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PURE STATES ON $\mathcal{O}_d$

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Dedicated to Professor Erling Størmer on the occasion of his sixtieth birthday

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ABSTRACT. We study representations of the Cuntz algebras $\mathcal{O}_d$ and their associated decompositions. In the case that these representations are irreducible, their restrictions to the gauge-invariant subalgebra UHF_d have an interesting cyclic structure. If S_i, $1 \leqslant i \leqslant d$, are representatives of the Cuntz relations on a Hilbert space $\mathcal{H}$, special attention is given to the subspaces which are invariant under S_i^*. The applications include wavelet multiresolutions corresponding to wavelets of compact support (to appear in the later paper [8]), and finitely correlated states on one-dimensional quantum spin chains.

KEYWORDS: *Cuntz algebra, representations of C^*-algebras, Hilbert space, endomorphism, completely positive maps, dilation, commutant, von Neumann algebras.*

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1. INTRODUCTION

The aim of the present paper was at the outset threefold:

(i) To develop further and simplify the theory of finitely (and infinitely) correlated states of the Cuntz algebra $\mathcal{O}_d$ given in [9].

(ii) To apply this theory to analyze in detail the representations of $\mathcal{O}_N$ coming from compactly supported wavelets constructed by multiresolution wavelet analysis of scale N ([10]). The main idea is that repeated applications of the adjoints of the Cuntz operators on any trigonometric polynomial in $L^2(\mathbb{T})$ in that case ultimately maps the polynomial into a fixed finite-dimensional subspace $\mathcal{K} \subset$

$L^2(\mathbb{T})$ of low-order polynomials, and thus the results of the present paper apply. This application will be postponed to the paper [8].

(iii) To understand better the connection between the theory of finitely correlated states on one-dimensional quantum spin chains developed in [24], [25] and the corresponding states on the Cuntz algebras.

The setting and results (especially Theorem 5.1) also serve as a generalization of the single-operator commutant lifting theorem ([19]) from one variable to several. In this setting, $\mathcal{O}_d$, for $d \geqslant 2$, is viewed as the multivariable version of the familiar C^*-algebra generated by a single isometry.

Recall that if $d \in \{2, 3, \ldots\}$, the *Cuntz algebra* $\mathcal{O}_d$ is the universal C^*-algebra generated by elements $s_1, \ldots, s_d$ subject to the relations

$$s_i^* s_j = \delta_{ij} \mathbb{1}, \quad \sum_{j=1}^{d} s_j s_j^* = \mathbb{1}.$$

There is a canonical action of the group $U(d)$ of unitary $d \times d$ matrices on $\mathcal{O}_d$ given by

$$\tau_g(s_i) = \sum_{j=1}^{d} \overline{g}_{ji} s_j$$

for $g = [g_{ij}]_{i,j=1}^d \in U(d)$. In particular the *gauge action* is defined by

$$\tau_z(s_i) = z s_i, \quad z \in \mathbb{T} \subset \mathbb{C}. \tag{1.1}$$

If UHF_d is the fixed point subalgebra under the gauge action, then UHF_d is the closure of the linear span of all Wick ordered monomials of the form $s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$. UHF_d is isomorphic to the UHF algebra of Glimm type d^∞:

$$\mathrm{UHF}_d \cong M_{d^\infty} = \bigotimes_1^\infty M_d$$

in such a way that the isomorphism carries the Wick ordered monomial above into the matrix element

$$e_{i_1 j_1}^{(1)} \otimes e_{i_2 j_2}^{(2)} \otimes \cdots \otimes e_{i_k j_k}^{(k)} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots.$$

The restriction of τ_g to UHF_d is then carried into the action $\mathrm{Ad}(g) \otimes \mathrm{Ad}(g) \otimes \cdots$ on $\bigotimes_1^\infty M_d$. We define the canonical endomorphism λ on UHF_d (or on $\mathcal{O}_d$) by

$$\lambda(x) = \sum_{j=1}^{d} s_j x s_j^* \tag{1.2}$$

and the isomorphism carries λ over into the one-sided shift

$$x_1 \otimes x_2 \otimes x_3 \otimes \cdots \to \mathbb{1} \otimes x_1 \otimes x_2 \otimes \cdots$$

on $\bigotimes_1^\infty M_d$. (See [16], [21], [7].)

If $s_i \mapsto S_i \in \mathcal{B}(\mathcal{H})$ is a representation of the Cuntz relations on a Hilbert space $\mathcal{H}$, we will consider the situation that there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $S_i^* \mathcal{K} \subset \mathcal{K}$ for $i \in \mathbb{Z}_d$, and $\mathcal{K}$ is cyclic for the representation. Thus, if $P : \mathcal{H} \to \mathcal{K}$ is the orthogonal projection onto $\mathcal{K}$, we have $PS_i^*P = S_i^*P$. In this situation, define $V_i \in \mathcal{B}(\mathcal{K})$ by $V_i = PS_i = PS_iP$. Then

$$\sum_{i \in \mathbb{Z}_d} V_i V_i^* = \mathbb{1}$$

so the map $\sigma : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ defined by

$$\sigma(X) = \sum_{i \in \mathbb{Z}_d} V_i X V_i^* \tag{1.3}$$

is completely positive and unital. We show that the representation can be completely recovered from $(\mathcal{K}, V_1, \ldots, V_d)$ in Theorem 2.1 and Theorem 5.1, and the commutant of the representation is isometrically order isomorphic to the fixed point set $\mathcal{B}(\mathcal{K})^\sigma = \{A \in \mathcal{B}(\mathcal{K}) \mid \sigma(A) = A\}$ by Proposition 4.1 and Theorem 5.1. This fixed point set is not an algebra in general, as is discussed in some detail in Section 3. In particular, the representation of $\mathcal{O}_d$ is irreducible if and only if σ is ergodic in the sense that $\mathcal{B}(\mathcal{K})^\sigma = \mathbb{C}\mathbb{1}$. In Section 6 we assume that the representation is irreducible and study its restriction to UHF_d in the case that there is a normal σ-invariant state φ on $\mathcal{B}(\mathcal{K})$. Such a state is automatically unique if it exists, and if $\mathcal{K}$ is finite-dimensional it always exists. In this case we replace $\mathcal{K}$ with the smaller S_i^*-invariant space $E\mathcal{H}$, where E is the support projection of φ, replace φ with its restriction to $E\mathcal{B}(\mathcal{K})E = \mathcal{B}(E\mathcal{K})$, and we define a state ψ on $\mathcal{O}_d$ by

$$\psi(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) = \varphi(ES_{i_1} \cdots S_{i_n} S_{j_m}^* \cdots S_{j_1}^* E).$$

Then $\psi \circ \lambda = \psi$. (Occasionally, we will identify ψ with its normal extension to $\mathcal{B}(\mathcal{H})$, defined by $\psi(X) = \varphi(EXE)$ for $X \in \mathcal{B}(\mathcal{H})$. This extension is a type I factor state with multiplicity $\dim(E)$.) We show in Theorem 6.3 that the set of $t \in \mathbb{T}$ such that $\psi \circ \tau_t = \psi$ is equal to the peripheral point spectrum $\mathrm{PSp}(\sigma) \cap \mathbb{T}$ of σ, and this set is a finite subgroup of $\mathbb{T}$. If k is the order of this subgroup, and $U \in \mathcal{B}(\mathcal{H})$ is the unitary operator such that $\tau_{\frac{1}{k}}$, corresponding to $z = \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}}$ in (1.1), satisfies $\tau_{\frac{1}{k}} = \mathrm{Ad}(U)$ with $U^k = \mathbb{1}$ (U is unique up to a phase factor in $\mathbb{Z}_k \subset \mathbb{T}$), and

$$U = \sum_{l \in \mathbb{Z}_k} \mathrm{e}^{\frac{2\pi \mathrm{i} l}{k}} E_k$$

is the spectral decomposition of U, then the subalgebra $\mathrm{UHF}_d \subset \mathcal{O}_d$ acts irreducibly on each of the subspaces $E_k\mathcal{H}$, the corresponding representations of UHF_d are irreducible and mutually disjoint, and are mapped cyclically into each other by the endomorphism λ.

In particular, this means that the restriction of the representation to UHF_d is irreducible if and only if the peripheral point spectrum $\mathrm{PSp}(\sigma) \cap \mathbb{T}$ of σ consists of the point 1 alone. It is remarkable that, if $\mathcal{K}$ is finite-dimensional, this is exactly the condition ensuring that the translation-invariant state defined by $\{\varphi, V_1, \dots, V_d\}$ on the two-sided one-dimensional quantum chain $\bigotimes_{-\infty}^{\infty} M_d = \bigotimes_{\mathbb{Z}} M_d$ is pure ([24], [25]). To be precise, this condition on $\{\varphi, V_1, \dots, V_d\}$ is sufficient to ensure purity of ω. It is not necessary for the given $\{\varphi, V_1, \dots, V_d\}$, but if ω is pure and finitely generated, there exists some $\{\varphi, V_1, \dots, V_d\}$ on a finite-dimensional $\mathcal{K}$, defining ω, such that the corresponding σ is ergodic and has trivial peripheral spectrum. One source of the nonuniqueness of $\{\varphi, V_1, \dots, V_d\}$, and the corresponding nonnecessity of the conditions on this set, is the following: if $\mathcal{K}$ is replaced by $\mathcal{K} \otimes \mathcal{K}'$, where $\mathcal{K}'$ is a Hilbert space of finite dimension $\geqslant 2$, V_k by $V_k \otimes \mathrm{id}$ and φ by $\varphi \otimes \varphi'$, where φ' is a faithful state on $\mathcal{B}(\mathcal{K}')$, then the new data define exactly the same state as the old, but the fixed point set of the new σ contains at least $\mathbb{1} \otimes \mathcal{B}(\mathcal{K}')$. To avoid this kind of degeneracy, we make in Section 7 the overall assumption that the operators $V_1, \dots, V_d$ on $\mathcal{K}$ (which does not need to be finite-dimensional) generate a factor $\mathcal{M}$ with a faithful normal σ-invariant state φ, and that $\mathcal{B}(\mathcal{K})^\sigma = \mathcal{M}'$. If in addition $\mathcal{M}$ is type I, we prove that the corresponding translationally invariant state ω on $\bigotimes_{\mathbb{Z}} M_d$ is pure if and only if $\mathrm{PSp}(\sigma|\mathcal{M}) \cap \mathbb{T} = \{1\}$. If $\mathcal{M}$ is a finite type I factor, this is exactly the result in [25]. If $\mathcal{M}$ is not type I, this equivalence is no longer true, but in that case we can prove that ω is pure if and only if ω is a factor state, i.e., if and only if ω has the clustering property $\lim_{|n| \to \infty} \omega(x\lambda^n(y)) = \omega(x)\omega(y)$ for each pair $x, y \in \bigotimes_{\mathbb{Z}} M_d$.

For more background material on the representations of $\mathcal{O}_d$, see, e.g., [12], [9], [11]. A representation (S_i) of $\mathcal{O}_d$ on $\mathcal{H}$ defines an endomorphism $\sigma(\,\cdot\,) = \sum_i S_i \cdot S_i^*$ of $\mathcal{B}(\mathcal{H})$, and conversely. Moreover, the connection between an endomorphism σ, corresponding to λ in (1.2), and the associated completely positive map σ in (1.3) above, is given by

$$P\sigma(X)P = \sigma(PXP), \quad X \in \mathcal{B}(\mathcal{H}).$$

The lifting problem, addressed in Section 2 below, then concerns the reconstruction of the endomorphism σ, or the associated $\mathcal{O}_d$-representation, from some given completely positive normal unital map σ of $\mathcal{B}(\mathcal{K})$.

Other somewhat related aspects of the representation theory of $\mathcal{O}_d$, and its restriction to UHF_d, have been considered in [34], [41], [28], [27], [17].

2. GENERAL STATES ON $\mathcal{O}_d$

First some notation: Let $d \in \{2, 3, \ldots\}$ and let $\mathbb{Z}_d$ be a set of d elements. (The group structure of $\mathbb{Z}_d$ is spurious for the purposes of this paper.) Let $\mathcal{I}$ be the set of finite sequences $(i_1, \ldots, i_m)$ where $i_k \in \mathbb{Z}_d$ and $m \in \{1, 2, \ldots\}$. We also include the empty sequence $\emptyset$ in $\mathcal{I}$, and denote elements in $\mathcal{I}$ by $I, J, \ldots$. If $I = (i_1, \ldots, i_m) \in \mathcal{I}$ and $i \in \mathbb{Z}_d$, we let Ii denote the element $(i_1, \ldots, i_m, i)$ in $\mathcal{I}$, and $s_I = s_{i_1} s_{i_2} \cdots s_{i_m} \in \mathcal{O}_d$ and $s_I^* = s_{i_m}^* s_{i_{m-1}}^* \cdots s_{i_1}^* \in \mathcal{O}_d$. In particular $s_\emptyset = \mathbb{1} = s_\emptyset^*$.

The following theorem is a version of a result of Popescu ([38]). It generalizes [7]. We give a streamlined proof which applies in this case.

THEOREM 2.1. *Let $d \in \{2, 3, \ldots\}$. There is a canonical one-one correspondence between the following objects:*

(2.1) *States $\widehat{\omega}$ on $\mathcal{O}_d$.*

(2.2) *Functions $C : \mathcal{I} \times \mathcal{I} \to \mathbb{C}$ with the following properties:*

(i) $C(\emptyset, \emptyset) = 1$*;*

(ii) *for any function $\lambda : \mathcal{I} \to \mathbb{C}$ with finite support we have*

$$\sum_{I,J \in \mathcal{I}} \overline{\lambda(I)} C(I, J) \lambda(J) \geqslant 0;$$

(iii) $\sum_{i \in \mathbb{Z}_d} C(Ii, Ji) = C(I, J)$ *for all $I, J \in \mathcal{I}$.*

(2.3) *Unitary equivalence classes of objects $(\mathcal{K}, \Omega, V_1, \ldots, V_d)$ where*

(i) *$\mathcal{K}$ is a Hilbert space;*

(ii) *Ω is a unit vector in $\mathcal{K}$;*

(iii) $V_1, \ldots, V_d \in \mathcal{B}(\mathcal{K})$*;*

(iv) *the linear span of vectors of the form $V_I^* \Omega$, where $I \in \mathcal{I}$, is dense in $\mathcal{K}$;*

(v) $\sum_{i \in \mathbb{Z}_d} V_i V_i^* = \mathbb{1}_{\mathcal{K}}$.

The correspondence is given by

$$\widehat{\omega}(s_I s_J^*) = C(I, J) = \langle V_I^* \Omega | V_J^* \Omega \rangle. \tag{2.4}$$

Proof. It is immediate that if either $\widehat{\omega}$ or $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ is given, and $C(\,\cdot\,,\cdot\,)$ is defined by the relation (2.4), then C satisfies (2.2). ((i) corresponds to the normalization $\|\widehat{\omega}\| = 1 = \widehat{\omega}(\mathbb{1})$, or $\|\Omega\| = 1$, (ii) corresponds to positivity, and (iii) to the relations $\sum_i s_i s_i^* = \mathbb{1}$, $\sum_i V_i V_i^* = \mathbb{1}$.)

To go from the positive definite function C in (2.2) to the object $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ one uses the usual Kolmogorov construction: one puts $\mathcal{K}$ equal to the completion of the free vector space $\mathcal{L}(\mathcal{I})$ of all formal finite linear combinations $\sum_{I \in \mathcal{I}} \lambda(I) I$ (alias all functions $\lambda : \mathcal{I} \to \mathbb{C}$ with finite support) with respect to the pre-inner product defined by sesquilinearity from $\langle I | J \rangle = C(I, J)$, after dividing out the vectors of zero norm. This gives a map $\Phi : \mathcal{L}(\mathcal{I}) \to \mathcal{K}$, and one defines V_i by$V_i^* \Phi(I) = \Phi(Ii)$. It is now routine to check the properties (i)–(v) in (2.3).

To go from the object $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ in (2.3) to the state $\widehat{\omega}$ on $\mathcal{O}_d$, we will actually prove more:

(There is also a simple direct way of establishing this direction which will be exhibited in Remark 5.2.)

Lemma 2.2. ([38]) *Assume that* $\mathcal{K}, \Omega, V_1, \dots, V_d$ *satisfy the properties* (i)–(v) *under* (2.3). *It follows that there exists a unique linear map* $R : \mathcal{O}_d \to \mathcal{B}(\mathcal{K})$ *such that*

$$R(s_I s_J^*) = V_I V_J^*$$

and this map is completely positive.

Proof. Let $\mathcal{T}_d$ be the Cuntz–Toeplitz algebra, realized on the unrestricted Fock space $\widehat{\mathcal{H}} = \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k}$ in the usual way $L_i : \xi \mapsto |i\rangle \otimes \xi$, where L_i, $i = 1, \dots, d$, are the operators mapping into s_i after dividing out by the compact operators ([21], [7]). Let $\lambda \in \mathbb{C}$, $|\lambda| < 1$, and define an operator $W_\lambda : \mathcal{K} \to \widehat{\mathcal{H}} \otimes \mathcal{K}$ by

$$W_\lambda \varphi = \sqrt{1 - |\lambda|^2} \bigoplus_{k=0}^{\infty} \lambda^k \sum_{I \in \mathcal{I}_k} |I\rangle \otimes V_I^* \varphi$$

where $\mathcal{I}_k$ denotes all sequences $I = (i_1, \dots, i_k)$ of length k with $i_j \in \mathbb{Z}_d$, and

$$|I\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_k\rangle.$$

One checks that W_λ is an isometry, and

$$(L_i^* \otimes \mathbb{1}_{\mathcal{K}}) W_\lambda = \lambda W_\lambda V_i^*.$$

From this intertwining relation, and its adjoint, it follows that

$$R_\lambda(L_I L_J^*) \equiv W_\lambda^* (L_I L_J^* \otimes \mathbb{1}_{\mathcal{K}}) W_\lambda = \overline{\lambda}^n \lambda^m V_I V_J^*$$

if $I \in \mathcal{I}_n$, $J \in \mathcal{I}_m$. It follows from this explicit Stinespring representation that the linear map defined from

$$L_I L_J^* \mapsto \overline{\lambda}^n \lambda^m V_I V_J^*$$

is then completely positive for all $|\lambda| < 1$, and taking the limit as $\lambda \to 1$, it follows further that R is completely positive as a map from $\mathcal{T}_n$ into $\mathcal{B}(\mathcal{K})$. To check that R, thus defined, defines a map from $\mathcal{O}_d$ into $\mathcal{B}(\mathcal{K})$, we have to show that R annihilates the ideal generated by the one-dimensional projection $p = \mathbb{1} - \sum_i L_i L_i^*$, i.e., that $R(XpY) = 0$ for all polynomials X, Y in the L_i's and the L_i^*'s. We may take X, Y to be Wick ordered monomials, i.e., of the form $L_I L_J^*$. Since $pL_i = 0$, we may assume that Y contains no factor L_I, and by the same token we may assume that X contains no factor L_J^*, and hence XpY has the form

$$L_I p L_J^* = L_I L_J^* - \sum_{i \in \mathbb{Z}_n} L_{Ii} L_{Ji}^*.$$

Using the definition of R_λ, and the relation $\sum_{i=1}^d V_i V_i^* = \mathbb{1}$, it now follows that $R_\lambda(XpY) = 0$. Hence $R(XpY) = 0$, and R defines a completely positive map from $\mathcal{O}_d$ into $\mathcal{B}(\mathcal{K})$. This ends the proof of the lemma. ∎

Proof of Theorem 2.1, *continued*: To go from the object $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ to the state $\widehat{\omega}$ is now clear: put $\widehat{\omega}(X) = \langle \Omega | R(X) \Omega \rangle$ where $R : \mathcal{O}_d \to \mathcal{B}(\mathcal{K})$ is the completely positive linear map defined in Lemma 2.2. Then $\widehat{\omega}(s_I s_J^*) = \langle \Omega | V_I V_J^* \Omega \rangle$ so (2.4) is fulfilled.

This establishes the one-one correspondence stated in Theorem 2.1. Of course, the system $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ is not unique, but determined only up to unitary equivalence. The argument for why this is so is exactly the same as the standard argument from representation theory ([13], Theorem 2.3.16), to the effect that a state on a C^*-algebra only determines a cyclic representation up to unitary equivalence. ∎

REMARK 2.3. Note also that there is a simple direct way of going from the state $\widehat{\omega}$ in (2.1) to the object $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ in (2.3). If $(\mathcal{H}, \Omega, \pi)$ is the cyclic representation of $\mathcal{O}_d$ defined by Ω, let $\mathcal{K}$ be the closure of the linear span of all vectors $S_I^* \Omega$, where $S_I = \pi(s_I)$. Let P be the projection from $\mathcal{H}$ onto $\mathcal{K}$, and put

$$V_i^* = P S_i^* P = S_i^* P.$$

The property $\sum_i V_i V_i^* = P = \mathbb{1}_{\mathcal{K}}$ follows immediately from $\sum_i S_i S_i^* = \mathbb{1}_{\mathcal{H}}$.

One can use Lemma 2.2 to prove stronger versions of Popescu's dilation theorem:

Corollary 2.4. *Let $\mathcal{K}$ be a Hilbert space, and $D \in \mathcal{B}(\mathcal{K})$ a positive operator, and $V_1, \dots, V_d \in \mathcal{B}(\mathcal{K})$ operators such that*

$$\sum_i V_i D V_i^* = D.$$

Then there exists a unique continuous linear map $R : \mathcal{O}_d \to \mathcal{B}(\mathcal{K})$ such that

$$R(s_I s_J^*) = V_I D V_J^*$$

and this map is completely positive.

Proof. Roughly, if R' is the completely positive map defined in Lemma 2.2 from the operators $V_i' = D^{-\frac{1}{2}} V_i D^{\frac{1}{2}}$, one verifies that

$$R'(s_I s_J^*) = V_I' V_J'^* = D^{-\frac{1}{2}} V_I D V_J^* D^{-\frac{1}{2}}.$$

Putting $R(\,\cdot\,) = D^{\frac{1}{2}} R'(\,\cdot\,) D^{\frac{1}{2}}$, we obtain the corollary. A more careful argument is given in Remark 5.3. ∎

3. ERGODIC THEORY OF COMPLETELY POSITIVE MAPS ON $\mathcal{B}(\mathcal{H})$

In this section we prove some more or less known results about completely positive unital normal maps φ of $\mathcal{B}(\mathcal{K})$, and we analyze the fixed-point set

$$\mathcal{B}(\mathcal{K})^\varphi := \{X \in \mathcal{B}(\mathcal{K}) \mid \varphi(X) = X\}.$$

We will need the arguments from the proofs here later in the paper.

Let $\mathcal{K}$ be a Hilbert space, and $\varphi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ a normal unital completely positive map. Then there exists a family of operators $V_i \in \mathcal{B}(\mathcal{K})$ such that

$$\sum_i V_i V_i^* = \mathbb{1} \quad \text{and} \quad \varphi(X) = \sum_i V_i X V_i^*$$

for all $X \in \mathcal{B}(\mathcal{K})$, where the sum converges in weak operator topology ([22]).

Lemma 3.1. *Let p be a projection in $\mathcal{B}(\mathcal{K})$. Then the following conditions are equivalent:*

(i) *there is a $\lambda \geqslant 0$ such that $\varphi(p) \leqslant \lambda p$;*
(ii) *$V_i p = p V_i p$ for all i;*
(iii) *$\varphi(p) \leqslant p$.*

REMARK 3.2. Condition (i) is of course equivalent to the condition that the weakly closed hereditary subalgebras $p\mathcal{B}(\mathcal{K})p$ of $\mathcal{B}(\mathcal{K})$ are invariant under φ. The property that there are no nontrivial weakly closed hereditary subalgebras of $\mathcal{B}(\mathcal{H})$ invariant under φ is called *irreducibility* of φ in [23], and since any such subalgebra is of the form $p\mathcal{B}(\mathcal{H})p$, irreducibility of φ is equivalent to the nonexistence of projections p with the property (i) or (ii). The proof of Lemma 3.1 is extracted from [23].

Proof. (i) $\Rightarrow$ (ii) Assume $\varphi(p) \leqslant \lambda p$. Then

$$0 \leqslant (\mathbb{1}-p)\varphi(p)(\mathbb{1}-p) \leqslant (\mathbb{1}-p)\lambda p(\mathbb{1}-p) = 0,$$

so

$$0 = \sum_i (\mathbb{1}-p)V_i p V_i^*(\mathbb{1}-p) = \sum_i ((\mathbb{1}-p)V_i p)((\mathbb{1}-p)V_i p)^*,$$

and hence

$$0 = (\mathbb{1}-p)V_i p,$$

which is (ii).

(ii) $\Rightarrow$ (iii) Assume that $V_i p = pV_i p$ for all i. Then

$$\varphi(p) = \sum_i V_i p V_i^* = \sum_i pV_i p V_i^* p = p\varphi(p)p \leqslant \|\varphi(p)\| p \leqslant p.$$

(iii) $\Rightarrow$ (i) is trivial. ∎

LEMMA 3.3. *Let p be a projection in $\mathcal{B}(\mathcal{K})$. Then the following conditions are equivalent:*

(i) $\varphi(p) = p$*;*

(ii) $p \in \{V_i, V_i^*\}_i'$*, i.e., $pV_i = V_i p$ for all i.*

Proof. Since $\sum_i V_i V_i^* = \mathbb{1}$, (ii) $\Rightarrow$ (i) is trivial. Conversely, assume $\varphi(p) = p$. Applying Lemma 3.1, (i) $\Rightarrow$ (ii), on p and $\mathbb{1}-p$, we obtain $V_i p = pV_i p$ and $V_i(\mathbb{1}-p) = (\mathbb{1}-p)V_i(\mathbb{1}-p)$, i.e., $pV_i = pV_i p$, so (ii) holds. ∎

If $\mathcal{B}(\mathcal{K})^\varphi = \{X \in \mathcal{B}(\mathcal{K}) \mid \varphi(X) = X\}$ were an algebra (which necessarily is weakly closed and closed under involution), it would follow from Lemma 3.3 that

$$\mathcal{B}(\mathcal{K})^\varphi = \{V_i, V_i^*\}'$$

(the inclusion $\supset$ is trivial, as mentioned before). There is one important special case where $\mathcal{B}(\mathcal{K})^\varphi$ is an algebra, namely when there is a faithful φ-invariant state:

Lemma 3.4. ([25]) *Assume that there is a faithful φ-invariant state ω on $\mathcal{B}(\mathcal{K})$. Then $\mathcal{B}(\mathcal{K})^{\varphi}$ is an algebra, and hence*

$$\mathcal{B}(\mathcal{K})^{\varphi} = \{V_i, V_i^*\}'.$$

Proof. We follow [25], proof of Proposition 2.2. By [15], Theorem 3.1, if φ is any 2-positive map on a C^*-algebra $\mathfrak{A}$, then

$$\{a \in \mathfrak{A} \mid \varphi(a^*a) = \varphi(a)^*\varphi(a)\} = \{a \in \mathfrak{A} \mid \varphi(xa) = \varphi(x)\varphi(a) \text{ for all } x \in \mathfrak{A}\}.$$

Going back to our case, assume that $\varphi(a) = a$ for some $a \in \mathcal{B}(\mathcal{K})$. By the generalized Schwarz inequality, we then have

$$\varphi(a^*a) - a^*a = \varphi(a^*a) - \varphi(a^*)\varphi(a) \geqslant 0.$$

But $\omega(\varphi(a^*a) - a^*a) = 0$ by invariance of ω, and as ω is faithful it follows that $\varphi(a^*a) = a^*a = \varphi(a^*)\varphi(a)$. By Choi's theorem, $\varphi(xa) = \varphi(x)a$ for all $x \in \mathcal{B}(\mathcal{H})$, and if in particular $x \in \mathcal{B}(\mathcal{H})^{\varphi}$, then $\varphi(xa) = xa$. Thus $xa \in \mathcal{B}(\mathcal{H})^{\varphi}$, and $\mathcal{B}(\mathcal{H})^{\varphi}$ is an algebra. ∎

Note that the map $\omega \mapsto \omega\circ\varphi$ is obviously a continuous map on the state space of $\mathcal{B}(\mathcal{H})$, and this space is compact in the weak*-topology from $\mathcal{B}(\mathcal{H})$. Hence it follows from the Schauder–Tychonoff fixed point theorem that there exists a state ω such that $\omega\circ\varphi = \omega$ ([44], [20], p. 456, Section V.10.5, Theorem 5). Unfortunately the state ω is not necessarily faithful. For example: let $\mathcal{K} = \mathbb{C}^n$, e_{ij} a full set of matrix units for $\mathcal{B}(\mathbb{C}^n) = M_n$, and put $V_i = e_{i1}$ for $i = 1, \dots, n$. Then

$$\sum_i V_i V_i^* = \mathbb{1} \quad \text{and} \quad \{V_i, V_i^*\}' = \mathbb{C}\,\mathbb{1},$$

but the unique invariant state for φ is the pure state $\omega\Big(\sum_{ij} X_{ij}e_{ij}\Big) = X_{11}$.

The states fixed by φ need not in general be normal either. We will discuss these states further in the beginning of Section 6.

Actually, there also exist examples where $\mathcal{B}(\mathcal{K})^{\varphi}$ is not an algebra. The following example is from [1], [2]: $\mathcal{K} = \mathbb{C}^3$,

$$\varphi\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & \frac{1}{2}(X_{11}+X_{22}) \end{pmatrix}.$$

Then φ is completely positive, and one checks that

$$\mathcal{B}(\mathcal{K})^{\varphi} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{2}(a+b) \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\},$$

which contains no nontrivial subalgebras, and hence $\{V_i, V_i^*\}' = \mathbb{C}\mathbb{1}$ whatever the choice of V_i's. One choice is

$$V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

The invariant states are all the convex combinations of the two states $(X_{ij}) \mapsto X_{11}$ or $(X_{ij}) \mapsto X_{22}$. Thus, in the general situation, the following theorem is the best possible.

Theorem 3.5. *Let $\varphi(\cdot) = \sum_i V_i \cdot V_i^*$ be a normal unital completely positive map of $\mathcal{B}(\mathcal{K})$. Then*

$$\{V_i, V_i^*\}' \subset \mathcal{B}(\mathcal{H})^{\varphi}.$$

Furthermore, the space $\mathcal{B}(\mathcal{H})^{\varphi}$ contains a largest $*$-subalgebra, and this algebra is $\{V_i, V_i^*\}'$.

Proof. Since $\sum_i V_i V_i^* = \mathbb{1}$, the first assertion is trivial. Next note that as φ is normal, if $\mathfrak{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})^{\varphi}$, then the weak*-closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ is contained in $\mathcal{B}(\mathcal{H})^{\varphi}$. But since $\overline{\mathfrak{A}}$ is the weak*-closure of the linear span of its projections, it follows from Lemma 3.3 that $\mathfrak{A} \subset \overline{\mathfrak{A}} \subset \{V_i, V_i^*\}'$. This proves the theorem. ∎

4. The commutant lifting theorem and pure states on $\mathcal{O}_d$

The main aim of this section is to decide which systems $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ give rise to pure states $\widehat{\omega}$ on $\mathcal{O}_d$. To this end it will be convenient to define a completely positive unital map σ of $\mathcal{B}(\mathcal{K})$ by

$$\sigma(A) = \sum_{i=1}^{d} V_i A V_i^*. \tag{4.1}$$

We will actually establish an order isomorphism between the order interval $[0, \widehat{\omega}]$ in the set of positive functionals on $\mathcal{O}_d$, and the set of operators $A \in \mathcal{B}(\mathcal{K})$ such that $0 \leqslant A \leqslant \mathbb{1}$ and $\sigma(A) = A$. This is a natural generalization of the commutant lifting theorem of [39], and another version of this result is Corollary 5.4 in [9]. The term "commutant lifting" is from single-operator theory ([43], [26], [19]) where it refers to the Sz.-Nagy lifting theorem, which for every contractive operator V in a given Hilbert space $\mathcal{K}$ yields a minimal coisometry, and in fact, by a second step, also a unitary operator U, acting on a bigger Hilbert space $\mathcal{H}$, and serving as a lifting of V. If P denotes the projection of $\mathcal{H}$ onto $\mathcal{K}$, i.e., $\mathcal{K} = P\mathcal{H}$, then Sz.-Nagy's dilation theorem states the existence of $(U, \mathcal{H})$ such that $UP = PUP$ and $V^n = PU^nP$ on $\mathcal{K}$ for all $n \in \mathbb{N}$. "Minimality" here is the requirement that the subspace $\mathcal{K}$ be cyclic for $\{U^n \mid n \in \mathbb{Z}\}$ in $\mathcal{H}$. If we have two contractions $V_i : \mathcal{K}_i \to \mathcal{K}_i$, $i = 1, 2$, with corresponding minimal coisometric (or unitary) dilations $(U_i, \mathcal{H}_i)$ and projections $P_i : \mathcal{H}_i \to \mathcal{K}_i$, $P_i\mathcal{H}_i = \mathcal{K}_i$, $i = 1, 2$, and if $Y : \mathcal{K}_1 \to \mathcal{K}_2$ is a bounded operator which is given to intertwine the two contractions, i.e., $YV_1 = V_2Y$, then Y lifts, by [19], to a bounded $X : \mathcal{H}_1 \to \mathcal{H}_2$ with the same operator norm, $\|X\| = \|Y\|$, and satisfying $XU_1 = U_2X$, and $P_2XP_1 = Y$ on $\mathcal{K}_1$.

The analogy to the present setting refers to an operator Y which intertwines two given V_i-systems $\{V_i\}_{i=1}^d$ and $\{W_i\}_{i=1}^d$, say, and its canonical lifting to an operator which intertwines the corresponding two representations of the Cuntz algebra $\mathcal{O}_d$.

Proposition 4.1 and Theorem 4.4 below represent our multivariable analogue of this lifting result, but only for the special case when $V_i = W_i$, while Theorem 5.1 is our general multivariable commutant lifting theorem.

Proposition 4.1. *Adopt the notation in Remark* 2.3. *Then the selfadjoint part of the commutant $\pi(\mathcal{O}_d)'$ is norm and order isomorphic to the space of selfadjoint fixed points of the completely positive map σ. This isomorphism takes $A \in \pi(\mathcal{O}_d)'$ into $PAP \in \mathcal{B}(\mathcal{K})^\sigma$.*

Proof. Let $X \in \pi(\mathcal{O}_d)'$. Then PXP is determined by the matrix elements

$$\langle V_I^*\Omega | PXPV_J^*\Omega\rangle = \langle S_I^*\Omega | XS_J^*\Omega\rangle.$$

Writing the same expression for $I \mapsto Ii$ and $J \mapsto Ji$, and summing over i, shows that $PXP \in \mathcal{B}(\mathcal{K})^\sigma$. (Why can we not conclude $PXP \in \{V_i, V_i^*\}'$? We get

$$\begin{aligned}\langle V_I^*\Omega | V_i PXPV_J^*\Omega\rangle &= \langle S_i^* S_I^*\Omega | XS_J^*\Omega\rangle = \langle S_I^*\Omega | XS_i S_J^*\Omega\rangle \\ &= \langle V_I^*\Omega | PXS_i PV_J^*\Omega\rangle \neq \langle V_I^*\Omega | PXPS_i S_J^*\Omega\rangle,\end{aligned}$$

because $S_i S_J^*\Omega \notin \mathcal{K}$ in general. See Theorem 3.5.) Conversely, assume that $D \in \mathcal{B}(\mathcal{K})^\sigma$, with $0 \leqslant D \leqslant \mathbb{1}$. That is, D satisfies the hypothesis of Corollary 2.4. Hence the linear functional on $\mathcal{O}_d$ defined by

$$\widetilde{\omega}(s_I s_J^*) = \langle V_I^*\Omega | DV_J^*\Omega\rangle$$

is positive, and, applying the same argument to $\mathbb{1} - D$, we find that $0 \leqslant \widetilde{\omega} \leqslant \widehat{\omega}$. Since Ω is cyclic for $\mathcal{O}_d$, there is an $X \in \pi(\mathcal{O}_d)'$ (with $0 \leqslant X \leqslant \mathbb{1}$), which is uniquely determined by the equation

$$\widetilde{\omega}(s_I s_J^*) = \langle \Omega | XS_I S_J^*\Omega\rangle.$$

But since $X \in \mathcal{O}_d'$, we have $\langle \Omega | XS_I S_J^*\Omega\rangle = \langle V_I^*\Omega | XV_J^*\Omega\rangle = \langle V_I^*\Omega | DV_J^*\Omega\rangle$. That is to say $PXP = D$.

Since $\mathbb{1} \in \mathcal{B}(\mathcal{K})^\sigma$, the real linear span of this positive cone in $\mathcal{B}(\mathcal{K})^\sigma$ is all of the selfadjoint part, and hence the map $X \mapsto PXP$ is onto, and (by scaling with suitable positive factors) the above arguments show that the map is an order isomorphism between the respective selfadjoint parts of $\pi(\mathcal{O}_d)'$ and $\mathcal{B}(\mathcal{K})^\sigma$. The selfadjoint subspaces are also order unit spaces, i.e.,

$$\|A\| = \inf\{\alpha \geqslant 0 \mid -\alpha\mathbb{1} \leqslant A \leqslant \alpha\mathbb{1}\}.$$

(For $\mathcal{B}(\mathcal{K})^\sigma$, this formula is inherited from $\mathcal{B}(\mathcal{K})$, using, of course, crucially that $\mathbb{1} \in \mathcal{B}(\mathcal{K})^\sigma$.) From this it is evident that the isomorphism is also isometric. ∎

Having now identified $\mathcal{B}(\mathcal{K})^\sigma$ with $P\pi(\mathcal{O}_d)'P$, let us return to the question raised in Theorem 3.5 and the preceding remarks on when $\mathcal{B}(\mathcal{K})^\sigma$ is an algebra.

Proposition 4.2. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and let P be a projection in $\mathcal{H}$ such that $X \mapsto PXP$ is an isometry on the selfadjoint part of $\mathcal{M}$. Then the following are equivalent:*

(i) *$P\mathcal{M}P$ is an algebra;*

(ii) *$X \mapsto PXP$ is a homomorphism on $\mathcal{M}$;*

(iii) *$P \in \mathcal{M}'$.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. (ii) $\Rightarrow$ (iii) follows from the observation that the homomorphism property implies $PX^*(\mathbb{1}-P)XP = 0$, i.e., $(\mathbb{1}-P)XP = 0$, and $XP = PXP = PX$. Note that these steps do not even depend on the isometry property.

The nontrivial bit, (i) $\Rightarrow$ (ii), is essentially contained in the proof of Theorem 3.5. Here is a slightly different way of putting it: The isometry property means that the unit interval of $\mathcal{M}$ is isometrically mapped onto that of the algebra $P\mathcal{M}P$. In particular, extremal points correspond to extremal points, which in a von Neumann algebra means that projections go into projections, and orthogonality of projections is preserved. By the spectral theorem, we find that the compression map is a Jordan isomorphism, and hence the direct sum of a homomorphism and an anti-homomorphism. Because it is completely positive, it is a homomorphism. ∎

Remark 4.3. Note that it is not enough to require isometry on the whole (complex) vector space $\mathcal{M}$, since the norms alone do not give enough information. A counterexample can be made with a two-dimensional abelian algebra. ($\mathcal{M} = \mathbb{C}Q + \mathbb{C}(\mathbb{1} - Q)$, and QPQ has both eigenvalues 0 and 1.)

Note also that if Propositions 4.1 and 4.2 are applied to the example of Arveson discussed prior to Theorem 3.5, it follows that the dilation of $\{\mathbb{C}^3, V_1, \dots, V_4\}$ to a representation π of $\mathcal{O}_4$ decomposes into two disjoint irreducible representations, and P is not contained in $\pi(\mathcal{O}_4)''$. Note that the nontrivial projections in $\pi(\mathcal{O}_4)'$ then cannot be Popescu dilations of anything in $\mathcal{B}(\mathbb{C}^3)$.

We are now ready to state the characterization of pure states $\widehat{\omega}$ on $\mathcal{O}_d$. If $\widehat{\omega}$ is any state on $\mathcal{O}_d$, let again $(\mathcal{H}, \Omega, S_1, \dots, S_d)$ be the corresponding representation, and $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ the corresponding Popescu system, and define the corresponding endomorphism λ of $\mathcal{B}(\mathcal{H})$ by $\lambda(\,\cdot\,) = \sum_{i=1}^{d} S_i \,\cdot\, S_i^*$, and the unital completely positive map σ of $\mathcal{B}(\mathcal{K})$ by $\sigma(\,\cdot\,) = \sum_{i=1}^{d} V_i \,\cdot\, V_i^*$.

THEOREM 4.4. *If $\widehat{\omega}$ is a state on $\mathcal{O}_d$, the following conditions are equivalent:*
(i) *$\widehat{\omega}$ is pure;*
(ii) *$\lambda(X) = X$ implies $X \in \mathbb{C}\mathbb{1}_{\mathcal{H}}$, $X \in \mathcal{B}(\mathcal{H})$;*
(iii) *$\sigma(Y) = Y$ implies $Y \in \mathbb{C}\mathbb{1}_{\mathcal{K}}$, $Y \in \mathcal{B}(\mathcal{K})$;*
(iv) *$\{V_i, V_i^*\}$ acts irreducibly on $\mathcal{K}$, and $P \in \pi(\mathcal{O}_d)''$.*

Proof. In general the fixed point algebra for λ is $\pi(\mathcal{O}_d)'$ (see, e.g., [12], formula (3.5) or [33], Proposition 3.1) and hence (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) follows from Proposition 4.1. If $\widehat{\omega}$ is pure, then $P \in \pi(\mathcal{O}_d)''$ and $\{V_i, V_i^*\}$ acts irreducibly on $\mathcal{K}$ because $\{V_i, V_i^*\}' \subset \mathcal{B}(\mathcal{K})^\sigma = \mathbb{C}\mathbb{1}_{\mathcal{H}}$ by (iii), hence (i) $\Rightarrow$ (iv). Conversely if $P \in \pi(\mathcal{O}_d)''$ it follows, by applying Proposition 4.2 on $\mathcal{M} = \pi(\mathcal{O}_d)'$, that $P\pi(\mathcal{O}_d)'P$ is an algebra. But this algebra is $\mathcal{B}(\mathcal{K})^\sigma$ by Proposition 4.1, and if $\{V_i, V_i^*\}$ acts irreducibly, it follows from Theorem 3.5 that $\mathcal{B}(\mathcal{K})^\sigma = \mathbb{C}\mathbb{1}_{\mathcal{K}}$. Thus (iv) $\Rightarrow$ (iii), and Theorem 4.4 is proved. ∎

5. REPRESENTATIONS OF $\mathcal{O}_d$

For the wavelet applications described in Section 1, we will need versions of Theorem 2.1, Proposition 4.1 and Theorem 4.4 where the state $\widehat{\omega}$ is replaced merely by the system $(\mathcal{K}, V_1, \dots, V_d)$.

THEOREM 5.1. *Let $\mathcal{K}$ be a Hilbert space, and let $V_1, \dots, V_d \in \mathcal{B}(\mathcal{K})$ be operators satisfying*

$$\sum_{i \in \mathbb{Z}_d} V_i V_i^* = \mathbb{1}.$$

Then $\mathcal{K}$ can be embedded into a larger Hilbert space $\mathcal{H} = \mathcal{H}_V$ carrying a representation $S_1, \dots, S_d$ of the Cuntz algebra $\mathcal{O}_d$ such that if $P : \mathcal{H} \to \mathcal{K}$ is the projection onto $\mathcal{K}$ we have

$$V_i^* = S_i^* P$$

(i.e., $S_i^*\mathcal{K} \subset \mathcal{K}$ and $S_i^* P = P S_i^* P = V_i^*$) and $\mathcal{K}$ is cyclic for the representation. The system $(\mathcal{H}, S_1, \dots, S_d, P)$ is unique up to a unitary equivalence, and if $\sigma : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ is defined by

$$\sigma(A) = \sum_i V_i A V_i^*,$$

then the commutant of the representation $\{S_1, \dots, S_d\}'$ is isometrically order isomorphic to the fixed point set $\mathcal{B}(\mathcal{K})^\sigma = \{A \in \mathcal{B}(\mathcal{K}) \mid \sigma(A) = A\}$ by the map

$A' \mapsto PA'P$. More generally, if $W_1, \dots, W_d \in \mathcal{B}(\mathcal{K})$ is another set of operators satisfying

$$\sum_{i \in \mathbb{Z}_d} W_i W_i^* = \mathbb{1}$$

and $T_1, \dots, T_d$ are the corresponding representatives of $s_1, \dots, s_d$, then there is an isometric linear isomorphism between intertwiners $U : \mathcal{H}_V \to \mathcal{H}_W$, i.e., operators satisfying

$$US_i = T_i U,$$

and operators $V \in \mathcal{B}(\mathcal{K})$ such that

$$\sum_{i \in \mathbb{Z}_d} W_i V V_i^* = V, \tag{5.1}$$

given by the map $U \mapsto V = PUP$.

Proof. Inspecting the proof of Lemma 2.2, we see that the vector Ω plays no role in the proof, so the map $R : \mathcal{O}_d \to \mathcal{B}(\mathcal{K})$ defined by $R(s_I s_J^*) = V_I V_J^*$ is well defined and completely positive. The representation $S_1, \dots, S_d$ of $\mathcal{O}_d$ on $\mathcal{H}$ thus may be taken to be the Stinespring dilation of R ([1], [14], p. 229, Notes and Remarks to Chapter 5, [42]), and uniqueness up to unitary equivalence follows from uniqueness of the Stinespring representation.

The commutant lifting property is established as in Proposition 4.1, using Corollary 2.4.

To establish the final intertwiner lifting property, one considers the direct sum representation of $\mathcal{O}_d$ on $\mathcal{H}_V \oplus \mathcal{H}_W$ given by $s_i \mapsto S_i \oplus T_i$. Note that some operator $U : \mathcal{H}_V \to \mathcal{H}_W$ is an intertwiner if and only if $\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}$ is in the commutant of this sum representation. But the operators corresponding to V_i of this latter representation, relative to the subspace $\mathcal{K} \oplus \mathcal{K} \subset \mathcal{H}_V \oplus \mathcal{H}_W$, are

$$\begin{pmatrix} V_i & 0 \\ 0 & W_i \end{pmatrix}, \quad i \in \mathbb{Z}_d,$$

so, using the commutant lifting property of the direct sum representation, one verifies that U intertwines the S_i's and the T_i's if and only if $V = PUP$ is fixed under the map $\sum_i W_i \cdot V_i^*$. Specifically, if $\beta(\,\cdot\,) := \sum_i T_i \cdot S_i^*$, then we have the identity

$$P\beta(X)P = \sum_i W_i PXPV_i^*,$$

valid for all operators $X : \mathcal{H}_V \to \mathcal{H}_W$. Now note that U intertwines the two $\mathcal{O}_d$-representations, if and only if $\beta(U) = U$, and the assertion follows from this. ∎

REMARK 5.2. Another more direct way of constructing the representation of $\mathcal{O}_d$ in Theorem 5.1 is the following: let $\mathcal{I}_n$ be the set of finite sequences $I = (i_1, \dots, i_m)$ where $m \leqslant n$ and $i_k \in \mathbb{Z}_d$ for all k (including the empty sequence), and let $\mathbb{C}\mathcal{I}_n$ be the complex linear space of formal linear combinations of elements in $\mathcal{I}_n$. Put $\mathcal{I} = \bigcup_n \mathcal{I}_n$ as in Section 2, and define

$$H_n = \mathbb{C}\mathcal{I}_n \otimes \mathcal{K}$$

(algebraic tensor product). For each $I \in \mathcal{I}$, define a linear operator S_I on $H = \bigcup_n H_n$ by

$$S_I(J \otimes \xi) = IJ \otimes \xi$$

and linearity. Define a semi-inner product on H by requiring

$$\begin{aligned}\langle I \otimes \xi | IJ \otimes \eta \rangle &= \langle \xi | V_J \eta \rangle, \\ \langle IJ \otimes \xi | I \otimes \eta \rangle &= \langle V_J \xi | \eta \rangle\end{aligned}$$

for all $I, J \in \mathcal{I}$, $\xi, \eta \in \mathcal{K}$, and

$$\langle I \otimes \xi | J \otimes \eta \rangle = 0$$

if the pair I, J does not have one of the forms above. To show that this sesquilinear form is indeed positive and well defined, we proceed by induction: this is true for $H_0 = \mathcal{K}$. Suppose this is proved for H_{n-1} and let $\zeta \in H_n$. We express ζ as

$$\zeta = \sum_{j \in \mathbb{Z}_d} S_j \zeta_j + \zeta_0$$

where $\zeta_j \in H_{n-1}$ and $\zeta_0 \in H_0 = \mathcal{K}$. Then

$$\begin{aligned}\langle \zeta | \zeta \rangle &= \Big\langle \sum_j S_j \zeta_j + \zeta_0 \Big| \sum_k S_k \zeta_k + \zeta_0 \Big\rangle \\ &= \sum_j \langle \zeta_j | \zeta_j \rangle + \sum_j \langle \zeta_j | V_j^* \zeta_0 \rangle + \sum_k \langle V_k^* \zeta_0 | \zeta_k \rangle + \langle \zeta_0 | \zeta_0 \rangle \\ &= \sum_j \| \zeta_j + V_j^* \zeta_0 \|^2 \geqslant 0.\end{aligned}$$

Let $\mathcal{H}$ be the completion of H modulo zero-vectors and $\Lambda : H \to \mathcal{H}$ the canonical map. We define a bounded operator S_i on $\mathcal{H}$ by

$$S_i \Lambda(\zeta) = \Lambda(S_i \zeta),$$

and, using $\sum_j V_j V_j^* = \mathbb{1}$, one easily verifies that $s_i \mapsto S_i$ is a representation of $\mathcal{O}_d$ satisfying the required properties.

REMARK 5.3. Note that Corollary 2.4 can also be proved along the lines in Remark 5.2, but now one defines the semi-inner product $\langle \cdot | \cdot \rangle_D$ on $\mathcal{H}$ by requiring

$$\begin{aligned} \langle I \otimes \xi | IJ \otimes \eta \rangle_D &= \langle V_J^* \xi | D\eta \rangle, \\ \langle IJ \otimes \xi | I \otimes \eta \rangle_D &= \langle \xi | D V_J^* \eta \rangle, \quad \text{etc.} \end{aligned}$$

REMARK 5.4. In comparing the single-operator commutant lifting ([19]) with our Theorem 5.1, we note that the naive (or natural) multivariable generalization of the intertwining property for an operator Y on the "small" Hilbert space $\mathcal{K}$ would be $YV_i = W_i Y$. But this property is slightly different from the present one, $\sum_i W_i Y V_i^* = Y$, i.e., (5.1), used in Theorem 5.1.

There is naturally a variety of ways of generalizing the classical single-operator commutant lifting theorem to several variables, each serving different purposes. In addition to ours and the others mentioned above, there are related, but different, approaches (to the multivariable theory) in recent papers by Arveson ([3]) and Bhat ([4]).

6. IRREDUCIBLE REPRESENTATIONS OF $\mathcal{O}_d$ AND THEIR RESTRICTION TO UHF_d

Consider an irreducible representation $S_1, \dots, S_d$ of $\mathcal{O}_d$ on a Hilbert space $\mathcal{H}$, and let $\mathcal{K}$ be a cyclic subspace of $\mathcal{H}$ invariant under $S_1^*, \dots, S_d^*$. Define again $V_1, \dots, V_d \in \mathcal{B}(\mathcal{K})$ by

$$V_i^* P = S_i^* P = P S_i^* P$$

where $P : \mathcal{H} \to \mathcal{K}$ is the projection onto $\mathcal{K}$. By Theorem 5.1, irreducibility on $\mathcal{H}$ is equivalent to ergodicity of the completely positive map σ defined on $\mathcal{B}(\mathcal{K})$ by $\sigma(\,\cdot\,) = \sum_i V_i \cdot V_i^*$. Since $\sigma(\mathbb{1}) = \mathbb{1}$, σ maps the state space of $\mathcal{B}(\mathcal{K})$ into itself, and hence there is a σ-invariant state φ. If $\mathcal{K}$ is finite-dimensional, we will show that φ is unique. The state φ is automatically normal since $\mathcal{K}$ is finite-dimensional. Let E be the support of φ.

LEMMA 6.1. *$S_i^* E\mathcal{K} \subset E\mathcal{K}$ for all $i \in \mathbb{Z}_d$.*

Proof. Since $\varphi(\sigma(E)) = \varphi(E) = \mathbb{1}$ and $0 \leqslant \sigma(E) \leqslant \mathbb{1}$, it follows that$\sigma(E) \geqslant E$. Applying Lemma 3.1, (i) $\Rightarrow$ (ii), on $p = \mathbb{1} - E$ gives $V_i^* E = E V_i^* E$. This proves Lemma 6.1. ∎

But from [9], Lemma 6.3, it follows that there is only one σ-invariant state with support inside E. So, if φ_1, φ_2 are two σ-invariant states with respective support projections E_1, E_2, then $\frac{1}{2}(\varphi_1 + \varphi_2)$ is a σ-invariant state with support $E_1 \vee E_2$, and hence $\varphi_1 = \frac{1}{2}(\varphi_1 + \varphi_2) = \varphi_2$ by the argument above. We have proved:

Lemma 6.2. *If $\mathcal{K}$ is finite-dimensional, then $\mathcal{B}(\mathcal{K})$ has a unique σ-invariant state when σ is ergodic.*

The example after the proof of Lemma 3.4 shows that this σ-invariant state need not be faithful. However, replacing P by the support E of φ, and using Lemma 6.1, the following theorem is applicable to general irreducible representations when $\mathcal{K}$ is finite-dimensional, replacing P by E.

Theorem 6.3. *Consider an irreducible representation of $\mathcal{O}_d$ on $\mathcal{H}$, and let $\mathcal{K}, V_1, \dots, V_d, P, \sigma$ be as in the introduction to this section. Assume that there exists a normal faithful σ-invariant state φ on $\mathcal{B}(\mathcal{K})$. Let ψ be the state of $\mathcal{O}_d$ defined by*

$$\psi(s_I s_J^*) = \varphi(V_I V_J^*).$$

The following three subsets of the circle group $\mathbb{T}$ are equal:

(i) $\{t \in \mathbb{T} \mid \psi \circ \tau_t = \psi\}$*, where τ is the gauge action;*

(ii) $\{t \in \mathbb{T} \mid \psi \circ \tau_t$ *is quasi-equivalent to* $\psi\}$*;*

(iii) $\mathrm{PSp}(\sigma) \cap \mathbb{T}$*, where* $\mathrm{PSp}(\sigma)$ *is the set of eigenvalues of* σ*.*

Furthermore, this set is a finite subgroup of $\mathbb{T}$. If k is the order of this subgroup, the restriction of the representation to UHF_d *decomposes into k mutually disjoint irreducible representations, and these are mapped cyclically into each other by the one-sided shift $\lambda(\,\cdot\,) = \sum_i s_i \cdot s_i^*$.*

Remark 6.4. Since the normal states on $\mathcal{B}(\mathcal{K})$ are given by density matrices, it follows from [9], Lemma 6.3, (as in the proof of Lemma 6.2 above) that if there is a faithful σ-invariant normal state φ, then this is the unique σ-invariant normal state. Note that the state ψ defined in Theorem 6.3 is well defined by Lemma 2.2, and $\psi \circ \lambda = \psi$ since $\varphi \circ \sigma = \varphi$.

During the proof of Theorem 6.3 we will establish that

$$\mathrm{PSp}(\sigma) \cap \mathbb{T} = \mathrm{PSp}(\lambda) \cap \mathbb{T}$$

and that each of the corresponding eigenspaces is spanned by a unitary operator (in $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H})$, respectively), and this unitary operator in $\mathcal{B}(\mathcal{H})$ implements τ_t if $\bar{t}$ is the eigenvalue. In fact, if $U \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$ and $\lambda(U) = \bar{t}U$, then $\lambda(U^*U) =$

$\lambda(U)^*\lambda(U) = U^*U$, hence U^*U, and likewise UU^*, is a scalar multiple of $\mathbb{1}$. Thus, renormalizing U, we may take U to be unitary. But $\lambda(U) = \sum_i S_i U S_i^* = \bar{t}U$, so $S_i U = \bar{t} U S_i$ and hence $U S_i U^* = t S_i = \tau_t(S_i)$. Conversely, if U implements τ_t, then $\lambda(U) = \bar{t}U$, and we have shown

$$\mathrm{PSp}(\lambda) \cap \mathbb{T} = \{t \mid \tau_t \text{ is inner}\}.$$

In particular, this shows that $\mathrm{PSp}(\sigma) \cap \mathbb{T}$ is independent of the particular state φ chosen (with the required properties).

Proof of Theorem 6.3. We first prove the inclusion:

Lemma 6.5. $\{t \in \mathbb{T} \mid \psi \circ \tau_t = \psi\} \subset \mathrm{PSp}(\sigma) \cap \mathbb{T}$.

Proof. Since φ is normal on $\mathcal{B}(\mathcal{K})$, φ is a (possibly infinite) convex combination of vector states, and thus ψ is a convex combination of vector states. Since the given representation of $\mathcal{O}_d$ on $\mathcal{H}$ is irreducible, ψ is a type I factor state. If $\psi \circ \tau_t = \psi$, it follows that there is a unitary $\overline{U}_t \in \mathcal{B}(\mathcal{H})$ such that $\tau_t = \mathrm{Ad}(\overline{U}_t)$. But if $E = \mathrm{supp}\psi$, the invariance implies $\tau_t(E) = E$, and hence $E \in \overline{U}_t'$. Thus $U_t = E\overline{U}_t E = \overline{U}_t E = E\overline{U}_t$ is unitary. But $\overline{U}_t S_i \overline{U}_t^* = tS_i$, so multiplying to the left with E, we get $U_t V_i U_t^* = tV_i$. Multiplying to the right with $U_t V_i^*$, and summing over i, we then obtain $U_t = t\sigma(U_t)$, i.e., U_t is an eigenvector of σ with eigenvalue $\bar{t}$. Thus U_t^* is an eigenvector with eigenvalue t, and the lemma is proved. ∎

We next establish the converse inclusion.

Lemma 6.6. $\mathrm{PSp}(\sigma) \cap \mathbb{T} \subset \{t \in \mathbb{T} \mid \psi \circ \tau_t = \psi\}$.

Proof. If $t \in \mathrm{PSp}(\sigma) \cap \mathbb{T}$, let $U^* = U_t^*$ be a corresponding eigenvector, and assume that $\|U^*\| = 1$. We argue that U is unitary by using the argument employed in the proof of Lemma 3.4: by the generalized Schwarz inequality,

$$\sigma(U^*U) \geqslant \sigma(U^*)\sigma(U) = t\bar{t}U^*U = U^*U,$$

so $\sigma(U^*U) - U^*U \geqslant 0$. But by σ-invariance of φ, $\varphi(\sigma(U^*U) - U^*U) = 0$, and as φ is faithful, $\sigma(U^*U) = U^*U$. Since σ is ergodic and $\|U^*\| = 1$, it follows that $U^*U = \mathbb{1}$. In the same way, one shows that $UU^* = \mathbb{1}$, so $U = U_t$ is unitary. But we have

$$U_t = t\sigma(U_t) = t\sum_i V_i U_t V_i^*.$$

Before continuing the proof of Lemma 6.6, we now prove

Lemma 6.7. *If U is a unitary operator in $\mathcal{B}(\mathcal{K})$ with $\sigma(U) = \bar{t}U$, where $t \in \mathbb{T} \subset \mathbb{C}$, then*

$$UV_iU^* = tV_i \tag{6.1}$$

for $i \in \mathbb{Z}_d$.

Proof. By [15], Theorem 3.1, we have

$$\sigma(XU) = \sigma(X)\sigma(U) = \sigma(X)\bar{t}U$$

for all $X \in \mathcal{B}(\mathcal{K})$. Define $X_i = UV_iU^* - tV_i$. Then

$$\begin{aligned}\sum_i X_iX_i^* &= U\Big(\sum_i V_iV_i^*\Big)U^* - t\sum_i V_iUV_i^*U^* - \bar{t}\sum_i UV_iU^*V_i^* + \sum_i V_iV_i^* \\ &= \mathbb{1} - \sigma(U)tU^* - \bar{t}U\sigma(U^*) + \mathbb{1} = \mathbb{1} - \sigma(UU^*) - \sigma(UU^*) + \mathbb{1} = 0.\end{aligned}$$

It follows that $X_i = UV_iU^* - tV_i = 0$, and Lemma 6.7 is proved. ∎

Continuation of the proof of Lemma 6.6. We may now finalize the proof of Lemma 6.6 by extending the unitary U_t on $\mathcal{K}$, to a unitary $\overline{U}_t$ on $\mathcal{H}$, through the definition

$$\overline{U}_t\Big(\sum_I \alpha_I S_I \xi_I\Big) = \sum_I \alpha_I t^{|I|} S_I U_t \xi_I$$

where I is a finite multi-index with elements from $\mathbb{Z}_d$, $\alpha_I \in \mathbb{C}$ and $\xi_I \in \mathcal{K}$. $\overline{U}_t$ is well defined and unitary by the following computation, where J, I are multi-indices related by $J = IJ'$, where J' is another multi-index. Lemma 6.7 is used in the computation.

$$\begin{aligned}\langle S_IU_t\xi_I | S_JU_t\xi_J\rangle &= \langle U_t\xi_I | S_{J'}U_t\xi_J\rangle = \langle U_t\xi_I | V_{J'}U_t\xi_J\rangle = t^{|J'|}\langle U_t\xi_I | U_tV_{J'}\xi_J\rangle \\ &= t^{|J'|}\langle \xi_I | V_{J'}\xi_J\rangle = \langle t^{|I|}S_I\xi_I | t^{|J|}S_J\xi_J\rangle.\end{aligned}$$

But, from the definition of $\overline{U}_t$, it follows that

$$\overline{U}_tS_i\overline{U}_t^* = tS_i, \tag{6.2}$$

so $\overline{U}_t$ implements τ_t. (In passing from (6.1) to (6.2) with the lifting $U \mapsto \overline{U}_t$, we note that this is a "scaled" version of the commutant lifting in Section 5.) Use now the same symbol τ_t to denote also the normal extension of τ_t to $\mathcal{B}(\mathcal{H})$. By construction of $\overline{U}_t$, we have $\overline{U}_tP = P\overline{U}_t$, so

$$\tau_t(P) = P.$$

We now argue that $\psi\circ\tau_t=\psi$. Put $\psi_t=\psi\circ\tau_t$. Since τ_t is unitarily implemented, ψ_t is normal in the given representation and extends to $\mathcal{B}(\mathcal{H})$. Since $\psi_t(P)=\psi(\tau_t(P))=\psi(P)=\mathbb{1}$, we have $\mathrm{supp}(\psi_t)\leqslant P$, and we may define a state φ_t on $\mathcal{B}(\mathcal{K})$ by $\varphi_t(PXP)=\psi_t(X)$ for $X\in\mathcal{B}(\mathcal{H})$. But

$$\lambda\tau_t(X)=\sum_i S_i\tau_t(X)S_i^*=\tau_t\lambda(X)$$

for $X\in\mathcal{B}(\mathcal{H})$, and, as $\psi\circ\lambda=\psi$, we deduce that

$$\begin{aligned}\varphi_t\circ\sigma(PXP)&=\psi_t\circ\lambda(X)=\psi\tau_t\lambda(X)=\psi\lambda\tau_t(X)\\&=\psi\tau_t(X)=\varphi\tau_t(PXP)=\varphi_t(PXP),\end{aligned}$$

so $\varphi_t\circ\sigma=\varphi_t$. Using the fact that $\mathcal{B}(\mathcal{K})$ has a unique σ-invariant normal state by assumption, we conclude that $\varphi_t=\varphi$, and hence $\psi\circ\tau_t=\psi_t=\psi$. This ends the proof of Lemma 6.6. ∎

We have now established that the sets (i) and (iii) in Theorem 6.3 are equal. Clearly set (i) is contained in set (ii), and to establish the converse, we have to show that, if ψ is τ_t-covariant for some $t\in\mathbb{T}$, then ψ is actually τ_t-invariant. To this end, note that, as

$$\psi\circ\tau_t\circ\lambda=\psi\circ\lambda\circ\tau_t=\psi\circ\tau_t,$$

this will follow once we can show the following lemma:

LEMMA 6.8. *Adopt the assumptions of Theorem* 6.3. *Then ψ is a unique λ-invariant normal state on $\mathcal{B}(\mathcal{H})$.*

Proof. If $X\in\mathcal{B}(\mathcal{H})$, then

$$\operatorname*{w^*\text{-}lim}_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\lambda^k(X)=\psi(X)\mathbb{1}$$

by the following reasoning: putting

$$X_N=\frac{1}{N+1}\sum_{k=0}^{N}\lambda^k(X),$$

we have $\lambda(X_N)-X_N=(\lambda^{N+1}(X)-X)/(N+1)$, and hence

$$\|\lambda(X_N)-X_N\|\leqslant\frac{2\|X\|}{N+1}.$$

It follows that any weak*-limit point of the sequence X_N is λ-invariant. But, as the representation is irreducible, the only λ-invariant elements in $\mathcal{B}(\mathcal{H})$ are the scalar multiples of $\mathbb{1}$ (see, e.g., (3.5) in [12]). Moreover, as $\psi \circ \lambda = \psi$, we have $\psi(X_N) = \psi(X)$, and the claim follows. Finally, if ω is a λ-invariant normal state and $X \in \mathcal{B}(\mathcal{H})$, it follows that $\omega(X) = \omega(X_N)$, and therefore

$$\omega(X) = \lim_{N\to\infty} \omega(X_N) = \omega(\psi(X)\mathbb{1}) = \psi(X). \quad ∎$$

Note that Lemma 6.8 could be used to simplify the last part of the proof of Lemma 6.6.

Next, we establish the finiteness of the three sets in Theorem 6.3:

Lemma 6.9. *$\{t \in \mathbb{T} \mid \psi \circ \tau_t = \psi\}$ is a finite subgroup of $\mathbb{T}$.*

Proof. The set is clearly a closed subgroup of $\mathbb{T}$, so if it is not finite it is equal to $\mathbb{T}$. But in that case the automorphism group $t \mapsto \tau_t$ extends to the weak closure $\pi_\psi(\mathcal{O}_d)''$ of $\pi_\psi(\mathcal{O}_d)$ in the GNS representation defined by ψ. Since the original representation of $\mathcal{O}_d$ on $\mathcal{H}$ is irreducible, and ψ is a normal state in this representation, $\pi_\psi(\mathcal{O}_d)''$ is a type I factor and π_ψ extends canonically to a $*$-isomorphism from $\mathcal{B}(\mathcal{H}) = \mathcal{O}_d''$ to this factor. Transporting τ_t back by this isomorphism, it follows that there exists a unitary representation $t \mapsto U_t$ of $\mathbb{T}$ on $\mathcal{H}$ such that $\tau_t(x) = U_t x U_t^*$ for $x \in \mathcal{O}_d$. For this covariant representation, let

$$U_t = \sum_{n\in\mathbb{Z}} t^n E_n$$

be the Stone-Naimark-Ambrose-Godement (SNAG) decomposition ([35]) of U. As

$$U_t S_i = \tau_t(S_i) U_t = t S_i U_t,$$

we obtain $E_n S_i = S_i E_{n-1}$, and thus $\lambda(E_n) = E_{n+1}$. But

$$\psi(\,\cdot\,) = \psi \circ \int_{\mathbb{T}} \operatorname{Ad} U_t(\,\cdot\,)\,\mathrm{d}t = \sum_n \psi(E_n \,\cdot\, E_n),$$

where we identify ψ with the vector state it defines on the bounded operators on the representation Hilbert space. Therefore, if $\psi_n(X) = \psi(E_n X E_n)$, then

$$\psi_n(\lambda(X)) = \psi(E_n \lambda(X) E_n) = \psi(\lambda(E_{n-1} X E_{n-1})) = \psi_{n-1}(X),$$

since $\psi \circ \lambda = \psi$. (Here we use implicitly the facts that both ψ and λ extend by weak*-continuity to $\mathcal{B}(\mathcal{H}) = \mathcal{O}_d''$, and that the invariance $\psi \circ \lambda = \psi$ is preserved in the extension. If the original representation were not irreducible, this point would be problematic.) But then

$$\psi(\mathbb{1}) = \sum_n \psi_n(\mathbb{1}) = \sum_n \psi_0(\mathbb{1}) = \infty,$$

since $\lambda(\mathbb{1}) = \mathbb{1}$. This is impossible, so Lemma 6.9 is established. ∎

It remains to prove the last statements of Theorem 6.3. To this end, define $k \in \mathbb{N}$ such that the finite group in Lemma 6.9 is $\{\exp(2\pi i l/k) \mid l = 0, 1, \dots, k-1\}$.

Lemma 6.10. *With k defined as above and ψ as in Theorem* 6.3*, we have*

$$\pi_\psi(\mathcal{O}_d^{\tau_{1/k}})'' = \pi_\psi(\mathrm{UHF}_d)''$$

and

$$\pi_\psi(\mathcal{O}_d)'' \cap \pi_\psi(\mathcal{O}_d^{\tau_{1/k}})' \cong \mathbb{C}^k,$$

where $\mathcal{O}_d^{\tau_{1/k}}$ denotes the fixed point algebra in $\mathcal{O}_d$ under the gauge automorphism $\tau_{\exp(2\pi i/k)}$.

Proof. Since π_ψ is merely a multiple of the given irreducible representation on $\mathcal{H}$ (by normality of ψ), we only need to show

$$(\mathcal{O}_d^{\tau_{1/k}})'' = (\mathrm{UHF}_d)''$$

and

$$(\mathcal{O}_d^{\tau_{1/k}})' \cong \mathbb{C}^k.$$

But, if U is the unitary on $\mathcal{H}$ implementing $\tau_{\frac{1}{k}}$, we have shown that U is an eigenunitary of λ with eigenvalue $e^{-\frac{2\pi i}{k}}$. As $\lambda(U^k) = e^{-\frac{2\pi i k}{k}} U^k = U^k$, we have $U^k \in \mathbb{C}\mathbb{1}$; and we may assume $U^k = \mathbb{1}$ by changing U by a phase factor. Thus U will have a spectral decomposition

$$U = \sum_{l \in \mathbb{Z}_k} e^{\frac{i2\pi l}{k}} E_l$$

where E_l, $l \in \mathbb{Z}_k$, are mutually orthogonal projections summing up to $\mathbb{1}$. Moreover, as

$$\lambda(U) = \sum_{l \in \mathbb{Z}_k} e^{\frac{i2\pi l}{k}} \lambda(E_l) = e^{-\frac{i2\pi}{k}} U = \sum_{l \in \mathbb{Z}_k} e^{\frac{i2\pi(l-1)}{k}} E_l,$$

we see that

$$\lambda(E_l) = E_{l+1}$$

for $l \in \mathbb{Z}_k$. It follows that all the projections E_l are nonzero. Thus

$$(\mathcal{O}_d^{\tau_{1/k}})'' = (\mathcal{O}_d)'' \cap \{U, U^*\}' = \{E_l \mid l \in \mathbb{Z}_k\}' = \bigoplus_{l \in \mathbb{Z}_k} E_l \mathcal{B}(\mathcal{H}) E_l,$$

so

$$(\mathcal{O}^{\tau_{1/k}})' = \sum_{l \in \mathbb{Z}_k} \mathbb{C}\, E_l \cong \mathbb{C}^k$$

as asserted.

To prove that $(\mathrm{UHF}_d)'' = (\mathcal{O}^{\tau_{1/k}})''$ we first note that if τ_t is restricted to $t \in [0, \frac{1}{k}\rangle$, then τ defines a representation of $\mathbb{T}$ in $\mathrm{Aut}(\mathcal{O}^{\tau_{1/k}})$. Now consider the direct integral representation

$$\pi = \int_{[0,\frac{1}{k}\rangle}^{\oplus} \mathrm{d}t \, \pi_{\psi|_{\mathcal{O}_d^{\tau_{1/k}}}} \circ \tau_t$$

of $\mathcal{O}_d^{\tau_{1/k}}$ on $\mathcal{H}_{\psi|\mathcal{O}_d^{\tau_{1/k}}} \otimes L^2([0, \frac{1}{k}\rangle)$. (See [6].)

We establish the following observation concerning this representation before finalizing the proof of Lemma 6.10:

Lemma 6.11. $\mathbb{1} \otimes L^\infty([0, \frac{1}{k}\rangle) \subset \pi(\mathcal{O}_d^{\tau_{1/k}})''$.

Proof. Note that

$$\lambda(\mathcal{O}_d^{\tau_{1/k}}) \subset \mathcal{O}_d^{\tau_{1/k}},$$

since $\lambda\tau_t = \tau_t\lambda$. If $t_1, t_2 \in [0, \frac{1}{k}\rangle$ and $t_1 \neq t_2$, it follows from the already proved part of Theorem 6.3 that there exists an $x \in \mathcal{O}_d$ with $\psi(\tau_{t_1}(x)) \neq \psi(\tau_{t_2}(x))$. Replacing x with its mean over $\mathbb{Z}_k$, $\frac{1}{k} \sum_{l \in \mathbb{Z}_k} \tau_{\frac{l}{k}}(x)$, we may assume that $x \in \mathcal{O}_d^{\tau_{1/k}}$. Since

$$\operatorname*{w-lim}_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \lambda^n(\tau_t(x)) = \psi(\tau_t(x))\mathbb{1},$$

by the reasoning in the proof of Lemma 6.8, it follows that

$$\operatorname*{w-lim}_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \pi(\lambda^n(x)) = \mathbb{1} \otimes f,$$

where $f(t) = \psi(\tau_t(x))$. Lemma 6.11 follows, as these f's separate points. ∎

Continuation of the proof of Lemma 6.10. It follows from Lemma 6.11 that

$$\pi(\mathcal{O}_d^{\tau_{1/k}})'' = \pi_\psi(\mathcal{O}_d^{\tau_{1/k}})'' \otimes L^\infty([0, \tfrac{1}{k}\rangle).$$

But π is clearly τ-covariant, $\mathbb{T}$ acting by translation, and therefore

$$\pi(\mathrm{UHF}_d)'' = \pi(\mathcal{O}_d^{\tau_{1/k}})''^{\tau} = \pi_\psi(\mathcal{O}_d^{\tau_{1/k}})'' \otimes \mathbb{1}.$$

This equality then also holds on fibers, so

$$\pi_\psi(\mathrm{UHF}_d)'' = \pi_\psi(\mathcal{O}_d^{\tau_{1/k}})'',$$

and this ends the proof of Lemma 6.10. ∎

End of proof of Theorem 6.3. We finally observe from the proof that this means that UHF_d acts irreducibly on each of the subspaces $E_l\mathcal{H}$, that these representations are mutually disjoint, and since $\lambda(E_l) = E_{l+1}$, the endomorphism λ maps these representations of UHF_d cyclically one into another, i.e.,

$$\pi_0 \to \pi_1 \to \cdots \to \pi_{k-1} \to \pi_0,$$

where π_l is the "cut down" of π_ψ by E_l, $\pi_l(\,\cdot\,) = \pi_\psi(\,\cdot\,)E_l$, $l \in \mathbb{Z}_k$. ∎

7. Translationally invariant states on the two-sided quantum chain

Let us recall the definition of finitely correlated pure states from [24], [25]. These are translationally invariant states defined on the one-dimensional quantum chain $\bigotimes_{\mathbb{Z}} M_d$ as follows: let $\mathcal{K}$ be a finite-dimensional Hilbert space and let $V : \mathcal{K} \to \mathcal{K} \otimes \mathbb{C}^d$ be an isometry. Define

$$\mathbb{E} : \mathcal{B}(\mathcal{K}) \otimes M_d \to \mathcal{B}(\mathcal{K})$$

by $\mathbb{E}(X) = V^*XV$. Let φ be a state on $\mathcal{B}(\mathcal{K})$ such that $\varphi(\mathbb{E}(B \otimes \mathbb{1})) = \varphi(B)$ for all $B \in \mathcal{B}(\mathcal{K})$. Define

$$\mathbb{E}_A : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$$

by $B \mapsto \mathbb{E}(B \otimes A)$ for $A \in M_d$. Then

$$\omega(A_1 \otimes A_2 \otimes \cdots \otimes A_m) = \varphi(\mathbb{E}_{A_1} \circ \mathbb{E}_{A_2} \circ \cdots \circ \mathbb{E}_{A_m}(\mathbb{1}_{\mathcal{K}}))$$

defines a translation-invariant state on $\bigotimes_{\mathbb{Z}} M_d$. It is proved in [25], Theorem 1.5, that this state is pure if the completely positive map $\sigma = \mathbb{E}_{\mathbb{1}}$ has trivial peripheral spectrum, i.e., the only eigenvectors of $\mathbb{E}_{\mathbb{1}}$ with eigenvalue of modulus one are the scalar multiples of $\mathbb{1}$. Conversely, if ω is pure, there does exist a realization of ω as above such that σ has trivial peripheral spectrum (but it might not be the given one; see the remarks at the end of Section 1). Now $V : \mathcal{K} \to \mathcal{K} \otimes \mathbb{C}^d = \bigoplus_1^d \mathcal{K}$ has the matrix form $V = \begin{bmatrix} V_1^* \\ \vdots \\ V_d^* \end{bmatrix}$ and the property that V is an isometry translates into

$$V^*V = \sum_{k \in \mathbb{Z}_d} V_k V_k^* = \mathbb{1}.$$

We check that

$$\sigma(X) = \mathbb{E}(X \otimes \mathbb{1}) = \sum_{k \in \mathbb{Z}_d} V_k X V_k^*.$$

Using Theorem 5.1 we can thus associate a representation π_V of $\mathcal{O}_d$ to V, and since φ is normal by finite-dimensionality of $\mathcal{K}$, we can associate a state ψ on $\mathcal{O}_d$ to (V, φ) which is normal in the given representation. We next verify that the restriction of ψ to the one-sided tensor product $\mathrm{UHF}_d = \bigotimes_{\mathbb{N}} M_d$ is equal to the restriction of ω to $\bigotimes_{\mathbb{N}} M_d$:

$$\begin{aligned}
\omega(e_{i_1 j_1} \otimes \cdots \otimes e_{i_m j_m}) &= \varphi(\mathbb{E}_{e_{i_1 j_1}} \circ \mathbb{E}_{e_{i_2 j_2}} \circ \cdots \circ \mathbb{E}_{e_{i_m j_m}}(\mathbb{1}_{\mathcal{K}})) \\
&= \varphi(\mathbb{E}_{e_{i_1 j_1}} \circ \mathbb{E}_{e_{i_2 j_2}} \circ \cdots \circ \mathbb{E}_{e_{i_{m-1} j_{m-1}}}(V_{i_m} V_{j_m}^*)) \\
&= \varphi(V_{i_1} \cdots V_{i_m} V_{j_m}^* \cdots V_{j_1}^*) \\
&= \psi(s_{i_1} \cdots s_{i_m} s_{j_m}^* \cdots s_{j_1}^*) = \psi(e_{i_1 j_1}^{(1)} \otimes \cdots \otimes e_{i_m j_m}^{(m)}).
\end{aligned}$$

Note that finite-dimensionality of $\mathcal{K}$ and normality of φ do not play any role in the computation above.

The main theorem in this section is the following.

THEOREM 7.1. *Let $\mathcal{M}$ be a factor, φ a faithful normal state on $\mathcal{M}$, $V_1, \dots, V_d$ operators in $\mathcal{M}$ satisfying*

$$\sum_{k \in \mathbb{Z}_d} V_k V_k^* = \mathbb{1},$$

and σ the completely positive unital normal map of $\mathcal{B}(\mathcal{K})$ defined by

$$\sigma(X) = \sum_{k \in \mathbb{Z}_d} V_k X V_k^*$$

for $X \in \mathcal{B}(\mathcal{K})$, and assume that

$$\mathcal{B}(\mathcal{K})^\sigma = \mathcal{M}'.$$

If $\mathcal{M}$ is type I, *the following two conditions are equivalent:*

(i) *the translationally invariant state ω defined by $\{\varphi, V_1, \dots, V_d\}$ on $\bigotimes_{\mathbb{Z}} M_d$ is pure;*

(ii) $\mathrm{PSp}(\sigma|\mathcal{M}) \cap \mathbb{T} = \{1\}$.

If $\mathcal{M}$ is not assumed to be type I, *the condition* (i) *is nevertheless equivalent to each of the following two conditions:*

(iii) *ω is a factor state, i.e.,*

$$\lim_{|n| \to \infty} \omega(x \lambda^n(y)) = \omega(x)\omega(y)$$

for all $x, y \in \bigotimes_{\mathbb{Z}} M_d$*, where* λ *is the shift;*

(iv) *the Connes spectrum of* $\tau|H$ *on* $\pi_\psi(\mathcal{O}_d)''$ *is* $\widehat{H}$*, where* H *is the subgroup of* $t \in \mathbb{T}$ *such that* τ_t *extends from* $\mathcal{O}_d$ *to the weak closure* $\pi_\psi(\mathcal{O}_d)''$ *and* ψ *is defined in Theorem* 6.3.

Remark 7.2. The condition $\mathcal{B}(\mathcal{K})^\sigma = \mathcal{M}'$ implies in particular that the von Neumann algebra generated by $V_1, \dots, V_d$ is $\mathcal{M}$. See Section 3 for a further discussion.

Note that the condition $\mathcal{B}(\mathcal{K})^\sigma = \mathcal{M}'$ does not depend on the particular normal representation of $\mathcal{M}$ (when σ is defined by the representatives for V_i). The reason for this is that any normal representation of $\mathcal{M}$ is a product of a spatial isomorphism, an induction $\mathcal{M} \ni X \mapsto XP$ where P is a projection in $\mathcal{M}'$, and an amplification $\mathcal{M} \ni X \mapsto X \otimes \mathbb{1}$, and applying these three types of maps on the V_i's, one verifies that the condition remains the same; see [18], Théorème I.4.3 for details on normal representations. When developing a duality theory later, we will use the representation where φ is defined by a separating and cyclic vector.

The rest of this section will be devoted to a proof of Theorem 7.1. To this end we have to develop a certain duality theory for the objects $(\mathcal{M}, \varphi, V_1, \dots, V_d)$. But before that we will mention some more pedestrian results on translationally invariant states.

Recall from [13], Example 4.3.24, that any translationally invariant factor state of $\bigotimes_{\mathbb{Z}} M_d$ is extremal among the invariant states, i.e., is ergodic. Conversely, an ergodic state need not be a factor state: if, for example, ω_1, ω_2 are distinct pure states on M_d, the mean of the pure product state $\cdots \otimes \omega_1 \otimes \omega_2 \otimes \omega_1 \otimes \omega_2 \otimes \cdots$ on $\bigotimes_{\mathbb{Z}} M_d$ and its shift is extremally invariant, but not a factor state (see [13], Example 4.3.26).

The difference between factor states and ergodic states is reflected in the fact that if ω is a translationally in variant state on $\bigotimes_{\mathbb{Z}} M_d$, then ω is a factor state if and only if it is strongly clustering,

$$\lim_{|n| \to \infty} \omega(x\lambda^n(y)) = \omega(x)\omega(y)$$

(see [40]), while ω is ergodic if and only if it is clustering in the mean

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \omega(x\lambda^n(y)) = \omega(x)\omega(y)$$

(see [13], Example 4.3.5 and Theorem 4.3.17). However, the following is true:

PROPOSITION 7.3. *There is a canonical one-one correspondence between the following three sets:*

(i) *the set of extremal translationally invariant states on* $\bigotimes_{\mathbb{Z}} M_d$*;*

(ii) *the set of states on* $\bigotimes_{\mathbb{N}} M_d$ *which are extremal among the states invariant under the one-sided shift* λ*;*

(iii) *the set of orbits under the gauge action* τ *in the sets of states* ψ *on* $\mathcal{O}_d$ *such that*

$$\psi \circ \lambda = \psi$$

and ψ *is a factor state with an ergodic restriction to* UHF_d.

The maps giving the correspondence are defined by the restriction maps from Set (i) *and Set* (iii) *to Set* (ii)*, using the inclusions*

$$\bigotimes_{\mathbb{N}} M_d \subset \bigotimes_{\mathbb{Z}} M_d$$

and

$$\bigotimes_{\mathbb{N}} M_d = \mathrm{UHF}_d \subset \mathcal{O}_d.$$

Proof. If we also use λ to denote the two-sided shift on $\bigotimes_{\mathbb{Z}} M_d$, the new λ extends the old, and

$$\bigotimes_{\mathbb{Z}} M_d = \overline{\bigcup_{n=1}^{\infty} \lambda^{-n}\Big(\bigotimes_{\mathbb{N}} M_d\Big)}^{\|\cdot\|},$$

so the one-one correspondence between Set (i) and Set (ii) is trivial.

It is clear that the map from Set (iii) to Set (ii) is well defined. To prove that it is injective, let ω' be an extremal invariant state on $\bigotimes_{\mathbb{N}} M_d$, and consider the set

$$K = \{\psi \mid \psi \text{ is a state of } \mathcal{O}_d \text{ such that } \psi \circ \lambda = \psi \text{ and } \psi|\mathrm{UHF}_d = \omega'\}.$$

By applying an invariant mean on an extension of ω' to $\mathcal{O}_d$ it is clear that K is nonempty, and K is clearly convex and compact, and a face in the set of λ-invariant states since ω' is extremal. We finish the proof of Proposition 7.3 by proving:

LEMMA 7.4. $\psi \in K$ *is an extremal point in* K *if and only if* ψ *is a factor state, and then all other extremal points have the form* $\psi \circ \tau_t$ *for some* $t \in \mathbb{T}$.

Proof. If ψ is not factorial, there is a nontrivial projection $E \in \pi_\psi(\mathcal{O}_d)'' \cap \pi_\psi(\mathcal{O}_d)'$. But $\psi_E(x) = \langle E\Omega_\psi | \pi_\psi(x) E\Omega_\psi\rangle$. Then $\psi_E \leqslant \psi$ and as $\lambda(E) = E$ we have $\psi_E \circ \lambda = \psi_E$. But $\psi_E|\mathrm{UHF}_d \leqslant \omega'$ and it follows from extremality of ω' that there is a scalar c such that $\psi_E|\mathrm{UHF}_d = c\,\omega'$. But as $\psi = \psi_E + \psi_{\mathbb{1}-E}$, and ψ_E and $\psi_{\mathbb{1}-E}$ are disjoint, this contradicts the extremality of ψ. Thus the extremal points in K are factor states.

Conversely, if $\psi \in K$ is a factor state, it follows as in the proof of Lemma 6.8 that

$$\operatorname*{w\text{-}lim}_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \pi_\psi(\lambda^k(x)) = \psi(x)\mathbb{1},$$

and hence ψ is ergodic by [13], Theorems 4.3.17 and 4.3.23. (Strictly speaking, these theorems are proved under the assumption that λ is an automorphism, but extending λ to an automorphism of the inductive limit

$$\mathcal{O}_d \xrightarrow{\lambda} \mathcal{O}_d \xrightarrow{\lambda} \mathcal{O}_d \xrightarrow{\lambda} \cdots$$

and extending ψ by requiring λ-invariance, one still has the clustering

$$\lim_{N\to\infty} \psi\Big(y\Big(\frac{1}{N+1}\sum_{k=0}^{N}\lambda^k(x)\Big)z\Big) = \psi(yz)\psi(x),$$

so the extended ψ is ergodic, and thus the original ψ is so, since there is a one-one correspondence between the λ-invariant states on $\mathcal{O}_d$ and those on the inductive limit.)

Finally, let ψ be a given extremal point in the face K in the invariant states. It follows from [13], Theorem 4.3.19, and the previous paragraph that any two translates $\psi\circ\tau_{t_1}$, $\psi\circ\tau_{t_2}$ of ψ are either equal or disjoint. Put

$$G = \{t \in \mathbb{T} \mid \psi\circ\tau_t = \psi\}$$

and define

$$\psi_0 = \omega' \circ \int_{\mathbb{T}} \tau_t \,\mathrm{d}t = \int_{\mathbb{T}} (\psi\circ\tau_t)\,\mathrm{d}t.$$

Then

$$\pi_{\psi_0} = \int_{\mathbb{T}/G}^{\oplus} (\pi_\psi \circ \tau_{t'})\,\mathrm{d}t'$$

is the central decomposition of π_{ψ_0} by Lemma 6.11 and its proof. If now ψ' is an extremal point in K, i.e., ψ' is a factorial λ-invariant state with $\psi'|\mathrm{UHF}_d = \omega'$, then

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} (\psi'\circ\tau_t)\,\mathrm{d}t \leqslant \frac{1}{\varepsilon}\int_{\mathbb{T}} (\psi'\circ\tau_t)\,\mathrm{d}t = \frac{1}{\varepsilon}\psi_0$$

and hence, since the left-hand side of the above inequality is λ-invariant, by Segal's Radon-Nikodym theorem ([13], Theorem 2.3.19), there is a function $g_\varepsilon \in L^\infty(\mathbb{T}/G)$ such that

$$\frac{1}{\varepsilon}\int_0^\varepsilon (\psi' \circ \tau_t)\,\mathrm{d}t = \int_{\mathbb{T}/G} (\psi \circ \tau_t) g_\varepsilon(t)\,\mathrm{d}t.$$

Letting $\varepsilon \to 0$, we find a measure μ on $\mathbb{T}/G$ such that

$$\psi' = \int_{\mathbb{T}/G} (\psi \circ \tau_t)\,\mathrm{d}\mu(t).$$

But as ψ' is extremal in K, this must be a Dirac measure, and $\psi' = \psi \circ \tau_t$ for some t.

This ends the proof of Lemma 7.4 and Proposition 7.3. ∎

In order to prove Theorem 7.1, we need to develop a duality theory for the objects $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ somewhat different from the duality theory in [30]. The starting point is more restrictive in that the normal state φ is assumed to be faithful. We assume that $V_1, \dots, V_d \in \mathcal{M}$ and

$$\sum_{j=1}^d V_j V_j^* = \mathbb{1},$$

and assume invariance $\varphi \circ \sigma = \varphi$, where σ is the unital completely positive map on $\mathcal{M}$ defined by

$$\sigma(X) = \sum_{j=1}^d V_j X V_j^*.$$

We will construct a dual object $(\widetilde{\mathcal{M}}, \widetilde{\varphi}, \widetilde{V}_1, \dots, \widetilde{V}_d, \widetilde{\sigma})$ satisfying the same axioms. To this end we assume $\mathcal{M}$ is acting on a Hilbert space $\mathcal{K}$ with a cyclic vector Φ such that $\varphi(X) = \langle \Phi | X\Phi \rangle$. Note that Φ is then separating for $\mathcal{M}$ by faithfulness of φ. In the application to Theorem 7.1, the system $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ will roughly correspond to a state on a Cuntz algebra $\mathcal{O}_d$ with associated UHF algebra $\bigotimes_1^\infty M_d$, and the dual object to a state on an isomorphic Cuntz algebra $\widetilde{\mathcal{O}}_d$ with associated UHF algebra $\bigotimes_{-\infty}^0 M_d$, and $\bigotimes_1^\infty M_d$ and $\bigotimes_{-\infty}^0 M_d$ will be embedded into $\bigotimes_{-\infty}^\infty M_d$ in the obvious manner. This statement will be made more precise in Lemma 7.15.

Then to the definitions: we let $\widetilde{\mathcal{M}} = \mathcal{M}'$. The cyclic and separating vector Φ for $\mathcal{M}$ defines the associated Tomita modular conjugation J, and modular operator

Δ; see, e.g., [13], Theorem 2.5.14. Let σ_t be the modular automorphism group of $\mathcal{M}$:

$$\sigma_t(X) = \Delta^{\mathrm{i}t} X \Delta^{-\mathrm{i}t}$$

for $X \in \mathcal{M}$. We put

$$\widetilde{V}_j = (J\sigma_{\frac{\mathrm{i}}{2}}(V_j^*)J)^{\overline{}} = (J\Delta^{-\frac{1}{2}}V_j^*\Delta^{\frac{1}{2}}J)^{\overline{}}$$

where $\overline{}$ denotes closure of the respective operators. To show that this is a well-defined operator in $\mathcal{M}'$, define a positive sesquilinear form Q_j on $J\mathfrak{A}\Phi$, where $\mathfrak{A}$ is the $*$-algebra of σ_t-entire elements in $\mathcal{M}$, by

$$\begin{aligned} Q_j(JX\Phi, JY\Phi) &= \langle \widetilde{V}_j^* JX\Phi | \widetilde{V}_j^* JY\Phi\rangle = \langle J\sigma_{-\frac{\mathrm{i}}{2}}(V_j)X\Phi | J\sigma_{-\frac{i}{2}}(V_j)Y\Phi\rangle \\ &= \langle J\Delta^{\frac{1}{2}}V_j\sigma_{\frac{\mathrm{i}}{2}}(X)\Phi | J\Delta^{\frac{1}{2}}V_j\sigma_{\frac{\mathrm{i}}{2}}(Y)\Phi\rangle \\ &= \langle \sigma_{-\frac{\mathrm{i}}{2}}(X^*)V_j^*\Phi | \sigma_{-\frac{\mathrm{i}}{2}}(Y^*)V_j^*\Phi\rangle. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j\in\mathbb{Z}_d} Q_j(JX\Phi, JY\Phi) &= \varphi(\sigma(\sigma_{\frac{\mathrm{i}}{2}}(X)\sigma_{-\frac{\mathrm{i}}{2}}(Y^*))) = \varphi(\sigma_{\frac{\mathrm{i}}{2}}(X)\sigma_{-\frac{\mathrm{i}}{2}}(Y^*)) \\ &= \langle \sigma_{-\frac{\mathrm{i}}{2}}(X^*)\Phi | \sigma_{-\frac{\mathrm{i}}{2}}(Y^*)\Phi\rangle = \langle \Delta^{\frac{1}{2}}X^*\Phi | \Delta^{\frac{1}{2}}Y^*\Phi\rangle \\ &= \langle J\Delta^{\frac{1}{2}}Y^*\Phi | J\Delta^{\frac{1}{2}}X^*\Phi\rangle = \langle Y\Phi | X\Phi\rangle = \langle JX\Phi | JY\Phi\rangle. \end{aligned}$$

It follows both that

$$\|\widetilde{V}_j^* JX\Phi\| \leqslant \|JX\Phi\|,$$

i.e., $\widetilde{V}_j^*$ is bounded, and that

$$\sum_{j=1}^{d} \widetilde{V}_j\widetilde{V}_j^* = \mathbb{1}.$$

We now naturally define a completely positive map $\widetilde{\sigma}$ on $\widetilde{\mathcal{M}} = \mathcal{M}'$ by

$$\widetilde{\sigma}(X) = \sum_{j\in\mathbb{Z}_d} \widetilde{V}_j X \widetilde{V}_j^*$$

for $X \in \widetilde{\mathcal{M}}$, and a faithful normal state $\widetilde{\varphi}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\varphi}(X) = \langle \Phi | X\Phi\rangle$ for $X \in \mathcal{M}'$. We introduce the terminology $(\mathcal{M}', \widetilde{\varphi}, \widetilde{V}_1, \dots, \widetilde{V}_d, \widetilde{\sigma})$ for the *dual system* of $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$. Note that $\widetilde{\varphi} \circ \widetilde{\sigma} = \widetilde{\varphi}$, since, for $X \in \mathcal{M}'$,

$$\begin{aligned} \widetilde{\varphi}\widetilde{\sigma}(X) &= \sum_j \langle \Phi | J\sigma_{\frac{\mathrm{i}}{2}}(V_j^*)JXJ\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\Phi\rangle = \sum_j \langle V_j^*\Phi | XV_j^*\Phi\rangle \\ &= \Big\langle \Phi \,\Big|\, X\Big(\sum_j V_jV_j^*\Big)\Phi\Big\rangle = \widetilde{\varphi}(X). \end{aligned}$$

The term "dual system" is justified by the fact that the dual system of $(\mathcal{M}', \widetilde{\varphi}, \widetilde{V}_1, \dots, \widetilde{V}_d, \widetilde{\sigma})$ is $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ again. For this, we just need to check $\widetilde{\widetilde{V}}_j = V_j$. But this follows from the computation

$$\widetilde{\widetilde{V}}_j = J\Delta^{\frac{1}{2}}\widetilde{V}_j^*\Delta^{-\frac{1}{2}}J = J\Delta^{\frac{1}{2}}(J\Delta^{\frac{1}{2}}V_j\Delta^{-\frac{1}{2}}J)\Delta^{-\frac{1}{2}}J = V_j$$

where we used that J and Δ^{-1} are the modular conjugation and modular operator associated to the pair $(\mathcal{M}', \Phi)$, $J\Delta = \Delta^{-1}J$ and $J^2 = \mathbb{1}$.

This duality has several nice properties. For example σ is ergodic if and only if $\widetilde{\sigma}$ is, and $\mathrm{PSp}(\sigma) \cap \mathbb{T} = \mathrm{PSp}(\widetilde{\sigma}) \cap \mathbb{T}$. These properties will be discussed in Section 8. For the moment we return to the proof of Theorem 7.1. So let $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ be as in the hypothesis of the theorem, put $\mathcal{K} = \mathcal{H}_\varphi$ and identify V_i with its representative $\pi_\omega(V_i)$ on $\mathcal{K}$. If $(\widetilde{\mathcal{M}}, \widetilde{\varphi}, \widetilde{V}_1, \dots, \widetilde{V}_d, \widetilde{\sigma})$ is the dual system, we have the canonical identification $\mathcal{H}_{\widetilde{\varphi}} = \mathcal{K}$, and $\widetilde{\varphi}$ is the vector state on $\widetilde{\mathcal{M}}$ defined by the same vector Φ as φ. By Theorem 5.1 there are Hilbert spaces $\mathcal{H}_0$, $\widetilde{\mathcal{H}}_0$ containing $\mathcal{K}$, with projectors $P_0 : \mathcal{H}_0 \to \mathcal{K}$, $\widetilde{P}_0 : \widetilde{\mathcal{H}}_0 \to \mathcal{K}$ and representations S_i, $\widetilde{S}_i$ of the Cuntz relations on $\mathcal{H}_0$, $\widetilde{\mathcal{H}}_0$, respectively such that $\mathcal{K}$ is cyclic for both representations and

$$\begin{aligned} P_0 S_I S_J^* P_0 &= V_I V_J^*, \\ \widetilde{P}_0 \widetilde{S}_I \widetilde{S}_J^* \widetilde{P}_0 &= \widetilde{V}_I \widetilde{V}_J^*. \end{aligned}$$

We will now form a sort of amalgamated tensor product of $\mathcal{H}_0$ and $\widetilde{\mathcal{H}}_0$ over the joint subspace $\mathcal{K}$ and thus obtain a Hilbert space $\mathcal{H}$ carrying two commuting representations of $\mathcal{O}_d$. To this end we generalize the construction in Remark 5.2. $\mathcal{H}$ is the completion of the quotient of

$$H = \mathbb{C}\mathcal{I} \otimes \mathbb{C}\widetilde{\mathcal{I}} \otimes \mathcal{K},$$

where $\mathcal{I}, \widetilde{\mathcal{I}}$ both consist of all finite sequences in $\mathbb{Z}_d$, by the equivalence relation defined by a semi-inner product defined on H by requiring

$$\begin{aligned} \langle I \otimes \widetilde{I} \otimes \xi | IJ \otimes \widetilde{I}\widetilde{J} \otimes \eta \rangle &= \langle \xi | V_J \widetilde{V}_{\widetilde{J}} \eta \rangle, \\ \langle I \otimes \widetilde{I}\widetilde{J} \otimes \xi | IJ \otimes \widetilde{I} \otimes \eta \rangle &= \langle \widetilde{V}_{\widetilde{J}} \xi | V_J \eta \rangle, \end{aligned}$$

etc., all inner products that cannot be put in these forms being zero. Since the V_J's and $\widetilde{V}_{\widetilde{J}}$'s commute along with all combinations of their adjoints, we see that this gives rise to two commuting representations of $\mathcal{O}_d$ on $\mathcal{H}_0$ as follows:

$$S_I \Lambda(J \otimes \widetilde{J} \otimes \xi) = \Lambda(IJ \otimes \widetilde{J} \otimes \xi), \tag{7.1}$$

$$\widetilde{S}_{\widetilde{I}} \Lambda(J \otimes \widetilde{J} \otimes \xi) = \Lambda(J \otimes \widetilde{I}\widetilde{J} \otimes \xi), \tag{7.2}$$

where $\Lambda : H \to \mathcal{H}$ is the quotient map. This is a slight abuse of notation as the earlier $S_I, \widetilde{S}_{\widetilde{I}}$ identify with the restriction of the present S_I, $\widetilde{S}_{\widetilde{I}}$ to the subspaces $\mathcal{H}_0, \widetilde{\mathcal{H}}_0$ of $\mathcal{H}$ spanned by vectors $\Lambda(I \otimes \{\emptyset\} \otimes \xi)$ and $\Lambda(\{\emptyset\} \otimes \widetilde{I} \otimes \xi)$, respectively.

All the previous statements are easy to check. For example the positivity of the sesquilinear form on $H \times H$ is checked by induction as follows, where the operators S_i and $\widetilde{S}_i$ on H are defined in the obvious manner: if

$$\zeta = \sum_{ij} S_i \widetilde{S}_j \zeta_{ij} + \sum_i S_i \zeta_{i\,0} + \sum_j \widetilde{S}_j \zeta_{0\,j} + \zeta_0$$

is a general element in $H_n = \mathbb{C}\,\mathcal{I}_n \otimes \mathbb{C}\,\widetilde{\mathcal{I}}_n \otimes \mathcal{K}$ where $\zeta_{ij} \in H_{n-1}$, $\zeta_{i\,0} \in \mathbb{C}\,\mathcal{I}_{n-1} \otimes \{\emptyset\} \otimes \mathcal{K}$, $\zeta_{0\,j} \in \{\emptyset\} \otimes \mathbb{C}\,\widetilde{\mathcal{I}}_{n-1} \otimes \mathcal{K}$, $\zeta_0 \in \mathcal{K}$, and we assume the form is positive on $H_{n-1} \times H_{n-1}$, we compute

$$\begin{aligned}
\langle \zeta | \zeta \rangle &= \sum_{ij} \|\zeta_{ij}\|^2 + \sum_i \|\zeta_{i\,0}\|^2 + \sum_j \|\zeta_{0\,j}\|^2 + \|\zeta_0\|^2 \\
&\quad + \sum_{ij} \{ \langle \widetilde{V}_j \zeta_{ij} | \zeta_{i\,0} \rangle + \langle \zeta_{i\,0} | \widetilde{V}_j \zeta_{ij} \rangle \} + \sum_{ij} \{ \langle V_i \zeta_{ij} | \zeta_{0\,j} \rangle + \langle \zeta_{0\,j} | V_i \zeta_{ij} \rangle \} \\
&\quad + \sum_{ij} \{ \langle V_i \widetilde{V}_j \zeta_{ij} | \zeta_0 \rangle + \langle \zeta_0 | V_i \widetilde{V}_j \zeta_{ij} \rangle \} + \sum_i \{ \langle V_i \zeta_{i\,0} | \zeta_0 \rangle + \langle \zeta_0 | V_i \zeta_{i\,0} \rangle \} \\
&\quad + \sum_j \{ \langle \widetilde{V}_j \zeta_{0\,j} | \zeta_0 \rangle + \langle \zeta_0 | \widetilde{V}_j \zeta_{0\,j} \rangle \} + \sum_{ij} \{ \langle V_i \zeta_{i\,0} | \widetilde{V}_j \zeta_{0\,j} \rangle + \langle \widetilde{V}_j \zeta_{0\,j} | V_i \zeta_{i\,0} \rangle \} \\
&= \sum_{ij} \|\zeta_{ij} + \widetilde{V}_j^* \zeta_{i\,0} + V_i^* \zeta_{0\,j} + V_i^* \widetilde{V}_j \zeta_0\|^2 \geqslant 0.
\end{aligned}$$

Note that $\mathcal{K} = \mathcal{H}_\varphi$ identifies with a subspace of $\mathcal{H}$ through the map

$$\mathcal{K} \ni \xi \mapsto \Lambda\{\{\emptyset\} \otimes \{\emptyset\} \otimes \xi\}.$$

Then $\mathcal{K} = \mathcal{H}_0 \cap \widetilde{\mathcal{H}}_0$, so $\mathcal{H}$ may be viewed as an amalgamated tensor product

$$\mathcal{H} = \mathcal{H}_0 \otimes_{\mathcal{K}} \widetilde{\mathcal{H}}_0.$$

Let P be the projection from $\mathcal{H}$ onto $\mathcal{K}$. Then

$$\begin{aligned}
S_i^* P &= P S_i^* P = V_i^*, \\
\widetilde{S}_i^* P &= P \widetilde{S}_i^* P = \widetilde{V}_i^*.
\end{aligned}$$

We can thus define states $\psi, \widetilde{\psi}$ on $\mathcal{O}_d$ through the requirement

$$\begin{aligned}
\psi(s_I s_J^*) &= \varphi(V_I V_J^*), \\
\widetilde{\psi}(s_I s_J^*) &= \widetilde{\varphi}(\widetilde{V}_I \widetilde{V}_J^*).
\end{aligned}$$

Let E be the support projection of ψ as a state on $\mathcal{O}_d''$ on the amalgamated tensor product, and similarly let $\widetilde{E}$ be the support projection of $\widetilde{\psi}$. Here $\mathcal{O}_d''$ and $\widetilde{\mathcal{O}}_d''$ denote the von Neumann algebras generated by $\{S_1, \dots, S_d\}$ and $\{\widetilde{S}_1, \dots, \widetilde{S}_d\}$ of (7.1)–(7.2), respectively. (The amalgamated tensor product thus carries a representation of $\mathcal{O}_d \otimes \widetilde{\mathcal{O}}_d$, where $\widetilde{\mathcal{O}}_d \cong \mathcal{O}_d$, and the states $\psi, \widetilde{\psi}$ identify with the restriction of the vector state $\langle \Phi | \cdot \Phi \rangle$ to each of the two tensor factors.)

Lemma 7.5. *$\mathcal{H}_0$ is an invariant subspace for $\mathcal{O}_d$ and $P|\mathcal{H}_0 = E|\mathcal{H}_0$.*

Proof. $\mathcal{H}_0$ is obviously an invariant subspace for $\mathcal{O}_d$ by construction. But the map

$$(\mathcal{O}_d|\mathcal{H}_0)' \to \mathcal{B}(\mathcal{K})^\sigma = \mathcal{M}' : Q \mapsto PQP$$

is an order isomorphism onto $\mathcal{M}'$ by Proposition 4.1 and the assumptions of Theorem 7.1. Hence, as $\mathcal{M}'$ is a factor by assumption and $(\mathcal{O}_d|\mathcal{H}_0)'$ is a von Neumann algebra, the map $Q \mapsto PQP$ is either an isomorphism or anti-isomorphism by [13], Proposition 3.22, or [29], [31], [32]. But as the map $Q \mapsto PQP$ is clearly completely positive, it is an isomorphism. Hence $P \in ((\mathcal{O}_d|\mathcal{H}_0)')' = \mathcal{O}_d''|\mathcal{H}_0$ by Proposition 4.2. But as P is the support projection of the normal state ψ on $\mathcal{O}_d''|\mathcal{H}_0$, it follows that P is the image of E under the map $\mathcal{O}_d'' \ni A \mapsto A|\mathcal{H}_0$, i.e.,

$$P|\mathcal{H}_0 = E|\mathcal{H}_0. \quad \blacksquare$$

Lemma 7.6. *$E\widetilde{E} = P$.*

Proof. Clearly $E \geqslant P$, $\widetilde{E} \geqslant P$, so $E\widetilde{E} \geqslant P$. The converse inequality follows by using Lemma 7.5, $E\widetilde{E} = \widetilde{E}E$, and $E\widetilde{S}_{\widetilde{I}} = \widetilde{S}_{\widetilde{I}}E$:

$$\begin{aligned} \widetilde{E}E\Lambda(I \otimes \widetilde{I} \otimes \xi) &= \widetilde{E}\widetilde{S}_{\widetilde{I}}E\Lambda(I \otimes \{\emptyset\} \otimes \xi) = \widetilde{E}\widetilde{S}_{\widetilde{I}}P\Lambda(I \otimes \{\emptyset\} \otimes \xi) \\ &= \widetilde{E}\widetilde{S}_{\widetilde{I}}PS_I\xi = \widetilde{E}\widetilde{S}_{\widetilde{I}}V_I\xi = \widetilde{E}\Lambda(\{\emptyset\} \otimes \widetilde{I} \otimes V_I\xi) \\ &= P\Lambda(\{\emptyset\} \otimes \widetilde{I} \otimes V_I\xi) = \widetilde{V}_{\widetilde{I}}V_I\xi \in \mathcal{K}, \end{aligned}$$

so $\widetilde{E}E \leqslant P$. $\blacksquare$

Lemma 7.7. *$\mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d'' = (\mathcal{O}_d \cup \widetilde{\mathcal{O}}_d)'' = \mathcal{B}(\mathcal{H})$.*

Proof. We have $P = E\widetilde{E} \in \mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d''$. But by Lemma 7.5 (applied both to $\mathcal{O}_d$ and $\widetilde{\mathcal{O}}_d$), $P(\mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d'')P$ contains both $\mathcal{M}$ and $\mathcal{M}'$, and as $\mathcal{M}$ is a factor, we have

$$P(\mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d'')P = \mathcal{B}(\mathcal{K}).$$

Since $\mathcal{K}$ is cyclic for $\mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d''$, the lemma follows. $\blacksquare$

Lemma 7.8. $\mathcal{O}_d' = \widetilde{\mathcal{O}}_d''$.

Proof. Since $\mathcal{O}_d$ and $\widetilde{\mathcal{O}}_d$ mutually commute, it follows from Lemma 7.7 that $\mathcal{O}_d'$ and $\widetilde{\mathcal{O}}_d''$ are factors, and $\mathcal{O}_d' \supset \widetilde{\mathcal{O}}_d''$. The projection $E \in \mathcal{O}_d''$ commutes with $\mathcal{O}_d'$ and thus also with $\widetilde{\mathcal{O}}_d''$, and hence it suffices to show that $\mathcal{O}_d'E = \widetilde{\mathcal{O}}_d''E$. Introduce $\mathcal{N}_1 = \mathcal{O}_d'E$ and $\mathcal{N}_2 = \widetilde{\mathcal{O}}_d''E$. Then $P \in \mathcal{N}_2 \subset \mathcal{N}_1$, and $P\mathcal{N}_1'P = \mathcal{M}$, $P\mathcal{N}_2P = \mathcal{M}'$, so

$$P\mathcal{N}_1P = P\mathcal{N}_2P = \mathcal{M}'.$$

The relations $P \in \mathcal{N}_2 \subset \mathcal{N}_1$, and $P\mathcal{N}_1P = P\mathcal{N}_2P$, imply $\mathcal{N}_1 = \mathcal{N}_2$. (We can find a type I subfactor M of $\mathcal{N}_2$ such that P dominates a minimal projection in M, and the above conditions imply that $\mathcal{N}_1 \cap M' = \mathcal{N}_2 \cap M'$, which again implies $\mathcal{N}_1 = \mathcal{N}_2$.) ∎

Denote the gauge action of $\mathbb{T}$ on $\mathcal{O}_d$, respectively $\widetilde{\mathcal{O}}_d$, by τ, respectively $\widetilde{\tau}$. Define

$$H = \{z \in \mathbb{T} \mid \tau_z \text{ extends to an automorphism of } \mathcal{O}_d''\}.$$

As in Theorem 6.3, it follows from $\psi \circ \lambda = \psi$ that

$$H = \{z \in \mathbb{T} \mid \psi \circ \tau_z = \psi\},$$

and hence H is a closed subgroup of $\mathbb{T}$. Define a subgroup $\widetilde{H}$ in the same way as H by using $\widetilde{\tau}$ instead of τ.

As mentioned before, the algebra $\mathcal{O}_d \otimes \widetilde{\mathcal{O}}_d$ is naturally represented on $\mathcal{H}$. Define

$$G = \{(z_1, z_2) \in \mathbb{T}^2 \mid \tau_{z_1} \otimes \widetilde{\tau}_{z_2} \text{ extends to an automorphism of } \mathcal{O}_d'' \vee \widetilde{\mathcal{O}}_d'' = \mathcal{B}(\mathcal{H})\}.$$

Lemma 7.9. *$\widetilde{H} = H$ and $\{(z, z) \mid z \in H\} \subset G \subset H \times H$.*

Proof. Once we can show $\{(z, z) \mid z \in H\} \subset G$ it follows that $H \subset \widetilde{H}$, and then it follows by symmetry that $\widetilde{H} \subset H$, so $H = \widetilde{H}$. But then $G \subset H \times H$ is obvious. So it remains to show

$$\{(z, z) \mid z \in H\} \subset G.$$

For this, let $z \in H$, i.e., $\psi \circ \tau_z = \psi$. Thus $\tau_z(E) = E$ (where still $E = \operatorname{supp}\psi$), and one can defne a unitary operator U_0 on $P\mathcal{H} = \mathcal{K}$ by $U_0Q\Phi = \tau_z(Q)\Phi$ for $Q \in \mathcal{M}$. (We are now working in the cyclic representation defined by φ, and τ_z

also denotes the extension of τ_z to $\mathcal{O}_d''$.) If J, Δ are the modular conjugation and modular operator associated to $(\mathcal{M}, \Phi)$, it follows from τ_z-invariance that

$$U_0 J U_0^* = J,$$
$$U_0 \Delta U_0^* = \Delta.$$

Thus

$$\operatorname{Ad} U_0(\widetilde{V}_j) = U_0(J\Delta^{-\frac{1}{2}} V_j^* \Delta^{\frac{1}{2}} J) U_0^* = z\widetilde{V}_j.$$

Using this, we can extend U_0 to a unitary operator U on $\mathcal{H}$ by the definition

$$U\Lambda(I \otimes \widetilde{I} \otimes \xi) = z^{|I|+|\widetilde{I}|}\Lambda(I \otimes \widetilde{I} \otimes U_0\xi).$$

This operator U is indeed well defined and unitary because

$$\langle \Lambda(I \otimes \widetilde{I} \otimes U_0\xi) | \Lambda(J \otimes \widetilde{J} \otimes U_0\eta)\rangle = z^{|I|+|\widetilde{I}|-|J|-|\widetilde{J}|}\langle \Lambda(I \otimes \widetilde{I} \otimes \xi) | \Lambda(J \otimes \widetilde{J} \otimes \eta)\rangle.$$

We have

$$US_i\Lambda(I \otimes \widetilde{I} \otimes \xi) = z^{|I|+1+|\widetilde{I}|}\Lambda(iI \otimes \widetilde{I} \otimes U_0\xi) = zS_i\Lambda(I \otimes \widetilde{I} \otimes \xi),$$

and similarly $U\widetilde{S}_i = z\widetilde{S}_i U$. Hence, $\operatorname{Ad} U | \mathcal{O}_d \otimes \widetilde{\mathcal{O}}_d = \tau_z \otimes \widetilde{\tau}_z$, so $(z, z) \in G$. ∎

We next prove an analogue of Theorem 6.3 in this situation.

Lemma 7.10. *If $z \in \mathbb{T}$, the following conditions are equivalent:*

(i) *$(z, 1) \in G$;*

(ii) *τ_z extends to an inner automorphism of $\mathcal{O}_d''$;*

(iii) *$z \in \operatorname{PSp}(\sigma|\mathcal{M})$.*

Proof. (i) $\Rightarrow$ (ii) If $(z, 1) \in G$, there is a unitary U on $\mathcal{H}$ such that $\tau_z \otimes \mathrm{id} = \mathrm{Ad}U$. But then $\mathrm{Ad}U|\widetilde{\mathcal{O}}_d = \mathrm{id}$, i.e., $U \in \widetilde{\mathcal{O}}_d' = \mathcal{O}_d''$ (by Lemma 7.8) and hence $\tau_z = \mathrm{Ad}U|\mathcal{O}_d$ extends to an inner automorphism of $\mathcal{O}_d''$.

(ii) $\Rightarrow$ (iii) If (ii) holds, then $\psi \circ \tau_z = \psi$ by the comment prior to Lemma 7.9, and hence $UEU^* = E$ for a unitary $U \in \mathcal{O}_d''$ with $\tau_z = \mathrm{Ad}U$. Thus $U_0 = UE\widetilde{E} = UP$ is a unitary in $\mathcal{M}$ with $\sigma(U_0) = \overline{z}U_0$ and thus $\sigma(U_0^*) = zU_0^*$. Thus $z \in \operatorname{PSp}(\sigma|\mathcal{M})$.

(iii) $\Rightarrow$ (i) Since $\mathcal{M}^\sigma = \mathbb{C}\,\mathbb{1}$, it follows from (iii) and the beginning of the proof of Lemma 6.6 that there exists a unitary operator $U_0 \in \mathcal{M}$ with $\sigma(U_0) = zU_0$, and from Lemma 6.7 it follows that $\operatorname{Ad} U_0(V_i) = \overline{z}V_i$. Proceeding as in the final parts of the proofs of Lemma 6.6 and Lemma 7.9, we extend U_0 to a unitary U on $\mathcal{H}$ by

$$U\Lambda(I \otimes \widetilde{I} \otimes \xi) = \overline{z}^{|I|}\Lambda(I \otimes \widetilde{I} \otimes U_0\xi).$$

We check as there that U is a well-defined unitary, and that $\mathrm{Ad}(U^*) = \tau_z \otimes \mathrm{id}$. Thus $(z, 1) \in G$. ∎

We now show that the representation of $\mathrm{UHF}_d \subset \mathcal{O}_d$ on $\mathcal{H}$ is quasi-equivalent to the subrepresentation on $[\mathrm{UHF}_d\Phi]$:

Lemma 7.11. *The representation of* UHF_d *on* $\mathcal{H}$ *is quasi-equivalent to* $\pi_{\psi|\mathrm{UHF}_d}$.

Proof. Since Φ is cyclic for the representation of $\mathcal{O}_d \otimes \widetilde{\mathcal{O}}_d$ on $\mathcal{H}$ by Lemma 7.7, the vectors

$$\xi = S_I S_J^* \widetilde{S}_{\widetilde{I}} \widetilde{S}_{\widetilde{J}}^* \Phi$$

span a dense subspace of $\mathcal{H}$, and thus it suffices to show that ω_ξ is normal in $\pi_\psi|\mathrm{UHF}_d$. For this, if $n \geqslant |I|$ and $x \in \mathrm{UHF}_d$, we have, using the Cuntz relations, on the two representations (i)–(ii),

$$\begin{aligned}\langle \xi | \lambda^n(x)\xi \rangle &= \Big\langle S_I \widetilde{S}_{\widetilde{I}} S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi \,\Big|\, \sum_{|I'|=n} S_{I'} x S_{I'}^* S_I \widetilde{S}_{\widetilde{I}} S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi \Big\rangle \\ &= \langle \widetilde{S}_{\widetilde{I}} S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi | \lambda^{n-|I|}(x) \widetilde{S}_{\widetilde{I}} S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi \rangle = \langle S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi | \lambda^{n-|I|}(x) S_J^* \widetilde{S}_{\widetilde{J}}^* \Phi \rangle.\end{aligned}$$

Since $\widetilde{S}_{\widetilde{J}}^*$ commute with the other factors in this inner product and $\widetilde{S}_{\widetilde{J}} \widetilde{S}_{\widetilde{J}}^*$ is a projection, it follows for positive x that

$$\begin{aligned}\langle \xi | \lambda^n(x)\xi \rangle &\leqslant \langle \Phi | S_J \lambda^{n-|I|}(x) S_J^* \Phi \rangle \leqslant \sum_{|J'|=|J|} \langle \Phi | S_{J'} \lambda^{n-|I|}(x) S_{J'}^* \Phi \rangle \\ &= \langle \Phi | \lambda^{n-|I|+|J|}(x) \Phi \rangle = \psi \circ \lambda^{n-|I|+|J|}(x) = \psi(x),\end{aligned}$$

where the last identity follows from Remark 6.4. Hence, $\omega_\xi \circ \lambda^n|\mathrm{UHF}_d$ is a vector state in the $\psi|\mathrm{UHF}_d$-representation, and since

$$\mathrm{UHF}_d \cong M_{d^n} \otimes \lambda^n(\mathrm{UHF}_d)$$

in a canonical fashion, it follows that $\omega_\xi|\mathrm{UHF}_d$ is normal in $\psi|\mathrm{UHF}_d$. See [12], proof of Lemma 5.2. ∎

Lemma 7.12. *Let* $\mathcal{N}$ *be a factor and* α *an action of a group* G *on* $\mathcal{N}$. *Assume that* G *is the circle group or a finite cyclic group. If the fixed point algebra* $\mathcal{N}^\alpha$ *is a factor, then* $\mathcal{N} \cap (\mathcal{N}^\alpha)'$ *is the abelian von Neumann algebra generated by a unitary operator* V *such that*

$$\alpha_g(V) = \langle g | \gamma_0 \rangle V$$

for some $\gamma_0 \in \widehat{G}$, *and*

$$\operatorname{Ad} V | \mathcal{N} = \alpha_h$$

for some $h \in G$.

Proof. Since $\mathcal{N}^\alpha$ is a factor, $\alpha|\mathcal{N} \cap (\mathcal{N}^\alpha)'$ is ergodic, and therefore $\operatorname{Sp}(\alpha|\mathcal{N} \cap (\mathcal{N}^\alpha)')$ is a subgroup of $\widehat{G}$, and thus $\operatorname{Sp}(\alpha|\mathcal{N} \cap (\mathcal{N}^\alpha)')$ is either a finite cyclic group or $\mathbb{Z}$. In any case, $\operatorname{Sp}(\alpha|\mathcal{N} \cap (\mathcal{N}^\alpha)')$ has a generator γ_0, and by ergodicity the corresponding eigen-subspace is the linear span of a unitary operator V ([13], Section 3.2.3, [36], [5]). Also V generates $\mathcal{N} \cap (\mathcal{N}^\alpha)'$ as a von Neumann algebra, so this algebra is abelian. Let $\beta = \operatorname{Ad} V|\mathcal{N}$. Since $\alpha_g(V) = \langle g|\gamma_0\rangle V$ for all $g \in G$, we have $\beta\alpha_g = \alpha_g\beta$ for all $g \in G$, and hence β fixes each spectral subspace of α in $\mathcal{N}$. As $V \in (\mathcal{N}^\alpha)'$, we have $\beta|\mathcal{N}^\alpha = \mathrm{id}$. Since $\mathcal{N}^\alpha$ is a factor, each spectral subspace $\mathcal{N}^\alpha(\gamma)$ for $\gamma \in \widehat{G}$ either is 0, or has the form $\mathcal{N}^\alpha(\gamma) = \mathcal{N}^\alpha V(\gamma)$ for an isometry $V(\gamma) \in \mathcal{N}^\alpha(\gamma)$, or the form $\mathcal{N}^\alpha(\gamma) = V(\gamma)\mathcal{N}^\alpha$ for a coisometry $V(\gamma) \in \mathcal{N}^\alpha(\gamma)$. We may assume that G acts faithfully, and this excludes the case $\mathcal{N}^\alpha(\gamma) = 0$.

Now consider the case that $V(\gamma)$ is an isometry. Since $\beta(V(\gamma)) \in \mathcal{N}^\alpha(\gamma)$, there is an operator $U(\gamma) \in \mathcal{N}^\alpha$ such that $\beta(V(\gamma)) = U(\gamma)V(\gamma)$. Since the projection $Q = V(\gamma)V(\gamma)^*$ is in $\mathcal{N}^\alpha$, we may replace $U(\gamma)$ by $U(\gamma)Q$ without changing the equation above, and then $U(\gamma)Q = U(\gamma)$. Then since $U(\gamma) = \beta(V(\gamma))V(\gamma)^*$, we have likewise

$$\begin{aligned} QU(\gamma) &= V(\gamma)V(\gamma)^*\beta(V(\gamma))V(\gamma)^* = \beta(V(\gamma)V(\gamma)^*V(\gamma))V(\gamma)^* \\ &= \beta(V(\gamma))V(\gamma)^* = U(\gamma), \end{aligned}$$

so $U(\gamma) \in Q\mathcal{N}^\alpha Q$. If $A \in Q\mathcal{N}^\alpha Q \subset \mathcal{N}^\alpha$, then $AV(\gamma) \in \mathcal{N}^\alpha(\gamma)$, and we have on one side

$$\beta(AV(\gamma)) = \beta(A)\beta(V(\gamma)) = AU(\gamma)V(\gamma)$$

and on the other side, since $V(\gamma)^*AV(\gamma) \in \mathcal{N}^\alpha$,

$$\begin{aligned} \beta(AV(\gamma)) &= \beta(V(\gamma)V(\gamma)^*AV(\gamma)) = \beta(V(\gamma))\beta(V(\gamma)^*AV(\gamma)) \\ &= U(\gamma)V(\gamma)(V(\gamma)^*AV(\gamma)) = U(\gamma)AV(\gamma). \end{aligned}$$

Comparing the last two expressions, we see that $AU(\gamma) = U(\gamma)A$ for all $A \in Q\mathcal{N}^\alpha Q$. Hence $U(\gamma) \in Q((\mathcal{N}^\alpha)' \cap \mathcal{N}^\alpha)Q = \mathbb{C}\,Q$, so $U(\gamma)$ is a scalar multiple $f(\gamma) \in \mathbb{T}$ of Q, and $\beta(B) = f(\gamma)B$ for all $B \in \mathcal{N}^\alpha(\gamma) = \mathcal{N}^\alpha V(\gamma)$. If $V(\gamma)$ is a coisometry, the verification of this relation is analogous. Since β is an automorphism, one verifies that $f \in \widehat{\widehat{G}} = G$, so there exists an $h \in G$ with $\beta(B) = \langle h|\gamma\rangle B = \alpha_h(B)$ for $B \in \mathcal{N}^\alpha(\gamma)$ and $\gamma \in \widehat{G}$; and hence $\beta = \alpha_h$. ∎

Lemma 7.13. *If ω is a translationally invariant state on $\mathfrak{A} = \bigotimes_{\mathbb{Z}} M_d$ such that $\pi_\omega(\mathfrak{A})''$ is a type* I *factor, then ω is pure.*

Proof. Let π be an irreducible representation quasi-equivalent to π_ω. There is a density matrix ρ on $\mathcal{H}_\pi$ such that$\omega(x) = \mathrm{Tr}(\pi(x)\rho)$ for $x \in \mathfrak{A}$. Since π_ω and thus π are translationally covariant, there is a unitary operator U on $\mathcal{H}_\pi$ such that $U\pi(x)U^* = \pi(\lambda(x))$ for all $x \in \mathfrak{A}$, where λ is the translation automorphism. Since $\omega \circ \lambda = \omega$, we obtain that $U^*\rho U = \rho$. Assume *ad absurdum* that ρ is not a one-dimensional projection. Then there are at least two orthogonal eigenvectors ξ_1, ξ_2 of U. Thus, for any $x \in A$,

$$\langle \xi_i | \pi(\lambda^n(x)) \xi_i \rangle = \langle U^{*\,n} \xi_i | \pi(x) U^{*\,n} \xi_i \rangle = \langle \xi_i | \pi(x) \xi_i \rangle.$$

But any weak*-limit point of $\pi(\lambda^n(x))$, as $n \to \infty$, is in $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$, and it follows that

$$\langle \xi_1 | \pi(x) \xi_1 \rangle = \langle \xi_2 | \pi(x) \xi_2 \rangle$$

for all $x \in \mathfrak{A}$. But as π is irreducible, this is a contradiction. Thus ρ must be a one-dimensional projection, and ω is pure. ∎

Lemma 7.14. *Let $\mathcal{N}$ be a type* I *von Neumann algebra and α an action of a group G on $\mathcal{N}$. Assume that G is the circle group or a finite cyclic group. Then the fixed point subalgebra $\mathcal{N}^\alpha$ is of type* I.

Proof. By considering the action α on the center $\mathcal{N} \cap \mathcal{N}'$, the von Neumann algebra decomposes into algebras of the form $\mathcal{M} \otimes L^\infty(G/H)$, where $\mathcal{M}$ is a type I factor, H is a closed subgroup of G, and G acts on $L^\infty(G/H)$ by translation, until reaching the end of the orbit, in which case the action may be modified by an automorphism of $\mathcal{M}$ (if $H \neq \{0\}$). The latter automorphism is inner and of finite order, except in the case $H = G = \mathbb{T}$, in which case we have an inner action of $\mathbb{T}$ on the type I factor $\mathcal{M}$. In any case, it is clear that the fixed point subalgebra of $\mathcal{M} \otimes L^\infty(G/H)$ under the action is a type I von Neumann algebra, and the lemma follows. ∎

Lemma 7.15. *The vector state ω_Φ on*

$$\widetilde{\mathrm{UHF}}_d \otimes \mathrm{UHF}_d \cong \Big(\bigotimes_{-\infty}^{0} M_d \Big) \otimes \Big(\bigotimes_{1}^{\infty} M_d \Big) \cong \bigotimes_{\mathbb{Z}} M_d$$

is equal to ω.

Proof. Since

$$\widetilde{S}_j^* \Phi = \widetilde{V}_j^* \Phi = J \Delta^{\frac{1}{2}} V_j \Delta^{-\frac{1}{2}} J \Phi = V_j^* \Phi = S_j^* \Phi,$$

it follows that

$$\langle\Phi|\widetilde{S}_{i\widetilde{I}}\widetilde{S}^*_{j\widetilde{J}}S_IS_J^*\Phi\rangle = \langle\widetilde{S}_i^*\Phi|\widetilde{S}_{\widetilde{I}}\widetilde{S}^*_{\widetilde{J}}S_IS_J^*\widetilde{S}_j^*\Phi\rangle = \langle\Phi|\widetilde{S}_{\widetilde{I}}\widetilde{S}^*_{\widetilde{J}}S_{iI}S_{jJ}^*\Phi\rangle.$$

This proves the lemma since the vector state ω_Φ on UHF_d is the restriction of ω to $\mathrm{UHF}_d = \bigotimes_{\mathbb{N}} M_d$. ∎

Proof of Theorem 7.1. We first merely assume that $\mathcal{M}$ is a factor.

(i) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (i) By Lemma 7.7, $\mathcal{O}_d''$ is a factor, and by assumption (iii), $\pi_\omega\Big(\bigotimes_{\mathbb{Z}} M_d\Big)''$ is a factor. By viewing $\mathrm{UHF}_d = \bigotimes_{\mathbb{N}} M_d$ as a subalgebra of both $\bigotimes_{\mathbb{Z}} M_d$ and $\mathcal{O}_d$, we have

$$\omega\Big| \bigotimes_{\mathbb{N}} M_d = \psi|\mathrm{UHF}_d.$$

Hence, by Lemma 7.11, UHF_d'' is a factor. Recall from the remark prior to Lemma 7.9 that the set

$$H = \{z \in \mathbb{T} \mid \tau_z \text{ extends to an automorphism of } \mathcal{O}_d''\}$$

is a closed subgroup of $\mathbb{T}$, and, as in the proof of Lemma 6.11, we have

$$\mathcal{O}_d''^{\tau_H} = \mathrm{UHF}_d''.$$

By Lemma 7.12, the algebra $\mathcal{O}_d'' \cap \mathrm{UHF}_d'$ is abelian, and thus

$$(\mathcal{O}_d'' \cap \mathrm{UHF}_d')' = \mathcal{O}_d' \vee \mathrm{UHF}_d'' = \widetilde{\mathcal{O}}_d'' \vee \mathrm{UHF}_d''$$

is of type I, where we used Lemma 7.8 for the last identity. For any $z \in \mathbb{T}$,

$$\widetilde{\tau}_z \otimes \mathrm{id}|\widetilde{\mathcal{O}}_d \otimes \mathrm{UHF}_d = \widetilde{\tau}_z \otimes \tau_z|\widetilde{\mathcal{O}}_d \otimes \mathrm{UHF}_d$$

extends to an automorphism of $\widetilde{\mathcal{O}}_d'' \vee \mathrm{UHF}_d''$ if and only if $z \in H$. Then, again invoking the argument in the proof of Lemma 6.11, we obtain

$$(\widetilde{\mathcal{O}}_d'' \vee \mathrm{UHF}_d'')^{\widetilde{\tau}_H \otimes \mathrm{id}} = \widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d''.$$

Since H is either $\mathbb{T}$ or a closed subgroup of $\mathbb{T}$, it follows from Lemma 7.14 that $\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d''$ is of type I. By Lemma 7.15, $\omega = \omega_{\Phi|\widetilde{\mathrm{UHF}}_d \otimes \mathrm{UHF}_d}$, and by assumption (iii), the restriction of $\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d''$ to the closed subspace $[(\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d'')\Phi]$ is a factor. Thus it follows from Lemma 7.13 that ω is pure. This ends the proof of (iii) $\Rightarrow$ (i).

From now on, assume that $\mathcal{M}$ is a type I factor.

(ii) $\Rightarrow$ (i) Since $\mathcal{M} \simeq P\mathcal{O}_d''P \simeq E\mathcal{O}_d''E$ and $\mathcal{O}_d''$ is a factor, it follows that $\mathcal{O}_d''$ is a type I factor. Since any automorphism of a type I factor is inner, it follows from Lemma 7.10 that $H = \{0\}$. Thus $\mathcal{O}_d'' = \mathrm{UHF}_d''$ as in Lemma 6.10. Similarly $\widetilde{\mathcal{O}}_d'' = \widetilde{\mathrm{UHF}}_d''$, and it follows from Lemma 7.7 that $\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d'' = \mathcal{B}(\mathcal{H})$, so ω is pure.

(i) $\Rightarrow$ (ii) If $\mathcal{M}$ is of type I, it follows again that $\mathcal{O}_d''$ and $\widetilde{\mathcal{O}}_d''$ are type I factors, and hence $\widetilde{\mathcal{O}}_d'' \vee \mathcal{O}_d'' \simeq \widetilde{\mathcal{O}}_d'' \otimes \mathcal{O}_d''$. Recall that $\mathcal{O}_d''^{\tau_H} = \mathrm{UHF}_d''$, and that τ_H is inner. It then follows that $\mathcal{O}_d'' \cap \mathrm{UHF}_d'$ is abelian, and we conclude that

$$\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d'' \simeq \widetilde{\mathrm{UHF}}_d'' \otimes \mathrm{UHF}_d''$$

and further that the commutant of this algebra is again abelian. If (i) holds, it follows from Lemma 7.11 and Proposition 7.3 that UHF_d'' and $\widetilde{\mathrm{UHF}}_d''$ are factors, and hence

$$\widetilde{\mathrm{UHF}}_d'' \vee \mathrm{UHF}_d'' = \mathcal{B}(\mathcal{H}).$$

This implies that the subgroup H of $\mathbb{T}$ must be trivial, and hence

$$\mathrm{PSp}(\sigma|\mathcal{M}) \cap \mathbb{T} = \{1\}$$

by Lemma 7.10, so (ii) holds.

Finally, we argue that (iii) $\Leftrightarrow$ (iv). If $\mathcal{N} = \pi_\psi(\mathcal{O}_d)'' \cong \mathcal{O}_d''$, then $\mathcal{N}^{\tau|H} = \mathrm{UHF}_d''$ by Lemma 6.10. But, by Corollary 8.10.5 in [37], it follows that $\mathcal{N}^{\tau|H} = \mathrm{UHF}_d''$ is a factor if and only if the Connes spectrum of the extension of $\tau|H$ to $\mathcal{O}_d''$ is equal to $\widehat{H}$. The rest of the proof of (iii) $\Leftrightarrow$ (iv) is as above. ∎

Remark 7.16. If $\mathcal{N}$ is a type I factor in the last paragraph of the preceding proof, then $H = \{1\}$ since H is $\mathbb{T}$ or a cyclic group, and we get the trivial peripheral spectrum. If $\mathcal{N}$ is a type III factor, the subgroup H could in principle be nontrivial, with some nontrivial τ_t inner, i.e., $\mathrm{PSp}(\sigma|\mathcal{M}) \neq \{1\}$. This is because τ_t could be implemented by a unitary in $\mathcal{N}$ which would then not be fixed by H, and, in this case, the Connes spectrum still could be $\widehat{H}$. See, e.g., [41] for examples of that.

8. REMARKS ON DUALITY

In this section we will establish several more properties of the duality theory of the objects $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ introduced between Lemmas 7.4 and 7.5, and thereby also extend [9], Lemma 6.3. So we assume that $(\mathcal{M}, \varphi, V_1, \dots, V_d, \sigma)$ satisfies the general hypotheses in Theorem 7.1, and let the dual object

$$(\widetilde{\mathcal{M}}, \widetilde{\varphi}, \widetilde{V}_1, \dots, \widetilde{V}_d, \widetilde{\sigma})$$

be defined as after Lemma 7.4, i.e.,

$$\widetilde{\mathcal{M}} = \mathcal{M}', \quad \widetilde{V}_j = \overline{(J\sigma_{\frac{\mathrm{i}}{2}}(V_j^*)J)}, \quad \widetilde{\varphi}(X) = \langle \Phi | X\Phi \rangle,$$

where Φ is the separating and cyclic vector for $\mathcal{M}$ defining the state φ.

Recall the notation

$$\mathcal{M}^\sigma = \{X \in \mathcal{M} \mid \sigma(X) = X\},$$
$$\mathcal{M}_*^\sigma = \{\eta \in \mathcal{M}_* \mid \eta \circ \sigma = \eta\},$$

where $\mathcal{M}_*$ is the predual of $\mathcal{M}$.

Lemma 8.1. *If $\mathcal{M}^\sigma = \mathbb{C}\mathbb{1}$, then $\mathcal{M}_*^\sigma = \mathbb{C}\,\varphi$.*

Proof. Since $\varphi \circ \sigma = \varphi$, and σ is ergodic, this is established as in Lemma 6.8 and its proof. ∎

We use Lemma 8.1 to establish:

Proposition 8.2. *The map σ is ergodic if and only if the dual map $\widetilde{\sigma}$ is ergodic.*

Proof. Assume that $X' \in \mathcal{M}'$ and that $\widetilde{\sigma}(X') = X'$. This means that

$$\sum_j V_j^* \Delta^{\frac{1}{2}} JX'J\Delta^{\frac{1}{2}} V_j = \Delta^{\frac{1}{2}} JX'J\Delta^{\frac{1}{2}},$$

for example as sesquilinear forms on $\mathcal{M}\Phi$. Define $X = JX'J \in \mathcal{M}$. If $A \in \mathcal{M}$ is an entire element for the modular group σ_t, we have

$$\begin{aligned}
&\langle \Delta^{\frac{1}{2}} X^*\Phi | \sigma(A)\Phi \rangle \\
&= \Big\langle \Delta^{\frac{1}{2}} X^*\Phi \,\Big|\, \sum_j V_j A V_j^* \Phi \Big\rangle = \Big\langle X^*\Phi \,\Big|\, \sum_j \sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)\Delta^{\frac{1}{2}} V_j^*\Phi \Big\rangle \\
&= \sum_j \langle \Phi | X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)\Delta^{\frac{1}{2}} V_j^*\Phi \rangle = \sum_j \langle \Phi | X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)J^2\Delta^{\frac{1}{2}} V_j^*\Phi \rangle \\
&= \sum_j \langle \Phi | X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)JV_jJ\Phi \rangle = \sum_j \langle JV_j^*\Phi | X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)\Phi \rangle \\
&= \sum_j \langle J\Delta^{\frac{1}{2}}\sigma_{\frac{\mathrm{i}}{2}}(V_j^*)\Phi | X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)\Phi \rangle = \sum_j \langle \Phi | \sigma_{\frac{\mathrm{i}}{2}}(V_j^*)X\sigma_{-\frac{\mathrm{i}}{2}}(V_j)\sigma_{-\frac{\mathrm{i}}{2}}(A)\Phi \rangle \\
&= \sum_j \langle \Phi | \Delta^{-\frac{1}{2}} V_j^* \Delta^{\frac{1}{2}} X \Delta^{\frac{1}{2}} V_j A\Phi \rangle.
\end{aligned}$$

But $\sum_j V_j^* \Delta^{\frac{1}{2}} X \Delta^{\frac{1}{2}} V_j = \Delta^{\frac{1}{2}} X \Delta^{\frac{1}{2}}$ by the identity above, so furthermore:

$$\langle \Delta^{\frac{1}{2}} X^* \Phi | \sigma(A) \Phi \rangle = \langle \Phi | X \Delta^{\frac{1}{2}} A \Phi \rangle = \langle \Delta^{\frac{1}{2}} X^* \Phi | A \Phi \rangle.$$

Thus the functional $\eta \in \mathcal{M}_*$ defined by

$$\eta(A) = \langle \Delta^{\frac{1}{2}} X^* \Phi | A \Phi \rangle$$

is in $\mathcal{M}_*^\sigma$. Since σ is ergodic, it follows from Lemma 8.1 that $\eta \in \mathbb{C}\varphi$, and so $\Delta^{\frac{1}{2}} X^* \Phi \in \mathbb{C}\Phi$, which implies $X\Phi \in \mathbb{C}\Phi$, or $X \in \mathbb{C}\mathbb{1}$. Thus $X' \in \mathbb{C}\mathbb{1}$, which shows that $\widetilde{\sigma}$ is ergodic. Since $\widetilde{\widetilde{\sigma}} = \sigma$, the other implication follows. ∎

Our next aim is to show that the two sets $\mathrm{PSp}(\sigma) \cap \mathbb{T}$ and $\mathrm{PSp}(\widetilde{\sigma}) \cap \mathbb{T}$ are equal. First we need a lemma.

LEMMA 8.3. *If $\eta \in \mathcal{M}_* \setminus \{0\}$ and*

$$\eta \circ \sigma = t\eta$$

for some $t \in \mathbb{T}$, then t is an eigenvalue of σ.

Proof. Again we invoke the technique in the proof of Lemma 6.8: pick an $X \in \mathcal{M}$ such that $\eta(X) \neq 0$, and take a limit point L of the sequence

$$X_n = \frac{1}{n} \sum_{k=0}^{n-1} t^{-k} \sigma^k(X)$$

in the weak operator topology. Then it follows that $\eta(L) = \eta(X) \neq 0$, and $\sigma(L) = tL$. Thus t is an eigenvalue for σ. ∎

PROPOSITION 8.4. $\mathrm{PSp}(\sigma) \cap \mathbb{T} = \mathrm{PSp}(\widetilde{\sigma}) \cap \mathbb{T}$.

Proof. Let $t \in \mathrm{PSp}(\widetilde{\sigma}) \cap \mathbb{T}$, and let $X' \in \mathcal{M}'$ be a corresponding eigenvector: $\widetilde{\sigma}(X') = tX'$. This means that, with $X = JX'J \in \mathcal{M}$,

$$\sum_j V_j^* \Delta^{\frac{1}{2}} X \Delta^{\frac{1}{2}} V_j = \bar{t} \Delta^{\frac{1}{2}} X \Delta^{\frac{1}{2}}.$$

In the same way as in the proof of Proposition 8.2, one uses this identity to establish

$$\langle \Delta^{\frac{1}{2}} X^* \Phi | \sigma(A) \Phi \rangle = \bar{t} \langle \Delta^{\frac{1}{2}} X^* \Phi | A \Phi \rangle$$

for all $A \in \mathcal{M}$. Thus the linear functional $\eta(A) = \langle \Delta^{\frac{1}{2}} X^* \Phi | A \Phi \rangle$ satisfies $\eta \circ \sigma = \bar{t}\eta$, and therefore $\bar{t} \in \mathrm{PSp}(\sigma) \cap \mathbb{T}$ by Lemma 8.3. Since $\mathrm{PSp}(\sigma)$ is invariant under complex conjugation, we obtain that $\mathrm{PSp}(\widetilde{\sigma}) \cap \mathbb{T} \subset \mathrm{PSp}(\sigma) \cap \mathbb{T}$. As $\widetilde{\widetilde{\sigma}} = \sigma$, the other implication follows. ∎

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