J. OPERATOR THEORY
42(1999), 371–377

BISHOP'S PROPERTY (β) FOR TENSOR PRODUCT TUPLES OF OPERATORS

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Communicated by Florian-Horia Vasilescu

Abstract. In this paper, we prove that a tensor product tuple $R = (S \otimes I, I \otimes T)$ possesses Bishop's property (β), supposed that the commuting tuples S and T of Hilbert space operators have property (β). As an application, we show that the Hardy space $H^2(\mathbb{D}^n)$ over the polydisc in $\mathbb{C}^n$ is quasicoherent.

Keywords: *Bishop's property* (*beta*), *Hardy spaces, Toeplitz operator.*

MSC (2000): 47A13.

1. INTRODUCTION

Let H be a Hilbert space and let $T = (T_1, \dots, T_n) \in L(H)^n$ be a commuting tuple of bounded linear operators on H. We say that T has Bishop's property (β), if a certain transversality relation holds, namely:

$$\widehat{\mathrm{Tor}}_p^{\,\mathcal{O}(\mathbb{C}^n)}(H, \mathcal{O}(U)) \begin{cases} \text{is Hausdorff}, & \text{for } p = 0, \\ = 0, & \text{for } p \geqslant 1; \end{cases}$$

for all Stein open sets $U \subseteq \mathbb{C}^n$ (cf. [2], [3] and [6]). In the situation of this paper, a different characterization of property (β) is more useful.

Definition 1.1. We call a finite exact sequence

$$0 \to H \xrightarrow{\ \varepsilon\ } H^0 \xrightarrow{\ d^0\ } H^1 \xrightarrow{\ d^1\ } \cdots \tag{1.1}$$

consisting of Hilbert spaces H, H^j ($j = 0, 1, \dots$) and bounded linear operators $\varepsilon : H \to H^0$, $d^j : H^j \to H^{j+1}$ a *decomposable resolution* of T, if for every

$j = 0, 1, \ldots$ there is a decomposable tuple $T^{(j)} = (T_1^{(j)}, \ldots, T_n^{(j)}) \in L(H^j)^n$, such that (1.1) intertwines the tuples $T, T^{(0)}, T^{(1)}, \ldots$ componentwise.

We refer to [2] for an introduction into the theory of decomposable tuples. The connection between property (β) and decomposability is illustrated in the next theorem.

THEOREM 1.2. (i) *The tuple T is decomposable if and only if both T and the tuple $T^* = (T_1^*, \ldots, T_n^*)$ of Hilbert space adjoints have property (β).*

(ii) *The tuple T has property (β) if and only if it admits a decomposable resolution.*

Now let $S = (S_1, \ldots, S_m) \in L(H)^m$ and $T = (T_1, \ldots, T_n) \in L(K)^n$ be two commuting tuples of bounded linear Hilbert space operators and let $H\widehat{\otimes}\hat{K}$ denote the Hilbert space tensor product of H and K. We then have:

THEOREM 1.3. (Satz 6.3 in [1]) *The tensor product tuple*

$$R \in L(H\widehat{\otimes}K)^{m+n},$$

$$R = (S \otimes I, I \otimes T) := (S_1 \otimes I, \ldots, S_m \otimes I, I \otimes T_1, \ldots, I \otimes T_n)$$

is decomposable if and only if S and T are decomposable.

In Section 2 of this paper, a corresponding result for Bishop's property (β) is derived by constructing a decomposable resolution for R from the given resolutions of S and T.

2. PROPERTY (β) FOR TENSOR PRODUCT TUPLES

We now state and prove the main theorem of this paper.

THEOREM 2.1. *Suppose that H and K are Hilbert spaces and suppose that $S = (S_1, \ldots, S_m) \in L(H)^m$ and $T = (T_1, \ldots, T_n) \in L(K)^n$ are two commuting tuples of bounded linear operators. If S and T have property (β), then the commuting tuple $R \in L(H\widehat{\otimes}K)^{m+n}$,*

$$R = (S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_m \otimes I, I \otimes T_1, \ldots, I \otimes T_n)$$

has property (β).

We postpone the proof of Theorem 2.1 for a moment in order to mention a result that is more useful in applications (cf. Theorem 3.2). This corollary can be proved by a straightforward induction over N.

Corollary 2.2. *Let $H_1, \ldots, H_N$ be Hilbert spaces and for $1 \leqslant j \leqslant N$ let $T_j \in L(H_j)$ be a bounded linear operator on H_j. If all the operators T_j have property (β), then the commuting tuple $T \in L(H)^N$,*

$$H = H_1 \widehat{\otimes} \cdots \widehat{\otimes} H_N$$

and

$$T = (T_1 \otimes I \otimes \cdots \otimes I, \ldots, I \otimes \cdots \otimes I \otimes T_N)$$

has property (β).

Proof of Theorem 2.1. Because S and T have property (β), they admit decomposable resolutions

$$0 \to H \xrightarrow{\varepsilon_{|}} H^0 \xrightarrow{d_{|}^0} H^1 \xrightarrow{d_{|}^1} \cdots \tag{2.1}$$

and

$$0 \to K \xrightarrow{\varepsilon_{\|}} K^0 \xrightarrow{d_{\|}^0} K^1 \xrightarrow{d_{\|}^1} \cdots \tag{2.2}$$

by Theorem 1.2. From these resolutions, we construct the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle I \otimes d_{\|}^1} & & \uparrow{\scriptstyle I \otimes d_{\|}^1} & & \\
0 & \to & H^0 \widehat{\otimes} K^1 & \xrightarrow{d_{|}^0 \otimes I} & H^1 \widehat{\otimes} K^1 & \xrightarrow{d_{|}^1 \otimes I} & \cdots \\
 & & \uparrow{\scriptstyle I \otimes d_{\|}^0} & & \uparrow{\scriptstyle I \otimes d_{\|}^0} & & \\
0 & \to & H^0 \widehat{\otimes} K^0 & \xrightarrow{d_{|}^0 \otimes I} & H^1 \widehat{\otimes} K^0 & \xrightarrow{d_{|}^1 \otimes I} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

in which all the rows and columns are exact in all degrees except 0. Let $(X^\bullet, d^\bullet)$ denote the totalization of this double complex. With

$$\varepsilon : H \widehat{\otimes} K \to X^0 = H^0 \widehat{\otimes} K^0, \quad \varepsilon = \varepsilon_{|} \otimes \varepsilon_{\|}$$

we obtain the following finite sequence

$$0 \to H \widehat{\otimes} K \xrightarrow{\varepsilon} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \tag{2.3}$$

consisting of Hilbert spaces and bounded linear operators, and it remains to show that (2.3) is a decomposable resolution of R.

Lemma 2.3. *The sequence* (2.3) *is exact.*

Proof. The operator ε is injective because $\varepsilon_{|}$ and $\varepsilon_{\|}$ are injective and have closed ranges. Because $\operatorname{im} d_{|}^0$, $\operatorname{im} d_{\|}^j$ are closed and the diagram

$$\begin{array}{ccc} \ker(d_{|}^0 \otimes I) & \stackrel{\sim}{=\!=} & (\ker d_{|}^0)\widehat{\otimes} K^{j+1} \\ \uparrow{\scriptstyle I\otimes d_{\|}^0} & & \uparrow{\scriptstyle I\otimes d_{\|}^0} \\ \ker(d_{|}^0 \otimes I) & \stackrel{\sim}{=\!=} & (\ker d_{|}^0)\widehat{\otimes} K^{j} \end{array}$$

is commutative, we obtain for all $j \geqslant 0$ the canonical isomorphisms:

$$\begin{aligned} &\ker\big(I \otimes d_{\|}^j : \ker(d_{|}^0 \otimes I) \to \ker(d_{|}^0 \otimes I)\big) \\ &\qquad \cong \ker\big(I \otimes d_{\|}^j : (\ker d_{|}^0)\widehat{\otimes} K^j \to (\ker d_{|}^0)\widehat{\otimes} K^{j+1}\big) \cong (\ker d_{|}^0)\widehat{\otimes}(\ker d_{\|}^j) \end{aligned}$$

and

$$\begin{aligned} (I \otimes d_{\|}^j)\big(\ker(d_{|}^0 \otimes I)\big) &\cong (I \otimes d_{\|}^j)\big((\ker d_{|}^0)\widehat{\otimes} K^j\big) \\ &\cong (\ker d_{|}^0)\widehat{\otimes}(\operatorname{im} d_{\|}^j) = (\ker d_{|}^0)\widehat{\otimes}(\ker d_{\|}^{j+1}). \end{aligned}$$

In particular, we have for $j \geqslant 1$ the identity:

$$\ker\big(I \otimes d_{\|}^j : \ker(d_{|}^0 \otimes I) \to \ker(d_{|}^0 \otimes I)\big) = (I \otimes d_{\|}^{j-1})\big(\ker(d_{|}^0 \otimes I)\big).$$

The proof of Lemma 2.3 is now completed by using the fact that the homology spaces of (2.3) can be identified with certain iterated homology spaces (cf. Lemma A2.6 in [2]).

If $j = 0$, then we have

$$\begin{aligned} \ker d^0 = H^0(X) &= \ker\big(I \otimes d_{\|}^0 : \ker(d_{|}^0 \otimes I) \to \ker(d_{|}^0 \otimes I)\big) \\ &\cong (\ker d_{|}^0)\widehat{\otimes}(\ker d_{\|}^0) = (\operatorname{im} \varepsilon_{|})\widehat{\otimes}(\operatorname{im} \varepsilon_{\|}) \cong \operatorname{im}(\varepsilon_{|} \otimes \varepsilon_{\|}) = \operatorname{im} \varepsilon, \end{aligned}$$

and for $j \geqslant 1$ the j-th homology space satisfies:

$$H^j(X) \cong \frac{\ker\big(I \otimes d_{\|}^j : \ker(d_{|}^0 \otimes I) \to \ker(d_{|}^0 \otimes I)\big)}{(I \otimes d_{\|}^{j-1})\big(\ker(d_{|}^0 \otimes I)\big)} = 0. \quad \blacksquare$$

Let $S^{(p)} \in L(H^p)^m$ and $T^{(q)} \in L(K^q)^n$ $(p, q = 0, 1, \ldots)$ be the decomposable tuples such that the resolutions (2.1) and (2.2) intertwine $S, S^{(0)}, S^{(1)}, \ldots$ and $T, T^{(0)}, T^{(1)}, \ldots$ componentwise. In order to construct a decomposable resolution of $R \in L(H \widehat{\otimes} K)^{m+n}$, we define:

$$R^{(p,q)} := (S^{(p)} \otimes I, I \otimes T^{(q)}) \in L(H^p \widehat{\otimes} K^q)^{m+n}$$

and

$$R^{(j)} := \bigoplus_{p+q=j} R^{(p,q)} \in L(X^j)^{m+n}.$$

Because decomposability is stable under forming direct sums, every tuple $R^{(j)}$ is decomposable by Theorem 1.3. In view of Theorem 1.2, the following lemma completes the proof of Theorem 2.1.

LEMMA 2.4. *The sequence* (2.3) *is a decomposable resolution of* R.

Proof. We only have to show that the resolution (2.3) intertwines the tuples $R, R^{(0)}, R^{(1)}, \dots$ componentwise. On $H\widehat{\otimes}K$ we have:

$$\begin{aligned}\varepsilon \circ R &= (\varepsilon_{|} \otimes \varepsilon_{\|}) \circ (S \otimes I, I \otimes T) = \big((\varepsilon_{|} \circ S_1) \otimes \varepsilon_{\|}, \dots, \varepsilon_{|} \otimes (\varepsilon_{\|} \circ T_n)\big)\\ &= \big((S_1^{(0)} \circ \varepsilon_{|}) \otimes \varepsilon_{\|}, \dots, \varepsilon_{|} \otimes (T_n^{(0)} \circ \varepsilon_{\|})\big) = R^{(0)} \circ \varepsilon,\end{aligned}$$

and on $H^p\widehat{\otimes}K^q$ with $p + q = j$ and $1 \leqslant \mu \leqslant m$ the following holds:

$$\begin{aligned}d^j \circ (S_\mu^{(p)} \otimes I) &= \big((d_{|}^p \otimes I) + (-1)^p(I \otimes d_{\|}^q)\big) \circ (S_\mu^{(p)} \otimes I)\\ &= (d_{|}^p \circ S_\mu^{(p)}) \otimes I + (-1)^p(S_\mu^{(p)} \otimes d_{\|}^q)\\ &= (S_\mu^{(p+1)} \circ d_{|}^p) \otimes I + (-1)^p(S_\mu^{(p)} \otimes d_{\|}^q)\\ &= (S_\mu^{(p+1)} \otimes I) \circ (d_{|}^p \otimes I) + (S_\mu^{(p)} \otimes I) \circ \big((-1)^p(I \otimes d_{\|}^q)\big).\end{aligned}$$

The operators $d^j \circ (I \otimes T_\nu^{(q)})$ with $1 \leqslant \nu \leqslant n$ are treated in an analogues manner, and from this we conclude:

$$d^j \circ R^j = R^{j+1} \circ d^j \qquad \text{for } j \geqslant 0. \quad \blacksquare$$

3. APPLICATION TO THE HARDY SPACE OVER THE POLYDISC

We denote the unit disc in $\mathbb{C}$ by $\mathbb{D}$ and the unit polydisc in $\mathbb{C}^n$ by $\mathbb{D}^n$. $\mathbb{T}$ is the topological boundary of $\mathbb{D}$ and $\mathbb{T}^n$ is the n-fold cartesian product of $\mathbb{T}$, i.e. $\mathbb{T}^n$ is the distinguished boundary of $\mathbb{D}^n$.

The Hardy space $H^2(\mathbb{D}^n)$ is defined as the space of all holomorphic functions u in $\mathbb{D}^n$ that satisfy:

$$\|u\|_2 := \sup_{0<r<1} \Big(\int_{\mathbb{T}^n} |u(rz)|^2 \, \mathrm{d}\lambda_n(z) \Big)^{1/2} < \infty.$$

Here, λ_n denotes the Lebesgue measure on $\mathbb{T}^n$. It is well-known, that the expression $\| \cdot \|_2$ defines a norm on $H^2(\mathbb{D}^n)$ that turns this space into a Hilbert space (cf. [4] and [5]). Via non-tangential limits, the Hardy space $H^2(\mathbb{D}^n)$ can be isometrically embedded into $L^2(\mathbb{T}^n)$. $H^\infty(\mathbb{D}^n)$ is the Banach algebra of all bounded holomorphic functions in $\mathbb{D}^n$, equipped with the supremum norm. If $f \in H^\infty(\mathbb{D}^n)$, then multiplication with f defines a bounded linear operator T_f on $H^2(\mathbb{D}^n)$, a so called Toeplitz operator (cf. [6] and [7]).

Now suppose that $u_1, \dots, u_n$ belong to $H^2(\mathbb{D})$. We then define the holomorphic function U in $\mathbb{D}^n$ by

$$U(z_1, \dots, z_n) = u_1(z_1)u_2(z_2)\cdots u_n(z_n).$$

By Fubini's theorem, we have for all $0 < r < 1$:

$$\int_{\mathbb{T}^n} |U(rz)|^2 \,\mathrm{d}\lambda_n(z) = \prod_{j=1}^{n} \int_{\mathbb{T}} |u_j(rz)|^2 \,\mathrm{d}\lambda_1(z)$$

and therefore:

$$U \in H^2(\mathbb{D}^n) \quad \text{and} \quad \|U\|_2 = \|u_1\|_2 \cdots \|u_n\|_2.$$

Here, we used the fact that the supremum defining the norm $\|\cdot\|_2$ is in fact a monotone limit.

LEMMA 3.1. *The map*

$$\Phi_0 : H^2(\mathbb{D}) \times \cdots \times H^2(\mathbb{D}) \to H^2(\mathbb{D}^n), \quad \Phi_0(u_1, \dots, u_n) = U$$

induces an isometric isomorphism:

$$H^2(\mathbb{D})\widehat{\otimes} \cdots \widehat{\otimes} H^2(\mathbb{D}) \cong H^2(\mathbb{D}^n).$$

Proof. The multilinear map Φ_0 factors through the algebraic tensor product and the induced linear map $\widehat{\Phi}_0 : \bigotimes H^2(\mathbb{D}) \to H^2(\mathbb{D}^n)$ satisfies by Fubini's theorem:

$$\langle \widehat{\Phi}_0(u_1 \otimes \cdots \otimes u_n), \widehat{\Phi}_0(v_1 \otimes \cdots \otimes v_n) \rangle = \langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle.$$

Hence, $\widehat{\Phi}_0$ extends to an isometric operator $\Phi : \widehat{\otimes} H^2(\mathbb{D}) \to H^2(\mathbb{D}^n)$. Because Φ is isometric, it is injective and has a closed range and since the holomorphic polynomials are dense in $H^2(\mathbb{D}^n)$, its range is dense. Thus, Φ is surjective. ∎

We now can use this identification to derive the following result from Theorem 2.1.

THEOREM 3.2. *Suppose that* $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ *and define the functions* $F_j \in H^\infty(\mathbb{D}^n)$ *for* $1 \leqslant j \leqslant n$ *by:*

$$F_j(z_1, \dots, z_n) = f_j(z_j).$$

Then the commuting tuple $(T_{F_1}, \dots, T_{F_n})$ *of bounded linear operators on* $H^2(\mathbb{D}^n)$ *has property* (β).

Proof. Because the Hardy space $H^2(\mathbb{D})$ can be regarded as a closed subspace of $L^2(\mathbb{T})$, the Toeplitz operators T_{f_j} are subnormal and therefore have property (β). The statement of Theorem 3.2 now follows from Corollary 2.2, because we have:

$$I \otimes \cdots T_{f_j} \cdots \otimes I = T_{F_j}. \quad \blacksquare$$

In [3] and [6] it was shown that the Hardy space $H^2(b\Omega)$ over a bounded weakly pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ has a localization property known as quasi-coherence. It follows directly from the definitions that a Hardy space is quasi-coherent if and only if the tuple $(T_{z_1}, \dots, T_{z_n})$ of multiplication operators with the coordinate functions has property (β). We therefore obtain the following corollary.

Corollary 3.3. *The Hardy space $H^2(\mathbb{D}^n)$ over the unit polydisc in $\mathbb{C}^n$ is quasi-coherent.*

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Receved October 22, 1997; revised March 23, 1998.