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COHOMOLOGY AND EXTENSIONS OF HYPO-ŠILOV MODULES OVER UNIT MODULUS ALGEBRAS

XIAOMAN CHEN and KUNYU GUO

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ABSTRACT. This paper is a study of cohomology and extensions of hypo-Šilov modules over unit modulus algebras. We first prove that every $C(\partial A_U)$-extension of a hypo-Šilov module, viewed as a Hilbert module over A_U, is projective and injective. It follows that some interesting results are derived, especially so-called "Hom-Isomorphism" theorem. By using "Hom-Ext" sequences, we can compute Ext_{A_U}-groups for hypo-Šilov modules and cohypo-Šilov modules. Finally, these results are applied to the discussion of rigidity and extensions of Hardy submodules over polydisk algebras.

KEYWORDS: *Extension, hypo-Šilov module, projective module, rigidity.*

MSC (2000): Primary 47D; Secondary 47C.

1. INTRODUCTION

A few years ago, Douglas and Paulsen ([5]) introduced the notion of a Hilbert module as a Hilbert space together with the action of a function algebra A. They began to study the application of the methods and techniques of homological algebra to the category $\mathcal{H}(A)$ of all Hilbert A-modules. This coordinate free approach to multivariable operator theory has some remakable consequences. In fact, numerous problems from operator theory can be expressed in terms of homological constructions such as extensions and extension groups, $\mathrm{Ext}_A(-,-)$ introduced by Carlson and Clark in [2]. As we have seen from [2], $\mathrm{Ext}_A(-,-)$ is a fruitful object of study and a useful tool in operator theory. Also many propositions on "Ext" in the pure algebraic setting have analogues in the context of Hilbert modules.

However, the lack of projective (injective) modules in the category $\mathcal{H}(A)$ seems to make the computation of $\mathrm{Ext}_A(-,-)$ very difficult.

One of the main contributions of this paper is to prove that every $C(\partial A_U)$-extension of a hypo-Šilov module over a unit modulus algebra A_U is projective and injective as an A_U-Hilbert module. This allows us to characterize $\mathrm{Ext}_{A_U}(-,\mathcal{U}_0)$ and $\mathrm{Ext}_{A_U}(\mathcal{U}\ominus\mathcal{U}_0,-)$ by using "Hom-Ext"- sequences. We also prove a "Hom-Isomorphism" theorem which states that if $\mathcal{U}_1$ is a minimal $C(\partial A_U)$-extension of a hypo-Šilov module M_1, $\mathcal{U}_2$ a $C(\partial A_U)$-extension of another hypo-Šilov module M_2 with M_2 being pure, then $\mathrm{Hom}_{A_U}(M_1,M_2)\cong\mathrm{Hom}_{A_U}(\mathcal{U}_1\ominus M_1,\mathcal{U}_2\ominus M_2)$. When this theorem is applied to the case of Hardy submodules over polydisk algebras, some results in operator theory appear to be connected to the question of the commutants of analytic Toeplitz-type operators.

In Section 2, we introduce some standard notation and definitions and explore some basic properties of hypo-Šilov modules over unit modulus algebras. The duality principle for Hilbert modules is developed in this section. In Section 3 we consider the $C(\partial A_U)$-extensions of hypo-Šilov modules over unit modulus algebra A_U and show that every $C(\partial A_U)$-extension is projective and injective as an A_U-Hilbert module. Some interesting results are obtained, especially the so-called "Hom-Isomorphism" theorem. By using "Hom-Ext"-sequences we show how to compute Ext_{A_U}-groups for hypo-Šilov modules and cohypo-Šilov modules in Section 4. In particular, one of the main results in [2] is derived. In Section 5 the results in the previous sections are applied to the discussion of rigidity and extensions of Hardy submodules over polydisk algebras $A(\mathbb{D}^n)$. Of particular importance is the fact that the computation of $\mathrm{Ext}_{A(\mathbb{D}^n)}$-groups is closely related to function space theory of polydisks, and can reveal rigidity of Hardy submodules.

2. PRELIMINARY NOTATIONS AND DEFINITIONS

In [2] Carlson and Clark introduced one of the central concepts from homological algebra, the Ext-functor, into the discussion of Hilbert modules. Basically, they considered the following problem of classifying extensions in the category $\mathcal{H}(A)$ of all Hilbert modules over A. Suppose that H and K are in $\mathcal{H}(A)$. Let $S(K,H)$ be the set of all short exact sequences

$$E:0\longrightarrow H\stackrel{\alpha}{\longrightarrow}J\stackrel{\beta}{\longrightarrow}K\longrightarrow 0$$

where α,β are Hilbert-module maps. We call two elements E,E' *equivalent* if there exists a Hilbert module map θ such that the diagram

$$\begin{array}{ccccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ E': & 0 & \longrightarrow & H & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & K & \longrightarrow & 0 \end{array}$$

commutes. The set of equivalence classes of $S(K,H)$ under this relation is defined to be *the cohomology group*, $\mathrm{Ext}_A(K,H)$. In fact, $\mathrm{Ext}_A(-,-)$ is a bifunctor from the category $\mathcal{H}(A)$ to the category of abelian groups, it is contravariant in the first and covariant in the second variables (see [2]). Since the category of Hilbert modules lacks enough projective and injective objects, it is impossible to define the functor Ext as the derived functor of Hom as in [6]. Nevertheless, many propositions on "Ext" in module theory have analogues in the context of Hilbert modules. The following theorem proved in [2] is useful for the present paper, and it is well known in the pure algebraic setting.

Theorem 2.1. ([2]) $\mathrm{Ext}_A(K,H) \cong \mathcal{B}_1/\mathcal{B}_2$, *where* $\mathcal{B}_1 = \mathcal{B}_1(K,H)$ *is the set of all continuous* (*in both variables*) *bilinear maps* $\sigma : A \times K \to H$ *such that*

$$\sigma(ab,k) = a\sigma(b,k) + \sigma(a,bk)$$

with $a,b \in A$ *and* $k \in K$, *and* $\mathcal{B}_2 = \mathcal{B}_2(K,H)$ *is the set of all* $\sigma \in \mathcal{B}_1$ *having the form* $\sigma(a,k) = aTk - Tak$, *with* $T : K \to H$ *is a bounded linear operator.*

By definition, the elements of $\mathcal{B}_1$ are called cocycles and the elements of $\mathcal{B}_2$ are called coboundaries (see [2]). With the aid of Theorem 2.1, Carlson and Clark studied the extensions of Hilbert modules over the disk algebra $A(\mathbb{D})$. Their methods seem to be valid only in the case of the disk algebra. For the purposes of this paper, the duality principle for Hilbert modules should be kept in our mind.

Let $A \subseteq C(X)$ be a function algebra. For a Hilbert module M over A, we may also consider M to be a Hilbert module over $\overline{A}$ by setting $\overline{f} \cdot h = {T_f}^{(M)^*} h$, $f \in A$, $h \in M$, where $T_f^{(M)} : M \to M$ is the linear map defined by $T_f^{(M)} h = fh$. For emphasis, we denote this $\overline{A}$-module by M_*. When this is done, the opposite category of $\mathcal{H}(A)$ is naturally identified with the category $\mathcal{H}(\overline{A})$. Therefore, the following proposition is basic.

Proposition 2.2. *For any* M_1, M_2 *in* $\mathcal{H}(A)$, *we have*

(i) $\mathrm{Hom}_A(M_1,M_2)^* = \mathrm{Hom}_{\overline{A}}(M_{2*}, M_{1*})$ *where* $\mathrm{Hom}_A(M_1,M_2)$ *is the set of all Hilbert-module maps from* M_1 *to* M_2.

(ii) *The cohomology group* $\mathrm{Ext}_A(M_1,M_2)$ *is naturally isomorphic to* $\mathrm{Ext}_{\overline{A}}(M_{2*}, M_{1*})$ *by* $\widetilde{\sigma} \mapsto \widetilde{\overline{\sigma}}$, *where* $\widetilde{\sigma}(\widetilde{\overline{\sigma}})$ *is the cohomology class of the cocycle* $\sigma(\overline{\sigma})$, *and* $\overline{\sigma}$ *is defined by* $\overline{\sigma}(\overline{a}, \cdot\,) := \sigma(a, \cdot\,)^*$, $\overline{a} \in \overline{A}$.

Proof. Part (i) results from the above explanations.

For Part (ii), let $\sigma \in \mathcal{B}_1(M_1, M_2)$; we may consider $\sigma(a, \cdot) : M_1 \to M_2$ as a bounded linear operator from M_1 to M_2 by $\sigma(a, \cdot)h = \sigma(a, h)$ for $a \in A$, $h \in M_1$. Define the linear operator $\overline{\sigma}(\overline{a}, \cdot)$ from M_2 to M_1 by $\sigma(a, \cdot)^*$. A simple computation shows that the continuous bilinear map $\overline{\sigma} : \overline{A} \times M_2 \to M_1$ is a cocycle, i.e. $\overline{\sigma} \in \mathcal{B}_1(M_{2*}, M_{1*})$. Furthermore, $\overline{\sigma}$ is a coboundary if σ is a coboundary. We thus see that the map $\sigma \mapsto \overline{\sigma}$ sends cocycles to cocycles and coboundaries to coboundaries. Part (ii) thus holds from Theorem 2.1. ∎

DEFINITION 2.3. Let A be a function algebra, $A \subseteq C(\partial A)$, where ∂A is the Šilov boundary of A. If $\mathcal{U}$ is a Hilbert module over $C(\partial A)$, a closed subspace $M \subseteq \mathcal{U}$ which is invariant for A is called *a hypo-Šilov module over* A and $\mathcal{U}$ is called *a* $C(\partial A)$*-extension of* M. A hypo-Šilov module over A is *reductive* if it is invariant for $C(\partial A)$ and *pure* if no non-zero subspace of it is reductive. Furthermore, if $\mathcal{U}$ is contractive over $C(\partial A)$, we also call M to be *a Šilov module over A*.

DEFINITION 2.4. A function algebra $A \subseteq C(\partial A)$ is called *a unit modulus algebra* if $U_A = \{f \in A : |f(x)| = 1, \text{all } x \in \partial A\}$ generates A. In the following, we write A_U for a unit modulus algebra.

From Definition 2.4, it follows readily that the polydisk algebra $A(\mathbb{D}^n)$ on $\mathbb{T}^n$ is a unit modulus algebra.

In the present paper, we will concentrate on the case A_U. Let $\mathcal{U}_0$ be a hypo-Šilov module over A_U and $\mathcal{U}$ be a $C(\partial A_U)$-extension of $\mathcal{U}_0$. It follows that we have an exact sequence of Hilbert modules:

$$E_{U_0} : 0 \longrightarrow \mathcal{U}_0 \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U} \ominus \mathcal{U}_0 \longrightarrow 0 \tag{2.1}$$

where i is the inclusion map and π the quotient map; that is, π is the orthogonal projection $P_{\mathcal{U}\ominus\mathcal{U}_0}$ from $\mathcal{U}$ onto $\mathcal{U} \ominus \mathcal{U}_0$. As usual, the action of A_U on $\mathcal{U} \ominus \mathcal{U}_0$ is given by the formula $a \cdot h = P_{\mathcal{U}\ominus\mathcal{U}_0} T_a^{(\mathcal{U})} h$ for $a \in A_U$ and $h \in \mathcal{U} \ominus \mathcal{U}_0$. We indicate two "Hom-Ext"-sequences of E_{U_0} which will be used in the sequel. According to the Hom-Ext-sequence ([2], Proposition 2.1.5), for each M in $\mathcal{H}(A_U)$ the induced sequences

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{A_U}(M, \mathcal{U}_0) &\xrightarrow{i_*} \operatorname{Hom}_{A_U}(M, \mathcal{U}) \xrightarrow{\pi_*} \operatorname{Hom}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0) \\ \xrightarrow{\delta} \operatorname{Ext}_{A_U}(M, \mathcal{U}_0) &\xrightarrow{i_*} \operatorname{Ext}_{A_U}(M, \mathcal{U}) \xrightarrow{\pi_*} \operatorname{Ext}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0) \end{aligned} \tag{2.2}$$

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{A_U}(\mathcal{U} \ominus \mathcal{U}_0, M) &\xrightarrow{\pi^*} \operatorname{Hom}_{A_U}(\mathcal{U}, M) \xrightarrow{i^*} \operatorname{Hom}_{A_U}(\mathcal{U}_0, M) \\ \xrightarrow{\delta} \operatorname{Ext}_{A_U}(\mathcal{U} \ominus \mathcal{U}_0, M) &\xrightarrow{\pi^*} \operatorname{Ext}_{A_U}(\mathcal{U}, M) \xrightarrow{i^*} \operatorname{Ext}_{A_U}(\mathcal{U}_0, M) \end{aligned} \tag{2.3}$$

are exact, where δ denotes the connecting homomorphisms.

To develop further the properties of hypo-Šilov modules, we need the following terminology. Let $\mathcal{U}$ be any $C(\partial A_U)$-extension of a hypo-Šilov module $\mathcal{U}_0$ over A_U. We call $\mathcal{U}$ to be a *minimal* $C(\partial A_U)$-extension of $\mathcal{U}_0$ if $C(\partial A_U)\cdot\mathcal{U}_0$ is dense in $\mathcal{U}$. The next proposition tells us that minimal $C(\partial A_U)$-extensions of a hypo-Šilov module over A_U are similar as $C(\partial A_U)$-Hilbert modules.

Proposition 2.5. *Let M_i be hypo-Šilov modules over A_U and $\mathcal{U}_i$ be the $C(\partial A_U)$-extension of M_i, $i = 1, 2$. Then each $\theta \in \mathrm{Hom}_{A_U}(M_1, M_2)$ can lift to a $C(\partial A_U)$-Hilbert module map $\theta' : \mathcal{U}_1 \to \mathcal{U}_2$. Furthermore, if $\mathcal{U}_1$ is a minimal $C(\partial A_U)$-extension of M_1, this lifting is unique.*

Proof. By Proposition 2.19 in [5] we see that A_U is convexly approximating in modulus on ∂A_U. It follows that the proof is completed by using Theorem 1.9 and Theorem 2.20 in [5]. ∎

From Proposition 2.5, one finds that it is independent from the choice of $C(\partial A_U)$-extensions of $\mathcal{U}_0$ whether a hypo-Šilov module $\mathcal{U}_0$ over A_U is reductive or pure.

Let us again consider the short exact sequence (2.1) in the category $\mathcal{H}(A_U)$. By duality, this short exact sequence gives the short exact sequence:

$$(2.4) \qquad E^*_{\mathcal{U}_0}: \quad 0 \longrightarrow (\mathcal{U}\ominus\mathcal{U}_0)_* \xrightarrow{\pi^*} \mathcal{U}_* \xrightarrow{i^*} \mathcal{U}_{0*} \longrightarrow 0$$

in the category $\mathcal{H}(\overline{A}_U)$. So $(\mathcal{U}\ominus\mathcal{U}_0)_*$ is a hypo-Šilov module over $\overline{A}_U$. We may thus call $\mathcal{U}\ominus\mathcal{U}_0$ a cohypo-Šilov module over A_U. From Proposition 2.5 we know that a hypo-Šilov module over A_U is cohypo-Šilov if and only if it is reductive. From Proposition 2.2 and the sequence (2.4) we see that the sequence (2.3) is essentially the duality version of (2.2).

3. HYPO-ŠILOV MODULES OVER UNIT MODULUS ALGEBRAS

To prove the main theorem in this section, we need the following notation. Let G be a semigroup. *An invariant mean* of G is a state μ on $l^\infty(G)$ such that $\mu(F) = \mu({}_gF)$, where ${}_gF(g') := F(gg')$ for all $g \in G$ and $F \in l^\infty(G)$. A basic fact is that every abelian semigroup has an invariant mean (see [8]).

THEOREM 3.1. *Let $\mathcal{U} \in \mathcal{H}(A_U)$ be a $C(\partial A_U)$-Hilbert module. Then, for every Hilbert module K over A_U,* $\mathrm{Ext}_{A_U}(K,\mathcal{U}) = 0$ *and* $\mathrm{Ext}_{A_U}(\mathcal{U}, K) = 0$.

Proof. By using Theorem 2.1, we must prove that for every cocycle $\sigma \in \mathcal{B}_1(K,\mathcal{U})$ there exists a bounded linear operator $T : K \to \mathcal{U}$ such that $\sigma(a,k) = \sigma_T(a,k) = Tak - aTk$, $a \in A$, $k \in K$. To do this, we write $B_1(\mathcal{U},K)$ for all trace class operators from $\mathcal{U}$ to K, $B(K,\mathcal{U})$ for all bounded linear operators from K to $\mathcal{U}$, and identify $B(K,\mathcal{U})$ with $B_1^*(\mathcal{U},K)$ by setting $\langle T,C\rangle = \mathrm{tr}\,(TC)$, $T \in B(K,\mathcal{U})$, $C \in B_1(\mathcal{U},K)$.

Let μ be an invariant mean on the multiplication semigroup U_{A_U}, where U_{A_U} is $\{\eta \in A_U : |\eta(x)| = 1, \text{ for all } x \in \partial A_U\}$. We define $T \in B(K,\mathcal{U}) = B_1^*(\mathcal{U},K)$ by setting $\langle T,C\rangle = \mu_\eta\big(\big\langle T_{\overline{\eta}}^{(\mathcal{U})}\sigma(\eta,\cdot\,),C\big\rangle\big)$, that is, $\langle T,C\rangle$ is the mean of the bounded complex function $\eta \mapsto \big\langle T_{\overline{\eta}}^{(\mathcal{U})}\sigma(\eta,\cdot\,),C\big\rangle$. For each $\eta' \in U_{A_U}$, we have

$$\begin{aligned}
\big\langle T_{\eta'}^{(\mathcal{U})}T - TT_{\eta'}^{(K)},C\big\rangle &= \big\langle T, CT_{\eta'}^{(\mathcal{U})} - T_{\eta'}^{(K)}C\big\rangle\\
&= \mu_\eta\big(\big\langle T_{\overline{\eta}}^{(\mathcal{U})}\sigma(\eta,\cdot\,), CT_{\eta'}^{(\mathcal{U})} - T_{\eta'}^{(K)}C\big\rangle\big)\\
&= \mu_\eta\big(\big\langle T_{\overline{\eta}\eta'}^{(\mathcal{U})}\sigma(\eta,\cdot\,) - T_{\overline{\eta}}^{(\mathcal{U})}\sigma(\eta,\cdot\,)T_{\eta'}^{(K)},C\big\rangle\big)\\
&= \mu_\eta\big(\big\langle T_{\overline{\eta}\eta'}^{(\mathcal{U})}\sigma(\eta,\cdot\,) - T_{\overline{\eta}}^{(\mathcal{U})}(\sigma(\eta\eta',\cdot\,) - T_{\eta}^{(\mathcal{U})}\sigma(\eta',\cdot\,)),C\big\rangle\big)\\
&= \mu_\eta(\langle\sigma(\eta',\cdot\,),C\rangle) + \mu_\eta\big(\big\langle T_{\overline{\eta}\eta'}^{(\mathcal{U})}\sigma(\eta,\cdot\,) - T_{\overline{\eta}}^{(\mathcal{U})}\sigma(\eta\eta',\cdot\,),C\big\rangle\big)\\
&= \langle\sigma(\eta',\cdot\,),C\rangle + \mu_\eta\big(\big\langle T_{\overline{\eta}\eta'}^{(\mathcal{U})}\sigma(\eta,\cdot\,),C\big\rangle\big) - \mu_{\eta\eta'}\big(\big\langle T_{\overline{\eta\eta'}\eta'}^{(\mathcal{U})}\sigma(\eta\eta',\cdot\,),C\big\rangle\big)\\
&= \langle\sigma(\eta',\cdot\,),C\rangle
\end{aligned}$$

for all $C \in B_1(\mathcal{U},K)$, so that $\sigma(\eta',\cdot\,) = T_{\eta'}^{(\mathcal{U})}T - TT_{\eta'}^{(K)}$. Since U_{A_U} generates A_U, we see that $\sigma = \sigma_T$. This is just what is needed. Thus we get $\mathrm{Ext}_{A_U}(K,\mathcal{U}) = 0$. The proof of $\mathrm{Ext}_{A_U}(\mathcal{U},K)$ being zero follows from Proposition 2.2 in Section 2. ∎

The following important corollary comes immediately from Theorem 3.1 and [2], Proposition 2.1.5.

COROLLARY 3.2. *Let $A_U \subseteq C(\partial A_U)$ be a unit modulus algebra and $\mathcal{U}$ a $C(\partial A_U)$-Hilbert module. Then $\mathcal{U}$, viewed as a A_U-Hilbert module, is projective and injective.*

REMARK 3.3. (i) In their book ([5]), Douglas and Paulsen asked whether there is any function algebra, other than $C(X)$, with any (non-zero) projective module (see Problem 4.6). In [3], it is proved that every unitary $C(\partial D)$-Hilbert module, viewed as a Hilbert module over the disk algebra $A(\mathbb{D})$, is projective and injective. However, from Corollary 3.2, we see that there exist non-zero projective modules over every unit modulus algebra. Clearly, our method is different from that of [3].

(ii) In the purely algebraic setting, one knows from [6] that there is no non-zero module which is projective and injective over every principal ideal domain (other than a field). Hence, Corollary 3.2 points out a very different character of Hilbert modules.

The next corollary tells us that injective hypo-Šilov modules (projective cohypo-Šilov modules) on A_U appear only in an extreme case.

Corollary 3.4. *Let $\mathcal{U}_0$ be a hypo-Šilov module over A_U and $\mathcal{U}$ any $C(\partial A_U)$-extension of $\mathcal{U}_0$. Then the following statements are equivalent:*

(i) *$\mathcal{U}_0$ is injective;*

(ii) *$\mathcal{U} \ominus \mathcal{U}_0$ is projective;*

(iii) *$\mathcal{U}_0$ is reductive;*

(iv) *the short exact sequence $E_{\mathcal{U}_0} : 0 \longrightarrow \mathcal{U}_0 \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U} \ominus \mathcal{U}_0 \longrightarrow 0$ is split.*

Proof. Since $\mathcal{U}$ is projective and injective, this implies that (i), (ii) and (iv) are equivalent. From Corollary 3.2, it is easy to see that (iii) leads to (i). If $E_{\mathcal{U}_0}$ is split, then there is a split map $\sigma : \mathcal{U} \ominus \mathcal{U}_0 \to \mathcal{U}$ such that $\pi\sigma = 1_{\mathcal{U}\ominus\mathcal{U}_0}$. For any $\xi \in \mathcal{U}_0$ and any $\eta \in A_U$ with unit modulus, we write $T_{\bar{\eta}}^{(\mathcal{U})}\xi = \xi_1 + \xi_2$, $\xi_1 \in \mathcal{U}_0$, $\xi_2 \in \mathcal{U} \ominus \mathcal{U}_0$. Hence

$$\xi = T_\eta^{(\mathcal{U})}\xi_1 + T_\eta^{(\mathcal{U})}\xi_2.$$

So

$$\pi(T_\eta^{(\mathcal{U})}\xi_2) = T_\eta^{(\mathcal{U}\ominus\mathcal{U}_0)}\xi_2 = 0.$$

This induces the following

$$\sigma(T_\eta^{(\mathcal{U}\ominus\mathcal{U}_0)}\xi_2) = T_\eta^{(\mathcal{U})}\sigma(\xi_2) = 0$$

i.e. $\sigma(\xi_2) = 0$. Since σ is an injective Hilbert module map, it follows easily that

$$\xi_2 = 0.$$

So $\mathcal{U}_0$ is reductive. This completes the proof of the corollary. ∎

According to Corollary 3.2, we have the following interesting "Hom-Isomorphism" theorem which states that there exists a natural isomorphism between Hom of hypo-Šilov modules and that of the corresponding cohypo-Šilov modules.

THEOREM 3.5. *Let M_1, M_2 be hypo-Šilov modules over A_U with M_2 being pure. If $\mathcal{U}_1$ is the minimal $C(\partial A_U)$-extension of M_1, and $\mathcal{U}_2$ a $C(\partial A_U)$-extension of M_2, then the following are isomorphic as A_U-modules:*

$$\mathrm{Hom}_{A_U}(M_1, M_2) \cong \mathrm{Hom}_{A_U}(\mathcal{U}_1 \ominus M_1, \mathcal{U}_2 \ominus M_2).$$

The isomorphism is given by $\beta(\theta) = P_{\mathcal{U}_2 \ominus M_2}\theta' | \mathcal{U}_1 \ominus M_1$ for $\theta \in \mathrm{Hom}_{A_U}(M_1, M_2)$, where θ' is uniquely determined from θ by Proposition 2.5.

Proof. Theorem 3.5 can be expressed as the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i_1} & \mathcal{U}_1 & \xrightarrow{\pi_1} & \mathcal{U}_1 \ominus M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\beta(\theta)} & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{i_2} & \mathcal{U}_2 & \xrightarrow{\pi_2} & \mathcal{U}_2 \ominus M_2 & \longrightarrow & 0 \end{array}$$

where i_1, i_2 are the inclusion maps and π_1, π_2 the quotient Hilbert module maps. By Proposition 2.5, it is easy to see that $\beta : \mathrm{Hom}_{A_U}(M_1, M_2) \to \mathrm{Hom}_{A_U}(\mathcal{U}_1 \ominus M_1, \mathcal{U}_2 \ominus M_2)$ is a A_U-module homomorphism, where the module structure of $\mathrm{Hom}_{A_U}(M_1, M_2)$ is given by $(f \cdot \theta)(h) = \theta(f \cdot h)$ for $f \in A_U$, $h \in M_1$, and the module structure of $\mathrm{Hom}_{A_U}(\mathcal{U}_1 \ominus M_1, \mathcal{U}_2 \ominus M_2)$ is similar to that of $\mathrm{Hom}_{A_U}(M_1, M_2)$. Since M_2 is pure, Proposition 2.5 implies that β is injective. Because $\mathcal{U}_1$ is projective, β is surjective. This completes the proof of the theorem. ∎

To conclude this section we apply Theorem 3.5 to the discussion of Hardy submodules on polydisk algebra $A(\mathbb{D}^n)$. To do this, let Γ be a subset of $L^2(\mathbb{T}^n, \frac{1}{(2\pi)^n} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n)$; we shall say that a Borel set $E \subseteq \mathbb{T}^n$ is the *support* of Γ, denoted by $S(\Gamma)$, if each function from Γ vanishes on $\mathbb{T}^n \setminus E$, and for any Borel subset E' of E with $\sigma(E') > 0$, there exists a function $f \in \Gamma$ such that $f|E' \neq 0$, where we denote the measure $\frac{1}{(2\pi)^n} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n$ by σ. For a Hilbert submodule M of $L^2(\mathbb{T}^n)$ over $A(\mathbb{D}^n)$ it is not difficult to prove that $\chi_{S(M)} L^2(\mathbb{T}^n)$ is its minimal $C(\mathbb{T}^n)$-extension, where $\chi_{S(M)}$ is the characteristic function of $S(M)$. We also note that a Hilbert submodule M' of $L^2(\mathbb{T}^n, \mathrm{d}\sigma)$ is pure if and only if $\sigma(S(M'^{\perp})) = 1$. It is easy to check that Theorem 3.5 implies the following:

COROLLARY 3.6. *Let M_1 and M_2 be submodules of $L^2(\mathbb{T}^n)$ over $A(\mathbb{D}^n)$, with $\sigma(S(M_1)) = \sigma(S(M_2^{\perp})) = 1$. Then*

$$\mathrm{Hom}_{A(\mathbb{D}^n)}(M_1, M_2) \cong \mathrm{Hom}_{A(\mathbb{D}^n)}(L^2(\mathbb{T}^n) \ominus M_1, L^2(\mathbb{T}^n) \ominus M_2).$$

The isomorphism is given by $\varphi \mapsto H_\varphi^{[M_2]} | L^2(\mathbb{T}^n) \ominus M_1$, where $H_\varphi^{[M_2]}$ is defined by $H_\varphi^{[M_2]} f = P_{L^2(\mathbb{T}^n) \ominus M_2}(\varphi f)$ for all $f \in L^2(\mathbb{T}^n)$.

Example 3.7. Let $H^2(\mathbb{D}^n)$ be the usual Hardy module over $A(\mathbb{D}^n)$, and $H^2(\mathbb{D}^n)^{\perp}(= L^2(\mathbb{T}^n) \ominus H^2(\mathbb{D}^n))$ the quotient module. Then by Corollary 3.6, we have

$$\operatorname{Hom}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n)^{\perp}, H^2(\mathbb{D}^n)^{\perp}) \cong H^{\infty}(\mathbb{D}^n).$$

For any f in $L^{\infty}(\mathbb{T}^n)$, define $A_f : H^2(\mathbb{D}^n)^{\perp} \to H^2(\mathbb{D}^n)^{\perp}$ by $A_f(h) = P_{H^2(\mathbb{D}^n)^{\perp}}(fh)$, $h \in H^2(\mathbb{D}^n)^{\perp}$. Then the commutant of $\{A_{z_1}, \ldots, A_{z_n}\}$ is equal to $\{A_f : f \in H^{\infty}(\mathbb{D}^n)\}$.

4. COHOMOLOGY AND EXTENSIONS OF HYPO-ŠILOV MODULES OVER UNIT MODULUS ALGEBRAS

Let $\mathcal{U}_0$ be a hypo-Šilov module over A_U and $\mathcal{U}$ any $C(\partial A_U)$-extension of $\mathcal{U}_0$. For $M \in \mathcal{H}(A_U)$ and $\theta \in \operatorname{Hom}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0)$, we define a cocycle $r_\theta \in \mathcal{B}_1(M, \mathcal{U}_0)$ by $r_\theta(a, h) = P_{\mathcal{U}_0} T_a^{(\mathcal{U})} \theta(h)$, where $P_{\mathcal{U}_0}$ is the orthogonal projection from $\mathcal{U}$ to $\mathcal{U}_0$. Also for any $\theta \in \operatorname{Hom}_{A_U}(\mathcal{U}_0, M)$, we define a cocycle $\sigma_\theta \in \mathcal{B}_1(\mathcal{U} \ominus \mathcal{U}_0, M)$ by $\sigma_\theta(a, h) = \theta(P_{\mathcal{U}_0} T_a^{(\mathcal{U})} h)$. Then from [2], Propositions 1.1.5 and 2.2.3, one sees easily that the connecting homomorphism $\delta_1 : \operatorname{Hom}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0) \longrightarrow \operatorname{Ext}_{A_U}(M, \mathcal{U}_0)$ is given by $\delta_1(\theta) = \widetilde{r}_\theta$, and $\delta_2 : \operatorname{Hom}_{A_U}(\mathcal{U}_0, M) \longrightarrow \operatorname{Ext}_{A_U}(\mathcal{U} \ominus \mathcal{U}_0, M)$ is given by $\delta_2(\theta) = \widetilde{\sigma}_\theta$. From the sequences (2.2), (2.3) and Theorem 3.1, we immediately obtain:

Theorem 4.1. *Let $\mathcal{U}_0$ be a hypo-Šilov module and $\mathcal{U}$ any $C(\partial A_U)$-extension of $\mathcal{U}_0$. Then, for each Hilbert module M over A_U, we have*

(i) $\operatorname{Ext}_{A_U}(M, \mathcal{U}_0) \cong \operatorname{coker}(\pi_* : \operatorname{Hom}_{A_U}(M, \mathcal{U}) \to \operatorname{Hom}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0))$
the correspondence being given by $\delta_1(\widetilde{\theta}) = \widetilde{r}_\theta$ for $\theta \in \operatorname{Hom}_{A_U}(M, \mathcal{U} \ominus \mathcal{U}_0)$ and

(ii) $\operatorname{Ext}_{A_U}(\mathcal{U} \ominus \mathcal{U}_0, M) \cong \operatorname{coker}(i^* : \operatorname{Hom}_{A_U}(\mathcal{U}, M) \to \operatorname{Hom}_{A_U}(\mathcal{U}_0, M))$
the correspondence being given by $\delta_2(\widetilde{\theta}) = \widetilde{\sigma}_\theta$ for $\theta \in \operatorname{Hom}_{A_U}(\mathcal{U}_0, M)$.

Remark 4.2. Theorem 4.1 provides us a very useful method to calculate cohomology groups of Hilbert modules over A_U. In particular, if one of M, $\mathcal{U}_0$ and $\mathcal{U} \ominus \mathcal{U}_0$ is cyclic or co-cyclic, then by Proposition 2.2 (ii) and Theorem 4.1, the characterization of $\operatorname{Ext}_{A_U}(-,-)$ may be summed up as the actions of homomorphisms on cyclic vectors, or co-cyclic vectors, where we use the concept of co-cyclic Hilbert modules, which says that M is a co-cyclic Hilbert module over A if and only if M_* is a cyclic Hilbert module over $\overline{A}$.

The next corollary yields a simple proof for one of the main results in [2].

COROLLARY 4.3. ([2]) *Let $A(\mathbb{D})$ be the disk algebra on the unit circle $\mathbb{T}$ and $H^2(\mathbb{D})$ the usual Hardy module over $A(\mathbb{D})$. Then for any Hilbert module K over $A(\mathbb{D})$, $\mathrm{Ext}_{A(\mathbb{D})}(K, H^2(\mathbb{D}))$ is characterized as an $A(\mathbb{D})$-module by the following*

$$\mathrm{Ext}_{A(\mathbb{D})}(K, H^2(\mathbb{D})) \cong K_1/K_0,$$

where K_1 is given by $\{\theta^*(\overline{z}) : \theta \in \mathrm{Hom}_{A(\mathbb{D})}(K, L^2(\mathbb{T}) \ominus H^2(\mathbb{D}))\}$, and $K_0 = \{L^*(\overline{z}) : L \in \mathrm{Hom}_{A(\mathbb{D})}(K, L^2(\mathbb{T}))\}$. The action of $A(\mathbb{D})$ on K_1 is given by $f \cdot \theta^*(\overline{z}) := \theta^*(\overline{fz})$ and the action of $A(\mathbb{D})$ on K_0 is similar to that of $A(\mathbb{D})$ on K_1.

Proof. By using Theorem 4.1 and the fact that every cocycle $\sigma : A(\mathbb{D}) \times K \to H^2(\mathbb{D})$ is completely determined by $\sigma(z, \cdot)$, one sees that $\delta(\widetilde{\theta})$ is the cohomology class determined by r_θ for all $\theta \in \mathrm{Hom}_{A(\mathbb{D})}(K, L^2(\mathbb{T}) \ominus H^2(\mathbb{D}))$, where $r_\theta(z, k) = P_{H^2(\mathbb{D})} T_z^{(L^2(\mathbb{T}))} \theta(k) = \langle k, \theta^*(\overline{z}) \rangle$ for all $k \in K$. If there exists a Hilbert module map $L : K \to L^2(\mathbb{T})$ such that $\theta^*(\overline{z}) = L^*(\overline{z})$, then it is easy to see that r_θ is a coboundary. In other words, if r_θ is a coboundary, then there is a bounded linear operator $A : K \to H^2(\mathbb{D})$ such that $r_\theta(f, k) = AT_f^{(K)} k - T_f^{(H^2(\mathbb{D}))} Ak$. Define an operator $L : K \to L^2(\mathbb{T})$ by $Lk = Ak + \theta k$, $k \in K$. Since

$$\begin{aligned} fLk &= T_f^{(H^2(\mathbb{D}))} Ak + T_f^{(L^2(\mathbb{T}))} \theta k \\ &= T_f^{(H^2(\mathbb{D}))} Ak + P_{H^2(\mathbb{D})} T_f^{(L^2(\mathbb{T}))} \theta k + P_{H^2(\mathbb{D})^\perp} T_f^{(L^2(\mathbb{T}))} \theta k \\ &= T_f^{H^2(\mathbb{D})} Ak + r_\theta(f, k) + \theta(fk) = AT_f^{(K)} k + \theta(fk) \end{aligned}$$

it follows that L is a Hilbert module map from K to $L^2(\mathbb{T})$. Moreover, for any $k \in K$, we see

$$\langle L(k), \overline{z} \rangle = \langle \theta(k), \overline{z} \rangle.$$

This leads to $\theta^*(\overline{z}) = L^*(\overline{z})$. The proof of the corollary is thus completed. ∎

5. APPLICATIONS TO RIGIDITY AND EXTENSIONS OF HARDY SUBMODULES OVER POLYDISK ALGEBRAS

In this section, we apply the results in previous sections to the discussion of rigidity and extensions of Hardy submodules over polydisk algebras. For an $A(\mathbb{D}^n)$-Hilbert submodule M of $L^2(\mathbb{T}^n, \frac{1}{(2\pi)^n} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n)$, we define a function space $B(M)$ by the following statements: $\varphi \in L^2(\mathbb{T}^n)$ is in $B(M)$ if the densely-defined Hankel operator $H_\varphi^{(M)} : H^2(\mathbb{D}^n) \to L^2(\mathbb{T}^n) \ominus M$ can be continuously extended onto $H^2(\mathbb{D}^n)$, where $H_\varphi^{(M)} f := P_{L^2(\mathbb{T}^n) \ominus M}(\varphi f)$, $f \in A(\mathbb{D}^n)$. It is easy to check that for every $\varphi \in B(M)$, $H_\varphi^{(M)}$ is a Hilbert-module map from $H^2(\mathbb{D}^n)$ to $L^2(\mathbb{T}^n) \ominus M$, and each Hilbert-module map β from $H^2(\mathbb{D}^n)$ to $L^2(\mathbb{T}^n) \ominus M$ has such a form; that is, there exists a $\varphi \in B(M)$ such that $\beta = H_\varphi^{(M)}$. In particular, if $M = H^2(\mathbb{D}^n)$, then $B(H^2(\mathbb{D}^n))$ is equal to $\mathrm{BMO_r} + H^2(\mathbb{D}^n)$, where $\mathrm{BMO_r}$ is restricted BMO space introduced in [4]. Furthermore, for a non-zero Hardy submodule M_0, another function space $B(M_0, M)$ is defined by $\varphi \in B(M_0, M)$ if $\varphi \in B(M)$ and $\ker H_\varphi^{(M)} \supseteq M_0$. From these explanations, Theorem 4.1 and Proposition 2.5, we have the following.

Proposition 5.1. (i) $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), M) \cong B(M)/(L^\infty(\mathbb{T}^n) + M)$;
(ii) $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), H^2(\mathbb{D}^n)) \cong (BMO_r + H^2(\mathbb{D}^n))/(L^\infty(\mathbb{T}^n) + H^2(\mathbb{D}^n))$;
(iii) $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_0, M) \cong B(M_0, M)/M$.

Remark 5.2. It is well known that in the case $n = 1$, $\mathrm{BMO_r} + H^2(\mathbb{D}) = L^\infty(\mathbb{T}) + H^2(\mathbb{D})$. This implies that $\mathrm{Ext}_{A(\mathbb{D})}(H^2(\mathbb{D}), H^2(\mathbb{D})) = 0$. However for $n > 1$, by [4], it is easy to check that $\mathrm{BMO_r} + H^2(\mathbb{D}^n) \supsetneqq L^\infty(\mathbb{T}^n) + H^2(\mathbb{D}^n)$. One concludes thus that $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), H^2(\mathbb{D}^n)) \neq 0$.

Let two Hardy submodules M_1, M_2 satisfy $0 \neq M_1 \subseteq M_2 \neq H^2(\mathbb{D}^n)$. It follows that 1 is in $B(M_1, M_2)$. This implies that $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) \neq 0$ by Proposition 5.1. The above discussion explains thus that for Hardy submodules M_1, M_2, and $M_1 \neq 0$, if $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) = 0$, then there is no proper Hardy submodule M_3 which satisfies that $M_3 \supseteq M_1$, and M_3 is similar to M_2. The next example may give us some information on rigidity of Hardy submodules.

Example 5.3. Let F be a multiplication of polynomials with forms $z^m \pm \omega^n$ ($m, n \geqslant 0$ and $m + n \neq 0$). Write $M_F = \overline{FH^2(\mathbb{D}^2)}$; then there is no proper Hardy submodule M such that $M_F \subseteq M$ and M is similar to $H^2(\mathbb{D}^2)$.

In fact, by the preceding explanation, we must point out that $\mathrm{Ext}_{A(\mathbb{D}^2)}(H^2(\mathbb{D}^2) \ominus M_F, H^2(\mathbb{D}^2)) = 0$. Let $\varphi \in B(M_F, H^2(\mathbb{D}^2))$. What we do is to prove that $\varphi \in H^2(\mathbb{D}^2)$ by Proposition 5.1. Now if $f \in L^2(\mathbb{T}^2)$ and $(z^m \pm \omega^n)f \in H^2(\mathbb{D}^2)$, then

$$(z^m \pm \omega^n)f = \sum_{k=-\infty}^{+\infty} (z^m f_k(z) \pm f_{k-n}(z))\omega^k$$

where $f = \sum_{k=-\infty}^{+\infty} f_k(z)\omega^k$ is its expansion relative to ω. Thus, in the case $k < 0$, we have

$$z^m f_k(z) \pm f_{k-n}(z) = 0.$$

In this case, if $n = 0$, then $f_k = 0$. If $n \neq 0$, then for any positive integer l, the following is true:

$$\|f_k\|^2 + \|f_{k-n}\|^2 + \cdots + \|f_{k-ln}\|^2 = (l+1)\|f_k\|^2 \leqslant \|f\|^2.$$

This again implies that f_k is equal to zero. Considering the expansion of f relative to z, it follows that $f \in H^2(\mathbb{D}^2)$. Thus $(z^m \pm \omega^n)f \in H^2(\mathbb{D}^2)$ implies $f \in H^2(\mathbb{D}^2)$. This induces that $\varphi F \in H^2(\mathbb{D}^2)$ implies $\varphi \in H^2(\mathbb{D}^2)$. Thus

$$\mathrm{Ext}_{A(\mathbb{D}^2)}(H^2(\mathbb{D}^2) \ominus M_F, H^2(\mathbb{D}^2)) = 0.$$

Moreover, we shall prove the following by using the techniques of [1].

PROPOSITION 5.4. *For $n > 1$, let M_1 be of finite codimension in $H^2(\mathbb{D}^n)$ and $M_2 \subseteq H^2(\mathbb{D}^n)$. Then*

$$M_2 \subseteq B(M_1, M_2) \subseteq H^2(\mathbb{D}^n).$$

In particular, if $M_1 \subseteq M_2$, then $B(M_1, M_2) = H^2(\mathbb{D}^n)$.

Proof. For any $\varphi \in B(M_1, M_2)$, write

$$\varphi = \sum_{s=-\infty}^{+\infty} f_s(z_2, \ldots, z_n) z_1^s.$$

Since M_1 has finite codimension in $H^2(\mathbb{D}^n)$, for sufficiently large integer l, some non-zero linear combination $\sum_{j=0}^{l} c_j z_2^j$ of the functions $1, z_2, z_2^2, \ldots, z_2^l$ belongs to M_1. Thus $\varphi\Big(\sum_{j=0}^{l} c_j z_2^j\Big)$ is in M_2. However,

$$\varphi\Big(\sum_{j=0}^{l} c_j z_2^j\Big) = \sum_{s=-\infty}^{+\infty} \Big[f_s\Big(\sum_{j=0}^{l} c_j z_2^j\Big)\Big] z_1^s \in M_2.$$

The above equation implies that $f_s = 0$ for $s < 0$. Considering the expansions of φ relative to the other variables $z_2, z_3, \ldots, z_n$, it follows that φ is in $H^2(\mathbb{D}^n)$. The relation $M_2 \subseteq B(M_1, M_2)$ is clear. In particular, if $M_1 \subseteq M_2$, we see that $H^\infty(\mathbb{D}^n) \subseteq B(M_1, M_2)$. Since M_2 is of finite codimension in $H^2(\mathbb{D}^n)$, it is easy to prove that $H^2(\mathbb{D}^n) \ominus M_2$ is contained in $H^\infty(\mathbb{D}^n)$ by using the methods of [7]. We thus conclude that $B(M_1, M_2) = H^2(\mathbb{D}^n)$. ∎

COROLLARY 5.5. *For $n > 1$, let M_1 be of finite codimension in $H^2(\mathbb{D}^n)$ and $M_1 \subseteq M_2 \subseteq H^2(\mathbb{D}^n)$. Then*

$$\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) \cong H^2(\mathbb{D}^n) \ominus M_2.$$

From Corollary 5.5, one finds that for $n > 1$, a finite codimensional Hardy submodule $M(\neq H^2(\mathbb{D}^n))$ is never similar to $H^2(\mathbb{D}^n)$. The reason is that $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M, H^2(\mathbb{D}^n)) = 0$, and $\mathrm{Ext}_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M, M) \cong H^2(\mathbb{D}^n) \ominus M$. Of course, this is not a new observation; we may compare it with that of [1].

Finally, we point out that partial results in this section have been previously obtained using a different method by K.Y. Guo and X.M. Chen, see "Ext of Hardy modules over polydisk algebras", *Chinese Ann. Math. Ser. B* **20**(1999), 103–110.

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XIAOMAN CHEN
Institute of Mathematics
Fudan University
Shanghai, 200433
P.R. CHINA
E-mail: xchen@ms.fudan.edu.cn

KUNYU GUO
Institute of Mathematics
Fudan University
Shanghai, 200433
P.R. CHINA
E-mail: kyguo@fudan.edu.cn

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