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COMPLETELY AND ALTERNATINGLY HYPEREXPANSIVE OPERATORS

V.M. SHOLAPURKAR and AMEER ATHAVALE

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ABSTRACT. Special classes of functions on the classical semigroup $\mathbb{N}$ of non-negative integers, arising through the mechanism of difference operators, can be associated in a natural way with special classes of bounded linear operators on Hilbert spaces. The interplay among the theories of special classes of functions on $\mathbb{N}$ gets mirrored in the interaction among the associated classes of operators and leads to a fruitful synthesis of the theory of harmonic analysis on semigroups with operator theory. The present paper is a specific illustration of that theme.

KEYWORDS: *completely monotone, completely alternating, absolutely monotone, subnormal, completely hyperexpansive, alternatingly hyperexpansive.*

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The purpose of this paper is threefold:

(1) to continue the study of completely hyperexpansive operators on complex Hilbert spaces initiated in [6];

(2) to introduce and discuss a new class of operators on complex Hilbert spaces to be referred to as the class of alternatingly hyperexpansive operators; and

(3) to explore the various intra- and inter- relationships of the following three classes of operators: subnormal, completely hyperexpansive, and alternatingly hyperexpansive.

The classical backward and forward difference operators ∇ and Δ allow one to define various real-valued special functions on the semigroup $\mathbb{N}$ of non-negative integers. Prominent among these are the functions referred to in the literature as completely monotone, completely alternating, absolutely monotone, and absolutely decreasing. As was emphasized in [3], completely monotone functions on $\mathbb{N}$, which are a subset of the set of positive definite functions on $\mathbb{N}$, are closely tied to the theory of contractive subnormals. It was seen in [6] that completely alternating functions on $\mathbb{N}$, which are a subset of the set of negative definite functions on $\mathbb{N}$, give rise to completely hyperexpansive operators. As will be seen in the sequel, absolutely monotone functions on $\mathbb{N}$ get associated with alternatingly hyperexpansive operators and absolutely decreasing functions on $\mathbb{N}$ with isometries.

It is but natural to expect that the interplay among the theories of special functions on $\mathbb{N}$ would be reflected to some extent in the interaction among the associated classes of operators. As was observed in [6], the interplay between the theories of completely monotone and completely alternating functions on $\mathbb{N}$ forges some interesting connections between contractive subnormals and completely hyperexpansive operators. Both the classes of operators come with a rich supply of examples and include some classical examples of weighted shifts. As will be seen later, the class of alternatingly hyperexpansive operators encompasses a wide variety of examples and has substantial intersections with the class of subnormals and the class of p-isometries. At this stage, the reader is urged to take a look at the tables in Section 4 to gain an overall perspective of the various classes of operators under consideration. While the study of any of these classes is rewarding in itself, we will attempt to emphasize the basic unity of several of these classes in the sense of their being operator-theoretic manifestations of the various special classes of functions on $\mathbb{N}$, and also try to advocate the merit of treating these classes in conjunction with each other.

In Section 1, we fix the notation and record a few definitions. In Section 2, we present several examples of weighted shifts associated with various classes of operators. These examples have been chosen carefully with a view to highlighting the salient features of each of the classes. The examples should carry some of the flavor of the underlying unity of the different classes of operators. Section 3 is devoted to the study of completely hyperexpansive operators and takes over from where the second-named author left off in [6]. Besides attempting a rather fine analysis of the spectral properties of a completely hyperexpansive operator, we present a probabilistic interpretation of the conditions characterizing a completely hyperexpansive operator. We also observe in that section that the context of completely hyperexpansive operators allows for a few enhanced versions of the results in [2] established for 2-isometries. In Section 4, we discuss alternatingly hyperexpansive operators and their links to subnormals and completely hyperexpansive operators. Further, the theory of p-isometries as developed in [2] is made to bear on a subspecies of p-isometries, viz. alternatingly hyperexpansive p-isometries. Toward the end of Section 4, we dwell briefly upon unbounded subnormal weighted shifts.

1. PRELIMINARIES

All the Hilbert spaces occurring below are complex and separable and for any such Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$. Unless stated otherwise, the Hilbert spaces are infinite-dimensional, and the operators bounded and linear. For the definitions and discussions related to the spectrum $\sigma(T)$, left spectrum $\sigma_{\mathrm{l}}(T)$, right spectrum $\sigma_{\mathrm{r}}(T)$, essential spectrum $\sigma_{\mathrm{e}}(T)$, left essential spectrum $\sigma_{\mathrm{le}}(T)$, right essential spectrum $\sigma_{\mathrm{re}}(T)$, point spectrum $\sigma_{\mathrm{p}}(T)$, approximate point spectrum $\sigma_{\mathrm{ap}}(T)$, etc. of an operator T in $\mathcal{B}(\mathcal{H})$, as well as the Fredholm theory of T, the reader is referred to [12]. An operator S in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $\mathcal{B}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N|\mathcal{H} = S$. The one and only choice for a comprehensive and nearly up-to-date account of the theory of subnormals is [13]. An operator T in $\mathcal{B}(\mathcal{H})$ will be referred to as a contraction or an expansion according as $I - T^*T \geqslant 0$ or $I - T^*T \leqslant 0$.

The symbol $\mathbb{N}$ stands for the set of non-negative integers which forms a semigroup under addition. A real-valued map ϕ on $\mathcal{N}$ is said to be *positive definite* if $\sum\limits_{i,j=1}^{n} c_i c_j \varphi(s_i + s_j) \geqslant 0$ for all $n \geqslant 1, \{s_1, \dots, s_n\} \subset \mathbb{N}$ and $\{c_1, \dots, c_n\} \subset \mathbb{R}$, the field of reals. A real-valued map ψ on $\mathbb{N}$ is said to be *negative definite* if $\sum\limits_{i,j=1}^{n} c_i c_j \psi(s_i + s_j) \leqslant 0$ for all $n \geqslant 2, \{s_1, \dots, s_n\} \subset \mathbb{N}$ and for $\{c_1, \dots, c_n\} \subset \mathbb{R}$ such that $\sum\limits_{i=1}^{n} c_i = 0$.

For a real-valued map φ on $\mathbb{N}$ we define (backward and forward) difference operators ∇ and Δ as follows: $(\nabla\varphi)(n) = \varphi(n) - \varphi(n+1)$ and $(\Delta\varphi)(n) = \varphi(n+1) - \varphi(n)$. The operators ∇^n and Δ^n are inductively defined for all $n \geqslant 0$ through the relations $\nabla^0\varphi = \Delta^0\varphi = \varphi$, $\nabla^n\varphi = \nabla(\nabla^{n-1}\varphi)\,(n \geqslant 1)$, $\Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)(n \geqslant 1)$. A non-negative map φ on $\mathbb{N}$ is said to be *completely monotone* if $(\nabla^k\varphi)(n) \geqslant 0$ for all $k, n \geqslant 0$. A real-valued map ψ on $\mathbb{N}$ is said to be *completely alternating* if $(\nabla^k\psi)(n) \leqslant 0$ for all $n \geqslant 0$, $k \geqslant 1$. Completely monotone maps on $\mathbb{N}$ form an extreme subset of the set of positive definite functions on $\mathbb{N}$, while completely alternating functions form an extreme subset of the set of negative definite functions on $\mathbb{N}$ (see [9]). A non-negative map φ on $\mathbb{N}$ is said to be *absolutely monotone* if

$(\Delta^k\varphi)(n) \geqslant 0$ for all $k, n \geqslant 0$. A real-valued map ψ on $\mathbb{N}$ is said to be *absolutely decreasing* if $(\Delta^k\psi)(n) \leqslant 0$ for all $n \geqslant 0$, $k \geqslant 1$. Later we will have occasions to discuss the notions of "completely monotone" and "absolutely monotone" with reference to $\mathbb{R}_+$ and $\mathbb{R}_-$, the non-negative real axis and the non-positive real axis, respectively.

Jim Agler proved in [1] that T in $\mathcal{B}(\mathcal{H})$ is a subnormal contraction if and only if

$$\sum_{0\leqslant k\leqslant n} (-1)^k \binom{n}{k} T^{*k}T^k \geqslant 0 \quad \text{for all } n \geqslant 1. \tag{1.1}$$

It was observed in [3] that condition (1.1) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\varphi_h(n) = \|T^n h\|^2$ to be completely monotone on $\mathbb{N}$. Completely hyperexpansive operators were introduced in [6]. An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *completely hyperexpansive* if

$$\sum_{0\leqslant k\leqslant n} (-1)^k \binom{n}{k} T^{*k}T^k \leqslant 0 \quad \text{for all } n \geqslant 1. \tag{1.2}$$

It was observed in [6] that condition (1.2) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\psi_h(n) = \|T^n h\|^2$ to be completely alternating on $\mathbb{N}$. The symbyotic relationship between completely monotone and completely alternating maps carries over to subnormal contractions and completely hyperexpansive operators and this theme was focussed upon in [6]. We now proceed to define a class of operators related to the notion of "absolutely monotone".

Definition 1.1. An operator T in $\mathcal{B}(\mathcal{H})$ is *alternatingly hyperexpansive* if

$$\sum_{0\leqslant k\leqslant n} (-1)^{n-k} \binom{n}{k} T^{*k}T^k \geqslant 0 \quad \text{for all } n \geqslant 1. \tag{1.3}$$

Remark 1.2. If $\varphi_h(n) = \|T^n h\|^2 (h \in \mathcal{H})$ and $\langle\cdot\,,\cdot\rangle$ denotes the inner product of $\mathcal{H}$, then one has

$$(\Delta^p\varphi_h)(n) = \Big\langle\Big(\sum_{0\leqslant k\leqslant p} (-1)^{p-k} \binom{p}{k} T^{*k}T^k\Big)T^n h, T^n h\Big\rangle$$

for all $p, n \geqslant 0$, so that T is alternatingly hyperexpansive if and only if $n \to \|T^n h\|^2$ is absolutely monotone on $\mathbb{N}$ for every h in $\mathcal{H}$.

Remark 1.3. One may be tempted to define a class of operators in $\mathcal{B}(\mathcal{H})$ corresponding to the notion of "absolutely decreasing" in the spirit of (1.1), (1.2) and (1.3) (that is, by reversing the inequalities in (1.3)). This would in particular force the conditions $I - T^*T \geqslant 0$ and $I - 2T^*T + T^{*2}T^2 \leqslant 0$, where I is the identity operator on $\mathcal{H}$. In view of a result of S. Richter ([20]), however, the condition $I - 2T^*T + T^{*2}T^2 \leqslant 0$ would force $I - T^*T \leqslant 0$, leading to $I - T^*T = 0$ and forcing T to be an isometry. Recall that T in $\mathcal{B}(\mathcal{H})$ is said to be a *p-isometry* ($p \geqslant 1$) if

$$\sum_{0\leqslant k\leqslant p} (-1)^k \binom{p}{k} T^{*k}T^k = 0. \tag{1.4}$$

An isometry is nothing but a 1-isometry while any p-isometry is a $(p+1)$-isometry. It is trivial to see that any isometry satisfies conditions (1.1), (1.2) and (1.3), so that any isometry is subnormal, completely hyperexpansive, and alternatingly hyperexpansive. Any 2-isometry satisfies (1.2) and (1.3) in view of the result of S. Richter mentioned above so that any 2-isometry T is completely hyperexpansive as well alternatingly hyperexpansive; in particular T is an expansion.

It was first observed in [15] that the subnormality of an operator T in $\mathcal{B}(\mathcal{H})$ is equivalent to requiring, for any $h \in \mathcal{H}$, the existence of a positive regular Borel measure μ_h on the interval $[0, \|T\|^2]$ such that

$$\|T^n h\|^2 = \int\limits_{[0,\|T\|^2]} x^n \,\mathrm{d}\mu_h(x) \quad \text{for all } n \geqslant 0. \tag{1.5}$$

The case of a subnormal contraction thus corresponds to stipulating $\{\|T^n h\|^2\}$ to be a *Hausdorff Moment Sequence* which in turn is equivalent to stipulating $\{\|T^n h\|^2\}$ to be completely monotone on $\mathbb{N}$. (See [9], Chapter 4, Proposition 6.11.)

In the case of a completely hyperexpansive operator, the condition that $n \to \|T^n h\|^2$ be completely alternating on $\mathbb{N}$ forces, for every h in $\mathcal{H}$, the Levy–Khinchin representation

$$\|T^n h\|^2 = \|h\|^2 + n\mu_h\{1\} + \int\limits_{[0,1)} (1 - x^n) \frac{\mathrm{d}\mu_h(x)}{1 - x} \quad \text{for } n \geqslant 0, \tag{1.6}$$

where μ_h is a positive regular Borel measure on $[0, 1]$. (See [6], Proposition 2 and Remark 2; and [9], Chapter 4, Proposition 6.12.)

The special manifestations of (1.5) and (1.6) as relevant for weighted shift operators occur as Example 2.1 and Example 2.2 in the next section.

2. EXAMPLES

If $\{e_n\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$, then a weighted shift operator T on $\mathcal{H}$ with the weight sequence $\{\alpha_n\}_{n\geqslant 0}$ is defined through the relations $Te_n = \alpha_n e_{n+1}$ $(n \geqslant 0)$. It will always be assumed that $\alpha_n > 0$ for all n. For the basic properties of weighted shifts, the reader is referred to [13] and [22]. We will often use the notation $T : \{\alpha_n\}$ to indicate a weighted shift. Crucial for the discussion of $T : \{\alpha_n\}$ is the associated sequence $\{\beta_n = \beta_n(T)\}_{n\geqslant 0}$ defined by $\beta_0 = 1$, $\beta_n = \prod\limits_{k=0}^{n-1} \alpha_k^2$ $(n \geqslant 1)$. Note that one has $\beta_n = \|T^n e_0\|^2$ and $\alpha_n = \sqrt{\beta_{n+1}/\beta_n}$ for $n \geqslant 0$.

Example 2.1. For a regular Borel probability measure μ on [0,1], the relations

$$\beta_n = \int\limits_{[0,1]} x^n \,\mathrm{d}\mu(x), \quad n \geqslant 0 \tag{2.1}$$

characterize a contractive subnormal weighted shift. If μ is the unit point mass at 1, one obtains the unilateral shift $U : \{1\}$; and if $\mathrm{d}\mu(x)$ is the Lebesgue measure $\mathrm{d}x$ on [0,1], then one obtains the Bergman shift $B : \left\{\sqrt{\frac{n+1}{n+2}}\right\}$. In [1], Jim Agler discussed a sequence of functional Hilbert spaces $\mathcal{H}_n$ and the corresponding multiplication operators $M_z^{(n)}$ which fitted very naturally into his treatment of the model theory for operators through complete positivity considerations. In fact, $M_z^{(1)} = U$ and $M_z^{(2)} = B$. It follows from the observations in [4] that $M_z^{(k)}$, for $k \geqslant 2$, can be identified with weighted shift operators in such a way that, corresponding to $\beta_n = \beta_n(M_z^{(k)})$, one has the measure $\mathrm{d}\mu(x) = \mathrm{d}\mu_k(x) = (k-1)(1-x)^{k-2}\mathrm{d}x$ in (2.1). (In view of the requirement $\beta_0(M_z^{(k)}) = 1$, the expression for $\mathrm{d}\mu_k(x)$ in Remark 3 of [6] requires a minor correction in the normalization factor as indicated here; the gist of the related observations in [6] remains totally unaffected.)

Example 2.2. For a regular positive Borel measure $\mathrm{d}\mu$ on [0,1] the relations

$$\begin{aligned} \beta_n &= 1 + n\mu\{1\} + \int\limits_{[0,1)} (1-x^n)\,\frac{\mathrm{d}\mu(x)}{1-x} \\ &\Big(= 1 + \int\limits_{[0,1]} (1 + x + \cdots + x^{n-1})\,\mathrm{d}\mu(x)\Big) \quad \text{for } n \geqslant 1 \end{aligned} \tag{2.2}$$

characterize a completely hyperexpansive weighted shift. If μ is chosen to be the zero measure, one obtains the unilateral shift $U : \{1\}$ and if μ is the unit point mass at $\{1\}$, one obtains the Dirichlet shift $D : \left\{\sqrt{\frac{n+2}{n+1}}\right\}$. In [6], the second-named author discussed completely hyperexpansive weighted shifts $N^{(k)}$ associated with $M_z^{(k)}$ of Example 2.1 through the use of the difference operator Δ. The Dirichlet shift D is a well-known example of a 2-isometry.

Example 2.3. Consider the weighted shift

$$T_\lambda : \left\{\sqrt{\frac{n+\lambda}{n+1}}\right\}, \quad \lambda > 0. \tag{2.3}$$

Let

$$S_p = \sum_{0\leqslant k\leqslant p} (-1)^k \binom{p}{k} T_\lambda^{*k} T_\lambda^k, \quad p \geqslant 0.$$

It can be established by induction that $\langle S_p e_m, e_m\rangle = \prod_{k=1}^{p} (k-\lambda)/(m+k)$ for all $m \geqslant 0$, $p \geqslant 1$. It is now clear that $\langle S_p e_m, e_m\rangle \geqslant 0$ for all $m \geqslant 0$, $p \geqslant 1$ if $0 < \lambda \leqslant 1$ so that, for that range of λ, T_λ is a contractive subnormal. On the other hand, $\langle S_p e_m, e_m\rangle \leqslant 0$ for all $m \geqslant 0$, $p \geqslant 1$ if $1 \leqslant \lambda \leqslant 2$ so that, for that range of λ, T_λ is completely hyperexpansive. One notes that T_1 is the unilateral shift U and T_2 the Dirichlet shift D. Further, for any positive integral value of λ, say $\lambda = m$, T_λ is an m-isometry but not an $(m-1)$-isometry. (The last fact was verified in [4].) We will later return to T_λ to consider in particular the range $\lambda > 2$.

Example 2.4. If T satisfies the inequalities in (1.1) for $1 \leqslant n \leqslant m$, then T is said to be an *m-hypercontraction* ([1]). It follows from the work of Agler in [1] that, for any $m \geqslant 1$, the adjoints $M_z^{(m)*}$ of $M_z^{(m)}$ referred to in Example 2.1 is an m-hypercontraction, but not an $(m+1)$-hypercontraction (see also [17]). One may similarly refer to T as an *m-hyperexpansion* if the inequalities in (1.2) are required to hold for $1 \leqslant n \leqslant m$.

Consider, for $1 \leqslant x < \sqrt{2}$, the weighted shift operator

$$T_x : \{\alpha_n\}, \quad \text{where } \alpha_0 = \sqrt{2},\ \alpha_1 = x, \text{ and } \alpha_n = 1 \text{ for } n \geqslant 2. \tag{2.4}$$

We leave it to the reader to check that, for $\sqrt{\frac{3}{2}} < x \leqslant \sqrt{2}$, T_x is a 1-hyperexpansion but not a 2-hyperexpansion, and for $\sqrt{\frac{2m+1}{2m}} < x \leqslant \sqrt{\frac{2m-1}{2m-2}}$ $(m \geqslant 2)$, T_x is an m-hyperexpansion which is not an $(m+1)$-hyperexpansion. Note that T_1 is completely hyperexpansive.

Example 2.5. We begin by pointing out that $T : \{\alpha_n\}$ is alternatingly hyperexpansive if and only if $n \to \beta_n$ is absolutely monotone. (One simply has to argue as in Proposition 3 of [6].)

Now let x be any real. In keeping with the classical combinatorial theory, we define $(x)_0 = 1$, $(x)_1 = x$, and $(x)_k = x(x-1)\cdots(x-k+1)$ for any integer $k \geqslant 2$. Note that $\Delta(n)_k = k(n)_{k-1}$, $\Delta^k(n)_k = k!$, and $\Delta^m(n)_k = 0$ if $m > k$. For any real sequence $\{\beta_n\}_{n\geqslant 0}$, one has

$$\beta_n = \sum_{k=0}^{\infty} \frac{\Delta^k \beta_0}{k!}\,(n)_k = \sum_{k=0}^{n} \frac{\Delta^k \beta_0}{k!}\,(n)_k$$

(Newton's Interpolation Formula). Thus one can always write

$$\beta_n = \sum_{k=0}^{\infty} a_k (n)_k \tag{2.5}$$

for any real $\{\beta_n\}$; and $\Delta^p \beta_n \geqslant 0$ for all $p \geqslant 0$, $n \geqslant 0$ (that is, $\{\beta_n\}$ is absolutely monotone) if and only if $a_k \geqslant 0$ for all $k \geqslant 0$. In case the sequence $\{a_k\}$ contains only finitely many non-zero terms, one has $\beta_n = \sum\limits_{k\leqslant p} a_k(n)_k$ for some $p \geqslant 0$, and $\Delta^{p+1}(\beta_n) = 0$ for all $n \geqslant 0$. In fact, any real sequence $\{\beta_n\}$ satisfying $\Delta^{p+1}(\beta_n) = 0$ must be of the form $\beta_n = \sum\limits_{k\leqslant p} a_k(n)_k$ ([14]). Of particular interest is the situation $\beta_n = \beta_n(T)$, where T is in $\mathcal{B}(\mathcal{H})$ so that $a(0) = \beta_0 = 1$ and $\{\beta_{n+1}/\beta_n\}$ is a bounded sequence. Clearly, with the indicated restrictions on β_n, T is an alternatingly hyperexpansive weighted shift if $a_k \geqslant 0$ for all $k \geqslant 0, T$ is a p-isometry ($p \geqslant 1$) if $a_k = 0$ for $k \geqslant p$, and T is an alternatingly hyperexpansive p-isometry ($p \geqslant 1$) if $a_k \geqslant 0$ for $0 \leqslant k \leqslant p-1$ and $a_k = 0$ for $k \geqslant p$. We leave it to the reader to verify that if $\{\beta_n\}$ is the sequence given by $\beta_n = \sum\limits_{k=0}^{\infty} b_k n^k$ with $b_k \geqslant 0$, then β_n can be written as $\beta_n = \sum\limits_{k=0}^{\infty} a_k(n)_k$ with $a_k \geqslant 0$.

Example 2.6. We now look at some special manifestations of (2.5) as relevant for the class of alternatingly hyperexpansive operators. It follows from our discussion in Example 2.5 that certain special p-isometries form a subclass of the class of alternatingly hyperexpansive operators. We will presently see that certain subnormal expansions are also alternatingly hyperexpansive. Consider

$$\beta_n = \int\limits_{[1,a]} x^n \,\mathrm{d}\mu(x), \tag{2.6}$$

where $a \geqslant 1$ and μ is a positive regular Borel measure on $[1, a]$. Note that $\Delta^p \beta_n = \int\limits_{[1,a]} x^n (x-1)^p \,\mathrm{d}\mu(x) \geqslant 0$ for $p \geqslant 0$ and $n \geqslant 0$. In particular, if $\beta_n = \beta_n(T)$ (so that $\mu[1, a] = \beta_0 = 1$), then T is alternatingly hyperexpansive; further it is easy to see from (2.6) that T is a subnormal expansion. If μ is chosen to be the unit point mass at $\{a\}$, then one obtains the weighted shift $A : \{a\}$. Here $\beta_n = a^n$, which is $\sum\limits_{k=0}^{\infty} b_k n^k$ with $b_k = (\log a)^k/k! \geqslant 0$. In Section 4, we will elaborate upon the special significance of such examples by referring to the theory of "absolutely monotone functions" on $\mathbb{R}_+$. It may be noted that, for two absolutely monotone sequences $\{\beta_n\}$ and $\{\gamma_n\}$ on $\mathbb{N}, \{\beta_n + \gamma_n\}$ is clearly absolutely monotone, as is $\{\beta_n\gamma_n\}$ in view of $\Delta^p(\beta_n\gamma_n) = \sum\limits_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \Delta^k \beta_{n+p-k} \Delta^{p-k}\gamma_n$. As the reader can easily check, the values of $a\,(> 0)$ in $\beta_n = \beta_n(T) = \frac{1}{2}\Big(1 + an + \int\limits_1^2 x^n \,\mathrm{d}x\Big)$ or in $\beta_n = \beta_n(T) = (1+an)e^n$ can be so chosen that $\alpha_0^2 = \beta_1/\beta_0 > \beta_2/\beta_1 = \alpha_1^2$, so that T, though alternatingly hyperexpansive, is not hyponormal (see [13]) and hence, in particular, not subnormal; further T is also not a p-isometry for any $p \geqslant 1$.

Example 2.7. Examples 2.1, 2.2, and 2.3 dealt with subnormal contractions and completely hyperexpansive operators. In Example 2.4 we considered operators which were "partially contractively subnormal" and "partially completely hyperexpansive". Having dwelt upon the instances of alternatingly hyperexpansive operators in Examples 2.5 and 2.6, we finally look at an example which is "partially alternatingly hyperexpansive", say, an "m-alternating hyperexpansion". For this we return to the weighted shift T_λ of Example 2.3. Consider then

$$T_\lambda : \left\{ \sqrt{\frac{n+\lambda}{n+1}} \right\}, \quad \lambda \geqslant 1. \tag{2.7}$$

If we define $S'_p = (-1)^p \sum\limits_{k=0}^{p} (-1)^k \begin{pmatrix} p \\ k \end{pmatrix} T_\lambda^{*k} T_\lambda^k$, then $\langle S'_p e_n, e_n\rangle \geqslant 0$ for all $n \geqslant 0$ and for $1 \leqslant p \leqslant m$; but $\langle S'_{m+1} e_n, e_n\rangle < 0$ if $m < \lambda$.

We conclude this section by tabulating the myriad avatars of the weighted shift $T_\lambda : \left\{ \sqrt{\frac{n+\lambda}{n+1}} \right\}$, $\lambda > 0$ of (2.3):

$\lambda = 1 :$ Unilateral shift (isometry)

$\lambda = 2 :$ Dirichlet shift (2-isometry)

$\lambda = m :$ m-isometry

$0 < \lambda \leqslant 1 :$ Contractive subnormal

$1 \leqslant \lambda \leqslant 2 :$ Completely hyperexpansive

$m \leqslant \lambda < m + 1 :$ m-alternating hyperexpansion.

3. COMPLETELY HYPEREXPANSIVE OPERATORS

In this section we undertake a study of the structural and spectral properties of completely hyperexpansive operators building upon the results in [2] and [6]. We begin by proving the generalized versions of Propositions 1.6, 1.25, and Theorem 1.26 in [2] in the context of k-hyperexpansions. Then follows a rather meticulous discussion of the spectral parts of a completely hyperexpansive operator. We further show that, as in the case of subnormals (refer to [11], [26] and [13]), quasisimilar completely hyperexpansive operators have equal spectra and equal essential spectra. The section concludes by trying to provide a probabilistic interpretation of conditions (1.2) in the light of Hoeffding inequalities (refer to [9]).

For any T in $\mathcal{B}(\mathcal{H})$, define

$$\Delta_m(T) = \sum_{k=0}^{m} (-1)^r \binom{m}{r} T^{*m-r} T^{m-r}, \quad m \geqslant 1.$$

Note that $T^*\Delta_m(T)T - \Delta_m(T) = \Delta_{m+1}(T)$, and if T is an m-hyperexpansion, then in particular $(-1)^{m+1}\Delta_m(T) \geqslant 0$. For any S in $\mathcal{B}(\mathcal{H})$, we define the operator $M_S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $M_S(T) = S^*TS$ so that $M_S^p(T) = S^{*p}TS^p$ ($p \geqslant 1$). Note that conditions (1.1), (1.2) and (1.3) can respectively be expressed in terms of M_T as

$$(I - M_T)^n(I) \geqslant 0, \quad \text{for } n \geqslant 1, \tag{1.1$'$}$$

$$(I - M_T)^n(I) \leqslant 0 \quad \text{for } n \geqslant 1, \tag{1.2$'$}$$

and

$$(-1)^n(I - M_T)^n(I) \geqslant 0 \quad \text{for } n \geqslant 1. \tag{1.3$'$}$$

The reader should compare condition (vi) in Proposition 3.1 below with (1.32) in [2].

Proposition 3.1. *Let T in $\mathcal{B}(\mathcal{H})$ be a k-hyperexpansion ($k \geqslant 2$). Then*

(a) $\ker(\Delta_1(T))$ *is invariant for T and $T|\ker(\Delta_1(T))$ is an isometry; further, if $M \subset \mathcal{H}$ is an invariant subspace for T and $T|M$ is an isometry, then $M \subset$* $\ker(\Delta_1(T))$*;*

(b) *if M is the smallest invariant subspace for T generated by the closure of* $\operatorname{ran}(\Delta_1(T))$ *then M reduces T and $M^\perp$ is the largest reducing subspace for T on which T is an isometry;*

(c) *if M is the largest reducing subspace for T on which T is an isometry, then T has the matrix representation*

$$T = \begin{bmatrix} V_0 & 0 & 0 \\ 0 & V & E \\ 0 & 0 & X \end{bmatrix}$$

with respect to the decomposition

$$\mathcal{H} = M \oplus (\ker(\Delta_1(T)) \ominus M) \oplus \operatorname{ran}(\Delta_1(T))^-,$$

where "$-$" denotes the closure and $\ominus$ the orthogonal complement. Further,

(i) *V_0 is an isometry;*

(ii) *V is a pure isometry (so that V does not have a unitary part);*

(iii) *$V^*E = 0$;*

(iv) *$\operatorname{ran} V$ is dense in $\ker V^*$;*

(v) *$\ker(E^*E + X^*X - I) = \{0\}$; and*

(vi) *if $S = E^*E + X^*X - I$, then $(I - M_X)^{m-1}(S) \geqslant 0$, $1 \leqslant m \leqslant k$.*

Proof. (a) We require a minor adaptation of the argument in Proposition 1.6 of [2]. Since T is in particular a 2-hyperexpansion, $I - 2T^*T + T^{*2}T^2 \leqslant 0$, that is, $T^*(\Delta_1(T))T - \Delta_1(T) \leqslant 0$. If x is in $\ker(\Delta_1(T))$, then one has $\langle \Delta_1(T)Tx, Tx\rangle = \langle T^*\Delta_1(T)Tx, x\rangle \leqslant \langle \Delta_1(T)x, x\rangle = 0$.

Since $\Delta_1(T) \geqslant 0$, however, one must have $\Delta_1(T)Tx = 0$ proving that $\ker(\Delta_1(T))$ is invariant for T. The rest of the proof is identical with that in [2].

(b) Use the proof of Proposition 1.25 in [2].

(c) For establishing the matrix representation of T and properties (i) through (v), one relies on the arguments in the proof of Theorem 1.26 in [2]. We next consider (vi). Using the matrix representation for T, one finds that

$$\Delta_1(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S \end{bmatrix},$$

where $S = E^*E + X^*X - I$. Clearly, $\Delta_1(T) \geqslant 0$ implies $S \geqslant 0$. Using $T^*\Delta_m(T)T - \Delta_m(T) = \Delta_{m+1}(T)$, $(1 \leqslant m \leqslant k)$, one finds that

$$(-1)^{m+1}\Delta_m(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (I - M_X)^{m-1}(S) \end{bmatrix} \geqslant 0, \quad 1 \leqslant m \leqslant k$$

leading to $(I - M_X)^{m-1}(S) \geqslant 0$ for $1 \leqslant m \leqslant k$. ∎

Corollary 3.2. *If T in $\mathcal{B}(\mathcal{H})$ is completely hyperexpansive then the statements in Proposition* 3.1 *hold with the conditions in* (vi) *replaced by* $(I-M_X)^m(S) \geqslant 0$, $(m \geqslant 0)$.

Proposition 3.3. *Let T in $\mathcal{B}(\mathcal{H})$ be a 2-hyperexpansive weighted shift. If $\mathbb{D}$ denotes the open unit disk in the complex plane centered at the origin and $\partial\mathbb{D}$ its boundary, then*

(i) $\sigma(T) = \overline{\mathbb{D}}$;

(ii) $\mathbb{D} \subset \sigma_{\mathrm{p}}(T^*)$, $\sigma_{\mathrm{p}}(T) = \phi$, $\sigma_{\mathrm{e}}(T) = \partial\mathbb{D}$;

(iii) $\sigma_{\mathrm{ap}}(T) = \partial\mathbb{D}$, $\mathbb{D} \subset \Gamma(T)$, *the compression spectrum of* T;

(iv) $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \partial\mathbb{D}$, *and* $\mathrm{ind}\,(T-\lambda) = -1$ *for* $\lambda \in \mathbb{D}$ *(with* ind *denoting the Fredholm index); and*

(v) $\sigma_{\mathrm{l}}(T) = \partial\mathbb{D}$, $\sigma_{\mathrm{r}}(T) = \overline{\mathbb{D}}$.

Proof. The statements in (i) and (ii) follow from the arguments in Proposition 5 of [6]. The rest of the assertions are easy to establish. ∎

Remark 3.4. If T in $\mathcal{B}(\mathcal{H})$ is a completely hyperexpansive operator (or simply a 2-hyperexpansion), then $\sigma(T) \subset \overline{\mathbb{D}}$ ([20]); while it is easy to see that $\sigma_{\mathrm{ap}}(T)$, and hence $\sigma_{\mathrm{p}}(T)$, is contained in $\partial\mathbb{D}$. If T is in addition non-invertible, then the reader may check that $\sigma_{\mathrm{r}}(T) = \sigma(T) = \overline{\mathbb{D}}, \sigma_{\mathrm{l}}(T) = \sigma_{\mathrm{ap}}(T) = \partial\mathbb{D}$, and $\sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) = \partial\mathbb{D} = \sigma_{\mathrm{le}}(T)$. If T in $\mathcal{B}(\mathcal{H})$ is an invertible completely hyperexpansive operator (or simply an invertible 2-hyperexpansion), then T must be unitary, as can be checked by using the result of S. Richter mentioned earlier. From these observations it is also easy to deduce that every 2-hyperexpansion has a non-trivial, closed, proper invariant subspace.

The reader is now referred to [13] for the notion of quasisimilarity of two operators. As already mentioned, quasisimilar subnormals have equal spectra and equal essential spectra. In the light of Proposition 3.5 below, a similar assertion holds for completely hyperexpansive operators, though part (ii) there is in contrast with the known facts for subnormals (see [13]). The proof of Proposition 3.5 is left to the reader.

Proposition 3.5. *If S in $\mathcal{B}(\mathcal{H})$ and T in $\mathcal{B}(\mathcal{K})$ are quasisimilar* 2*-hyperexpansions, then*

(i) $\sigma(S) = \sigma(T)$;

(ii) $\sigma_{\mathrm{ap}}(S) = \sigma_{\mathrm{ap}}(T)$;

(iii) $\sigma_{\mathrm{le}}(S) = \sigma_{\mathrm{le}}(T)$ *and* $\sigma_{\mathrm{re}}(S) = \sigma_{\mathrm{re}}(T)$ *(so that* $\sigma_{\mathrm{e}}(S) = \sigma_{\mathrm{e}}(T)$*)*;

(iv) $\sigma_{\mathrm{l}}(S) = \sigma_{\mathrm{l}}(T)$ *and* $\sigma_{\mathrm{r}}(S) = \sigma_{\mathrm{r}}(T)$.

The next proposition describes the intersections of the class of completely hyperexpansive operators with the class of subnormals and the class of p-isometries.

PROPOSITION 3.6. (i) *If T in $\mathcal{B}(\mathcal{H})$ is completely hyperexpansive as well as subnormal, then T is an isometry.*

(ii) *If T in $\mathcal{B}(\mathcal{H})$ is completely hyperexpansive and a p-isometry, then T must be a 2-isometry.*

Proof. (i) Since T is subnormal, we have, for any h in $\mathcal{H}$, a positive regular Borel measure μ_h on $[0, \|T\|^2]$ such that, for all $n \geqslant 0$,

$$\|T^n h\|^2 = \int\limits_{[0,\|T\|^2]} x^n \, \mathrm{d}\mu_h(x).$$

Since T is completely hyperexpansive as well, there exists a positive regular Borel measure ν_h on $[0, 1]$ such that

$$\|T^n h\|^2 = \|h\|^2 + n\nu_h\{1\} + \int\limits_{[0,1)} (1 - x^n) \frac{\mathrm{d}\nu_h(x)}{1 - x},$$

so that

$$\int\limits_{[0,\|T\|^2]} x^n \, \mathrm{d}\mu_h(x) = \|h\|^2 + n\nu_h\{1\} + \int\limits_{[0,1)} (1 - x^n) \frac{\mathrm{d}\nu_h(x)}{1 - x}.$$

Applying ∇^2 to both sides, one has

$$\int\limits_{[0,\|T\|^2]} x^n (1 - x)^2 \, \mathrm{d}\mu_h(x) = - \int\limits_{[0,1)} x^n (1 - x)^2 \frac{\mathrm{d}\nu_h(x)}{1 - x} = - \int\limits_{[0,1)} x^n (1 - x) \, \mathrm{d}\nu_h(x).$$

This implies that $\int_{[0,\|T\|^2]} x^n (1 - x)^2 \, \mathrm{d}\mu_h(x) = 0$ for all $n \geqslant 0$ so that μ_h is concentrated on $\{1\}$; in particular,

$$\|T^n h\|^2 = \int\limits_{[0,\|T\|^2]} 1 \cdot \mathrm{d}\mu_h(x) = \|h\|^2$$

for all n, that is, T is an isometry.

(ii) As in (i), one has

$$\|T^n h\|^2 = \|h\|^2 + n\nu_h\{1\} + \int\limits_{[0,1)} (1 - x^n) \frac{\mathrm{d}\nu_h(x)}{1 - x}.$$

If T is an isometry, then T is trivially a 2-isometry. Thus assume that T is a p-isometry with $p \geqslant 2$. Then $\nabla^p(\|T^n h\|^2) = 0$ implies

$$- \int\limits_{[0,1)} x^n (1 - x)^p \frac{\mathrm{d}\nu_h(x)}{1 - x} = 0,$$

that is, $\int_{[0,1]} x^n (1 - x)^{p-1} \, \mathrm{d}\nu_h(x) = 0$, forcing ν_h to be concentrated on $\{1\}$ and $\|T^n h\|^2 = \|h\|^2 + n\nu_h\{1\}$ for all n, which yields $\nabla^2(\|T^n h\|^2) = 0$ for all n. ∎

Proposition 3.7. *If $T : \{\alpha_n\}$ is a bilateral weighted shift (so that n varies over $\mathbb{Z}$, the set of integers) which is completely hyperexpansive, then T must be unitary (with the weights $\alpha_n = 1$).*

Proof. Use Proposition 6.8 in Chapter 2 of [13] and Corollary 2 in [6]. ∎

We now turn our attention to a probabilistic interpretation of conditions (1.2) by exploiting the theory of Hoeffding inequalities as expounded in [9]. The subject matter of Section 4 does not in any way depend on the ensuing discussion in this section.

Let S be a non-empty set and $(\Omega, \mathcal{A}, P)$ a probability space. (Thus $\mathcal{A}$ is a sigma-field of subsets of Ω and P a probability measure on $\mathcal{A}$.) A map $X : \Omega \to S$ is called an *elementary random variable* if $\{X = s\} \in \mathcal{A}$ for all s in S and $X(\Omega)$ has finite cardinality.

Definition 3.8. Let S be an abelian semigroup. A map $\psi : S \to \mathbb{R}$ is said to fulfil *Hoeffding's inequality of order* $n \geqslant 2$ if for every sequence $X_1, X_2, \ldots, X_n$ of n independent elementary S-valued random variables the inequality

$$E_{\overline{\mu}}(\psi(X_1 + \cdots + X_n) \leqslant E_{\mu_1,\ldots,\mu_n}(\psi(X_1 + \cdots + X_n))$$

holds, where, on the right, $E_{\mu_1,\ldots,\mu_n}$ denotes the expectation with respect to $(\mu_1, \ldots, \mu_n)$, μ_i being the distribution for X_i, and, on the left, $E_{\overline{\mu}}$ denotes the expectation with respect to $(\overline{\mu}, \ldots, \overline{\mu})$, where $\overline{\mu} = \frac{1}{n}\sum\limits_{i=1}^{n} \mu_i$.

Of special interest to us is the case when $S = \mathbb{N}$ and $\psi(n) = \psi_h(n) = \|T^n h\|^2$ $(h \in \mathcal{H})$. Recall that T in $\mathcal{B}(\mathcal{H})$ is completely hyperexpansive if and only if $\psi_h : \mathbb{N} \to \mathbb{R}$ is completely alternating for every h in $\mathcal{H}$. One says that a map $\psi : \mathbb{N} \to \mathbb{R}$ is completely negative definite if $n \to \psi(n + k)$ is negative definite for every k in $\mathbb{N}$. Since $\psi_h(n) = \|T^n h\|^2 \geqslant 0$ for all n, it follows from Theorem 1.10 in Chapter 7 of [9] that ψ_h is completely alternating if and only if ψ_h is completely negative definite. Now it is known that $\psi : \mathbb{N} \to \mathbb{R}$ is negative definite if and only if ψ fulfils Hoeffding's inequality of order 2 ([9], Chapter 7, Corollary 1.4), while ψ is completely negative definite if and only if ψ fulfils Hoeffding's inequality of order 3 (and in fact Hoeffding's inequality of all orders) ([9], Chapter 7, Theorem 1.7). The reader may now easily establish the following proposition.

Proposition 3.9. *Let T be in $\mathcal{B}(\mathcal{H})$, and let $\psi_h : \mathbb{N} \to \mathbb{R}$ be defined by $\psi_h(n) = \|T^n h\|^2$ $(h \in \mathcal{H})$. Then the following are equivalent:*

(i) *T is completely hyperexpansive;*

(ii) *ψ_h is negative definite for every h in $\mathcal{H}$;*

(iii) *ψ_h is completely negative definite for every h in $\mathcal{H}$.*

(iv) ψ_h *satisfies Hoeffding's inequality of order* 2 *for every* h *in* $\mathcal{H}$.

(v) ψ_h *satisfies Hoeffding's inequality of order* 3 *(or of all orders) for every* h *in* $\mathcal{H}$.

4. ALTERNATINGLY HYPEREXPANSIVE OPERATORS

Besides studying the properties of alternatingly hyperexpansive operators, we also try in this section to explore their links to subnormals and completely hyperexpansive operators. In the course of our development of this topic, it will be convenient at several stages to summarize the facts pertaining to special classes of functions, operators, weighted shifts, etc. in tabular forms. For this purpose we use the following abbreviations for different classes of operators:

S: subnormal, CS: contractive subnormal, CH: completely hyperexpansive, AH: alternatingly hyperexpansive, ISO: isometry, p-ISO: p-isometry.

The conceptual framework for the subject matter of the present paper is displayed by Table 4.1 below:

Nature of $n \to \|T^n h\|^2$ on $\mathbb{N}$ ($h \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$)	Class of operators in $\mathcal{B}(\mathcal{H})$
completely monotone	CS
completely alternating	CH
absolutely monotone	AH
absolutely decreasing	ISO

Table 4.1.

The next proposition points out the strong connection between subnormals and alternatingly hyperexpansive operators.

Proposition 4.1. (i) *Let* T *in* $\mathcal{B}(\mathcal{H})$ *be invertible. Then* T *is a subnormal contraction if and only if* T^{-1} *is alternatingly hyperexpansive.*

(ii) *If* T *is subnormal with* N *as the minimal normal extension of* T *and with* $\sigma(N) \cap \mathbb{D} = \varphi$, *where* $\mathbb{D}$ *is the open unit disk centered at the origin, then* T *is alternatingly hyperexpansive.*

Proof. (i) Let $\Delta_T^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k}T^k$. If T is a subnormal contraction, one has $\Delta_T^{(n)} \geqslant 0$ for $n \geqslant 1$. But then $T^{-n^*}\Delta_T^{(n)}T^{-n} \geqslant 0$ for $n \geqslant 1$, that is, $(-1)^n\Delta_{T^{-1}}^{(n)} \geqslant 0$ for $n \geqslant 1$, implying that T^{-1} is alternatingly hyperexpansive. The converse holds by a similar argument.

(ii) Let $E(\cdot)$ denote the spectral measure of N in $\mathcal{B}(\mathcal{K})$. For any Borel set $\sigma \subset \mathbb{R}$, define $E'(\sigma) = E(\varphi^{-1}(\sigma))$, where $\varphi(z) = |z|^2$. Consider $\rho(\cdot) = P_{\mathcal{H}}E'(\cdot)|\mathcal{H}$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of K onto $\mathcal{H}$. Since $\sigma(N) \cap \mathbb{D}$ is empty, E is supported on some annulus $\{z \in \mathbb{C} : 1 \leqslant |z| \leqslant \sqrt{a}\}$ $(a \geqslant 1)$, so that ρ is supported on $[1, a]$. It is easy to see that

$$\|T^n h\|^2 = \int_{[1,a]} x^n \,\mathrm{d}\langle \rho(x)h, h\rangle$$

(cf. (2.5)) so that

$$\Delta^k(\|T^n h\|^2) = \int_{[1,a]} x^n(x-1)^k \,\mathrm{d}\langle \rho(x)h, h\rangle \geqslant 0 \qquad \text{for } k, n \geqslant 0. \qquad \blacksquare$$

If T in $\mathcal{B}(\mathcal{H})$ is subnormal, then $\mathrm{e}^T/\mathrm{e}^{\|T\|}$ is a subnormal contraction and it follows from part (i) of Proposition 4.1 that $\mathrm{e}^{\|T\|}\mathrm{e}^{-T}$ is alternatingly hyperexpansive; also for any λ not in $\sigma(T)$, $\|T-\lambda I\|(T-\lambda I)^{-1}$ is alternatingly hyperexpansive. Note that, by the functional calculus for a subnormal operator, if T is an invertible subnormal, then so is T^{-1}. Thus part (i) associates with any invertible subnormal contraction a subnormal expansion which is in fact alternatingly hyperexpansive.

Remark 4.2. As part (ii) of Proposition 4.1 indicates, it is rather difficult to make very general statements regarding the spectral parts of an alternatingly hyperexpansive operator. If T in $\mathcal{B}(\mathcal{H})$ is an invertible, alternatingly hyperexpansive operator, then T^{-1} and hence T are subnormal so that T has a closed, proper, non-trivial invariant subspace. Suppose T is non-invertible and alternatingly hyperexpansive. Since T is an expansion, $\sigma_{\mathrm{ap}}(T)$ must be contained in the complement of $\mathbb{D}$. Further, the facts $\partial\sigma(T) \subset \sigma_{\mathrm{ap}}(T)$ and $0 \in \sigma(T)$ imply $\mathbb{D} \subset \sigma(T)$. But then, for $|\lambda| < 1$, $\operatorname{ran}(T-\lambda)$ is a closed, proper, non-trivial invariant subspace for T. Hence every alternatingly hyperexpansive operator has a closed, proper, non-trivial invariant subspace. If S in $\mathcal{B}(\mathcal{H})$ and T in $\mathcal{B}(\mathcal{K})$ are quasisimilar alternatingly hyperexpansive operators, then S and T are either both invertible or both non-invertible, as follows by using the fact that S and T are expansions. In case both S and T are invertible, they are subnormal as well so that $\sigma(S) = \sigma(T)$ and $\sigma_{\mathrm{e}}(S) = \sigma_{\mathrm{e}}(T)$.

QUESTION. Do non-invertible quasisimilar alternatingly hyperexpansive operators have equal spectra? equal essential spectra?

The next proposition provides a few concrete ways of switching from one special class of weighted shifts to another.

PROPOSITION 4.3. (i) *If* $T : \{\alpha_n\}$ *is completely hyperexpansive, then* $T' : \{1/\alpha_n\}$ *is subnormal.*

(ii) *If* $T : \{\alpha_n\}$ *is a subnormal contraction, then for any* $a > 1$,

$$T' : \left\{ \sqrt{\frac{(1 + (a-1)\nabla)^{n+1}\beta_0}{(1 + (a-1)\nabla)^n \beta_0}} \right\} \text{ is alternatingly hyperexpansive.}$$

(iii) *If* $T : \{\alpha_n\}$ *is completely hyperexpansive, then, for any* $a > 1$,

$$T' : \left\{ \sqrt{\frac{2 - (1 + (a-1)\nabla)^{n+1}\beta_0}{2 - (1 + (a-1)\nabla)^n \beta_0}} \right\} \text{ is alternatingly hyperexpansive.}$$

Proof. Part (i) is a special consequence of Proposition 6 in [6] as noted in Remark 4 there. In part (ii), $\triangle^k \beta_n(T') \geqslant 0$ $(k, n \geqslant 0)$ results from $\nabla^k \beta_0 \geqslant 0$ $(k \geqslant 0)$; while in part (iii), $\triangle^k \beta_n(T') \geqslant 0$ $(k, n \geqslant 0)$ results from $\nabla^k \beta_0 \leqslant 0$ $(k \geqslant 1)$. ∎

REMARK 4.4. As pointed out in [6], the connection between contractive subnormal weighted shfits and completely hyperexpansive ones, of which part (i) in Proposition 4.3 is a special instance, is intrinsically the connection between completely monotone and completely alternating maps on $\mathbb{N}$. As was noted in [6], under the correspondence considered here, the unilateral shift $U : \{1\}$ goes to itself, while the Dirichlet shift $D = \left\{\sqrt{\frac{n+2}{n+1}}\right\}$ goes to the Bergman shift $B : \left\{\sqrt{\frac{n+1}{n+2}}\right\}$. Some more light may be shed on the content of part (ii) and part (iii) of Proposition 4.3 by exploiting the integral representations for β_n as given in (2.1) and (2.2). With reference to part (iii), the reader may easily show that

$$(1 + (a-1)\nabla)^n \beta_0 = \int_{[0,1]} [1 + (a-1)(1-x)]^n \, \mathrm{d}\mu(x) = \int_{[1,a]} t^n \, \mathrm{d}\mu'(t)$$

for an appropriate μ' (cf. (2.6)). Under the correspondence considered here the shift U goes to itself and the Bergman shift to T' with

$$\beta_n(T') = \int_{[1,a]} t^n \, \mathrm{d}t = \frac{a^{n+1} - 1}{n+1}.$$

With reference to part (iii), the reader may check that

$$2-(1+(a-1)\nabla)^n\beta_0 = 1+\int_{[1,a]}(1+t+\cdots+t^{n-1})\,\mathrm{d}\theta(t)$$

for an appropriate θ defined on $[1,a]$. Under the correspondence considered here, U goes to U and D to D.

In Proposition 3.6 we treated the intersections $\mathrm{S}\cap\mathrm{CH}$ and $\mathrm{CH}\cap(p\text{-ISO})$. The next proposition looks at $\mathrm{S}\cap(p\text{-ISO})$.

Proposition 4.5. *If T in $\mathcal{B}(\mathcal{H})$ is a subnormal as well as a p-isometry, then T must be an isometry.*

Proof. If T is a p-isometry, one has

$$\sum_{k=0}^{p}(-1)^k\binom{p}{k}T^{*k}T^k=0$$

so that

$$\sum_{k=0}^{p}(-1)^k\binom{p}{k}T^{*k}T^k\|T\|^{2k}/\|T\|^{2k}=0.$$

If T is subnormal then $T/\|T\|$ is a subnormal contraction; and if N is the minimal normal extension of $T/\|T\|$, then it follows by Proposition 8 of [7] that $\sum_{k=0}^{p}(-1)^k\binom{p}{k}\|T\|^{2k}N^{*k}N^k=0$, that is, $(I-\|T\|^2N^*N)^p=0$. Hence one has $I-\|T\|^2N^*N=0$ leading to $I-T^*T=0$. ∎

It is time we take stock of our understanding of the intersections of various classes of operators under consideration. The intersection $\mathrm{AH}\cap(p\text{-ISO})$ defies an easy description in general and we have indicated this by putting a question mark against $\mathrm{AH}\cap(p\text{-ISO})$ in Table 4.2 below. (It is trivial to see, however, that $\mathrm{AH}\cap\mathrm{ISO}=\mathrm{ISO}$ and $\mathrm{AH}\cap(2\text{-ISO})=(2\text{-ISO})$.) We will later turn to characterizations of certain subclasses of $\mathrm{AH}\cap(p\text{-ISO})$ (for any p). The other entries in Table 4.2 are either easy deductions from (1.1), (1.2) and (1.3), or have been covered by

Propositions 3.6 and 4.5.

Intersection of classes	Resulting class
$\mathrm{CS} \cap \mathrm{CH} \cap \mathrm{AH}$	ISO
$\mathrm{CS} \cap \mathrm{CH} = \mathrm{S} \cap \mathrm{CH}$	ISO
$\mathrm{CS} \cap \mathrm{AH}$	ISO
$\mathrm{CH} \cap \mathrm{AH}$	2-ISO
$\mathrm{CS} \cap (p\text{-ISO}) = \mathrm{S} \cap (p\text{-ISO})$	ISO
$\mathrm{CH} \cap (p\text{-ISO})$ $(p \geqslant 2)$	2-ISO
$\mathrm{AH} \cap (p\text{-ISO})$	?

Table 4.2.

Capitalizing on the theory of p-isometries as developed in [2], one may provide certain characterizations for alternatingly hyperexpansive cyclic or finitely cyclic p-isometries (see [13] for the definitions of cyclic and finitely cyclic operators). For this, we quickly recapituate the relevant notation and theory of [2].

The symbol $\mathcal{D}$ stands for the Frechet space of infinitely differentiable functions on $\partial\mathbb{D}$, while $\mathcal{D}'$ is the dual of $\mathcal{D}$, the space of distributions on $\partial\mathbb{D}$. The linear operator $D : \mathcal{D} \to \mathcal{D}$ is defined as the derivative $D\varphi = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi$; $D^{(0)}$ is identity, $D^{(1)}$ is D, and $D^{(l)}$, for $l \geqslant 2$, is $D(D-1)\cdots(D-l+1)$. A distribution differential operator (DDO) is a map $L : \mathcal{D} \to \mathcal{D}'$ that has the form $L = \sum\limits_{l=0}^{m} \beta_l D^{(l)}$, where $\beta_0, \ldots, \beta_m$ are in $\mathcal{D}'$. (Note that $L(\varphi)(\psi)$ with φ, ψ in $\mathcal{D}$ is to be interpreted as $L(\varphi)(\psi) = \sum\limits_{l=0}^{m} \beta_l((D^{(l)}\varphi)\psi)$.) If $\beta_m \neq 0$, then the order of L is defined to be m. The subset D_{a} of $\mathcal{D}$ is given by $\mathcal{D}_{\mathrm{a}} = \{\varphi \in \mathcal{D} : \varphi(\mathrm{e}^{\mathrm{i}\theta}) = \sum\limits_{n=0}^{\infty} \widehat{\varphi}(n)\mathrm{e}^{\mathrm{i}n\theta}\}$ and, if $\gamma_n \in \mathcal{D}'$ is defined as $\gamma_n(\varphi) = \int\limits_0^{2\pi} \varphi \mathrm{e}^{\mathrm{i}n\theta}\,\frac{\mathrm{d}\theta}{2\pi}$, then the subset $\mathcal{D}'_{\mathrm{a}}$ of $\mathcal{D}'$ is given by $\mathcal{D}'_{\mathrm{a}} = \Big\{u \in \mathcal{D}' : u = \sum\limits_{n=0}^{\infty} \widehat{u}(n)\gamma_n\Big\}$. A distribution Toeplitz operator (DTO) is a linear map $A : \mathcal{D}_{\mathrm{a}} \to \mathcal{D}'_{\mathrm{a}}$ that has the form $A = PL|\mathcal{D}_{\mathrm{a}}$ for some DDO L, where P denotes the canonical projection of $\mathcal{D}'$ onto $\mathcal{D}'_{\mathrm{a}}$. The operator L associated with A is unique so that one can define the order $\operatorname{ord}(A)$ of A to be that of L. Also A is said to be positive ($A \geqslant 0$) if $A(\varphi)(\overline{\varphi}) \geqslant 0$ for all φ in $\mathcal{D}_{\mathrm{a}}$. If A is a DTO, H^2_A is defined to be the completion of $\mathcal{D}_{\mathrm{a}}$ with respect to the sesquilinear form $\langle \varphi, \psi \rangle = A(\varphi)(\overline{\psi})$. An analytic Dirichlet operator is a DTO A with the property that either there exists $c > 1$ such that $A - \overline{c}^2\mathrm{e}^{-\mathrm{i}\theta}A\mathrm{e}^{\mathrm{i}\theta} \geqslant 0$ or ord $(A) = 0$ and $A \geqslant 0$. (Here $(\mathrm{e}^{-\mathrm{i}\theta} A \mathrm{e}^{\mathrm{i}\theta})(\varphi)(\psi)$ with φ and ψ in $\mathcal{D}_{\mathrm{a}}$ is to be interpreted as $PL(\mathrm{e}^{\mathrm{i}\theta}\varphi)(\mathrm{e}^{-\mathrm{i}\theta}\psi)$). It turns out that, for an analytic Dirichlet operator A, the multiplication operator M_A defined by $(M_A\varphi)(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}\varphi(\mathrm{e}^{\mathrm{i}\theta})$ is a well defined and bounded operator on H_A^2. Finally, for a DTO A, $(\frac{\mathrm{d}}{\mathrm{d}D})A$ is defined as $(\frac{\mathrm{d}}{\mathrm{d}D})A = P\frac{\mathrm{d}}{\mathrm{d}D}L = P\sum_{l\geqslant 1} l\beta_l D^{(l-1)}$.

Proposition 4.6. *If T in $\mathcal{B}(\mathcal{H})$ is an alternatingly hyperexpansive cyclic m-isometry with cyclic vector γ, then there exists a unique analytic Dirichlet operator A with* ord $(A) \leqslant m-1$ *and $(\frac{\mathrm{d}}{\mathrm{d}D})^l A \geqslant 0$, $(l \geqslant 1)$ such that T is unitarily equivalent to M_A in $\mathcal{B}(H_A^2)$ under a unitary U with $U(\gamma) = 1$. Conversely, if A is an analytic Dirichlet operator with* ord $(A) = m-1$ *and $(\frac{\mathrm{d}}{\mathrm{d}D})^l(A) \geqslant 0$ for all $l \geqslant 1$ then M_A is an alternatingly hyperexpansive cyclic m-isometry with cyclic vector* 1. *(In fact, $(\frac{\mathrm{d}}{\mathrm{d}D})^l(A) = 0$ for $l \geqslant m$.)*

Proof. If T is a cyclic m-isometry, then Theorem 3.23 of [2] asserts the existence of a unique analytic Dirichlet operator A with ord $(A) \leqslant m-1$ such that T is unitarily equivalent to M_A under a unitary U and $U(\gamma) = 1$. Now by Proposition 3.22 in [2], for any integer $l \geqslant 1$ and φ, ψ in $\mathcal{D}_{\mathrm{a}}$, one has

$$(-1)^l \Big\langle \Big(\sum_{k=0}^{l} (-1)^k \binom{l}{k} M_A^{*k} M_A^k \Big) \varphi, \psi \Big\rangle_{H_A^2} = \Big(\frac{\mathrm{d}}{\mathrm{d}D}\Big)^l A(\varphi)(\overline{\psi}).$$

Since T is alternatingly hyperexpansive, however, so is M_A and this clearly implies that $(\frac{\mathrm{d}}{\mathrm{d}D})^l A(\varphi)(\overline{\varphi}) \geqslant 0$ for all $l \geqslant 1$ so that $(\frac{\mathrm{d}}{\mathrm{d}D})^l(A) \geqslant 0$ for $l \geqslant 1$. Conversely, if A is an analytic Dirichlet operator of order $m-1$ with $(\frac{\mathrm{d}}{\mathrm{d}D})^l(A) \geqslant 0$ for all $l \geqslant 1$, then in view of Theorem 3.23 and Proposition 3.22 of [2], M_A is clearly an alternatingly hyperexpansive cyclic m-isometry with cyclic vector 1. ∎

We leave it to the reader to formulate the analog of Proposition 4.6 in the case of finitely cyclic operators by appealing to Theorem 3.49 and Proposition 3.41 of [2]. It is rewarding to brood over the implications of Proposition 4.6 with reference to (2.5).

We now attempt to make part (ii) of Proposition 4.1 somewhat explicit in the context of weighted shifts. This forces us into the discussion of “absolutely monotone functions” and “completely monotone functions” on real intervals.

Definition 4.7. A real valued function f defined on an interval $I \subset \mathbb{R}$ is said to be *absolutely monotone on I* if f is continuous on I and the derivatives $f^{(k)}(x)$ $(k \geqslant 0)$ are all non-negative at x in the interior of I. A function f defined on I is said to be *completely monotone on I* if $f(-x)$ is absolutely monotone on $-I = \{x \in \mathbb{R} : -x \in I\}$.

REMARK 4.8. An excellent reference for the discussion of absolutely and completely monotone functions on real intervals is [25]. A function f on $[a,b)$ is absolutely monotone if and only if

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh) \geqslant 0$$

for all non-negative integers n and for all x and h such that $a \leqslant x < x+h < \cdots < x+nh < b$ ([25], Chapter IV, Theorem 7). The reader is urged to compare this condition to our discussion centred around (1.3), and with the identifications $a=0, b=\infty$ in mind. It is known that an absolutely monotone function for the interval $\mathbb{R}_+$ in the sense of Definition 4.7 is an analytic power series $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ with $a_k \geqslant 0$ (see [10], and [25], Chapter IV, Theorem 3a). Absolutely monotone functions on $\mathbb{R}_+$, when interpolated on $\mathbb{N}$, yield absolutely monotone functions on the semigroup $\mathbb{N}$. (It should be noted, however, that not all absolutely monotone functions on $\mathcal{N}$ arise this way; consider for example $\beta_n = 1-n+2n^2 = 1+n+2n(n-1)$.) In this connection we also point out that if f is completely monotone on $[a,\infty)$ and δ is any positive number, then the sequence $\{f(a+n\delta)\}$ is completely monotone on $\mathbb{N}$ ([25], Chapter IV, Theorem 11d).

PROPOSITION 4.9. *Let $T : \{\alpha_n\}$ be a weighted shift operator. Then T is subnormal with the minimal normal extension N of T satisfying $\sigma(N) \cap \mathbb{D} = \phi$ if and only if $\beta_n(T) = 1 + \sum\limits_{k=1}^{\infty} a_k n^k$ where $\sum\limits_{k=1}^{\infty} a_k x^k$ is analytic on $[0,\infty)$ (with $a_k \geqslant 0$) and $\{k! a_k\}$ is a $[0,b]$-moment sequence for some $b > 0$.*

Proof. Let $T : \{\alpha_n\}$ be subnormal with the minimal normal extension N satisfying $\sigma(N) \cap \mathbb{D} = \phi$. Then as observed in part (ii) of Proposition 4.1, $\beta_n(T) = \int\limits_{[1,a]} x^n \, \mathrm{d}\mu(x)$ for some $a \geqslant 1$ and for a Borel probability measure μ on $[1,a]$. If $f(t) = \int\limits_{[1,a]} x^t \, \mathrm{d}\mu(x)$, then $f^{(k)}(t) = \int\limits_{[1,a]} x^t (\log x)^k \, \mathrm{d}\mu(x)$, so that f is an absolutely monotone function on $[0,\infty)$. This means $f(t) = 1 + \sum\limits_{k=1}^{\infty} a_k t^k$ where $\sum\limits_{k=1}^{\infty} a_k t^k$ is analytic on $[0,\infty)$ and $a_k \geqslant 0$ for all k. Indeed, $a_k = f^k(0)/k! = \int\limits_{[1,a]} (\log x)^k \, \mathrm{d}\mu(x)/k!$, so that $k! a_k = \int\limits_{[0,b]} y^k \, \mathrm{d}\mu'(y)$ for $b = \log a$ and μ' an appropriate measure.

Conversely, if $\beta_n(T) = 1 + \sum\limits_{k=1}^{\infty} a_k n^k$, where $\sum\limits_{k=1}^{\infty} a_k t^k$ is analytic on $[0, \infty)$ and $k! a_k = \int\limits_{[0,b]} x^k \, \mathrm{d}\mu(x)$ for an appropriate μ, then one has

$$\beta_n(T) = 1 + \sum_{k=1}^{\infty} \Big(\frac{1}{k!} \int\limits_{[0,b]} x^k \, \mathrm{d}\mu(x) \Big) n^k = \int\limits_{[0,b]} \mathrm{e}^{nx} \, \mathrm{d}\mu(x) = \int\limits_{[1,a]} y^n \, \mathrm{d}\mu'(y)$$

for $a = \mathrm{e}^b$ and an appropriate μ'. But this clearly implies that T is subnormal with the minimal normal extension N satisfying $\sigma(N) \cap \mathbb{D} = \varphi$. ∎

We list $\beta_n(T)$ corresponding to different varieties of weighted shifts T.

Class of operators in $\mathcal{B}(\mathcal{H})$, dim $(\mathcal{H}) = \aleph_0$	Weighted shift $T : \{\alpha_n\}_{n \geqslant 0}$ with $\beta_0 = 1$, $\beta_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geqslant 1)$
CS	$\beta_n = \int\limits_{[0,1]} x^n \, \mathrm{d}\mu(x)$ (μ a regular Borel probability measure on $[0,1]$)
CH	$\beta_n = 1 + \eta\mu\{1\} + \int\limits_{[0,1)} (1 - x^n) \frac{\mathrm{d}\mu(x)}{1-x}$ (μ a regular positive Borel measure on $[0,1]$)
AH	$\beta_n = 1 + \sum\limits_{k=1}^{\infty} a_k (n)_k \Big(= 1 + \sum\limits_{k=1}^{n} a_k (n)_k \Big)$, $a_k \geqslant 0$ $(k = 1, 2, \ldots)$, $\{\beta_{n+1}/\beta_n\}$ is a bounded sequence

Table 4.3.

Further, Table 4.4 depicts the finite-dimensional situation.

Class of operators in $\mathcal{B}(\mathcal{H})$ $(\dim(\mathcal{H}) < \infty)$	Class of matrices
CS	Normal ($\sigma(N) \subset \overline{\mathbb{D}}$)
CH	Unitary
AH	Normal ($\sigma(N) \cap \mathbb{D} = \emptyset$)
$\mathrm{S} \cap (p\text{-ISO})$	Unitary
$\mathrm{CH} \cap (p\text{-ISO})$	Unitary
$\mathrm{AH} \cap (p\text{-ISO})$	Unitary

Table 4.4.

At this stage, the authors simply cannot resist the temptation of pointing out an intriguing connection between bounded subnormal weighted shifts and unbounded subnormal weighted shifts brought about by the interaction of completely monotone functions on $\mathbb{R}_+$ and absolutely monotone functions on $\mathbb{R}_-$. (We mention that a procedure for linking bounded subnormal weighted shifts with unbounded ones was discussed in [5]; the procedure to be discussed here is different and appears to be more conceptual.) Recall that a linear operator T defined on an invariant dense subspace $\mathrm{D}(T)$ of $\mathcal{H}$ is called subnormal if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a normal operator N defined on a dense subspace $\mathrm{D}(N)$ of $\mathcal{K}$ such that $\mathrm{D}(T) \subset \mathrm{D}(N)$ and $Nh = Th$ for any h in $\mathcal{H}$. The creation operator $a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{\mathrm{d}}{\mathrm{d}x})$ acting on the Schwartz space $\mathcal{S}\,(\subset H = L^2(\mathbb{R}))$ is the most famous example of an unbounded subnormal operator (see [16], [23], [24]) and occurs in connection with the 2-dimensional Schrodinger representation of Quantum Mechanics. We consider $a^\dagger|D$ where D is the linear span of Hermite functions in $L^2(\mathbb{R})$ and use the well-known identification of $a^\dagger|D$ with the multiplication operator M_z as restricted to the space D of analytic polynomials in $\mathcal{H} = H^2(\mu), H^2(\mu)$ being the completion of D in $L^2(\mu)$, where $\mathrm{d}\mu(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{-r^2} r\mathrm{d}r\mathrm{d}\theta$ (see [21], [8]). Then $a^\dagger|D$ may be interpreted as the weighted shift $T : \{\sqrt{n+1}\}$, to be referred to as the Bargmann shift. The associated sequence $\{\beta_n = n!\}$ is a *Stieltjes Moment Sequence* in the sense that $n! = \int_{[0,\infty)} x^n \,\mathrm{d}\mu(x)$, for a positive, Borel measure μ on $\mathbb{R}_+$; indeed $\mathrm{d}\mu(x) = \mathrm{e}^{-x}\mathrm{d}x$.

It was shown in [24] that the weighted shift $T : \{\alpha_n\}$ as restricted to the linear span of $e_n\,(n \geqslant 0)$ is an unbounded (which includes "bounded") subnormal if and only if $\{\beta_n = \|T^n e_0\|^2\}$ is a Stieltjes moment sequence. On the other hand we know that T is a contractive subnormal unilateral weighted shift if and only if

$\{\beta_n\}$ is a Hausdorff moment sequence so that $\beta_n = \int\limits_{[0,1]} x^n \, \mathrm{d}\mu(x)$ for an appropriate μ. Among such moment sequences $\{\beta_n\}$ we look at the ones which are minimal in the sense that the sequence $\beta_0 - \varepsilon \ (= 1-\varepsilon), \beta_1, \beta_2, \ldots$ is not completely monotone for any $\varepsilon > 0$ ([25], Chapter IV, Definition 14a). The subnormal weighted shifts corresponding to minimal $\{\beta_n\}$ will be referred to as "nice". It follows from Theorem 14a in Chapter IV of [25] that $\{\beta_n\}$ is a minimal completely monotone sequence if and only if $\beta_n = \int\limits_{[0,1]} t^n \, \mathrm{d}\alpha(t)$, $(n \geqslant 0)$ where $\alpha(t)$ is a non-decreasing bounded function continuous at 0. Further, as follows from Theorem 14b there, $\{\beta_n\}$ is a minimal completely monotone sequence if and only if there exists a completely monotone function f on $\mathbb{R}_+$ such that $f(n) = \beta_n$ $(n = 0, 1, 2, \ldots)$. On the other hand, it follows from Theorem 15 in Chapter IV of [25] that $\{\beta_n\}$ is a Stieltjes Moment Sequence if and only if $\beta_n = f^{(n)}(0-)$ $(n \geqslant 0)$ for an absolutely monotone function on $\mathcal{R}_-$. This suggests the following scheme for linking certain contractive subnormal weighted shifts to unbounded subnormal weighted shifts:

minimal completely monotone sequence
(read "nice contractive subnormal weighted shift") $\rightarrow$
completely monotone function on $\mathbb{R}_+$ $\rightarrow$
absolutely monotone function on $\mathbb{R}_-$ $\rightarrow$
Stieltjes moment sequence
(read "unbounded subnormal weighted shift").

We illustrate the above scheme with a concrete example. Consider the Bergman shift $M_z^{(2)} = B$ with $\beta_n = 1/(n+1)$. As is easy to see, $\{\beta_n\}$ is minimal. (Indeed, all of the sequences $\{\beta_n(M_z^{(k)})\}$ corresponding to $M_z^{(k)}$ of Example 2.1 are minimal.) Note that $f(x) = 1/(x+1)$ is a completely monotone function on $\mathbb{R}_+$, which is the same as saying $g(x) = 1/(1-x)$ is an absolutely monotone function on $\mathbb{R}_-$. But then $g^{(n)}(0-)$ is a Stieltjes moment sequence; indeed, $g^{(n)}(0-) = n!$, and we see that under the above scheme the Bergman shift gets associated with the Bargmann shift. It follows from our observations in Example 2.1 that, for $k \geqslant 2$, $\beta_n(M_z^{(k)}) = (k-1)!/(n+1)(n+2)\cdots(n+k-1)$. We leave it to the reader to try out the above scheme on $M_z^{(k)}$ $(k \geqslant 3)$. The authors found it rather pleasing to apply the scheme to the unilateral shift $M_z^{(1)} = U : \{1\}$.

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V.M. SHOLAPURKAR
Department of Mathematics
University of Pune
Pune - 411007
INDIA

E-mail: vms@math.unipune.ernet.in

AMEER ATHAVALE
Department of Mathematics
University of Pune
Pune - 411007
INDIA

E-mail: athavale@math.unipune.ernet.in

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