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COMMUTATIVE RADICAL OPERATOR ALGEBRAS

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ABSTRACT. Let A be a norm-closed operator algebra which is radical; that is, each element in A is quasinilpotent. We consider the case when such algebras satisfy the stronger condition of being uniformly topologically nil. In particular, we study this question when A is generated by a quasinilpotent weighted shift or by the Volterra operator.

KEYWORDS: *radical operator algebra, quasinilpotent, uniformly topologically nil, weighted shift, Volterra operator.*

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A Banach algebra $\mathfrak{A}$ is radical if every $T \in \mathfrak{A}$ has spectral radius zero. A radical Banach algebra is uniformly topologically nil ([4]) if the sequences $\|T^n\|^{1/n}$ converge to zero at a uniform rate as T ranges over the unit ball of $\mathfrak{A}$. The purpose of this note is to examine the implications of this definition in the context of operator algebras (i.e., not necessarily self-adjoint algebras of operators on Hilbert space). The simplest radical operator algebras are those generated by a single quasinilpotent operator. Such an algebra is perforce radical. While one might expect that the norm-closed operator algebra generated by a single quasinilpotent operator to be uniformly topologically nil, that turns out not to be the case, as examples with weighted shifts and the Volterra operator show (See Examples 2.5, 2.8). As these examples suggest, it is difficult to obtain positive results of the form that certain classes of operator algebras are uniformly topologically nil. We know this is the case if the algebra is finitely (algebraically) generated as an ideal in the unitization; however, we suspect that all such algebras are finite dimensional, and indeed we can prove that this is the case if the algebra is singly algebraically generated as an ideal in the unitization (cf. Proposition 2.2.). Another question is whether the norm-closed algebra $\mathfrak{A}_T$ generated by a quasinilpotent operator T is an integral domain, if T is not nilpotent. In some cases, e.g., for weighted shifts, $\mathfrak{A}_T$ is an integral domain; however in the case of the Volterra operator it fails. There is also an open question of whether a commutative radical Banach algebra which is an integral domain can have prime elements ([15]). Even in the most tractable cases the question of the (non-)existence of primes is difficult, though there are two cases for which something can be said. (See Remark 2.7, Proposition 2.15.)

Section 1 contains a review of facts concerning radical Banach algebras and a discussion of (not necessarily radical) normed algebras whose norm closures are radical. Variants of topological nilpotence, due to Dixon ([4]), are introduced.

Section 2 is devoted primarily to singly generated algebras: if T is a bounded, linear operator which is quasinilpotent, what can be said about the norm closed algebra $\mathfrak{A}_T$ which it generates:

(i) Is $\mathfrak{A}_T$ topologically nilpotent?

(ii) If (i) fails, can it happen that the normalized powers of the generator T form a uniformly quasinilpotent sequence?

(iii) Is $\mathfrak{A}_T$ an integral domain?

(iv) Is T a prime element in $\mathfrak{A}_T$?

The main examples we consider are the weighted unilateral shift, and the Volterra operator.

1. PRELIMINARIES

If $\mathfrak{A}$ is an algebra over $\mathbb{C}$ without unit, and $A \in \mathfrak{A}$, the spectrum is defined by $\mathrm{sp}\,(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \widetilde{\mathfrak{A}}\}$, where $\widetilde{\mathfrak{A}}$ is the unitization of $\mathfrak{A}$. An algebra norm $\|\cdot\|$ on $\mathfrak{A}$ is a *spectral norm* if $\|A\| \geqslant \rho(A)$, $A \in \mathfrak{A}$, where $\rho(A) := \sup\{|\lambda| : \lambda \in \mathrm{sp}\,(A)\}$ is the spectral radius. The algebra $\mathfrak{A}$ is then called a *spectral normed algebra*.

Definition 1.1. Let $\mathfrak{A}$ be an algebra over $\mathbb{C}$. *The Jacobson radical* is $\mathrm{rad}\,(\mathfrak{A}) = \{b \in \mathfrak{A} : \rho(ab) = 0 \text{ for all } a \in \mathfrak{A}\}$. The ideal $\mathrm{rad}\,(\mathfrak{A})$ is the largest ideal $\mathcal{I}$ satisfying $\rho(b) = 0$ for all $b \in \mathcal{I}$. $\mathfrak{A}$ is called a *radical algebra* if $\mathfrak{A} = \mathrm{rad}\,(\mathfrak{A})$.

Definition 1.2. An *operator algebra* $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the bounded, linear operators on a complex Hilbert space $\mathcal{H}$.

There is an abstract characterization of operator algebras. If $\mathfrak{A}$ is a unital Banach algebra with an operator algebra structure (i.e., an L^∞-matrix norm structure on $\mathfrak{A}$ with respect to which $M_\infty(\mathfrak{A})$ becomes a normed algebra), then $\mathfrak{A}$ is completely isometrically isomorphic to an operator algebra ([2]).

Note that an operator algebra need not be closed in any topology.

If $\mathfrak{A}$ is a normed algebra, and $A \in \mathfrak{A}$, then $\lim_n \|A^n\|^{1/n} \leqslant \rho(A)$; if the norm is spectral, then equality holds. Any Banach algebra norm is a spectral norm (cf. [12], 2.2.2, 2.2.8).

Recall that an element A in an algebra (or a ring) $\mathfrak{A}$ is *nilpotent* if there is a positive integer n such that $A^n = 0$, and the algebra $\mathfrak{A}$ is *nil* if each element $A \in \mathfrak{A}$ is nilpotent. Any nil algebra is a radical algebra. We are interested in comparing properties of a normed algebra $\mathfrak{A}$ with its norm closure.

Proposition 1.3. *Let $\mathfrak{A}$ be a commutative nil normed algebra. Then its norm-completion is a radical Banach algebra.*

As this result is subsumed by Proposition 1.4, the proof will be omitted. We note that commutativity is essential to the conclusion. There are noncommutative nil normed algebras (operator algebras) whose norm-completions are semisimple. (See [8]. Examples can also be obtained from [6].)

An element T in a normed algebra $\mathfrak{A}$ will be called *quasinilpotent* if $\lim_{n\to\infty} ||T^n||^{1/n} = 0$. While the term "topologically nilpotent" is also used for this property, that term will be reserved for other purposes here (cf. Definition 1.5).

Proposition 1.4. *Let $\{T_i\}_{i\in I}$ be a family of pairwise commuting, quasinilpotent elements of a commutative Banach algebra $\mathcal{B}$. Then the norm-closed subalgebra $\mathfrak{A}$ generated by $\{T_i\}_{i\in I}$ is a radical Banach algebra.*

Proof. Since $\mathfrak{A}$ is commutative, all quasinilpotent elements are in the radical. But the radical is a closed ideal, so the radical of $\mathfrak{A}$ is $\mathfrak{A}$. ∎

As an application, we note that the norm closure of a semisimple operator algebra can be radical. Consider the polynomial ring $\mathcal{P} = \mathbb{C}[x]$ and its subring (maximal ideal) $\mathcal{P}_0$ of polynomials vanishing at $x = 0$. Both are semisimple. If $T \in \mathcal{B}(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$, and if T is not algebraic, then the image $\mathfrak{A}_0$ (resp., $\widetilde{\mathfrak{A}}_0$) of $\mathcal{P}_0$ (resp., $\mathbb{C}[x]$) in $\mathcal{B}(\mathcal{H})$ under the algebra isomorphism $p \mapsto p(T)$ is semisimple since the Jacobson radical can be characterized purely algebraically. If in addition T is quasinilpotent, $\mathfrak{A}_0$ is a semisimple algebra of quasinilpotent operators. This anomaly reflects the fact that the operator norm on $\mathfrak{A}_0$ is not spectral. By the proposition, the norm closure $\mathfrak{A}$ of $\mathfrak{A}_0$ is radical.

Recently there has been considerable progress in the study of radical Banach algebras. (Cf. [4], [5], [15].) The following definitions are due to Peter Dixon (in the context of Banach algebras).

DEFINITION 1.5. Let $\mathfrak{A}$ be a normed algebra. Set

$$a_n = \sup\bigl\{\|T_1T_2\cdots T_n\|^{1/n} : T_k \in \operatorname{ball}(\mathfrak{A}), 1 \leqslant k \leqslant n\bigr\}$$
$$b_n = \sup\bigl\{\|T^n\|^{1/n} : T \in \operatorname{ball}(\mathfrak{A})\bigr\}, \quad n = 1, 2, \ldots.$$

(i) $\mathfrak{A}$ is topologically nilpotent if $\lim\limits_n a_n = 0$;

(ii) $\mathfrak{A}$ is uniformly topologically nil if $\lim\limits_n b_n = 0$;

(iii) $\mathfrak{A}$ is weakly topologically nilpotent if for every sequence $(T_n)_{n=1}^{\infty}$ in $\operatorname{ball}(\mathfrak{A})$, $\lim\limits_n \|T_1T_2\cdots T_n\|^{1/n} = 0$.

Clearly (i) implies (ii); but Dixon and Müller have a noncommutative Banach algebra which satisfies (ii) but not (i). However, for commutative algebras we have:

THEOREM 1.6. *Let* $\mathfrak{A}_0$ *be a commutative normed algebra,* $\mathfrak{A}$ *its Banach algebra completion. Consider the following conditions:*

(i) $\mathfrak{A}_0$ *is topologically nilpotent;*

(ii) $\mathfrak{A}$ *is topologically nilpotent;*

(iii) $\mathfrak{A}_0$ *is uniformly topologically nil;*

(iv) $\mathfrak{A}$ *is uniformly topologically nil;*

(v) $\mathfrak{A}_0$ *is weakly topologically nilpotent;*

(vi) $\mathfrak{A}$ *is weakly topologically nilpotent.*

Then (i), (ii), (iii), (iv), (vi) *are equivalent.* (v) *does not imply* (vi).

Proof. The proofs of the equivalence of (i) and (ii), as well as (iii) and (iv), follows from the fact that the suprema in Definition 1.5 taken over a dense subset of the unit ball are the same as the suprema over the unit ball. That (ii) implies (iv) is trivial, while the implication (iv) implies (ii) is given in [4]. Clearly, (ii) implies (vi), and that (vi) implies (ii) is given in [5]. (That does not use commutativity.) To complete the proof, we give an example that (v) does not imply (vi).

Let $T_n \in M_n$ be the matrix which is zero except on the superdiagonal, where all the entries are 1. Let $\mathcal{U}_n$ be the algebra generated by T_n, so $\mathcal{U}_n$ consists of all operators T of the form $T = \sum\limits_{k=1}^{n-1} a_k T_n^k$, $a_k \in \mathbb{C}$. Let $\mathfrak{A}_0$ be the algebraic direct sum $\bigoplus\limits_{n\geqslant 2} \mathcal{U}_n$, so that $A \in \mathfrak{A}_0$ is a sequence $A = (S_2, S_3, \ldots)$ with the property there is an integer N (which may depend on A) with $S_n = 0$, $n \geqslant N$. $\mathfrak{A}_0$ is an operator algebra with the norm: $\|A\| = \sup\limits_n \|S_n\|$, where $\|S_n\|$ is the operator norm of the matrix S_n. It is easy to see $\mathfrak{A}_0$ is not topologically nilpotent: the $\{a_n\}$ in

Definition 1.5 are all 1. Let $\mathfrak{A}$ be the norm completion of $\mathfrak{A}_0$. Since statements (i) and (vi) are equivalent, it follows that $\mathfrak{A}$ is not weakly topologically nilpotent.

To see that $\mathfrak{A}_0$ is weakly topologically nilpotent, let $(A_n)_{n=1}^{\infty}$ be a sequence in $\operatorname{ball}(\mathfrak{A}_0)$: $A_n = (S_{1n}, S_{2n}, S_{3n}, \ldots)$. There is an $N \in \mathbb{N}$ such that for $k > N$, $S_{k1} = 0$. Thus in any product, $A_1 \cdots A_m$, the components belonging to $\mathcal{U}_k$, for $k > N$, are all zero. So $A_1 \cdots A_m$ belongs to the subalgebra $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n$. This is nilpotent of degree $N - 1$, so that $A_1 \cdots A_m = 0$ if $m \geqslant N - 1$. ∎

Comment 1.7. Our reason for the rather trivial extension of the definitions in [4] from the Banach algebra to the normed algebra setting is that in subsequent examples of operator algebras in terms of generators it is convenient to work with polynomials in the generators.

Question 1.8. Let $\mathfrak{A}_0$ be a weakly topologically nilpotent normed algebra. Then each $T \in \mathfrak{A}_0$ is quasinilpotent. If $\mathfrak{A}_0$ is commutative, then by Proposition 1.4 the norm closure $\mathfrak{A}$ is radical. If $\mathfrak{A}_0$ is not commutative, does it follow that $\mathfrak{A}$ is radical?

Example 1.9. A class of noncommutative operator algebras $\mathfrak{A}_0$ for which $\mathfrak{A}_0$ weakly topologically nilpotent implies the norm-closure $\mathfrak{A}$ is radical can be obtained from the theory of triangular AF-algebras. Briefly, let $\mathcal{T}$ be a TAF algebra with diagonal $\mathcal{D} = \mathcal{T} \cap \mathcal{T}^*$, and $\mathcal{T}^0$ the maximal diagonally disjoint subalgebra. If $\mathcal{T}$ is obtained as an inductive limit with respect to a fixed system of matrix units (which is always the case), let $\mathfrak{A}_0$ be the span of the strictly upper triangular matrix units in $\mathcal{T}$. Then the norm-closure of $\mathfrak{A}_0$ is $\mathfrak{A} = \mathcal{T}^0$. If $\mathfrak{A}$ is not radical, then from [6], Theorem 3, there is a sequence $(T_n) \subset \operatorname{ball}(\mathfrak{A}_0)$ with $\|T_1 T_2 \cdots T_n\| = 1$ for all n. In fact, the T_k can be chosen as strictly upper triangular matrix units. Thus $\mathfrak{A}_0$ is not weakly topologically nilpotent.

2.

In this section we investigate some singly generated radical operator algebras. Let $\mathcal{P}$ denote the set of complex polynomials in one variable, and $\mathcal{P}_0$ the subset of polynomials which vanish at 0. For an operator $T \in \mathcal{B}(\mathcal{H})$, let $\mathcal{P}_0(T)$ denote the operator algebra T generates: $\mathcal{P}_0(T) = \{p(T) : p \in \mathcal{P}_0\}$. $\mathfrak{A}_T$ will denote the norm-closure of $\mathcal{P}_0(T)$ in $B(\mathcal{H})$. A commutative normed algebra $\mathfrak{A}$ is an integral domain if it satisfies: $S, T \in \mathfrak{A}$ both nonzero implies $0 \neq ST$. If $\mathfrak{A}$ is a nonunital commutative normed algebra (not necessarily an integral domain), $S, T \in \mathfrak{A}$, say S divides T if $T = SU$ for $U \in \widetilde{\mathfrak{A}}$, where $\widetilde{\mathfrak{A}}$ is the unitization of $\mathfrak{A}$. We say that T is *prime* if whenever T divides a product SR, then either T divides S or T divides

R. We will be considering the following questions for a quasinilpotent operator $T \in \mathcal{B}(\mathcal{H})$:

(i) Is $\mathfrak{A}_T$ topologically nilpotent?

(ii) If T is not nilpotent (hence $\mathcal{P}_0(T)$ is an integral domain), is $\mathfrak{A}_T$ an integral domain?

(iii) If T is not nilpotent (in which case T is prime in $\mathcal{P}_0(T)$), is T prime in $\mathfrak{A}_T$?

The answers to (i) or (ii) will depend on the choice of T. It is an open question whether a radical commutative Banach algebra can possess a prime element if it is an integral domain ([15]).

If a commutative radical Banach algebra has a sufficiently nice algebraic structure, then it is topologically nilpotent.

PROPOSITION 2.1. *Let $\mathfrak{A}$ be a commutative radical Banach algebra. If $\mathfrak{A}$ is finitely generated as an ideal in $\widetilde{\mathfrak{A}}$ (the unitization of $\mathfrak{A}$), then $\mathfrak{A}$ is topologically nilpotent.*

Proof. By hypothesis there exist elements $T_1, \dots, T_K \in \mathfrak{A}$ such that the ideal in $\widetilde{\mathfrak{A}}$ generated by $\{T_1, \dots, T_K\}$ is $\mathfrak{A}$. We may assume $||T_k|| \leqslant 1,\ 1 \leqslant k \leqslant K$. Thus, any element $T \in \mathfrak{A}$ has the form

$$T = \sum_{k=1}^{K} T_k U_k$$

where $U_k \in \widetilde{\mathfrak{A}}$, $1 \leqslant k \leqslant K$. Thus, $T^n = \sum_{\substack{i_1,\dots,i_K \geqslant 0 \\ i_1+\cdots+i_K=n}} \frac{n!}{i_1!\cdots i_K!} T^{i_1} U_1^{i_1} \cdots T_K^{i_K} U_K^{i_K}$. Let $M = \max\{\|U_1\|, \dots, \|U_K\|\}$. Then

$$\|T^n\| \leqslant \sum_{\substack{i_1,\dots,k_K \geqslant 0 \\ i_1+\cdots+i_K=n}} \frac{n!}{i_1! \cdots i_K!} \|T_1^{i_1}\| \cdots \|T_K^{i_K}\| \cdot M^n.$$

For $n = 1, 2, \dots$, set

$$a_n = \max\left\{ \left(\|T_1^{i_1}\| \cdots \|T_K^{i_K}\|\right)^{1/n} : i_1, \dots, i_K \geqslant 0,\ i_1 + \cdots + i_K = n \right\}.$$

Then $\{a_n\}$ converges to zero, since

$$a_n \leqslant \max\left\{ \|T_k^{[n/K]}\|^{1/n} : 1 \leqslant k \leqslant K \right\},$$

where $[n/K]$ is the greatest integer $\leqslant n/K$. The right side approaches zero as the elements T_k are quasinilpotent. Thus,

$$\begin{aligned} \|T^n\| &\leqslant M^n \cdot K^n a_n^n, \quad \text{or} \\ \|T^n\|^{1/n} &\leqslant M \cdot K a_n, \quad \text{for } n = 1, 2, \dots. \end{aligned}$$

Define $\Lambda_k = \{T \in \mathfrak{A} : \|T\| \leqslant 1, \text{ and } \|T^n\|^{1/n} \leqslant k \cdot a_n,\ n = 1, 2, \dots\}$. Now Λ_k is closed for each $k = 1, 2, \dots$, and each T in the unit ball of $\mathfrak{A}$ belongs to some Λ_k by the previous calculation. Thus $\operatorname{ball}(\mathfrak{A}) = \bigcup_{k=1}^{\infty} \Lambda_k$, and by the Baire Category Theorem some Λ_k has nonempty interior, say Λ_L. Let $B(T_0; \varepsilon)$ be a ball centered at T_0, radius ε, contained in interior of Λ_L. Thus, if $T \in B(T_0; \varepsilon)$,

$$\begin{aligned} \|(T - T_0)^n\| &\leqslant \sum_{k=0}^{n} \binom{n}{k} \|T^k\| \|T_0^{n-k}\| \leqslant \sum_{k=0}^{n} \binom{n}{k} L^k a_k^k L^{n-k} a_{n-k}^{n-k} \\ &\leqslant \sum_{k=0}^{n} \binom{n}{k} L^n b_n^n \leqslant 2^n L^n b_n^n \end{aligned}$$

where $b_n = \max_{0 \leqslant k \leqslant n} \{(a_k^k a_{n-k}^{n-k})^{1/n}\}$. Since $T - T_0$ is in $B(0; \varepsilon)$, the ball centered at 0 withradius ε, it follows that if $T \in \operatorname{ball}(\mathfrak{A})$, $T = (1/\varepsilon)T'$, $T' \in B(0; \varepsilon)$. Thus if $T \in \operatorname{ball}(\mathfrak{A})$,

$$\|T^n\|^{1/n} \leqslant \frac{2}{\varepsilon} L b_n.$$

Finally, if $c_n = (2/\varepsilon) L b_n$, $n = 1, 2, \dots$, we have that $\lim_n c_n = 0$, and for any $T \in \operatorname{ball}(\mathfrak{A})$,

$$\|T^n\|^{1/n} \leqslant c_n.$$

This shows that $\mathfrak{A}$ is uniformly topologically nil, which by Theorem 1.6 is equivalent to topological nilpotence. ∎

If the hypotheses of the proposition are weakened by assuming that $\mathfrak{A}$ is finitely generated as a *closed* ideal, the conclusion may fail. Indeed, if $\mathfrak{A}$ is the norm-closed algebra generated by a single quasinilpotent operator, $\mathfrak{A}$ may fail to be topologically nilpotent. This will be seen in the example with the Volterra operator, as well as for certain shifts.

We have no example of a Banach algebras satisfying the hypotheses of Proposition 2.1 which is not finite dimensional.

OPEN QUESTION. Let $\mathfrak{A}$ be a commutative radical Banach algebra which is finitely (algebraically) generated as an ideal in $\widetilde{\mathfrak{A}}$. Is $\mathfrak{A}$ necessarily finite dimensional?

It may be that this question is related to questions concerning prime elements. (See [15].) We can prove, however, that if $\mathfrak{A}$ is a commutative Banach algebra which is *singly* algebraically generated as an ideal, then $\mathfrak{A}$ is finite dimensional. In fact the following can be obtained without assuming commutativity of $\mathfrak{A}$.

PROPOSITION 2.2. *Let $\mathfrak{A}$ be a radical Banach algebra. Suppose for some T in the center of $\mathfrak{A}$*

$$T\mathfrak{A} + \mathbb{C}T = \mathfrak{A}.$$

Then T is nilpotent, and $\mathfrak{A}$ is finite dimensional.

Proof. Let $\Lambda : \mathfrak{A} \to \mathfrak{A}$, $\Lambda(A) = TA$.

Claim 1. Given $n \in \mathbb{Z}^+$, $B \in \mathfrak{A}$, there is $A \in \mathfrak{A}$ and a polynomial p of degree at most n vanishing at 0, such that $B = T^nA + p(T)$. Furthermore, if $\mathcal{P}_0^{(n)}$ denotes the set of polynomials of degree at most n, vanishing at zero, then, if $T^n \neq 0$,

$$\mathfrak{A} = \operatorname{Ran} \Lambda^n \oplus \mathcal{P}_0^{(n)}(T).$$

We first show $\mathfrak{A} = \operatorname{Ran} \Lambda^n + \mathcal{P}_0^{(n)}(T)$. For $n = 1$, this is by hypothesis. For $n > 1$ assume inductively that $B = T^{n-1}A_1 + q(T)$ for some $A_1 \in \mathfrak{A}$ and polynomial q of degree at most $n-1$, $q(0) = 0$. Expressing A_1 as $A_1 = TA + cT$, we have

$$B = T^{n-1}(TA + cT) + q(T) = T^nA + cT^n + q(T) = T^nA + p(T)$$

where $p(x) = cx^n + q(x)$ has degree at most n, and $p(0) = 0$.

Next, suppose $T^nA = p(T)$, some $p \in \mathcal{P}_0^{(n)}$. Write $p(t) = \alpha_n t^n + \cdots + \alpha_1 t$. Then $T^nA - p(T) = 0 = T(C_1 - \alpha_1)$ where $C_1 = T^{n-1}A - \alpha_n T^{n-2} - \cdots - \alpha_2 T$. If $\alpha_1 \neq 0$, since $C_1 - \alpha_1$ is invertible in $\widetilde{\mathfrak{A}}$, the unitization of $\mathfrak{A}$, it follows that $T = 0$, contrary to hypotheses. So $\alpha_1 = 0$. Thus $T^nA - p(T) = 0$ has the form $T^2(C_2 - \alpha_2) = 0$, for some $C_2 \in \mathfrak{A}$. If $\alpha_2 \neq 0$, $C_2 - \alpha_2$ is invertible in $\widetilde{\mathfrak{A}}$, so that $T^2 = 0$, again a contradiction. Thus $\alpha_2 = 0$. Continuing in this way, we obtain $\alpha_1 = \cdots = \alpha_n = 0$, so $p = 0$.

Claim 2. If T is not nilpotent, $\ker \Lambda \subset \operatorname{Ran} \Lambda^n$.

Let $B \in \ker \Lambda$, $B = T^n + p(T)$, $p \in \mathcal{P}_0^{(n)}$ by Claim 1. So $0 = TB = T^{n+1}A + Tp(T)$. Again by Claim 1, both summands are zero. Hence, $Tp(T) = 0$.

As $\mathrm{sp}(T) = 0$, and T is not nilpotent, it must be that $p = 0$, and hence $B = T^n A \in \operatorname{Ran}\Lambda^n$.

Claim 3. $\operatorname{Ran}\Lambda^n$ is closed and of $\operatorname{codim} n$ in $\mathfrak{A}$.

Begin with $n = 1$. That $\operatorname{Ran}\Lambda$ is closed is the same argument as in [15], Section 2, but it is done here for the sake of completeness. Suppose $\{A_k\}_{n=1}^{\infty} \subset \mathfrak{A}$, and $\{TA_k\}$ converges to cT for some $c \neq 0$. Replacing A_k by $(1/c)A_k$, we may suppose $TA_k \to T$. As above, let $\widetilde{\mathfrak{A}}$ be the Banach algebra obtained by adjoining the identity to $\mathfrak{A}$, and $\widetilde{\Lambda} : \widetilde{\mathfrak{A}} \to \mathfrak{A}$, $\widetilde{\Lambda}(A + cI) = TA + cT$. $\widetilde{\Lambda}$ is a continuous linear map from one Banach space onto another, so it is an open map. Thus

$$T - TA_k = T(\lambda_k - B_k)$$

where $\lambda_k \in \mathbb{C}$, $B_k \in \mathfrak{A}$, and $\lambda_k - B_k \to 0$. Since $\mathfrak{A}$ and $\mathbb{C}$ are closed in $\widetilde{\mathfrak{A}}$, $B_k \to 0$ and $\lambda_k \to 0$. In particular, $\lambda_k \neq 1$ for some k. From the equation

$$T(1 - \lambda_k) = T(A_k - B_k)$$

we obtain $T = (1 - \lambda_k)^{-1} T(A_k - B_k) \in \operatorname{Ran}\Lambda$, contradicting that $\operatorname{Ran}\Lambda$, $\mathbb{C}T$ are linearly independent subspaces.

Given $n > 1$, assume inductively that $\operatorname{Ran}\Lambda^{n-1}$ is closed. Let $\{T^n A_k\}_{k=1}^{\infty}$ be a sequence in $\operatorname{Ran}\Lambda^n$ converging to some element of $\mathfrak{A}$. Since by induction $\operatorname{Ran}\,\Lambda^{n-1}$ is closed, $\{T^n A_k\} \to T^{n-1}B$. Write $B = TA + \lambda T$. If $\lambda = 0$, we are done. Otherwise divide by λ and replace $(A_k - A)/\lambda$ by A_k to obtain $T^n A_k \to T^n$ as $k \to \infty$. Now $\widetilde{\Lambda}^n : \widetilde{\mathfrak{A}} \to \operatorname{Ran}\widetilde{\Lambda}^n = \operatorname{Ran}\Lambda^{n-1}$ is an open map, so $T^n(A_k - 1) = T^n(B_k - \lambda_k)$ where $B_k \in \mathfrak{A}$, $\lambda_k \in \mathbb{C}$, and $B_k - \lambda_k \to 0$. As above, this implies that $B_k \to 0$ and $\lambda_k \to 0$. We have that $T^n(A_k - B_k) = T^n(1 - \lambda_k)$, so if some $\lambda_k \neq 1$, then $T^n = (1 - \lambda_k)^{-1} T^n (A_k - B_k)$, contradicting the direct sum decomposition in Claim 1.

Claim 4. If T is not nilpotent, $\ker\Lambda \neq (0)$.

If $\ker\Lambda = (0)$, then $\Lambda : \mathfrak{A} \to \mathfrak{A}$ has closed range with codim 1, so is Fredholm. As $\|\Lambda^n\| \leqslant \|T^n\|$, Λ is quasinilpotent. But then the Fredholm spectrum of Λ is empty, so $\mathfrak{A}$ is finite dimensional and T is nilpotent.

To complete the proof of the theorem, let $\mathfrak{B} = \bigcap_{n=1}^{\infty} \operatorname{Ran}\Lambda^n$. If T is not nilpotent, since $\mathfrak{B} \supset \ker\Lambda$ by Claim 2, we have that $\mathfrak{B} \neq (0)$ by Claim 4. Set $\Lambda_{\mathfrak{B}}$ the restriction of Λ to $\mathfrak{B}$. Then $\Lambda_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$ is onto and quasinilpotent. By [13], Theorem 4.15, the adjoint map $\Lambda_{\mathfrak{B}}^* : \mathfrak{B}^* \to \mathfrak{B}^*$ is one-to-one with closed range. Thus, by the Open Mapping Theorem (or, [13], Theorem 4.13) there is a

$\delta > 0$ such that $||\Lambda^* B^*|| \geqslant \delta ||B^*||$ for all $B^* \in \mathfrak{B}^*$. Hence, $||\Lambda^{*n} B^*|| \geqslant \delta^n ||B^*||$, and so Λ^* is not quasinilpotent. On the other hand, $||\Lambda^{*n}|| = ||\Lambda^n||$, so that Λ^* is quasinilpotent. This contradiction implies that T is nilpotent. Say $T^n \neq 0$, $T^{n+1} = 0$. By Claim 1, $\mathfrak{A} = \operatorname{Ran} \Lambda^n \oplus \mathcal{P}_0^{(n)}(T)$. So if $A \in \mathfrak{A}$, write $A = \Lambda^n B + p(T)$, $p \in \mathcal{P}_0^{(n)}$. But B has the form $B = \Lambda B_1 + \lambda T$, for some $B_1 \in \mathfrak{A}$, $\lambda \in \mathbb{C}$, so $\Lambda^n B = \Lambda^{n+1} B_1 + \lambda \Lambda^n T = T^{n+1}(B_1 + \lambda) = 0$. Thus, $\operatorname{Ran} \Lambda^n = (0)$, and $\mathfrak{A} = \mathcal{P}_0^{(n)}(T)$ is the finite dimensional algebra of polynomials in T. ∎

Next we turn to examples of operator algebras; these are operator algebras $\mathfrak{A}_T$ which are singly generated in the sense that $\mathfrak{A}_T$ is the norm closure in $\mathcal{B}(\mathcal{H})$ of polynomials in T, where $T \in \mathcal{B}(\mathcal{H})$ is the generator. Our first result is that if T is a weighted shift with positive weights tending monotonically to zero (hence T is quasinilpotent), then $\mathfrak{A}_T$ is topologically nilpotent. First we require a lemma.

Fix an orthonormal basis $\{\xi_n : n = 0, 1, 2, \ldots\}$ for a Hilbert space $\mathcal{H}$, and a positive decreasing sequence $\{\alpha_n\}$, $n = 1, 2, \ldots$ with $\lim\limits_{n\to\infty} \alpha_n = 0$, and let T denote the weighted shift on $\mathcal{H}$ given by: $T\xi_n = \alpha_{n+1}\xi_{n+1}$. It is convenient to normalize T so $||T|| = \alpha_1 = 1$.

LEMMA 2.3. *Let $\mathcal{P}$ be the set of complex polynomials, and $\mathcal{P}_0 = \{p \in \mathcal{P} : p(0) = 0\}$. Denote by $\mathcal{P}_0(T) = \{p(T) : p \in \mathcal{P}_0\}$. Let $m \in \mathbb{Z}^+$, $m > 1$, $p_1, \ldots, p_m \in \mathcal{P}_0$ with $\|p_k(T)\xi_0\| \leqslant 1$, $1 \leqslant k \leqslant m$. Then*

$$\|p_1(T) \cdots p_m(T)\xi_0\| \leqslant \|T^m\|.$$

Proof. Let $\beta_k = \prod\limits_{j=1}^{k} \alpha_j$, $k \in \mathbb{Z}^+$.

Claim. Let i, k be integers such that $k \geqslant m$, and $1 \leqslant i < k$. Then $\beta_k/\beta_i \leqslant \alpha_m \beta_{k-i}$.

As $i \geqslant 1$ and $(\alpha_n)_{n\geqslant 1}$ is decreasing, $\alpha_{i+1} \leqslant \alpha_2$, $\alpha_{i+2} \leqslant \alpha_3$, ..., $\alpha_{k-1} \leqslant \alpha_{k-i}$, and $\alpha_k \leqslant \alpha_m$. Thus

$$\alpha_{i+1} \cdots \alpha_{k-1}\alpha_k \leqslant \alpha_2 \cdots \alpha_{k-i}\alpha_m.$$

But the product on the left is β_k/β_i, and the right side is $\alpha_m\beta_{k-i}$, using the fact $\alpha_1 = 1$.

Returning to the lemma, write $p_j(t) = \sum\limits_{i=1}^{\infty} a_i^{(j)} t^i$, where for each j only finitely many of the $a_i^{(j)}$ are nonzero. We have

$$\left\|p_j(T)\xi_0\right\|^2 = \left\| \sum_{i=1}^{\infty} a_i^{(j)} T^i \xi_0 \right\|^2 = \left\| \sum_{i=1}^{\infty} a_i^{(j)} \beta_i \xi_i \right\|^2.$$

Thus, $\sum\limits_{i=1}^{\infty}\left|a_i^{(j)}\beta_i\right|^2 \leqslant 1,\ 1 \leqslant j \leqslant m$.

The proof will be by induction on m. Note that for $m = 1$, $\|p_1(T)\xi_0\| \leqslant 1 = \|T\|$. Suppose that $m > 1$ and whenever $q_1, \dots, q_{m-1} \in \mathcal{P}_0$ and $\|q_j(T)\xi_0\| \leqslant 1$, then $\|q_1(T)\cdots q_{m-1}(T)\xi_0\| \leqslant \beta_{m-1} = \|T^{m-1}\|$. e With $\{p_j\}_{1\leqslant j\leqslant m}$ as above, we compute

$$\begin{aligned}
\Big\|\prod_{j=1}^{m} p_j(T)\xi_0\Big\|^2 &= \Big\|\sum_{i_1,\dots,i_m=1}^{\infty} a_{i_1}^{(1)}\cdots a_{i_m}^{(m)} T^{i_1+\cdots+i_m}\xi_0\Big\|^2 \\
&= \Big\|\sum_{k=m}^{\infty}\sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\dots,i_m\geqslant 1}} a_{i_1}^{(1)}\cdots a_{i_m}^{(m)}\beta_k\xi_k\Big\|^2 \\
&= \sum_{k=m}^{\infty}\Big|\sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\dots,i_m\geqslant 1}} a_{i_1}^{(1)}\cdots a_{i_m}^{(m)}\Big|^2\beta_k^2 \\
&= \sum_{k=m}^{\infty}\Big|\sum_{i_1=1}^{k-m+1} a_{i_1}^{(1)}\sum_{\substack{i_2+\cdots+i_m=k-i_1\\ i_2,\dots,i_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2\beta_k^2 \\
&\leqslant \sum_{k=m}^{\infty}\sum_{i_1=1}^{k-m+1}\big|a_{i_1}^{(1)}\big|^2\Big|\sum_{\substack{i_2+\cdots+i_m=k-i_1\\ i_2,\dots,i_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2\beta_k^2.
\end{aligned}$$

By the claim, we obtain

$$\begin{aligned}
&\leqslant \sum_{k=m}^{\infty}\sum_{i_1=1}^{k-m+1}\big|a_{i_1}^{(1)}\big|^2\beta_{i_1}^2\alpha_m^2\Big|\sum_{\substack{i_2+\cdots+i_m=k-i_1\\ i_2,\dots,i_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2\beta_{k-i_1}^2 \\
&\leqslant \sum_{i_1=1}^{\infty}\big|a_{i_1}^{(1)}\big|^2\beta_{i_1}^2\sum_{k=i_1+m-1}^{\infty}\alpha_m^2\Big|\sum_{\substack{i_2+\cdots+i_m=k-i_1\\ i_2,\dots,k_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2\beta_{k-i_1}^2.
\end{aligned}$$

Making the change of index $l = k - i_1$, we obtain

$$\leqslant \sum_{i_1=1}^{\infty}\big|a_{i_1}^{(1)}\big|^2\beta_{i_1}^2\sum_{l=m-1}^{\infty}\alpha_m^2\Big|\sum_{\substack{i_2+\cdots+i_m=l\\ i_2,\dots,i_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2\beta_l^2.$$

Now $\sum\limits_{i_1=1}^{\infty}\big|a_{i_1}^{(1)}\big|^2\beta_{i_1}^2 = \|p_1(T)\xi_0\|^2 \leqslant 1$. Furthermore, $\sum\limits_{l=m-1}^{\infty}\Big|\sum\limits_{\substack{i_2+\cdots+i_m=l\\ i_2,\dots,i_m\geqslant 1}} a_{i_2}^{(2)}\cdots a_{i_m}^{(m)}\Big|^2$ $\cdot\beta_l^2$ is the norm $\|p_2(T)\cdots p_m(T)\xi_0\|^2$, which by the induction hypothesis is at most β_{m-1}^2. Thus, $\|p_1(T)\cdots p_m(T)\xi_0\|^2 \leqslant \alpha_m^2\beta_{m-1}^2 = \beta_m^2 \leqslant \|T^m\|$. ∎

Note that the hypothesis that the weights α_n decrease to zero implies that $\beta_m^{1/m} = \|T^m\|^{1/m}$ converges to zero, so that T is quasinilpotent.

THEOREM 2.4. *Let T as above be a weighted shift with positive, decreasing weight sequence converging to zero. Let $\mathfrak{A}_T$ be the norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T. For any $m \in \mathbb{Z}^+$ and $T_1, \dots, T_m \in \mathrm{ball}\,(\mathfrak{A}_T)$,*

$$\|T_1 \cdots T_m\| \leqslant \|T^n\|.$$

In particular, $\mathfrak{A}_T$ is topologically nilpotent.

Proof. With $\mathcal{P}_0$ as in the Lemma 2.3, $\mathcal{P}_0(T) = \{p(T) : p \in \mathcal{P}_0\}$ is a dense subalgebra of $\mathfrak{A}_T$. By Theorem 1.6 it is enough to prove that $\mathcal{P}_0(T)$ is topologically nilpotent. Thus we may take $T_j = p_j(T),\ p_j \in \mathcal{P}_0,\ 1 \leqslant j \leqslant m$, with $\|p_j(T)\| = 1$. Now for any $S \in \mathcal{B}(\mathcal{H})$, $\|S\| = \sup\{\|S\xi\| : \|\xi\| \leqslant 1$, ξ a finite linear combination of basis vectors $\{\xi_n\}_{n=0}^{\infty}\} = \sup\big\{\|Sq(T)\xi_0\| : q$ a polynomial, $\|q(T)\xi_0\| \leqslant 1\big\}$. Thus, $\|p_1(T)\cdots p_m(T)\| = \sup\big\{\|p_1(T)\cdots p_m(T)q(T)\xi_0\| : q$ a polynomial, $\|q(T)\xi_0\| \leqslant 1\big\}$. Noting that $(p_m q)(0) = 0$, $\|p_m(T)q(T)\xi_0\| \leqslant \|p_m(T)\|\,\|q(T)\xi_0\| \leqslant 1$, and $\|p_j(T)\xi_0\| \leqslant \|p_j(T)\|\,\|\xi_0\| \leqslant 1$, we may apply the Lemma 2.3 with the polynomial p_m in the lemma replaced by $p_m q$. We obtain

$$\|p_1(T)\cdots p_m(T)q(T)\xi_0\| \leqslant \|T^m\|.$$

Hence, $\|p_1(T)\cdots p_m(T)\| \leqslant \|T^m\|$. As T is quasinilpotent, $\mathfrak{A}_T$ is topologically nilpotent. ∎

Next we present a quasinilpotent weighted shift T for which $\mathfrak{A}_T$ is not topologically nilpotent. This shows that the hypothesis in Proposition 2.1 that $\mathfrak{A}$ be "finitely generated as an ideal" cannot be weakened to "finitely generated as a closed ideal".

Recall that if T is a weighted shift with weights $\{w_n\}$, and if $\lim\limits_n w_n = 0$, then T is quasinilpotent. This follows from $||T^n|| = \sup\limits_k |w_{k+1}w_{k+2}\cdots w_{k+n}|$.

EXAMPLE 2.5. Define a sequence of weights in the following manner:

$$\overbrace{\frac{1}{2},\dots,\frac{1}{2}}^{2n_1},\ a_1,\ \overbrace{\frac{1}{2^2},\dots,\frac{1}{2^2}}^{3n_2},\ a_2,\ \overbrace{\frac{1}{2^3},\dots,\frac{1}{2^3}}^{4n_3},\ a_3,\ \overbrace{\frac{1}{2^4},\dots,\frac{1}{2^4}}^{5n_4},\ a_4,\ \dots$$

where: $n_1 = 2$, $n_{k+1} = 1+$(number of terms preceding a_k), $k \geqslant 1$, and $a_k = \min\Big\{\Big(\frac{1}{n_k}\Big)^{n_k}, b_k\Big\}$, where $b_k = \dfrac{(1/2)(\text{product of the } n_{k+1} \text{ terms following } a_k)}{\text{product of the terms preceding } a_k}$. From this, one can compute

$$\|T^{n_1}\|^{1/n_1} = \frac{1}{2},\ \|T^{n_2}\|^{1/n_2} = \frac{1}{2^2},\ \dots,\ \|T^{n_k}\|^{1/n_k} = \frac{1}{2^k}, \quad k \geqslant 1.$$

Since for any $T \in \mathcal{B}(\mathcal{H})$ the limit $\lim_n \|T^n\|^{1/n}$ exists, it follows in this case that the limit is zero, so T is quasinilpotent. Let $T_n = \big(1/\|T^n\|\big)T^n$, $n \geqslant 1$. Note that $\|T_{n_1}^2\| = (1/2)^4/(1/4)^2 = 1$, $\|T_{n_2}^3\| = 1, \ldots, \|T_{n_k}^{k+1}\| = 1$. Thus, $\mathfrak{A}_T$ is not uniformly topologically nil, or what is equivalent, not topologically nilpotent.

REMARK 2.6. Let $\{\xi_n\}_{n\geqslant 0}$ be an orthonormal basis for a Hilbert space, and T a weighted shift, $T\xi_n = \alpha_{n+1}\xi_{n+1}$, with weight sequence $\alpha_k \neq 0$ for all $k \geqslant 1$. It follows from the formal power series description that $\mathfrak{A}_T$ is an integral domain (see [14]).

REMARK 2.7. Here we consider the question of whether T is prime in $\mathfrak{A}_T$. As in Theorem 2.4 we suppose that the sequence $(\alpha_n)_1^\infty$ of positive weights is monotone decreasing to 0. Suppose further that the weighted shift T is strictly cyclic. By this we mean that $\mathcal{H} = \{A\xi_0 : A \in \mathfrak{A}_T\}$. Thus by the Closed Graph Theorem, the norm $||A\xi_0||$, $A \in \mathfrak{A}_T$ is equivalent to the operator norm. A necessary and sufficient condition is that

$$\sup_n \sum_{k=0}^{n} \left(\frac{\beta_n}{\beta_k\beta_{n-k}}\right)^2 < \infty,$$

where $\beta_0 = 1$, $\beta_n = \prod_{k=1}^{n} \alpha_k$ for $k \geqslant 1$ ([14], Proposition 32). In that case the algebra $\mathfrak{A}_T$ can be identified with l_2. Suppose that the sequence (β_n) is *regulated*; that is, for some $k \geqslant 1$, $\lim_n(\beta_{n+k}/\beta_n) = 0$. Then T is not prime in $\mathfrak{A}_T$ ([15], Corollary 3.15).

A weight sequence satisfying these conditions would be $\alpha_n = 1/n!$, $n \geqslant 1$. For general quasinilpotent weighted shifts we do not know if T is prime in $\mathfrak{A}_T$.

EXAMPLE 2.8. Next we turn to the Volterra integral operator, defined on $L^2[0,1]$ by

$$V\xi(x) = \int_0^x \xi(t)\,\mathrm{d}t, \quad \xi \in L^2[0,1].$$

In [1] there is a model for the weakly closed subalgebra of $\mathcal{B}(L^2[0,1])$ generated by V. We adapt this to obtain a model for the norm-closed subalgebra generated by V. First, some preliminaries.

Let $H^\infty = H^\infty(\mathbb{D})$ denote the subspace of the L^∞ functions on the unit circle analytic in the interior, $\mathcal{A} = \mathcal{A}(\mathbb{D})$ the disk algebra; i.e., $\mathcal{A}$ is the subspace of H^∞ of continuous functions on the circle analytic in the interior. Let $\mathcal{M}_1$ be the maximal ideal in $\mathcal{A}$ of functions vanishing at the point 1. Let $e_r(z)$ denote the function $e_r(z) = \exp(r(z+1)/(z-1))$.

Observations:

(i) $e_1\mathcal{M}_1 = e_1H^\infty \cap \mathcal{M}_1 = e_1H^\infty \cap \mathcal{A}$. That $e_1\mathcal{M}_1 \subset e_1H^\infty \cap \mathcal{M}_1 \subset e_1H^\infty \cap \mathcal{A}$ is clear. Suppose $\varphi \in H^\infty$ and $e_1\varphi \in \mathcal{A}$. Since for $t \in (0,1)$, $e_1(t) \to 0$ as $t \uparrow 1$, it must be the case that for $z \in \mathbb{T}$, $z = \mathrm{e}^{\mathrm{i}\theta}$, $\varphi(z) \to 0$ as $\theta \to 0$, since $|e_1(\mathrm{e}^{\mathrm{i}\theta})| = 1$, $\theta \neq 0$, and $e_1\varphi$ is continuous by hypothesis. Since e_1 is continuous on $\mathbb{T}$ except at 1, and $|e_1(z)| = 1$, $z \in \mathbb{T}$, $z \neq 1$, it follows that φ is continuous on $\mathbb{T}$, except possibly at $z = 1$. If necessary, redefine φ so $\varphi(1) = 0$, then φ is continuous on $\mathbb{T}$. But then $\varphi \in \mathcal{M}_1$.

(ii) Let P_s denote the Poisson kernel. Note that for $f \in H^\infty$, $(P_s * f)(z) = f(sz)$. Then for any $g \in \mathcal{M}_1$, $\|e_1g - (P_s * e_1)g\| \to 0$ as $s \uparrow 1$. Indeed,

$$\|e_1g - (P_s * e_1)g\| \leqslant \|e_1g - P_s * (e_1g)\| + \|P_s * (e_1g) - (P_s * e_1)g\|.$$

Now the first term goes to zero since $e_1g \in \mathcal{M}_1$, and hence $P_s * (e_1g)$ converges uniformly to e_1g as $s \uparrow 1$. For the other, note that $P_s * (e_1g) = (P_s * e_1)(P_s * g)$ so that

$$\|P_s * (e_1g) - (P_s * e_1)g\| = \|P_s * e_1(g - P_s * g)\| \leqslant \|g - P_s * g\|$$

and by continuity of g the last term approaches zero as $s \uparrow 1$.

LEMMA 2.9. *Let $f \in \mathcal{M}_1$, $\varepsilon > 0$. Then there exists $r > 0$ such that $\|f - e_rf\| < \varepsilon$.*

Proof. Let $\mathcal{U}$ be a neighborhood of 1 in the closed unit disk $\mathrm{cl}\,(\mathbb{D})$ so that $|f(z)| < \varepsilon/2$, $z \in \mathcal{U}$. We can choose r sufficiently small so that for all $z \in \mathrm{cl}\,(\mathbb{D}) \setminus \mathcal{U}$, $\quad |e_r(z) - 1| < \varepsilon/\|f\|$. Then

$$|f(z) - e_r(z)f(z)| \leqslant \begin{cases} \dfrac{\varepsilon}{2}|1 - e_r(z)|, & z \in \mathcal{U}; \\ |f(z)| \cdot |1 - e_r(z)|, & z \in \mathrm{cl}\,(\mathbb{D}) \setminus \mathcal{U}; \end{cases}$$

$$\leqslant \begin{cases} \dfrac{\varepsilon}{2}2 = \varepsilon, & z \in \mathcal{U}; \\ \|f\|\dfrac{\varepsilon}{\|f\|} = \varepsilon, & z \in \mathrm{cl}\,(\mathbb{D}) \setminus \mathcal{U}. \end{cases}$$ ∎

LEMMA 2.10. *Let $f \in \mathcal{M}_1$. Then*

$$\inf_{g \in e_1H^\infty} \|f - g\| = \inf_{g \in e_1\mathcal{M}_1} \|f - g\|.$$

Proof. With no loss of generality assume $\|f\| = 1$. By the previous lemma we may choose r, $0 < r < 1$, so that $\|f - e_rf\| < \varepsilon$. By [1], p. 41, the infimum

$\{\|f-g\| : g \in e_1 H^\infty\}$ is achieved for some g, $g = e_1\varphi$, $\varphi \in H^\infty$. As $\|f\| = 1$, $\|g\| = \|\varphi\| \leqslant 2$. With P_s the Poisson kernel as above,

$$P_s * e_r(1) = e_r(s) = \exp\Big(r\frac{s+1}{s-1}\Big).$$

As this approaches 0 as $s \uparrow 1$, find s_0 so that $e_r(s) < \varepsilon$ for $s_0 < s < 1$. Next, observe that the function $e_1[P_s * e_r - e_r(s)]P_s * \varphi \in e_1\mathcal{M}_1$. By observation (ii), there is s_1, $s_0 \leqslant s_1 < 1$, so that for $s_1 < s < 1$,

$$(*) \qquad \begin{aligned} &\|P_s(e_r f) - e_1[P_s * e_r - e_r(s)]P_s * \varphi\| \\ &\quad \leqslant \|P_s * (e_r f) - P_s * e_1[P_s * e_r - e_r(s)]P_s * \varphi\| + \varepsilon. \end{aligned}$$

Also, $\|e_1 e_r(s) P_s * \varphi\| \leqslant 2\varepsilon$, so

$$\begin{aligned} (*) &\leqslant \|P_s(e_r f) - P_s * e_1 P_s * e_r P_s * \varphi\| + 3\varepsilon \\ &\leqslant \|e_r f - e_1 e_r \varphi\| + 3\varepsilon \leqslant \|f - e_1\varphi\| + 3\varepsilon. \end{aligned}$$

Now

$$\begin{aligned} \|f - P_s * (e_r f)\| &\leqslant \|f - P_s * f\| + \|P_s * f - P_s * (e_r f)\| \\ &\leqslant \|f - P_s * f\| + \|f - e_r f\| < 2\varepsilon \end{aligned}$$

for $s_2 < s < 1$, where $s_1 \leqslant s_2 < 1$ and for r sufficiently small, by Lemma 2.9. Thus,

$$\begin{aligned} &\|f - e_1[P_s * e_r - e_r(s)]P_s * \varphi\| \\ &\quad \leqslant \|f - P_s * (e_r f)\| + \|P_s * (e_r f) - e_1[P_s * e_r - e_r(s)]P_s * \varphi\| \\ &\quad < 2\varepsilon + \|f - e_1\varphi\| + 3\varepsilon < 5\varepsilon + \|f - e_1\varphi\|. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this shows that the infimum on the right is less then or equal the left side. The other inequality is obvious, and the proof is complete. ∎

If $\mathcal{B}$ is a Banach algebra, $\mathfrak{A} \subseteq \mathcal{B}$ a Banach subalgebra, and $\mathcal{I}$ a norm-closed ideal of $\mathcal{B}$, then the map $\mathfrak{A}/\mathfrak{A} \cap \mathcal{I} \to \mathfrak{A} + \mathcal{I}/\mathcal{I}$, which takes the coset $a + \mathfrak{A} \cap \mathcal{I} \to a + \mathcal{I}$ $(a \in \mathfrak{A})$, is an algebraic isomorphism, but as $\mathfrak{A} + \mathcal{I}$ may not be closed in $\mathcal{B}$, the quotient $\mathfrak{A} + \mathcal{I}/\mathcal{I}$ may be merely a normed algebra. In the case $\mathcal{B} = H^\infty$, $\mathfrak{A} = \mathcal{M}_1$, $\mathcal{I} = e_1 H^\infty$ it follows from the previous lemma that the mapping $\mathfrak{A}/\mathfrak{A} \cap \mathcal{I} \to \mathfrak{A} + \mathcal{I}/\mathcal{I}$ is isometric, hence an isometric isomorphism of Banach algebras.

Again, let V be the Volterra operator on $L^2[0,1]$, $T = (I - V)(I + V)^{-1}$ (the "cogenerator"). By [1], p. 99, V is a norm limit of polynomials in $(I - T)$, and conversely, $I - T$ is a norm limit of polynomials in V.

By [1], p. 97, Lemma 3.18, T is unitarily equivalent to $S(e_1)$, the compression of the shift S on H^2 to $\mathcal{H}(e_1) = H^2 \ominus e_1 H^2$. The commutant $\{S(e_1)\}'$ (and hence $\{T\}'$) is isometrically isomorphic to the quotient algebra H^∞/e_1H^∞. This is implemented by $u \in H^\infty \to u(S(e_1))$ ([1], p. 40, Corollary 1.20).

Due to the unitary equivalence of T, $S(e_1)$, we write $u(T)$, $u \in H^\infty$. Denote by Φ the isometric isomorphism $\Phi : H^\infty/e_1H^\infty \to \{T\}', u \in H^\infty \mapsto u(T)$. The image of the subalgebra $\mathcal{M}_1 + e_1H^\infty/e_1H^\infty$, which is norm closed, contains $T - I$ and hence V. On the other hand, as observed, any norm closed subalgebra containing V contains $T-I$. Let $\mathfrak{A}_V$ denote the norm closed subalgebra of $\mathcal{B}(L^2[0,1])$ generated by V.

COROLLARY 2.11.

$$\begin{aligned} \mathfrak{A}_V &= \Phi(\mathcal{M}_1 + e_1H^\infty/e_1H^\infty) \cong \mathcal{M}_1 + e_1H^\infty/e_1H^\infty \\ &\cong \mathcal{M}_1/e_1H^\infty \cap \mathcal{M}_1 \cong \mathcal{M}_1/e_1\mathcal{M}_1. \end{aligned}$$

NOTE: $e_1H^\infty \cap \mathcal{M}_1 = e_1\mathcal{M}_1$ by observation (i). Also, the polynomials vanishing at 1 are dense in $\mathcal{M}_1$.

The benefits of having a functional model for $\mathfrak{A}_V$ can work both ways. On one hand we can use what is known about $\mathfrak{A}_V$ (that it is a radical algebra) to get a result about approximation of analytic functions. On the other hand, the model provides information about $\mathfrak{A}_V$ that cannot easily be extracted directly by operator-theoretic tools.

From the (well-known) fact that V is quasinilpotent, we obtain from Proposition 1.4 that the norm-closed algebra $\mathfrak{A}_V$ generated by V is radical. Let e_1, and $\mathcal{M}_1$ be as before.

PROPOSITION 2.12. *Let $f \in \mathcal{M}_1$, $\varepsilon > 0$. Then there is a positive integer N and $g_n \in \mathcal{M}_1$, $n \geqslant N$, so that $\|f^n - e_1g_n\| < \varepsilon^n$, $n \geqslant N$.*

Proof. $\mathfrak{A}_V$ is isometrically isomorphic with $\mathcal{M}_1/e_1\mathcal{M}_1$. Since $\mathfrak{A}_V$ is radical, so is $\mathcal{M}_1/e_1\mathcal{M}_1$. The inequality above is a restatement of the fact that the equivalence class of any $f \in \mathcal{M}_1$ is quasinilpotent. ∎

More interesting for our purposes is the information that $\mathcal{M}_1/e_1\mathcal{M}_1$ provides about $\mathfrak{A}_V$.

COROLLARY 2.13. *The nilpotent elements in $\mathfrak{A}_V$ are dense.*

Proof. It is enough to prove density of nilpotent elements in $\mathcal{M}_1/e_1\mathcal{M}_1$. Given an equivalence class $[f] \in \mathcal{M}_1/e_1\mathcal{M}_1$, let $\varepsilon > 0$ and choose a representative $f \in [f]$. By Lemma 2.9 there is r, $0 < r < 1$, such that $\|f - e_rf\| < \varepsilon$. Now $e_rf \in \mathcal{M}_1$, and $\|[f] - [e_rf]\| < \varepsilon$. But $[e_rf]^n = [e_r^nf^n] = [e_{nr}f^n] = [0]$ if $nr > 1$, as $e_{nr} = e_1e_{(nr-1)}$. ∎

COROLLARY 2.14. *$\mathfrak{A}_V$ is not an integral domain.*

Following the treatment of accretive operators in [11], pp. 173–174, one can show that the (maximal) accretive operator V has a square root which lies in $\mathfrak{A}_V$. The result is sketched here.

For $z = re^{i\theta}$, $r > 0$, $|\theta| < \pi/2$, define $z^{1/2} = r^{1/2}e^{i\theta/2}$. The functions $(1+\lambda)^{1/2}$, $(1-\lambda)^{1/2}$, (λ in the open unit disk $\mathbb{D}$) are holomorphic in $\mathbb{D}$ and extend to continuous functions on $\mathrm{cl}\,(\mathbb{D})$. That is, they belong to the disk algebra $\mathcal{A}(\mathbb{D})$. Let

$$w(\lambda) = \frac{(1+\lambda)^{1/2} - (1-\lambda)^{1/2}}{(1+\lambda)^{1/2} + (1-\lambda)^{1/2}}.$$

$w \in \mathcal{A}(\mathbb{D})$. As before, set $T = (I - V)(I + V)^{-1}$. So,

$$V^{1/2} = (I - w(T))^{-1}(I + w(T)) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n (I + w(T))^n.$$

Using [11], Equation 5.7, p. 174, $|w(\lambda)| < 1$ for $\lambda \in \mathrm{cl}\,(\mathbb{D})$, $\lambda \neq \pm 1$, $w(-1) = -1$, $w(1) = 1$, and so $\sigma(w(T)) = 1$. The second equality above follows from [1], p. 99, in the same way V is obtained as a power series in $(I - T)$, except here we have $(I - w(T))$. Since $\sigma(w(T)) = w(\sigma(T)) = w(1) = 1$, the series converges in norm. Note that since $w(0) = 0$, $I - w(T)$ is a limit of polynomials in $(I - T)$. Thus, $I - w(T) \in \mathfrak{A}_V$, and so is $V^{1/2}$.

PROPOSITION 2.15. *V is not prime in $\mathfrak{A}_V$.*

Proof. V divides $V^{1/2}V^{1/2}$, but V does not divide $V^{1/2}$, since $V^{-1/2}$ is unbounded, hence not in $\mathfrak{A}_V$. ∎

Next we observe that the functional model for $\mathfrak{A}_V$ can be used to show that $\mathfrak{A}_V$ is not uniformly topologically nil. Note that $f(z) = (1/2)(1 - z)$, as well as the principal nth roots of f for all positive integers n, belong to the unit ball of $\mathcal{M}_1$; hence the equivalence classes of these functions belong to the unit ball of the quotient $\mathcal{M}_1/e_1\mathcal{M}_1$. If $\mathcal{M}_1/e_1\mathcal{M}_1$ were uniformly topologically nil, there would be a sequence $\{b_n\}_{n=1}^{\infty}$, $b_n > 0$, with $\lim_n b_n = 0$, such that for $[g] \in$ unit ball $(\mathcal{M}_1/e_1\mathcal{M}_1)$, $\|[g]^n\|^{1/n} \leqslant b_n$, $n = 1, 2, \ldots$. Let f be as above; for some $m \in \mathbb{Z}^+$, $\|[f]\| > b_m$. Then, if g is the mth principal root of f,

$$\|[g]^m\|^{1/m} = \|[f]\|^{1/m} \geqslant \|[f]\| > b_m.$$

This proves

PROPOSITION 2.16. *$\mathfrak{A}_V$ is not uniformly topologically nil and hence not topologically nilpotent.*

We remark that it is possible to provide an alternative proof of Proposition 2.16 based on a result of [7], that the weakly closed algebra generated by the Volterra operator contains an operator whose spectrum has more than one point. However, that proof yields less insight than the one presented here.

The other example, 2.5, of a singly generated operator algebra which is not uniformly topologically nil was given by a quasinilpotent weighted shift T (in which the weight sequence was not monotonic). The algebra $\mathfrak{A}_T$ failed to be uniformly topologically nil because the sequence $\{T_n := T^n/\|T^n\|\}$ of normalized powers of the generator was not a uniformly topologically nil sequence.

This leads to the following question: suppose T is a quasinilpotent operator which satisfies the condition: there is a positive sequence $\{b_k\}_{k=1}^{\infty}$, converging to zero, and

$$||T_n^k||^{1/k} \leqslant b_k, \quad \text{for all positive integers } n, k$$

where $T_n = T^n/\|T^n\|$, $n \geqslant 1$. Does it then follow that the algebra $\mathfrak{A}_T$ is uniformly topologically nil?

EXAMPLE 2.17 For the Volterra operator V one has the estimates

$$\frac{1}{n!(2n+1)} \leqslant \|V^n\| \leqslant \frac{1}{(n-1)!}, \quad n \geqslant 1.$$

So with $V_n = V^n/\|V^n\|$ one has

$$\|V_n^k\| = \frac{\|V^{nk}\|}{\|V^n\|^k} \leqslant \frac{(n!)^k(2n+1)^k}{(nk-1)!}.$$

From Stirling's formula one obtains

$$\|V_n^k\|^{1/k} \leqslant C(2n+1)^{3/2}\frac{1}{k^n}, \quad k \geqslant 2,$$

where C does not depend on n, k. If

$$b_k = \sup_n C(2n+1)^{3/2}\frac{1}{k^n}, \quad k \geqslant 2,$$

so $b_k \to 0$ as $k \to \infty$. Thus the sequence $\{V_n\}_{n=1}^{\infty}$ of normalized powers of the generator is a uniformly topologically nil sequence, although the algebra $\mathfrak{A}_V$ is not uniformly topologically nil (by Proposition 2.16). This answers the question raised above in the negative.

REMARK 2.18. Here are some remarks concerning noncommutative radical operator algebras. It is known that for noncommutative radical Banach algebras, the condition that $\mathfrak{A}$ be uniformly topologically nil is strictly weaker than topological nilpotence ([5]). The question remains open in the category of operator algebras.

A possible candidate for a uniformly topologically nil but not topologically nilpotent operator algebra is the following: let $\{(e_{ij}^{(n)}) : 1 \leqslant i, j \leqslant 2^n\}$ be set of matrix units for M_{2^n}, and $\nu_n : M_{2^n} \hookrightarrow M_{2^{n+1}}$ the refinement embedding, given by

$$\nu_n\left(e_{ij}^{(n)}\right) = e_{2i-1\ 2j-1}^{(n+1)} + e_{2i\ 2j}^{(n+1)}.$$

Let $\mathcal{T}_n \subset M_{2^n}$ denote the upper triangular subalgebra, and $\mathcal{T}_n^0$ the strictly upper triangular subalgebra. The operator algebra inductive limit, $\varinjlim(\mathcal{T}_n, \nu_n)$ is called the refinement algebra, and its Jacobson radical is known to be $\varinjlim(\mathcal{T}_n^0, \nu_n)$ ([6]). Let $x_1 = e_{12}^{(1)}$, $x_2 = e_{34}^{(2)}, \ldots, x_n = e_{2^n-1 2^n}^{(n)}, \ldots$. Let $\mathfrak{A}$ be the norm closed (operator) algebra generated by the $\{x_n\}_{n=1}^{\infty}$. As $\mathfrak{A}$ is contained in the radical of the refinement algebra, $\mathfrak{A}$ is radical. As $\|x_1 x_2 \cdots x_n\| = 1$ for all n, $\mathfrak{A}$ is not topologically nilpotent.

For $I = \{i_1 < i_2 < \cdots < i_k\}$ a finite subset of positive integers, let $x_I = x_{i_1} x_{i_2} \cdots x_{i_k}$. It is not difficult to see that any $T \in \mathfrak{A}$ can be expressed uniquely as a sum $\sum\limits_I a_I x_I$ over the finite subsets $I \subset \mathbb{N}$. Let $T \in \mathfrak{A}$ be non-nilpotent. One can show that $\mathfrak{A}_T$ is an integral domain. We do not know if $\mathfrak{A}_T$ is uniformly topologically nil.

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