$. Construct an operator $\widetilde{S}$ on $\mathcal{A}$ as follows:

$$\begin{aligned}\mathrm{D}(\widetilde{S}) = \{a \in \mathcal{A} : \pi(a) \in \mathrm{D}(\overline{S_\pi})\, \forall\, \pi \in \widehat{\mathcal{A}}, \exists\, b \in \mathcal{A} \ni \pi(b) = \overline{S_\pi}\pi(a)\, \forall\, \pi \in \widehat{\mathcal{A}}\},\\ \widetilde{S} a = b.\end{aligned} \tag{6.3}$$

LEMMA 6.4. *$\widetilde{S}$ is a closed semiregular extension of S.*

Proof. From the definition of $\mathrm{D}(\widetilde{S})$ and the fact that $\widehat{\mathcal{A}}$ separates points of $\mathcal{A}$, it follows that $\widetilde{S}$ is well defined. To show it is closed, take $a_n \in \mathrm{D}(\widetilde{S})$ such that a_n converges to c and $\widetilde{S} a_n$ converges to d. Then for each $\pi \in \widehat{\mathcal{A}}$, $\pi(a_n)$ converges to $\pi(a)$ and $\pi(\widetilde{S} a_n)$ converges to $\pi(d)$. But $\pi(\widetilde{S} a_n) = \overline{S_\pi}\pi(a_n)$. Therefore by closedness of $\overline{S_\pi}$, $\pi(c) \in \mathrm{D}(\overline{S_\pi})$ and $\overline{S_\pi}\pi(c) = \pi(d)$. This means c is in $\mathrm{D}(\widetilde{S})$ and $\widetilde{S} c = d$, i.e. $\widetilde{S}$ is closed. The inclusion $S \subseteq \widetilde{S}$ is obvious. So $\widetilde{S}$ is densely defined.

Using the same arguments for S^*, we see that $\widetilde{(S^*)}$ is densely defined. It is routine to check that for any $a \in \mathrm{D}(\widetilde{S})$ and $b \in \mathrm{D}(\widetilde{(S^*)})$, one has $\langle \widetilde{S} a, b\rangle = \langle a, \widetilde{(S^*)} b\rangle$, so that $\widetilde{(S^*)} \subseteq (\widetilde{S})^*$. Hence $(\widetilde{S})^*$ is densely defined.

Thus $\widetilde{S}$ is a closed semiregular extension of S. ∎

Next we compute the adjoint of $\widetilde{S}$.

LEMMA 6.5 *Let $\widetilde{S}$ be as above. Then $(\widetilde{S})^* = \widetilde{(S^*)}$.*

Proof. Define an operator S_* by the prescription

$$\mathrm{D}(S_*) = \{a \in \mathcal{A} : \pi(a) \in \mathrm{D}({S_\pi}^*),\, \exists\, b \in \mathcal{A} \ni \pi(b) = {S_\pi}^*\pi(a)\, \forall\, \pi \in \widehat{\mathcal{A}}\},$$

$$S_* a = b.$$

We claim that

$$\widetilde{(S^*)} \subseteq S_* \subseteq (\widetilde{S})^*.$$

The first inclusion follows from the inclusion $(S^*)_\pi \subseteq {S_\pi}^*$. The second inclusion follows from the observation that for any $a \in \mathrm{D}(\widetilde{S})$ and $b \in \mathrm{D}(S_*)$, one has $\pi(\langle \widetilde{S}a, b\rangle) = \pi(\langle a, S_* b\rangle)$ for all $\pi \in \widehat{\mathcal{A}}$. Now, we also have, by the previous lemma, $S \subseteq \widetilde{S}$ and $S^* \subseteq \widetilde{(S^*)}$. Thus we have the chain of inclusions

$$\widetilde{(S^*)} \subseteq S_* \subseteq (\widetilde{S})^* \subseteq S^* \subseteq \widetilde{(S^*)},$$

which proves the lemma. ∎

LEMMA 6.6. *If S is regular, then so is $\widetilde{S}$, and $S = \widetilde{S}$.*

Proof. Let τ denote the operator $a \oplus b \mapsto b \oplus (-a)$ on $\mathcal{A} \oplus \mathcal{A}$. Then for any operator T, $\tau(\mathrm{G}(T^*)) \subseteq \mathrm{G}(T)^\perp$. Hence $\tau\big(\mathrm{G}((\widetilde{S})^*)\big) \subseteq \mathrm{G}(\widetilde{S})^\perp$. Now $\mathrm{G}(S) \subseteq \mathrm{G}(\widetilde{S})$, so that $\mathrm{G}(\widetilde{S})^\perp \subseteq \mathrm{G}(S)^\perp$. Since S is regular, we have, using the previous lemma,

$$\tau(\mathrm{G}((\widetilde{S})^*)) \subseteq \mathrm{G}(\widetilde{S})^\perp \subseteq \mathrm{G}(S)^\perp = \tau(\mathrm{G}(S^*)) \subseteq \tau(\mathrm{G}(\widetilde{S^*})) = \tau(\mathrm{G}((\widetilde{S})^*)).$$

So we actually have equality everywhere. Hence $\mathcal{A} \oplus \mathcal{A} = \mathrm{G}(S) \oplus \mathrm{G}(S)^\perp \subseteq \mathrm{G}(\widetilde{S}) \oplus \mathrm{G}(\widetilde{S})^\perp$. This implies $\mathrm{G}(\widetilde{S}) \oplus \mathrm{G}(\widetilde{S})^\perp = \mathcal{A} \oplus \mathcal{A}$. Consequently $\widetilde{S}$ is regular, and $S = S^{**} = (\widetilde{S})^{**} = \widetilde{S}$. ∎

Now we are ready for the main result in this section. Let S be a semiregular operator on a liminal C^*-algebra $\mathcal{A}$. Since $\pi(\mathcal{A}) = \mathcal{B}_0(\mathcal{H}_\pi)$, by Proposition 5.1, each $\overline{S_\pi}$ is regular. Let z_π be the corresponding z-transform.

THEOREM 6.7. *Let S and z_π be as above. Suppose $\{u_\pi\}_{\pi \in \widehat{\mathcal{A}}}$ is a family of coisometries, where $u_\pi \in \mathcal{B}(\mathcal{H}_\pi)$, such that*

$$\big(u_\pi(I - z_\pi {z_\pi}^*){u_\pi}^*\big)^{1/2} = u_\pi\big(I - z_\pi {z_\pi}^*\big)^{1/2},$$

and there exists an element $z \in M(\mathcal{A})$ for which $\pi(z) = u_\pi z_\pi$ for all $\pi \in \widehat{\mathcal{A}}$. Then S admits a regular extension.

Proof. Let us first of all show that z is indeed the z-transform of a regular operator. Clearly $\|z\| \leqslant 1$. By Proposition 6.3, each $\pi(z)$ is the z-transform of a regular operator on $\pi(\mathcal{A})$. Therefore $(I - \pi(z)^*\pi(z))^{1/2}\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$. Since $\pi((I - z^*z)^{1/2}) = (I - \pi(z)^*\pi(z))^{1/2}$, it follows from Proposition 2.5 in [11] that $(I - z^*z)^{1/2}\mathcal{A}$ is dense in $\mathcal{A}$. Thus z is the z-transform of a regular operator, say, T.

Our next job is to show that T is an extension of S. It is easy to see that T_π, defined by (5.1) on $\pi(\mathcal{A})$, is a regular operator with z-transform $\pi(z)$. Proposition 6.3 tells us that $\overline{S_\pi} \subseteq T_\pi$. From the definition of $\widetilde{S}$ it follows that $\widetilde{S} \subseteq \widetilde{T}$. But by Lemma 6.4, $S \subseteq \widetilde{S}$ and by Lemma 6.6, $T = \widetilde{T}$. Therefore T is an extension of S. ∎

REMARK 6.8. From the proof of the above theorem, it is clear that if S is a semiregular operator on any C^*-algebra $\mathcal{A}$ (not necessarily liminal) such that the closure of each fibre S_π is regular with z-transform z_π, and there is one single element z in $M(\mathcal{A})$ such that $\pi(z) = z_\pi$ for all π, then $\widetilde{S}$ is a regular operator.

It is now easy to see why the example in Section 2 fails to be regular. Each of the fibres $\mathbf{t}_\pi$'s is regular, acting on the same Hilbert space $L_2(0,1)$. But while all the $\mathbf{t}_\pi$'s are equal for $\pi > 0$, $\mathbf{t}_0$ is different. The same is therefore true for their z-transforms z_π's. Hence clearly there can not be any element in $\mathcal{L}(E)$ (which are precisely the $\mathcal{B}(L_2(0,1))$-valued functions on $[0,1]$ that are both strong and strong*-continuous) whose π-image is the z-transform of $\mathbf{t}_\pi$ for all π.

As a consequence of Theorem 6.7 and Remark 6.8, we now have the following proposition.

PROPOSITION 6.9. *Let S be a semiregular operator on a C^*-algebra $\mathcal{A}$. Suppose there exists a $\pi_0 \in \widehat{\mathcal{A}}$, a regular operator $\mathbf{t}$ on $\pi_0(\mathcal{A})$, and a family $\{U_\pi\}_{\pi\in\widehat{\mathcal{A}}}$ of unitary operators*

$$U_\pi : \mathcal{H}_{\pi_0} \mapsto \mathcal{H}_\pi, \quad \pi \in \widehat{\mathcal{A}},$$

satisfying the following conditions:

(i) *$U_{\pi_0} = I$;*

(ii) *$S_\pi \subseteq U_\pi \mathbf{t}\, U_\pi{}^*$ for all π;*

(iii) *for any $a \in \mathcal{B}(\mathcal{H}_{\pi_0})$, there is an element $b \in M(\mathcal{A})$ such that $\pi(b) = U_\pi a\, U_\pi{}^*$ for all $\pi \in \widehat{\mathcal{A}}$.*

Then S admits a regular extension.

Proof. Define operators T_π on $\pi(\mathcal{A})$ as follows:

$$\mathrm{D}(T_\pi) = U_\pi \mathrm{D}(\mathbf{t}) U_\pi{}^*, \quad T_\pi = U_\pi \mathbf{t}\, U_\pi{}^*.$$

It is routine to verify that each T_π is regular,

$$\mathrm{D}(T_\pi{}^*) = U_\pi \mathrm{D}(\mathbf{t}^*) U_\pi{}^*, \quad T_\pi{}^* = U_\pi \mathbf{t}^* U_\pi{}^*,$$

and if w is the z-transform of $\mathbf{t}$, then the z-transform of T_π is $U_\pi w U_\pi{}^*$. Condition (iii) now ensures the existence of an element z in $M(\mathcal{A})$ such that $\pi(z) = U_\pi w U_\pi{}^*$ for all π. By Remark 6.8 above, it follows that $\widetilde{T}$ constructed out of these T_π's by the prescription (6.3) (with T replacing S) is regular. Also, a direct consequence of condition (ii) above and the definition of $\widetilde{T}$ is that $S \subseteq \widetilde{T}$. ∎

As an immediate corollary of the above proposition, we can deduce the following:

Corllary 6.10. *Let $E = C[0,1] \otimes \mathcal{H}$, $\mathbf{t}$ a closed operator on $\mathcal{H}$, and $\{U_\pi\}_\pi$ a strongly continuous family of unitaries on $\mathcal{H}$. Let $T_\pi = U_\pi \mathbf{t} U_\pi{}^*$. Let*

$$\mathrm{D}(T) = \{f \in E : f_\pi \in \mathrm{D}(T_\pi)\ \forall\, \pi,\ \pi \mapsto T_\pi f_\pi \ \textit{continuous}\},$$

$$(Tf)_\pi = T_\pi f_\pi.$$

Then T is a regular operator on E.

Proof. Here $\mathcal{K}(E) = C[0,1] \otimes \mathcal{B}_0(\mathcal{H})$ is the relevant C^*-algebra. All we need to check is that the condition (iii) in the foregoing proposition is fulfilled. For any $u \in \mathcal{H}$, $\pi \mapsto U_\pi u$ is continuous. Hence for any finite-rank operator S on $\mathcal{H}$, the function $\pi \mapsto U_\pi S U_\pi{}^*$ is continuous in the norm topology. By approximating a compact operator by finite-rank operators, one can show that $\pi \mapsto U_\pi S U_\pi{}^*$ is norm continuous for compact S also. Next, take any $S \in \mathcal{B}(\mathcal{H})$. From the strong continuity of $\{U_\pi\}$ and $\{U_\pi{}^*\}$, it follows that for any $\mathcal{B}_0(\mathcal{H})$-valued norm continuous function $\pi \mapsto R_\pi$, the maps $\pi \mapsto U_\pi S U_\pi{}^* R_\pi$ and $\pi \mapsto R_\pi U_\pi S U_\pi{}^*$ are both norm continuous. This implies that the function $\pi \mapsto U_\pi S U_\pi{}^*$ is an element of the multiplier algebra $M(C[0,1] \otimes \mathcal{B}_0(\mathcal{H}))$. If we call it b, then $\pi(b) = U_\pi S U_\pi{}^*$. The third condition in Proposition 6.9 is thus satisfied. So T is regular. ∎

We shall now apply the above result to a specific example.

Example 6.11. Let $E = C[0,1] \otimes L_2(0,1)$, D be as in Section 2 and let T be an operator on E defined as follows:

$$\mathrm{D}(T) = \{f \in E : f_\pi \in D,\ f_\pi(1) = \mathrm{e}^{\mathrm{i}\pi} f_\pi(0),\ \pi \mapsto f'_\pi \ \text{continuous}\},$$

$$(Tf)_\pi = f'_\pi + \mathrm{i}\pi f_\pi.$$

Then T is regular.

Proof. Just take U_π to be multiplication by the function $x \mapsto \mathrm{e}^{\mathrm{i}\pi x}$, $\mathbf{t}$ to be the operator T_0 in Section 2, and apply the previous result. ∎

More generally, Let $g \in E$ obey the following properties:

(i) $g_\pi \equiv g(\pi, \cdot\,)$ is absolutely continuous;

(ii) $g'_\pi \in L_2(0,1)$;

(iii) $g(0,x) = 0$ for all x, and

(iv) $\pi \mapsto g'_\pi$ is continuous.

Then the operator T on E given by the following prescription is regular:

$$\mathrm{D}(T) = \big\{ f \in E : f_\pi \in D,\ f_\pi(1) = \exp\big(\mathrm{i}(g(\pi,1) - g(\pi,0))\big) f_\pi(0),\ \pi \mapsto f'_\pi \text{ continuous} \big\},$$

$$(Tf)_\pi = f'_\pi - \mathrm{i} g'_\pi f_\pi.$$

In this case one has to take U_π to be multiplication by the function $\exp(\mathrm{i}g(\pi, \cdot\,))$.

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