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A SIMPLE C^*-ALGEBRA ARISING FROM A CERTAIN SUBSHIFT

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Abstract. We present an example of a subshift whose associated C^*-algebra is simple, purely infinite and not isomorphic to any Cuntz-Krieger algebra and Cuntz algebra. The subshift is called the context free shift. We will compute the topological entropy for the subshift and show that the KMS-state for the gauge action on the associated C^*-algebra exists if and only if the inverse temperature is $\log(1+\sqrt{1+\sqrt{3}}) = 2.652\ldots =$ the topological entropy for the subshift, and the corresponding KMS-state is unique.

Keywords: *Simple C^*-algebra, subshift, entropy,* KMS-*state.*

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1. INTRODUCTION

In [19], the author has constructed and studied the C^*-algebra $\mathcal{O}_\Lambda$ from a general (two-sided) subshift Λ keeping in mind that the class of the topological Markov shifts is a subclass of the class of the subshifts. In his studies, many structural properties of the Cuntz-Krieger algebras have been generalized to the C^*-algebras $\mathcal{O}_\Lambda$ associated with subshifts (cf. [19], [20], [21], [22]). In particular, the simplicity argument for the Cuntz-Krieger algebras has been generalized to the C^*-algebras associated with subshifts. In [22], the notion of Condition (I), irreducibility and aperiodicity have been extendedly considered for general subshifts. Suppose that the corresponding one-sided subshift X_Λ for Λ satisfies Condition (I). The following theorem has been proved:

THEOREM A. ([19], [22]) *For a subshift* Λ, *the* C^**-algebra* $\mathcal{O}_\Lambda$ *is simple if and only if the corresponding one-sided subshift* X_Λ *is irreducible in past equivalence.*

Although the Cuntz-Krieger's simplicity argument has been generalized to general subshifts, an example of a subshift for which the associated C^*-algebra is simple, purely infinite and not isomorphic to any Cuntz-Krieger algebra and Cuntz algebra has not been found yet. As seen in recent development of classification theory of simple nuclear C^*-algebras ([10], [16], [25], [27], [28], ... etc.), it seems to be interesting to construct and study a new concrete simple C^*-algebra related to some topological dynamical system. If a subshift belongs to the class of the sofic subshifts which strictly contains the class of the topological Markov shifts, its associated C^*-algebra has the same K-theory as a Cuntz-Krieger algebra.

In this paper, we first present an example of a subshift whose associated C^*-algebra is simple, purely infinite and not stably isomorphic to any Cuntz-Krieger algebra. The subshift is called the context free shift, which is denoted by Z. It is motivated by the theory of context free languages related to automata theory (cf. [2]). We first show:

THEOREM 1.1. (Corollary 4.7 and Theorem 4.10) *The* C^**-algebra* $\mathcal{O}_Z$ *associated with the context free shift* Z *is simple and purely infinite. It is the universal concrete* C^**-algebra generated by two isometries* S_1, S_2 *and one partial isometry* S_3 *satisfying the following relations:*

(i) $\sum_{j=1}^{3} S_j S_j^* = 1$;

(ii) $S_3^* S_3 = 1 - \sum_{k=1}^{\infty} \sum_{m=0,\, k\neq 2m}^{k} S_{2^m 1^{k-m} 3} S_{2^m 1^{k-m} 3}^*$

where $S_{2^m 1^{k-m} 3}$ *denotes* $\underbrace{S_2 \cdots S_2}_{\text{m times}} \underbrace{S_1 \cdots S_1}_{\text{k−m times}} S_3$. *The infinite sum of the right hand side of the relation* (ii) *is taken under strong operator topology on a Hilbert space.*

The C^*-algebra $\mathcal{O}_Z$ is unital, simple, purely infinite, nuclear and satisfies the universal coefficient theorem for K-theory. Hence it is classifiable in terms of K-theory by a classification theorem of Kirchberg and Phillips ([16], [25]). We will compute K-theory for $\mathcal{O}_Z$ and determine its isomorphism class in the following way:

THEOREM 1.2. (Theorem 5.7 and Corollary 5.8) *The* K*-theory for the* C^**-algebra* $\mathcal{O}_Z$ *is as follows:*

$$\mathrm{K}_0(\mathcal{O}_Z) = \mathbb{Z} \quad \textit{and} \quad \mathrm{K}_1(\mathcal{O}_Z) = 0.$$

The position of the unit $\mathbb{1}$ *in* $\mathrm{K}_0(\mathcal{O}_Z) = \mathbb{Z}$ *is* 0. *Hence the* C^**-algebra* $\mathcal{O}_Z$ *is not stably isomorphic to any Cuntz-Krieger algebra and not isomorphic to the Cuntz algebra* $\mathcal{O}_\infty$.

In studying a topological dynamical system, the topological entropy is a very important number to measure "complexity" for the topological dynamical system. In [11], it was shown that the topological entropy for irreducible Markov shifts appear as the inverse temperature admitting KMS-state for the gauge action on the corresponding Cuntz-Krieger algebras. This result has been generalized to more general subshifts in [23] (cf. [15]). We define the gauge action α on $\mathcal{O}_Z$ as an action of the group $\mathbb{R}$ of all real numbers by $\alpha_t(S_j) = \mathrm{e}^{\sqrt{-1}t} S_j$, $t \in \mathbb{R}$. Corresponding to these results, we will compute the topological entropy for Z and prove the following:

THEOREM 1.3. (Theorem 6.8) *There is a* KMS*-state for the gauge action on the* C^**-algebra* $\mathcal{O}_Z$ *if and only if the inverse temperature is* $\log(1 + \sqrt{1 + \sqrt{3}}) =$ *the topological entropy for the context free shift* Z*. The admitted* KMS*-state is unique.*

We will finally mention an application of our discussions to the theory of symbolic dynamics. For each real number $\beta > 1$, there is a subshift called the β-shift whose topological entropy is $\log \beta$ (cf. [26], [24]). For $\beta = 1 + \sqrt{1 + \sqrt{3}}$ the β-shift is not sofic and has the same topological entropy as the context free shift. We will however know that the β-shift is not conjugate to the context free shift as a one-sided subshift (Proposition 7.2).

In [17], Kumjian-Pask-Raeburn-Renault have generalized the class of the Cuntz-Krieger algebras from a graph theoretical view point. The presented C^*-algebras from subshifts in our paper are different from their ones.

2. NOTATION AND SUBSHIFTS

In this section, we will treat subshifts in general. We fix a finite set $\Sigma = \{1, 2, \ldots, n\}$ for $n \geqslant 2$. Let $\Sigma^{\mathbb{Z}}$, $\Sigma^{\mathbb{N}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on $\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{N}}$ given by $(\sigma(x))_i = x_{i+1}$, $i \in \mathbb{Z}, \mathbb{N}$ is called the *(full) shift.* Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$, i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|\Lambda)$ is called a *subshift.* We denote $\sigma|\Lambda$ by σ for simplicity. We denote by X_Λ the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the *one-sided subshift* for Λ.

A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a *block* or a *word*. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, \dots, \mu_k)$ is said to occur or appear in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, \dots, x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$. For a subshift (Λ, σ) and a number $k \in \mathbb{N}$, let Λ^k be the set of all words of length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. Put $\Lambda_l = \bigcup_{k=0}^{l} \Lambda^k, \Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$. Set

$$\Lambda_l(x) = \{\mu \in \Lambda_l \mid \mu x \in X_\Lambda\} \quad \text{for } x \in X_\Lambda,\ l \in \mathbb{N}.$$

We define a nested sequence of equivalence relations in the space X_Λ. For $l \in \mathbb{N}$, two points $x, y \in X_\Lambda$ are said to be *l-past equivalent* if $\Lambda_l(x) = \Lambda_l(y)$ ([22]). We write this equivalence as $x \sim_l y$. We denote by $\Omega_l = X_\Lambda / \sim_l$ the l-past equivalence classes of X_Λ.

LEMMA 2.1. *For $x, y \in X_\Lambda$ and $\mu \in \Lambda^k$,*
(i) *if $x \sim_l y$, we have $x \sim_m y$ for $m < l$;*
(ii) *if $x \sim_l y$ and $\mu x \in X_\Lambda$, we have $\mu y \in X_\Lambda$ and $\mu x \sim_{l-k} \mu y$ for $l > k$.*

By the first statement of the above lemma, the identity map on X_Λ induces the following sequence of surjections:

$$\Omega_1 \leftarrow \Omega_2 \leftarrow \cdots \leftarrow \Omega_l \leftarrow \Omega_{l+1} \leftarrow \cdots. \tag{2.1}$$

We easily see that a subshift (Λ, σ) is a Markov subshift if and only if $\Omega_1 = \Omega_l$ for all $l \in \mathbb{N}$, and that (Λ, σ) is a sofic subshift if and only if $\Omega_l = \Omega_{l+1}$ for some $l \in \mathbb{N}$.

In [22], the author introduced the following dynamical properties for the one-sided subshifts. They are: Condition (I), irreducibility and aperiodicity in some sense. If a subshift is a topological Markov shift Λ_A determined by a matrix A with entries in $\{0, 1\}$, their properties coincide with those of the matrix (Condition (I) in the sense of Cuntz-Krieger (cf. [7]), irreducibility and aperiodicity) respectively.

DEFINITION 2.2. (i) A subshift (X_Λ, σ) satisfies *Condition* (I) if for any $l \in \mathbb{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$ and $y \sim_l x$.

(ii) A subshift (X_Λ, σ) is *irreducible in past equivalence* if for any $l \in \mathbb{N}$, $y \in X_\Lambda$ and a sequence $(x^k)_{k \in K}$ of X_Λ with $x^k \sim_k x^{k+1}$, there exist a number N and a word $\mu \in \Lambda^N$ in a sequence of X_Λ such that $y \sim_l \mu x^{l+N}$.

(iii) A subshift (X_Λ, σ) is *aperiodic in past equivalence* if for any $l \in \mathbb{N}$, there exists a number $N \in \mathbb{N}$ such that for any pair $x, y \in X_\Lambda$ there exists a word μ of length N in a sequence of X_Λ such that $y \sim_l \mu x$.

We know that if a subshift (X_Λ, σ) is aperiodic in past equivalence or irreducible in past equivalence with an aperiodic point, then it satisfies Condition (I) ([22]).

For a fixed $l \in \mathbb{N}$, let F_i^l, $i = 1, 2, \ldots, m(l)$ denote the l-past equivalence classes of X_Λ. Hence X_Λ is a disjoint union of the sets F_i^l, $i = 1, 2, \ldots, m(l)$. For $h \in \Sigma$ and $i = 1, 2, \ldots, m(l)$, $j = 1, 2, \ldots, m(l+1)$, we know $hx \in F_i^l$ for some $x \in F_j^{l+1}$ if and only if $hy \in F_i^l$ for all $y \in F_j^{l+1}$ by Lemma 2.1. Let $A_{l,l+1}(i,j)$ be the number of the set $\{h \in \Sigma \mid hx \in F_i^l$ for some $x \in F_j^{l+1}\}$. Hence we have a sequence $\left\{[A_{l,l+1}(i,j)]_{i=1,2,\ldots,m(l)}^{j=1,2,\ldots,m(l+1)}\right\}_{l\in\mathbb{N}}$ of $m(l) \times m(l+1)$-matrices with entries in non-negative integers. The sequence

$$\left\{[A_{l,l+1}(i,j)]_{i=1,2,\ldots,m(l)}^{j=1,2,\ldots,m(l+1)}\right\}_{l\in\mathbb{N}}$$

of matrices is said to be aperiodic if for any $l \in \mathbb{N}$, there exists a number $N \in \mathbb{N}$ such that all the entries of the product $A_{l,l+1} \cdot A_{l+1,l+2} \cdots A_{l+N-1,l+N}$ of the matrices are strictly positive (cf. [14]). We then easily have:

LEMMA 2.3. *(X_Λ, σ) is aperiodic in past equivalence if and only if the sequence $\left\{[A_{l,l+1}(i,j)]_{i=1,2,\ldots,m(l)}^{j=1,2,\ldots,m(l+1)}\right\}_{l\in\mathbb{N}}$ of matrices is aperiodic.*

3. THE C^*-ALGEBRAS ASSOCIATED WITH SUBSHIFTS

We will review the construction of the C^*-algebras associated with subshifts along [19]. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of the n-dimensional Hilbert space $\mathbb{C}^n$. Set:

$F_\Lambda^0 = \mathbb{C}e_0$ (e_0 : vacuum vector);

F_Λ^k = the Hilbert space spanned by the vectors $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$ for $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k$;

$F_\Lambda = \bigoplus_{k=0}^{\infty} F_\Lambda^k$ (Hilbert space direct sum).

We denote by T_ν, $(\nu \in \Lambda^*)$ the creation operator on F_Λ of $e_\nu, \nu \in \Lambda^*$ $(\nu \neq \emptyset)$ defined by

$$T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu, & (\nu\mu \in \Lambda^*), \\ 0 & \text{else;} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. Let $\mathbf{P}_0$ be the rank one projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^{n} T_i T_i^* + \mathbf{P}_0 = 1$. We then easily see that for $\mu, \nu \in \Lambda^*$, the operator $T_\mu \mathbf{P}_0 T_\nu^*$ is the rank one partial isometry from the vector e_ν to e_μ. Hence, the C^*-algebra generated by the elements of the form $T_\mu \mathbf{P}_0 T_\nu^*$, $\mu, \nu \in \Lambda^*$ is the C^*-algebra $\mathcal{K}(F_\Lambda)$ of all compact operators on F_Λ. Let $\mathcal{T}_\Lambda$ be the C^*-algebra on F_Λ generated by the elements T_ν, $\nu \in \Lambda^*$.

DEFINITION 3.1. ([19]) The C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift (Λ, σ) is defined as the quotient C^*-algebra $\mathcal{T}_\Lambda / \mathcal{K}(F_\Lambda)$ of $\mathcal{T}_\Lambda$ by $\mathcal{K}(F_\Lambda)$.

We denote by S_i, S_μ the quotient image of the operator T_i, $i \in \Sigma$, T_μ, $\mu \in \Lambda^*$. Hence $\mathcal{O}_\Lambda$ is generated by n partial isometries $S_1, \dots, S_n$ with relation $\sum_{i=1}^{n} S_i S_i^* = 1$. If (Λ, σ) is a topological Markov shift, the C^*-algebra $\mathcal{O}_\Lambda$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [7], [11], [13]).

We will present the notation and the basic facts for studying the C^*-algebra $\mathcal{O}_\Lambda$. Put $a_\mu = S_\mu^* S_\mu$, $\mu \in \Lambda^*$. Since $S_\nu S_\nu^*$ commutes with $S_\mu^* S_\mu$, $\mu, \nu \in \Lambda^*$, the following identities hold

$$a_\mu S_\nu = S_\nu a_{\mu\nu}, \quad \mu, \nu \in \Lambda^*. \tag{3.1}$$

We notice that for $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$,

$$S_\mu^* S_\nu \neq 0 \quad \text{if and only if} \quad \mu = \nu.$$

We will use the following notation. Let k, l be natural numbers with $k \leqslant l$.

$\mathcal{A}_l$ = the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements a_μ, $\mu \in \Lambda_l$.

$\mathcal{A}_\Lambda$ = the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements a_μ, $\mu \in \Lambda^*$.

$\mathcal{F}_k^l$ = the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_l$.

$\mathcal{F}_k^\infty$ = the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_\Lambda$.

$\mathcal{F}_\Lambda$ = the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements $S_\mu a S_\nu^*$,

$\mu, \nu \in \Lambda^*$, $|\mu| = |\nu|$, $a \in \mathcal{A}_\Lambda$.

The projections $\{S_\mu^* S_\mu \mid \mu \in \Lambda^*\}$ mutually commute so the C^*-algebras $\mathcal{A}_l$, $l \in \mathbb{N}$ are commutative. Thus, we easily see the following lemma (cf. [19], Section 3).

LEMMA 3.2. (i) *$\mathcal{A}_l$ is finite dimensional and commutative.*

(ii) *$\mathcal{A}_l$ is naturally embedded into $\mathcal{A}_{l+1}$ so that $\mathcal{A}_\Lambda = \varinjlim \mathcal{A}_l$ is a commutative* AF*-algebra.*

(iii) *Each element of $\mathcal{F}_k^l$ is a finite linear combination of elements of the form $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_l$. Hence $\mathcal{F}_k^l$ is finite dimensional.*

(iv) *There are two embeddings in* $\{\mathcal{F}_k^l\}_{k\leqslant l}$*:*
(iv)–(a) $\iota_l : \mathcal{F}_k^l \subset \mathcal{F}_k^{l+1}$ *through the embedding* $\mathcal{A}_l \subset \mathcal{A}_{l+1}$ *and*
(iv)–(b) $\eta_k : \mathcal{F}_k^l \subset \mathcal{F}_{k+1}^{l+1}$ *through the identity*

$$S_\mu a S_\nu^* = \sum_{j=1}^{n} S_{\mu j} S_j^* a S_j S_{\nu j}^*, \quad \mu, \nu \in \Lambda^k,\ a \in \mathcal{A}_l.$$

(v) *Both* $\mathcal{F}_k^\infty = \lim\limits_{l\to\infty} \mathcal{F}_k^l$ *and* $\mathcal{F}_\Lambda = \lim\limits_{k\to\infty} \mathcal{F}_k^\infty$ *are* AF*-algebras.*

In the preceding Hilbert space F_Λ, the transformation $e_\mu \to z^k e_\mu$, $\mu \in \Lambda^k$, $z \in \mathbb{T} = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ on each basis e_μ yields a unitary representation which leaves $\mathcal{K}(F_\Lambda)$ invariant. Thus it gives rise to an action α of $\mathbb{T}$ on the C^*-algebra $\mathcal{O}_\Lambda$, which is called the gauge action and satisfies $\alpha_z(S_i) = zS_i$, $i = 1, 2, \ldots, n$.

Each element X of the $*$-subalgebra of $\mathcal{O}_\Lambda$ algebraically generated by S_μ, S_ν^*, $\mu, \nu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|\nu|\geqslant 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu|\geqslant 1} S_\mu X_\mu \quad \text{for some } X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\Lambda$$

because of the relation (3.1). The map $E(X) = \int_{z\in\mathbb{T}} \alpha_z(X)\, \mathrm{d}z$, $X \in \mathcal{O}_\Lambda$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_\Lambda^\alpha$ under α. We then have (cf. [19], Proposition 3.11):

LEMMA 3.3. $\mathcal{F}_\Lambda = \mathcal{O}_\Lambda^\alpha$.

Let $\mathfrak{D}_\Lambda$ be the C^*-subalgebra of $\mathcal{F}_\Lambda$ generated by $S_\mu S_\mu^*$, $\mu \in \Lambda^*$, which is isomorphic to the C^*-algebra $C(X_\Lambda)$ of all complex valued continuous functions on the space X_Λ. Put $\phi_\Lambda(X) = \sum_{j=1}^{n} S_j X S_j^*$ for $X \in \mathfrak{D}_\Lambda$ which corresponds to the shift σ on X_Λ. Consider the following condition called (I_Λ) in [19]:

(I_Λ) For any $l, k \in \mathbb{N}$ with $l \geqslant k$, there exists a projection q_k^l in $\mathfrak{D}_\Lambda$ such that
(i) $q_k^l a \neq 0$ for any nonzero $a \in \mathcal{A}_l$;
(ii) $q_k^l \phi_\Lambda^m(q_k^l) = 0$, $1 \leqslant m \leqslant k$.

We define the operator λ_Λ on the algebra $\mathcal{A}_\Lambda$ by

$$\lambda_\Lambda(X) = \sum_{j=1}^{n} S_j^* X S_j \quad \text{for } X \in \mathcal{A}_\Lambda.$$

It is said to be *irreducible* if there exists no non-trivial ideal of $\mathcal{A}_\Lambda$ invariant under λ_Λ. It is also said to be *aperiodic* if for any number l, there exists $N \in \mathbb{N}$ such that $\lambda_\Lambda^N(p) \geqslant 1$ for any minimal projection $p \in \mathcal{A}_l$.

LEMMA 3.4. ([19]) *If the C^*-algebra $\mathcal{O}_\Lambda$ satisfies Condition* (I_Λ) *and λ_Λ is irreducible on $\mathcal{A}_\Lambda$, then $\mathcal{O}_\Lambda$ is simple. In addition, if λ_Λ is aperiodic, then $\mathcal{O}_\Lambda$ is purely infinite.*

Since it is easy to see that $\mathcal{A}_l$ is isomorphic to the algebra $C(\Omega_l)$ of all continuous functions on Ω_l for each $l \in \mathbb{N}$ and $\mathcal{A}_\Lambda$ is isomorphic to $C(\Omega_\Lambda)$, we have (cf. [22]):

LEMMA 3.5. ([22]) (i) *For a subshift Λ, Condition* (I) *is equivalent to Condition* (I_Λ).

(ii) (X_Λ, σ) *is irreducible in past equivalence if and only if λ_Λ is irreducible on $\mathcal{A}_\Lambda$.*

(iii) (X_Λ, σ) *is aperiodic in past equivalence if and only if λ_Λ is aperiodic on $\mathcal{A}_\Lambda$.*

4. THE CONTEXT FREE SHIFT AND THE C^*-ALGEBRA

In this section, we will study the C^*-algebra associated with a certain subshift called the context free shift. Let Σ be the set of symbols $\{1, 2, 3\}$. The context free shift is defined to be the subshift Z over Σ whose forbidden words are $\{32^m1^k3 \mid m \neq k\}$ where the word 32^m1^k3 means $3\,\underbrace{2\cdots2}_{\text{m times}}\,\underbrace{1\cdots1}_{\text{k times}}\,3$ (cf. [18]). It is not a sofic subshift and hence not a Markov subshift.

Let $\mathcal{O}_Z$ be the C^*-algebra associated with it. Let S_1, S_2, S_3 be the canonical generating partial isometries of $\mathcal{O}_Z$ corresponding to the symbols $\{1, 2, 3\}$ respectively. We then have:

LEMMA 4.1. *For any word $\mu \in Z^*$, we have:*

(i) $S_\mu^* S_\mu = S_{3\nu}^* S_{3\nu}$ *for $\mu = \gamma 3 \nu$ with $\gamma \in Z^*$. In particular, $S_\mu^* S_\mu = S_3^* S_3$ for $\mu = \gamma 3$;*

(ii) *if μ does not contain the symbol "3", $S_\mu^* S_\mu = 1$; hence, $S_1^* S_1 = S_2^* S_2 = 1$.*

In particular, we have:

COROLLARY 4.2.

$$\mathcal{A}_1 = \mathbb{C} S_3^* S_3 \oplus \mathbb{C}(1 - S_3^* S_3).$$

Hence we have $\dim(\mathcal{A}_1) = 2$.

For studying the structure of the C^*-algebra $\mathcal{A}_l$, we will consider its character space Ω_l for $l \geqslant 2$. Define some sequences of subsets of X_Z in the following way:

$$P_0 = \{1^k 2^\infty \mid k \geqslant 0\} \cup \{2^k 1^m 2y \in X_Z \mid k \geqslant 0,\, m \geqslant 1,\, y \in X_Z\}$$

and for $n, j = 0, 1, \dots,$

$$\begin{aligned} E_j &= \{1^j 3y \in X_Z \mid y \in X_Z\}, \\ Q_n &= \bigcup_{j>n} E_j, \\ F_j &= \{2^m 1^{m+j} 3y \in X_Z \mid m \geqslant 1,\, y \in X_Z\}, \\ R_n &= \{2^m 1^k 3y \in X_Z \mid m \geqslant 1,\, k \geqslant 0,\, m+j \neq k \text{ for } j = 0, 1, \dots, n\}. \end{aligned}$$

LEMMA 4.3. *For each $l \in \mathbb{N}$, the space X_Z is decomposed as a disjoint union:*

$$X_Z = P_0 \bigcup_{j=0}^{l-1} E_j \cup Q_{l-1} \bigcup_{j=0}^{l-1} F_j \cup R_{l-1}.$$

This decomposition of X_Z into $2l+3$-components corresponds to the l-past equivalence classes of X_Z.

The projection a_μ for $\mu = \mu_1 \cdots \mu_k \in \Lambda^k$ is identified with the characteristic function on X_Z corresponding to the set $\sigma^k(U_\mu) \subset X_Z$ where U_μ is the cylinder set $\{(x_i)_{i \in \mathbb{N}} \in X_Z \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$. Hence a projection P in the algebra $\mathcal{A}_l$ corresponds to a subset of X_Z that we denote by $\mathrm{supp}(P)$. Then we see:

LEMMA 4.4. (i) $\mathrm{supp}(S_{32^j}^* S_{32^j}) = E_j \cup F_j \cup P_0$ *for $j = 0, 1, 2, \dots;$*

(ii) $\mathrm{supp}(S_{31}^* S_{31}) = \bigcup_{j=0}^{n} F_j \cup R_n \cup P_0$ *for all $n \in \mathbb{N}$;*

(iii) $\mathrm{supp}(S_3^* S_3 \cdot S_{31}^* S_{31} \cdot S_{32}^* S_{32}) = P_0$.

Hence we know the structure of $\mathcal{A}_l$ as folows:

COROLLARY 4.5. *For each $l \in \mathbb{N}$, the decomposition of X_Z of $2l+3$-components:*

$$X_Z = P_0 \bigcup_{j=0}^{l-1} E_j \cup Q_{l-1} \bigcup_{j=0}^{l-1} F_j \cup R_{l-1}$$

corresponds to the direct sum decomposition of the C^*-algebra $\mathcal{A}_l$ as the set of all minimal projections in $\mathcal{A}_l$. Hence we obtain $\dim(\mathcal{A}_l) = 2l+3$ *for $l \geqslant 2$.*

LEMMA 4.6. *The context free shift Z is aperiodic in past equivalence.*

Proof. For any natural number l, set $N = l+4$. For a word $\nu \in Z^{l+2}$, put $\mu = \nu 12 \in Z^N$. Then we have for $x \in X_Z$

$$\begin{aligned} &\mu x \in E_k && \text{if} \quad \nu = 1^k 3^{l-k+2}, && k = 0, 1, \dots, l-1; \\ &\mu x \in Q_{l-1} && \text{if} \quad \nu = 1^l 3^2; && \\ &\mu x \in F_k && \text{if} \quad \nu = 21^{k+1} 3^{l-k}, && k = 0, 1, \dots, l-1; \\ &\mu x \in R_{l-1} && \text{if} \quad \nu = 2^2 13^{l-1}; && \\ &\mu x \in P_0 && \text{if} \quad \nu = 12^{l+1}. && \end{aligned}$$

Since the family $E_0, F_0, E_1, F_1, \dots, E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}, P_0$ represents the set of all l-past equivalence classes, the subshift Z is aperiodic in past equivalence. ∎

As aperiodicity in past equivalence implies Condition (I) and irreducibility in past equivalence, we conclude by Lemma 3.4 and Lemma 3.5 that

COROLLARY 4.7. *The C^*-algebra $\mathcal{O}_Z$ is simple and purely infinite.*

The C^*-algebra $\mathcal{O}_Z$ is generated by three operators: S_1, S_2 and S_3. As in Lemma 4.1 both operators S_1 and S_2 are isometries and S_3 is a partial isometry. We will realize $\mathcal{O}_Z$ as a universal C^*-algebra subject to some opeator relations among S_1, S_2 and S_3. It has been proved in [21] that for a subshift Λ in general, the associated C^*-algebra $\mathcal{O}_\Lambda$ can be realized as a universal C^*-algebra in the following way:

LEMMA 4.8. ([21]) *For a subshift Λ over $\Sigma = \{1, 2, \dots, n\}$, the C^*-algebra $\mathcal{O}_\Lambda$ associated with Λ is the universal concrete C^*-algebra generated by n partial isometries S_i, $i = 1, 2, \dots, n$ subject to the following relations:*

(i) $\sum_{j=1}^{n} S_j S_j^* = 1$;

(ii) $S_i^* S_i = 1 - \sum_{k=1}^{\infty} \sum_{\nu \in L_i^k} S_\nu S_\nu^*$;

where $L_i^k = \{\nu_1 \cdots \nu_k \in \Lambda^k \mid i\nu_1 \cdots \nu_{k-1} \in \Lambda^k,\ i\nu_1 \cdots \nu_{k-1}\nu_k$ does not belong to $\Lambda^*\}$. The infinite sum of the right hand side of the relation (ii) *is taken under strong operator topology on a Hilbert space.*

The above theorem means that there exists a representation of $\mathcal{O}_\Lambda$ as operators on a Hilbert space such that the canonical generators satisfy the relations (i) and (ii). Conversely, if there exist n partial isometries on a Hilbert space satisfying the above relations, then there exists a canonical surjective homomorphism from $\mathcal{O}_\Lambda$ to the C^*-algebra generated by them.

We apply Lemma 4.8 to our C^*-algebra $\mathcal{O}_Z$. The following lemma is clear.

LEMMA 4.9. $L_3^k = \{2^m 1^l 3 \mid m + l = k - 1,\ m \neq l\}$ *for* $k = 2, 3, \dots$.

Thus we obtain:

THEOREM 4.10. *The C^*-algebra $\mathcal{O}_Z$ associated with the context free shift Z is simple and purely infinite. It is the universal concrete C^*-algebra generated by two isometries S_1, S_2 and one partial isometry S_3 satisfying the following relations:*

(i) $\sum_{j=1}^{3} S_j S_j^* = 1$;

(ii) $S_3^* S_3 = 1 - \sum_{k=1}^{\infty} \sum_{m=0, k\neq 2m}^{k} S_{2^m 1^{k-m} 3} S_{2^m 1^{k-m} 3}^*$;

where $S_{2^m1^{k-m}3}$ *denotes* $\underbrace{S_2\cdots S_2}_{\text{m times}}\underbrace{S_1\cdots S_1}_{\text{k}-\text{m times}}S_3$. *The infinite sum of the right hand side of the relation* (ii) *is taken under strong operator topology on a Hilbert space.*

5. THE K-THEORY FOR $\mathcal{O}_Z$

In this section, we compute K-groups for $\mathcal{O}_Z$. The main tool is the following K-theory formulae proved in [20]. The formulae hold for the C^*-algebras associated with subshifts in general.

LEMMA 5.1. ([20]) (i) $\mathrm{K}_0(\mathcal{O}_Z) = \varinjlim(\mathrm{K}_0(\mathcal{A}_l)/(\lambda_{l-1} - \iota_{l-1})_*\mathrm{K}_0(\mathcal{A}_{l-1}), \iota_{l*})$*;*
(ii) $\mathrm{K}_1(\mathcal{O}_Z) = \varinjlim(\ker(\lambda_{l-1} - \iota_{l-1})_*$ *in* $\mathrm{K}_0(\mathcal{A}_l), \iota_{l*})$*;*
where λ_{l-1} *is defined to be the map* λ_Λ *from* $\mathcal{A}_{l-1}$ *to* $\mathcal{A}_l$ *and* ι_{l-1} *is the natural inclusion from* $\mathcal{A}_{l-1}$ *to* $\mathcal{A}_l$*.*

We represent the homomorphisms λ_{l*} and ι_{l*} as rectangular matrices. The homomorphism λ_{l*} is nothing but the transpose $A^{\mathrm{t}}_{l,l+1}$ of the matrix $A_{l,l+1}$ defined in Section 2. Recall that $\dim(\mathcal{A}_l) = 2l+3$ for $l \geqslant 2$. Let $p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}$ be the set of all minimal projections in $\mathcal{A}_l$ corresponding to the set of all l-past equivalence classes $P_0, E_0, F_0, E_1, F_1, \ldots, E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}$ respectively. Then we can represent the induced matrices on the K_0-groups

$$L_l(=\lambda_{l*}), I_l(=\iota_{l*}) : \mathrm{K}_0(\mathcal{A}_l) = \mathbb{Z}^{2l+3} \to \mathrm{K}_0(\mathcal{A}_{l+1}) = \mathbb{Z}^{2l+5}$$

of the operator λ_l and the inclusion ι_l from $\mathcal{A}_l$ to $\mathcal{A}_{l+1}$ as follows:

$$L_l = \begin{bmatrix}
2 & 1 & & & & & & & & & & & & & \\
 & 1 & & 1 & & & & & & & & & & & 1 \\
1 & 1 & & & & & & & & & & & & & 1 \\
 & & 1 & & & 1 & & & & & & & & & \\
1 & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & 1 & & & & & & & \\
1 & & & 1 & & & & & & & & & & & \\
 & & & & & & 1 & & & \ddots & & & & & \\
1 & & & & & & 1 & & & & & & & & \\
 & & & & & & & & 1 & & & \ddots & & & \\
1 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & \ddots & & & 1 & \\
\vdots & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & & 1 & 1 & \\
\vdots & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & 1 & 1 \\
1 & & & & & & & & & & & & & & 1
\end{bmatrix},$$

$$I_l = \begin{bmatrix}
1 & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & 1
\end{bmatrix}$$

along the following ordered basis

$$p_0, e_0, f_0, e_1, f_1, \dots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}$$

where in the above matrices blanks denote zeros. Since we know

$$L_l - I_l = \begin{bmatrix}
1 & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & 1 \\
1 & 1 & -1 & & & & & & & & & & & & 1 \\
 & & 1 & -1 & & 1 & & & & & & & & & \\
1 & & 1 & & -1 & & & & & & & & & & \\
 & & & & 1 & -1 & & 1 & & & & & & & \\
1 & & & & 1 & & -1 & & & & & & & & \\
 & & & & & & 1 & -1 & & \ddots & & & & & \\
1 & & & & & & 1 & & -1 & & & & & & \\
 & & & & & & & & 1 & -1 & & \ddots & & & \\
1 & & & & & & & & 1 & & \ddots & & & & \\
 & & & & & & & & & & \ddots & -1 & & 1 & \\
\vdots & & & & & & & & & & \ddots & & -1 & & \\
 & & & & & & & & & & & & 1 & & \\
\vdots & & & & & & & & & & & & 1 & & -1 \\
 & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & & & & & & &
\end{bmatrix},$$

we easily see that

LEMMA 5.2. $\ker(L_l - I_l) = 0$ *for* $2 \leqslant l \in \mathbb{N}$.

Thus we have by Lemma 5.1,

PROPOSITION 5.3.

$$\mathrm{K}_1(\mathcal{O}_Z) \cong 0.$$

We next compute $\mathrm{K}_0(\mathcal{O}_Z)$.

LEMMA 5.4. *Fix $l = 2, 3, \ldots$. For* $z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2l+3} \end{bmatrix} \in \mathbb{Z}^{2l+3}$, *put*

$$
\begin{aligned}
x_1 &= z_{2l+3}, \\
x_2 &= z_1 - z_{2l+3}, \\
x_3 &= z_1 - z_3 + z_{2l+2}, \\
x_4 &= z_2 - z_{2l+2}, \\
x_{2m-1} &= z_{2l+3} + x_{2m-3} - z_{2m-1} \qquad \text{for } 3 \leqslant m \leqslant l-1, \\
x_{2m} &= z_{2m-2} - x_{2m-3} + x_{2m-2} \qquad \text{for } 3 \leqslant m \leqslant l-1, \\
x_{2l-1} &= z_{2l}, \\
x_{2l} &= z_{2l-2} - x_{2l-3} + x_{2l-2}, \\
x_{2l+1} &= z_{2l+2}
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi_l\big([z_i]_{i=1}^{2l+3}\big) &= z_{2l-1} - x_1 - x_{2l-3} + x_{2l-1}, \\
\psi_l\big([z_i]_{i=1}^{2l+3}\big) &= z_{2l+1} - x_1 - x_{2l-1} + x_{2l+1}.
\end{aligned}
$$

Then we have

$$
\begin{bmatrix} z_1 \\ \vdots \\ z_{2l+3} \end{bmatrix} = (L_{l-1} - I_{l-1}) \begin{bmatrix} x_1 \\ \vdots \\ x_{2l+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_l(z) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \psi_l(z) \\ 0 \\ 0 \end{bmatrix}.
$$

LEMMA 5.5. *The map* $\xi_l : \big[(z_i)_{i=1}^{2l+3}\big] \in \mathbb{Z}^{2l+3} \mapsto (\varphi_l(z), \psi_l(z)) \in \mathbb{Z}^2$ *induces an isomorphism* $\mathbb{Z}^{2l+3}/(L_{l-1} - I_{l-1})\mathbb{Z}^{2l+1} \cong \mathbb{Z}^2$.

LEMMA 5.6. *The following diagram is commutative:*

$$
\begin{array}{ccc}
\mathbb{Z}^{2l+3}/(L_{l-1} - I_{l-1})\mathbb{Z}^{2l+1} & \xrightarrow{I_l} & \mathbb{Z}^{2l+5}/(L_l - I_l)\mathbb{Z}^{2l+3} \\
\xi_l \Big\downarrow & & \xi_{l+1} \Big\downarrow \\
\mathbb{Z}^2 & \xrightarrow{M} & \mathbb{Z}^2
\end{array}
$$

where $M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Therefore we conclude

THEOREM 5.7. (i) $\mathrm{K}_0(\mathcal{O}_Z) \cong Z$;
(ii) $\mathbb{1} = 0$ *in* $\mathrm{K}_0(\mathcal{O}_Z) \cong Z$;
(iii) $[S_3^* S_3 \cdot S_{31}^* S_{31} \cdot S_{32}^* S_{32}]$ *generates* $\mathrm{K}_0(\mathcal{O}_Z) \cong Z$.

Proof. (i) By Lemma 5.1, it follows that

$$\begin{aligned}\mathrm{K}_0(\mathcal{O}_Z) &= \varinjlim(\mathrm{K}_0(\mathcal{A}_l)/(\lambda_{l-1} - \iota_{l-1})_* \mathrm{K}_0(\mathcal{A}_{l-1}), \iota_{l*}) \\ &= \varinjlim(\mathbb{Z}^{2l+3}/(L_{l-1} - I_{l-1})\mathbb{Z}^{2l+1}, I_l) = \varinjlim(\mathbb{Z}^2, M) \cong \mathbb{Z}.\end{aligned}$$

(ii) The class of the unit $\mathbb{1}$ in $\mathrm{K}_0(\mathcal{A}_l)$ corresponds to the vector $[1, \dots, 1] \in \mathbb{Z}^{2l+3}$. Since $\varphi_l(\mathbb{1}) = \psi_l(\mathbb{1}) = 0$, we see $\xi_l(\mathbb{1}) = (0, 0)$. Thus the projection $\mathbb{1}$ represents 0 in $\mathbb{Z} \cong \mathrm{K}_0(\mathcal{O}_Z)$.

(iii) It is easy to see that the sequence $[-1, 0, \dots, 0] \in \mathbb{Z}^{2l+3}$, $l = 1, 2, \dots$ of vectors generates the group $\mathrm{K}_0(\mathcal{O}_Z) = \varinjlim(\mathbb{Z}^{2l+3}/(L_{l-1} - I_{l-1})\mathbb{Z}^{2l+1}, I_l)$. It corresponds to the characteristic function on the subset P_0 which is expressed as the class of the projection $[S_3^* S_3 \cdot S_{31}^* S_{31} \cdot S_{32}^* S_{32}]$. ∎

By [6], we know that the K-groups for the Cuntz algebra $\mathcal{O}_\infty$ of infinite order are $\mathrm{K}_0(\mathcal{O}_\infty) = \mathbb{Z}$, $\mathrm{K}_1(\mathcal{O}_\infty) = 0$ and that the position of $\mathbb{1}$ in $\mathrm{K}_0(\mathcal{O}_\infty)$ corresponds to 1 in $\mathbb{Z}$. By the classification theorem of purely infinite simple C^*-algebras proved by Kirchberg ([16]) and Phillips ([25]), we have:

COROLLARY 5.8. *The C^*-algebra $\mathcal{O}_Z$ is not isomorphic to any Cuntz-Krieger algebra as well as to the Cuntz algebra $\mathcal{O}_\infty$ of infinite order, but it is stably isomorphic to $\mathcal{O}_\infty$. In fact, $\mathcal{O}_Z$ is isomorphic to $(\mathcal{O}_\infty)_{1-s_1 s_1^*}$ where $\mathcal{O}_\infty$ is generated by a sequence s_i, $i = 1, 2, \dots$ of isometries with mutually orthogonal ranges.*

6. KMS-STATES FOR GAUGE ACTION AND TOPOLOGICAL ENTROPY

In this section, we study KMS-state for the gauge action on $\mathcal{O}_Z$ and topological entropy for the context free shift. In studying KMS-state for the gauge action on the C^*-algebras associated with subshifts, the lemma bellow proved in [23] plays a crucial role.

LEMMA 6.1. ([23]) *For a subshift Λ, if a state φ on $\mathcal{O}_\Lambda$ is a* KMS*-state for the gauge action on $\mathcal{O}_\Lambda$ at the inverse temperature $\log \beta$, it satisfies the condition $\varphi \circ \lambda_\Lambda = \beta\varphi$ on $\mathcal{A}_\Lambda$. Conversely, a state φ on $\mathcal{A}_\Lambda$ satisfying the condition $\varphi \circ \lambda_\Lambda = \beta\varphi$ can be uniquely extended to a* KMS*-state for the gauge action on $\mathcal{O}_\Lambda$.*

We will apply this lemma to our C^*-algebra $\mathcal{O}_Z$ for the context free shift Z. We will find a state on $\mathcal{A}_Z$ satisfying the condition $\varphi \circ \lambda_Z = \beta\varphi$ for some real number β.

Let $\widehat{p}_0, \widehat{e}_0, \widehat{f}_0, \widehat{e}_1, \widehat{f}_1, \dots, \widehat{e}_l, \widehat{f}_l, \widehat{q}_l, \widehat{r}_l$ be real numbers for $l = 2, 3, \dots$ satisfying the condition

$$\widehat{p}_0 + \sum_{j=0}^{l} (\widehat{e}_j + \widehat{f}_j) + \widehat{q}_l + \widehat{r}_l = 1.$$

We consider the following equations:

$$\begin{aligned} L_{l+3}^{\mathrm{t}} &\Big[\widehat{p}_0, \widehat{e}_0, \widehat{f}_0, \widehat{e}_1, \widehat{f}_1, \widehat{e}_2, \widehat{f}_2, \dots, \widehat{e}_{l-1}, \widehat{f}_{l-1}, \widehat{e}_l, \widehat{f}_l, \widehat{q}_l, \widehat{r}_l\Big]^{\mathrm{t}} \\ &= \beta\Big[\widehat{p}_0, \widehat{e}_0, \widehat{f}_0, \widehat{e}_1, \widehat{f}_1, \widehat{e}_2, \widehat{f}_2, \dots, \widehat{e}_{l-1}, \widehat{f}_{l-1}, \widehat{q}_{l-1}, \widehat{r}_{l-1}\Big]^{\mathrm{t}} \end{aligned}$$

and the equations

$$\widehat{e}_l + \widehat{q}_l = \widehat{q}_{l-1}, \quad \widehat{f}_l + \widehat{r}_l = \widehat{r}_{l-1}.$$

These equations can be uniquely solved in the following way by straightforward calculation.

LEMMA 6.2.

$$\begin{aligned} \widehat{p}_0 &= \frac{1}{(\beta-1)^2}, \\ \widehat{e}_n &= \frac{\beta-2}{\beta^{n+1}}, \\ \widehat{f}_0 &= \frac{(\beta-1)(\beta-2)}{\beta} - \frac{1}{(\beta-1)^2}, \\ \widehat{f}_n &= \beta^n \widehat{f}_0 - (\beta^{n-1}\widehat{e}_1 + \beta^{n-2}\widehat{e}_2 + \cdots + \beta\widehat{e}_{n-1} + \widehat{e}_n), \\ \widehat{q}_n &= \frac{\beta-2}{\beta^{n+1}(\beta-1)}, \\ \widehat{r}_n &= (\beta-2)\widehat{p}_0 - (\widehat{f}_0 + \widehat{f}_1 + \cdots + \widehat{f}_n), \quad n = 0, 1, 2 \dots. \end{aligned} \tag{6.1}$$

LEMMA 6.3. *Suppose that* $\beta > 2$. *Then* $\widehat{f}_n \geqslant 0$ *for all* $n = 0, 1, \dots,$ *if and only if*

$$(\beta^2 - 1)(\beta - 1)^2(\beta - 2) - 2(\beta^2 - \beta + 1) \geqslant 0.$$

If this is the case, we have $\widehat{f}_n > 0$ *for all* $n = 0, 1, \dots$.

Proof. Since we have

$$\widehat{f}_n = \beta^n \widehat{f}_0 - \beta^n \frac{\beta^{2n} - 1}{\beta^{2n+2} - \beta^{2n}} \cdot \frac{\beta - 2}{\beta},$$

we see $\widehat{f}_n > 0$ for all $n = 0, 1, \dots$ if and only if

$$(\beta^2 - 1)(\beta - 1)^2(\beta - 2) - 2(\beta^2 - \beta + 1) \geqslant -\frac{1}{\beta^{2n}}(\beta - 2)(\beta - 1).$$

Hence we get the assertion. ∎

LEMMA 6.4. *Suppose that $\beta > 2$. Then $\widehat{r}_n \geqslant 0$ for all $n = 0, 1, \ldots,$ if and only if*

$$(\beta^2 - 1)(\beta - 1)^2(\beta - 2) - 2(\beta^2 - \beta + 1) \leqslant 0.$$

If this is the case, we have $\widehat{r}_n > 0$ for all $n = 0, 1, \ldots$.

Proof. Since we have

$$\widehat{r}_n = \frac{\beta - 2}{(\beta - 1)^2} - \frac{(1 - \beta^{n+1})\{(1 - \beta)\widehat{f}_0 + \frac{\beta-2}{\beta(\beta+1)}\} - \frac{\beta-2}{\beta^{n+1}} + \frac{\beta-2}{\beta} + \frac{\beta-2}{\beta(\beta+1)} - \frac{\beta-2}{\beta^{n-2}(\beta+1)}}{(\beta - 1)^2},$$

we see $\widehat{r}_n \geqslant 0$ for all $n = 0, 1, \ldots$ if and only if

$$(1 - \beta)\widehat{f}_0 + \frac{\beta - 2}{\beta(\beta + 1)} \geqslant 0.$$

This implies that

$$(\beta^2 - 1)(\beta - 1)^2(\beta - 2) - 2(\beta^2 - \beta + 1) \leqslant 0. \quad \blacksquare$$

Therefore we have

COROLLARY 6.5. *Suppose that $\beta > 2$. Then $\widehat{f}_n, \widehat{r}_n > 0$ for all $n = 0, 1, \ldots,$ if and only if $\beta = 1 + \sqrt{1 + \sqrt{3}}$. If this is the case, we have $\widehat{f}_n, \widehat{r}_n > 0$ for all $n = 0, 1, \ldots$.*

Let $p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}$ be the basis of $\mathcal{A}_l$ considered in the previous section. By putting

$$\varphi(t) = \widehat{t} \quad \text{for } t = p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1},$$

we know that to give a state φ on $\mathcal{A}_Z$ such that $\varphi \circ \lambda_Z = \beta\varphi$ is equivalent to find non-negative real numbers $\widehat{p}_0, \widehat{e}_0, \widehat{f}_0, \widehat{e}_1, \widehat{f}_1, \ldots, \widehat{e}_{l-1}, \widehat{f}_{l-1}, \widehat{q}_{l-1}, \widehat{r}_{l-1}$ satisfying Equations (6.1) for all $l = 2, 3, \ldots$. By non-negative condition for $\widehat{e}_n$ and $\widehat{f}_0$ in (6.1), we have $\beta > 2$.

Hence we get:

PROPOSITION 6.6. *A state φ on $\mathcal{A}_Z$ satisfies the condition $\varphi \circ \lambda_Z = \beta\varphi$ for some real number β if and only if $\beta = 1 + \sqrt{1+\sqrt{3}}$. Moreover a state φ that satisfies the above condition is unique.*

For $\beta = 1+\sqrt{1+\sqrt{3}}$, the concrete values of the above state φ at the support projections a_μ of the partial isometries S_μ, $\mu \in Z^*$ are easily determined by the vectors $\widehat{p}_0, \widehat{e}_0, \widehat{f}_0, \widehat{e}_1, \widehat{f}_1, \dots, \widehat{e}_n, \widehat{f}_n, \widehat{q}_n, \widehat{r}_n$. We know, for instance, that

$$
\begin{aligned}
\varphi(a_3) &= \widehat{p}_0 + \widehat{e}_0 + \widehat{f}_0 = \beta - 2,\\
\varphi(a_{32}) &= \widehat{p}_0 + \widehat{e}_1 + \widehat{f}_1 = (\beta-1)(\beta-2) - \frac{1}{\beta-1},\\
\varphi(a_{32^k}) &= \widehat{p}_0 + \widehat{e}_k + \widehat{f}_k\\
&= \beta^{k-1}(\beta-1)(\beta-2) - \frac{\beta^k}{(\beta-2)^2} - \frac{\beta-2}{\beta^{k-1}}(\beta^{k-2} + \cdots + \beta + 1) + \frac{1}{(\beta-1)^2},\\
\varphi(a_{31}) &= \varphi(a_{31^k}) = \widehat{p}_0 + \widehat{f}_0 + \widehat{f}_1 + \widehat{r}_1 = \frac{1}{\beta-1},\\
\varphi(a_{32^k1}) &= \widehat{p}_0 + \widehat{e}_{k-1} + \widehat{f}_0 + \widehat{f}_1 + \widehat{r}_1 = \frac{1}{\beta-1} + \frac{\beta-2}{\beta^k},\\
\varphi(a_{312^k}) &= 1, \quad k = 1, 2, \dots.
\end{aligned}
$$

The values of φ at the range projections $S_\mu S_\mu^*$ are determined by the formulae: $\varphi(S_\mu S_\mu^*) = \frac{1}{\beta^{|\mu|}}\varphi(a_\mu)$ as in [23].

The topological entropy for the irreducible topological Markov shift Λ_A determined by a matrix A with entries in $\{0,1\}$ and for β-shift with a real number $\beta > 1$ are $\log r(A)$ and $\log\beta$ respectively where $r(A)$ denotes the spectral radius of the matrix A. Their topological entropy have appeared as the inverse temperature of the admitted KMS-state for the gauge action on the associated C^*-algebras ([12], [15]).

For a subshift (Λ, σ) over $\Sigma = \{1, 2, \dots, n\}$ and a natural number k, let $\theta_k(\Lambda)$ be the cardinality of the words of length k appearing in Λ^*. The *topological entropy* $h_{\text{top}}(\Lambda)$ for the subshift (Λ, σ) is given by

$$h_{\text{top}}(\Lambda) = \lim_{k\to\infty} \frac{1}{k} \log \theta_k(\Lambda) \quad \text{(cf. [8], [18])}.$$

For the context free shift Z we have:

LEMMA 6.7. *If there exists a $\log\beta$ KMS-state on $\mathcal{O}_Z$ for the gauge action for some $1 < \beta \in \mathbb{R}$, we have*

$$\log\beta = \log r(\lambda_Z) = h_{\text{top}}(Z)$$

where $r(\lambda_Z)$ denotes the spectral radius of the operator λ_Z on $\mathcal{A}_Z$.

Proof. A word μ in $\{1,2,\dots,n\}$ appears in the subshift Z if and only if $S_\mu \neq 0$. Let φ be a $\log\beta$ KMS-state on $\mathcal{O}_Z$ for the gauge action. For $k \in \mathbb{N}$, it follows that

$$\beta^k = \varphi\Big(\sum_{\mu\in Z^k} S_\mu^* S_\mu\Big) \leqslant \Big\|\sum_{\mu\in Z^k} S_\mu^* S_\mu\Big\| = \|\lambda_Z^k(1)\| \leqslant \sum_{\mu\in Z^k}\|S_\mu^* S_\mu\| = \theta_k(Z).$$

As λ_Z^k is a completely positive map on the unital C^*-algebra $\mathcal{A}_Z$, we have $\|\lambda_Z^k(1)\| = \|\lambda_Z^k\|$ so that we see

$$\beta^k \leqslant \|\lambda_Z^k\| \leqslant \theta_k(Z). \tag{6.2}$$

On the other hand, by the inequality $\beta^k \geqslant \theta_k(Z)\min_{\mu\in Z^k}\varphi(a_\mu)$, we obtain

$$\min_{\mu\in Z^k}\varphi(a_\mu)^{\frac{1}{k}}\cdot\theta_k(Z)^{\frac{1}{k}} \leqslant \beta \leqslant \theta_k(Z)^{\frac{1}{k}}.$$

Now we have $a_\mu \geqslant P_0$ for any word $\mu \in Z^*$. It follows that

$$\varphi(a_\mu) \geqslant \varphi(P_0) = \frac{1}{(\beta-1)^2} = \frac{\sqrt{3}-1}{2} \quad \text{for } \mu \in Z^*.$$

Hence we obtain

$$\lim_{k\to\infty}\min_{\mu\in Z^k}\varphi(a_\mu)^{\frac{1}{k}} = 1$$

and $\lim_{k\to\infty}\theta_k(Z)^{\frac{1}{k}} = \beta$. Thus we get the desired equalities from (6.2). ∎

(We have more general results in [23]).

Therefore we conclude

THEOREM 6.8. (i) *For a real number β, there exists a* $\log\beta$ KMS-*state for the gauge action on $\mathcal{O}_Z$ if and only if $\beta = 1+\sqrt{1+\sqrt{3}} = 2.652891\dots$.*

(ii) *The above* KMS-*state is unique.*

(iii) $\log(1+\sqrt{1+\sqrt{3}}) = h_{\mathrm{top}}(Z)$, *the topological entropy for the context free shift Z.*

7. REMARK

We will finally apply our discussions to settle a conjugacy result in symbolic dynamics. We note the following fact:

LEMMA 7.1. ([19], Proposition 5.8) *Let (Λ_1, σ) and (Λ_2, σ) be subshifts such that both the corresponding one-sided subshifts X_{Λ_1} and X_{Λ_2} satisfy the Condition* (I). *If they are conjugate as one-sided subshifts, we have an isomorphism between the C^*-algebras $\mathcal{O}_{\Lambda_1}$ and $\mathcal{O}_{\Lambda_2}$.*

For $\beta = 1 + \sqrt{1 + \sqrt{3}}$ the β-shift is not sofic and has the same topological entropy as the context free shift. Hence it might be possible that the context free shift is conjugate to the β-shift. We will however see:

PROPOSITION 7.2. *The context free shift is not conjugate to any β-shift for $1 < \beta \in \mathbb{R}$ as a one-sided subshift.*

Proof. Since topological entropy is a conjugacy invariant and the entropy for the β-shift is $\log \beta$, it is enough to consider the case $\beta = 1 + \sqrt{1 + \sqrt{3}}$. By [15], we know that the β-shift for $\beta = 1 + \sqrt{1 + \sqrt{3}}$ is aperiodic in our sense and the corresponding C^*-algebra is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$. Hence the context free shift is not conjugate to the β-shift by Corollary 5.8 and Lemma 7.1. ∎

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