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AFFINE TEMPERLEY-LIEB ALGEBRAS

SANTE GNERRE

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Abstract. Given a finite index inclusion of factors $N \overset{E}{\subset} M$, it is possible to define a representation of the Affine Temperley-Lieb algebra on the relative commutant $N' \cap M_n$, via the left and right multiplication by the e_i's, and the conditional expectations E_n and λE^{-1}, where $\lambda = \mathrm{Ind}(E)^{-1}$.

This result generalizes a theorem by Vaughan Jones (see [10]), where he introduces the definition of the Affine Temperley-Lieb algebra, and proves that a representation of it exists on the Hilbert spaces $N' \cap M_n$ constructed from a finite index and extremal inclusion of II_1 factors $N \subset M$.

Keywords: *Subfactors, von Neumann algebras.*

MSC (2000): Primary 46L37; Secondary 81R05.

1. INTRODUCTION

If $N \subset M$ is a finite index inclusion of II_1 factors, and if $N \subset M$ is extremal, then we can find a representation of the Affine Temperley-Lieb algebra on the relative commutant $N' \cap M_n$, where M_n is the nth step of the Jones tower.

Definition 1.1. (see [10]) The *affine Temperley-Lieb algebra* $\mathcal{T}(n, \lambda)$ is the algebra over $\mathbb{C}$ defined by generators $f_1, \dots, f_n$ and relations:

$$\begin{cases} f_i f_{i\pm1} f_i = \lambda f_i & i = 1, \dots, n \ (\mathrm{mod}\, n), \\ f_i f_j = f_j f_i & |i-j| \geqslant 0 \ (\mathrm{mod}\, n), \\ f_i^2 = f_i & i = 1, \dots, n. \end{cases}$$

To be more precise, if $\lambda = [M:N]^{-1}$, L_{e_i} is the left multiplication by e_i, R_{e_j} the right multiplication by e_j, and $E_{N'\cap M_{n-1}}$, $E_{M'\cap M_n}$ are the trace preserving conditional expectations onto $N' \cap M_n$ and $M' \cap M_{n-1}$, then:

$$(1.1)\qquad \begin{matrix} f_1 & f_2 & \cdots & f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_{2n+1} & f_{2n+2} \\ L_{e_1} & L_{e_2} & \cdots & L_{e_n} & E_{N'\cap M_{n-1}} & R_{e_n} & R_{e_{n-1}} & \cdots & R_{e_1} & E_{M'\cap M_n} \end{matrix}$$

is a representation of the Affine Temperley-Lieb algebra $\mathcal{T}(2n+2,\lambda)$ on the Hilbert space $N' \cap M_n$, with $2n+2$ generators and parameter λ (see [10]).

Consider now the more general setting $N \overset{E}{\subset} M$ of any inclusion of factors equipped with a normal, faithful conditional expectation E. By [11], [13], [16], we can define in this case the index $\mathrm{Ind}(E)$ of the inclusion $N \overset{E}{\subset} M$, to be a real number bigger than or equal to 1 (∞ is allowed).

If $\mathrm{Ind}(E) < \infty$, then it is possible to define a basic construction $M \overset{E_1}{\subset} M_1$, with E_1 normal faithful conditional expectation from M_1 onto M (the "dual" to E), and where M_1 is the von Neumann algebra generated by M and the Jones projection e_1. As in the finite case, the relative commutant $N' \cap M$ is finite dimensional (see [6], [7], [11], [12], [13],).

An important step now is to understand how E_1 is defined. Starting with the spatial theory by Connes (see [2]), and the operator-valued weights' theory by Haagerup (see [6], [7]), given a (normal faithful) conditional expectation E, it is possible to define a canonical operator-valued weight E^{-1}, which is the cornerstone in the definition of $\mathrm{Ind}(E)$ in the sense of Kosaki. If $\mathrm{Ind}(E) < \infty$, then λE^{-1} defines a normal faithful conditional expectation from N' onto M', where $\lambda = \mathrm{Ind}(E)^{-1}$ (here M acts standardly on a fixed Hilbert space $\mathcal{H}$). It is from λE^{-1} that we can define the dual conditional expectation E_1.

Now we can iterate the basic construction, and get a tower of factors

$$(1.2)\qquad N \overset{E}{\subset} M \overset{E_1}{\subset} M_1 \subset \cdots \overset{E_n}{\subset} M_n,$$

where each E_i is the dual of E_{i-1}. In this way we also get the sequence of Jones projections $e_1, \ldots, e_n$. Thus the main theorem:

THEOREM 1.2. *If $N \overset{E}{\subset} M$ is a finite index inclusion of factors and e_i, λ, E^{-1} and E_n are as above, then:*

$$(1.3)\qquad \begin{matrix} f_1 & f_2 & \cdots & f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_{2n+1} & f_{2n+2} \\ L_{e_1} & L_{e_2} & \cdots & L_{e_n} & E_n & R_{e_n} & R_{e_{n-1}} & \cdots & R_{e_1} & \lambda E^{-1} \end{matrix}$$

is a representation of $\mathcal{T}(2n+2,\lambda)$ onto the Hilbert space $N' \cap M_n$.

In other words, a representation of $\mathcal{T}(2n+2,\lambda)$ onto $N'\cap M_n$ exists also both in the infinite factors setting, and in the finite factors but not extremal one, the main idea being that to get the representation it suffices to "tune up" the correct conditional expectations.

2. PROOF OF THE MAIN THEOREM

In the following we will only consider σ-finite factors.

We start with a finite index inclusion (of σ-finite factors) $N \overset{E}{\subset} M \subset \mathcal{B}(\mathcal{H})$, where $E \in \mathcal{E}(M,N)$, the set of all the normal, faithful conditional expectations from M onto N.

Assume M acts standardly on $\mathcal{H}$, and fix $\varphi \in N_\star^+$, any normal faithful state on N. Then $\varphi\circ E$ is a normal faithful state on M, and we can find a unique cyclic and separating vector ξ in the standard cone, such that $\varphi\circ E = \omega_\xi$ (see [7]). This vector ξ determines uniquely the antilinear isometry J (the modular conjugation), by considering the polar decomposition $S = J\Delta^{1/2}$ of the closure S of the map $x\xi \to x^\star\xi$, $x \in M$.

Define e_N by $e_N(x\xi) = E(x)\xi$, $x \in M$. Kosaki proves in [11] that e_N can be extended to a bounded projection on $\mathcal{H}$ which commutes with N (still denoted by e_N), and that e_N does not depend on the choice of φ.

Lemma 2.1. (see [11]) $E^{-1}(e_N) = 1$. *In particular,* $\mathrm{Ind}(E) \geqslant 1$ *and* $\mathrm{Ind}(E) = 1$ *if and only if* $N = M$.

This projection satisfies several "natural" conditions:

Lemma 2.2. (see [11]) *Let* $N \overset{E}{\subset} M$ *be a finite index inclusion of factors, and assume* M *acts standardly on* $\mathcal{H}$. *Fix* ξ *a cyclic and separating vector for* M *such that* $\varphi\circ E = \omega_\xi$, *for any fixed* φ *normal faithful state on* N, *and call* J *its modular conjugation. If* e_N *is as above, then:*

(i) $e_N x e_N = E(x)e_N$, $x \in M$;

(ii) *if* $x \in M$, *then* $xe_N = e_N x \Leftrightarrow x \in N$;

(iii) $N' = \langle M', e_N\rangle$;

(iv) J *commutes with* e_N;

(v) $\langle M, e_N\rangle = JN'J$;

(vi) $\left\{a_0 + \sum_{i=1}^{n} a_i e_N b_i \;:\; a_i, b_j \in M'\right\}$ *is dense in* N';

(vii) $\left\{a_0 + \sum_{i=1}^{n} a_i e_N b_i \;:\; a_i, b_j \in M\right\}$ *is dense in* $\langle M, e_N\rangle$.

Assume now $N \overset{E}{\subset} M$ are factors with $\mathrm{Ind}(E) < \infty$, and perform the basic construction $N \overset{E}{\subset} M \subset M_1$. Remember M acts standardly on $\mathcal{H}$, and consider:

$$N \overset{E}{\subset} M \subset M_1$$

$$M_1' \subset M' \overset{\lambda E^{-1}}{\subset} N'$$

where $\lambda = \mathrm{Ind}(E)^{-1}$. If J is, as in the previous lemma, the modular conjugation relative to ξ:

DEFINITION 2.3. (see [11], [12]) E_1, the *conditional expectation dual to* E, is defined by

$$E_1(x) = \lambda J E^{-1}(JxJ)J, \quad x \in M_1.$$

$E_1 \in \mathcal{E}(M_1, M)$, and $E_1(e_N) = \lambda$. Also, $\mathrm{Ind}(E) = \mathrm{Ind}(E_1)$ (see [11]).

To iterate this construction, fix a new Hilbert space $\mathcal{H}_1$ standard for M_1, and consider the modular conjugation J_1 relative to the vector state ξ_1 of $\varphi \circ E \circ E_1$. Now we can proceed as above, and can define the basic construction M_2 of $M \overset{E_1}{\subset} M_1$. In this way we get a tower

$$N \overset{E}{\subset} M \overset{E_1}{\subset} M_1 \subset \cdots \overset{E_{n-1}}{\subset} M_{n-1} \overset{E_n}{\subset} M_n \tag{2.1}$$

satisfying (see [11]):

(i) E_i is the dual of E_{i-1}, and $\mathrm{Ind}(E_i) = \lambda^{-1}$;

(ii) $M_i = \langle M_{i-1}, e_i \rangle$;

(iii) $E_i(e_i) = \lambda$;

(iv) $e_i e_{\pm 1} e_i = \lambda e_i$;

(v) $e_i e_j = e_j e_i$, for $|i - j| \geqslant 2$.

A word about the Hilbert spaces in this extended version of the Jones tower: given a tower as in (iv), it is always possible to let it act on $\mathcal{H}$, where M is standard on $\mathcal{H}$. To do this, first of all consider the inclusion $M \subset M_n$.

If M and M_n are infinite, then it is well known it exists a faithful representation of the inclusion $M \subset M_n$ in which both factors are in standard form *with respect to the same vector*. If instead $M \subset M_n$ are finite factors, then the result is well known from [9].

Lemma 2.4. *Let* $N \overset{E}{\subset} M$ *be a finite index inclusion of factors with* $\mathrm{Ind}(E) = \lambda^{-1}$*, and consider* M_1 *its basic construction. If* e *is the Jones projection relative to* E*, and* E_1 *is the conditional expectation dual to* E*, then:*

(i) $eE_1(ex) = \lambda\, ex$;

(ii) $E_1(xe)e = \lambda\, xe$.

Proof. Since elements of the type $a_0 + \sum_{i=1}^{n} a_i e b_i$, $a_i, b_j \in M$ are dense in M_1, it is enough to check the statement for $x = a$, and $x = aeb$, with $a, b \in M$. The case $x = a$ is obvious. Let $x = aeb$, $a, b \in M$. Then $eE_1(ex) = eE_1(eaeb) = eE_1(E(a)eb) = eE(a)E_1(e)b = \lambda eE(a)b = \lambda eaeb = \lambda ex$. The other equality is proved in the same way. ∎

Lemma 2.5. *Let* $N \overset{E}{\subset} M$ *be a finite index inclusion of factors with* $\mathrm{Ind}(E) = \lambda^{-1}$*, and perform the basic construction up to the* n*-th step:*

$$N \overset{E}{\subset} M \overset{E_1}{\subset} M_1 \subset \cdots \overset{E_n}{\subset} M_n.$$

Let the tower act on the Hilbert space $\mathcal{H}$*, with* M *in standard form on* $\mathcal{H}$*, call* $e_1, e_2, \ldots$ *the Jones projections relative to* $E, E_1, \ldots$*, and consider:*

$$\begin{array}{ccc} N' \cap M_{n-1} & \subset & N' \cap M_n \\ \cup & & \cup \\ M' \cap M_{n-1} & \subset & M' \cap M_n. \end{array} \tag{2.2}$$

Then:

(i) (a) $E_n \in \mathcal{E}(N' \cap M_n, N' \cap M_{n-1}) \cap \mathcal{E}(M' \cap M_n, M' \cap M_{n-1})$,

(b) $\lambda E^{-1} \in \mathcal{E}(N' \cap M_n, M' \cap M_n) \cap \mathcal{E}(N' \cap M_{n-1}, M' \cap M_{n-1})$;

(ii) $e_n x e_n = E_{n-1}(x) e_n$, $\forall x \in N' \cap M_{n-1}$;

(iii) $e_1 x e_1 = \lambda (E_1)^{-1}(x) e_1$, $\forall x \in M' \cap M_n$;

(iv) $E_n \circ \lambda E^{-1}|_{N' \cap M_n} = \lambda E^{-1} \circ E_n|_{N' \cap M_n}$.

Proof. Let $x \in N' \cap M_n$. $E_n(x) \in M_{n-1}$, but also: $\forall y \in N$: $E_n(x)y = E_n(xy) = E_n(yx) = yE_n(x)$, thus $E_n(x) \in N'$. The other relations in (i) are proved in the same way.

(ii) is shown as in Lemma 2.2.

Part (iii) is clear from the fact that N' is the basic construction of $M_1' \overset{\lambda(E_1)^{-1}}{\subset} M'$, with Jones projection e_1.

To show (iv), fix $\{u_i\}_{i=1}^n \subset M$ a Watatani's quasibasis for E (see [16]). Then $E^{-1}(z) = \sum_{i=1}^{n} u_i z u_i^\star$, for $z \in N'$. Fix $x \in N' \cap M_n$. $E_n \circ \lambda E^{-1}(x) = \lambda E_n\Big(\sum_{i=1}^{n} u_i x u_i^\star\Big) = \lambda \sum_{i=1}^{n} u_i E_n(x) u_i^\star = \lambda E^{-1} \circ E_n(x)$. ∎

It is worthwhile to note that (2.2) is not a commuting square in the ordinary sense (see [3]), but it can be considered to be a generalized one, since it has several of the interesting properties of that one. For instance, by (iv) of the previous lemma, $E_n \circ \lambda E^{-1}|_{N'\cap M_n} = \lambda E^{-1} \circ E_n|_{N'\cap M_n}$, and it is clear that $\lambda E^{-1} \circ E_n(N' \cap M_n) = M' \cap M_{n-1}$. This could be the definition for a *generalized commuting square.*

THEOREM 2.6. *Let* $N \overset{E}{\subset} M$ *be an inclusion of factors with* $\mathrm{Ind}(E) = \lambda^{-1} < \infty$. *Consider*

$$N \overset{E}{\subset} M \overset{E_1}{\subset} M_1 \subset \cdots \overset{E_n}{\subset} M_n$$

the tower up to the step n, *and let it act over* $\mathcal{H}$ *with* M *in standard form. Then*

$$\begin{array}{cccccccccc} f_1 & f_2 & \cdots & f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_{2n+1} & f_{2n+2} \\ L_{e_1} & L_{e_2} & \cdots & L_{e_n} & E_n & R_{e_n} & R_{e_{n-1}} & \cdots & R_{e_1} & \lambda E^{-1} \end{array} \tag{2.3}$$

defines a representation of $\mathcal{T}(2n+2, \lambda)$ *over the Hilbert space* $N' \cap M_n$.

Proof. The only non obvious relations are (with $x \in N' \cap M_n$):
(1) $e_n E_n(e_n x) = \lambda e_n x$ (and similarly: $E_n(xe_n)e_n = \lambda x e_n$),
(2) $e_1 \lambda E^{-1}(e_1 x) = \lambda e_1 x$ (and similarly: $\lambda E^{-1}(xe_1)e_1 = \lambda x e_1$),
(3) $E_n \circ \lambda E^{-1}(x) = \lambda E^{-1} \circ E_n(x)$.

Part (1) is proved applying Lemma 2.4 to $M_{n-2} \overset{E_{n-1}}{\subset} M_{n-1} \overset{E_n}{\subset} M_n$.

To show Part (2), either apply Lemma 2.4 to $M_1' \overset{\lambda(E_1)^{-1}}{\subset} M' \overset{\lambda E^{-1}}{\subset} N'$, or use a direct argument: since $x \in N' \cap M_n$, then x is the weak limit of elements of the type $\sum_{i=1}^{n} a_i e_1 b_i$, with $a_i, b_i \in M'$ (see [11]). Fix $a, b \in M'$ and let $x = ae_1 b$.

$$\begin{aligned} e_1 \lambda E^{-1}(e_1 x) &= e_1 \lambda E^{-1}(e_1 a e_1 b) = e_1 J E_1(J e_1 a e_1 b J)J = e_1 J E_1(e_1 JaJ e_1 JbJ)J \\ &= e_1 J E_1(E(JaJ) e_1 JbJ)J = e_1 J E(JaJ) \lambda JbJJ = \lambda J e_1 JaJ e_1 JbJJ \\ &= \lambda e_1 a e_1 b = \lambda e_1 x. \end{aligned}$$

For Part (3) use Lemma 2.5. ∎

COROLLARY 2.7. *If* $N \subset M$ *is an extremal inclusion of* II_1 *factors, with finite index, then the construction in* (1.1) *coincides with the one in* (2.3). *In other words*

$$E_n|_{N'\cap M_n} = E^{N'\cap M_n}_{N'\cap M_{n-1}},$$

and

$$\lambda E^{-1}|N' \cap M_n = E^{N'\cap M_n}_{M'\cap M_n},$$

where E, $E^{N'\cap M_n}_{N'\cap M_{n-1}}$, and $E^{N'\cap M_n}_{M'\cap M_n}$ are the usual trace preserving conditional expectations; and the E_i's are the sequence of conditional expectations constructed inductively by duality, as in [11].

Proof. Remember that as a general matter of fact (both in the finite and infinite factors case), the basic construction does not depend on the choice of the conditional expectation, because if $N \overset{E}{\subset} M$ is any inclusion of factors (finite or infinite) with $\mathrm{Ind}(E) < \infty$, then $M_1 = JN'J$, and J does not depend on E.

Moreover, if E is trace preserving, then its dual E_1 is also trace preserving (simply because $E_1(e_1) = \lambda$), so that E_n (the dual to E_{n-1}) coincides with $E^{M_n}_{M_{n-1}}$ (the trace preserving conditional expectation) for all n. It remains now to show

$$\lambda E^{-1}|N' \cap M_n = E^{N'\cap M_n}_{M'\cap M_n},$$

and to do that we need some more terminology. First of all let the tower act on $\mathcal{H} = \mathcal{L}^2(M)$, then call τ the usual trace on M_∞, and τ' the unique trace on $N' \subset \mathcal{B}(\mathcal{H})$. We will denote by F the τ' preserving conditional expectation of N' onto M'. Recall that elements of the type ae_1b, with $a, b \in M'$, span N'. We have:

$$\tau'(\lambda E^{-1}(ae_1b)) = \tau'(a\lambda E^{-1}(e_1)b) = \tau'(\lambda ab).$$

On the other side, using the fact E_1 coincides with the trace preserving conditional expectation:

$$\tau'(ae_1b) = \tau(JaJe_1JbJ) = \tau(\lambda JabJ) = \tau'(\lambda ab).$$

In other words $\lambda E^{-1} = F$.

The rest is now clear: $N \subset M$ extremal implies $N \subset M_n$ is also extremal (see for instance [4]), so that

$$\tau|_{N'\cap M_n} = \tau'|_{N'\cap M_n}.$$

This is enough to guarantee

$$E^{N'\cap M_n}_{M'\cap M_n} = F|_{N'\cap M_n},$$

which concludes the proof. ∎

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SANTE GNERRE
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720
USA

E-mail: gnerre@math.berkeley.edu

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