SUB-RIEMANNIAN METRICS FOR QUANTUM HEISENBERG MANIFOLDS

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ABSTRACT. Every Heisenberg manifold carries a natural "sub-Riemannian" metric with interesting properties. We describe the corresponding noncommutative metric structure for Rieffel's quantum Heisenberg manifolds ([12]).

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The purpose of this paper is to study an analog, for quantum Heisenberg manifolds, of the natural sub-Riemannian metrics on classical Heisenberg manifolds.

Quantum Heisenberg manifolds were defined in [12] and they have been further investigated in [1], [2], [3], and [4]. They are interesting for several reasons, one being just because they are tractable examples of noncommutative manifolds. This means that, like the related but simpler noncommutative tori, quantum Heisenberg manifolds provide a nice concrete setting in which to explore noncommutative geometry.

Our treatment of noncommutative metrics is based on Connes' approach ([9]). But we prefer to work with abstract derivations rather than the concretely presented derivations implicit in Connes' unbounded Fredholm modules. See [15] for further discussion.

Noncommutative metric structure usually arises via an analog of the classical exterior derivative d on a Riemannian manifold. Classically, this map may be

realized as a derivation from $\mathrm{Lip}\,(X) \subset L^\infty(X)$ into the module of bounded measurable 1-forms. The graph of this derivation is weak*-closed, a property which is characteristic of the domain being a Lipschitz algebra ([15]). In some sense the differentiable structure resides in the map d, while the metric structure resides in its domain $\mathrm{Lip}\,(X)$. There are some examples where one has the latter sort of structure but not the former ([16], [17]).

An interesting feature of the present work is that from the noncommutative or algebraic point of view, sub-Riemannian metrics are very close to genuine Riemannian metrics. In the language of the previous paragraph, the exterior derivative corresponding to a sub-Riemannian metric is given by composing d with orthogonal projection onto a closed submodule of $\Omega^1(X)$.

This work followed a suggestion by Marc Rieffel, and was helped by discussions about the Heisenberg group with Daniel Allcock. I also wish to thank the referee for suggesting a number of minor improvements.

We adopt the following notational conventions: c is a fixed positive integer; $\hbar, \mu$, and ν are fixed real numbers; and H is the Hilbert space $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$.

1. SUB-RIEMANNIAN STRUCTURE FOR CLASSICAL HEISENBERG MANIFOLDS

Let M be a connected Riemannian manifold. It is well-known that M has a natural metric such that the distance between two points x and y satisfies

$$d(x,y) = \inf\{l(p) : p \text{ is a path from } x \text{ to } y\},$$

where $l(p)$ denotes the length of p.

Now let B be a subbundle of the tangent bundle $\mathrm{T}M$. We can use it to define a new metric d_B by setting

$$d_B(x,y) = \inf\{l(p) : p \text{ is a path from } x \text{ to } y$$
$$\text{which is everywhere tangent to } B\}.$$

This is a *sub-Riemannian* or *Carnot-Carathéodory* metric. A good general reference on this topic is [6]. Note that we must either require that any two points can be connected by a path which is tangent to B, or else allow distances to be infinite.

The simplest non-trivial example of a sub-Riemannian metric arises on the Heisenberg group. This example is discussed at length in [10]. We now give a brief account of the corresponding construction for Heisenberg manifolds.

The (*continuous*) *Heisenberg group* G is the set of all real 3×3 matrices of the form

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

with the product inherited from the matrix ring $M_3(\mathbb{R})$. For any positive integer c the set H_c of elements for which x, y, and cz are integers constitutes a discrete subgroup of G, and the quotient construction yields the *Heisenberg manifold* $M_c = G/H_c$. G acts on M_c from the left.

G can be identified with $\mathbb{R}^3$ and so it carries a natural differentiable manifold structure. However, the Euclidean metric on $\mathbb{R}^3$ is not compatible with the group structure of G. Instead, we give G the unique right-invariant Riemannian metric which agrees with the Euclidean metric at the origin. Concretely, the three vectors

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial z}$$

define an orthonormal basis at each point $(x, y, z) \in G$.

Since this Riemannian metric is right-invariant, it descends to M_c. The span of the two vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ is then a bundle B of tangent planes over M_c. (In fact this is a contact subbundle of TM_c, the kernel of the contact 1-form $\eta = \mathrm{d}z - x\,\mathrm{d}y$.) We use it to give M_c a sub-Riemannian metric d_B by the procedure described above.

Interestingly, this metric is finite. That is, even though M_c is three-dimensional, any two points can be joined by a path whose tangent vector at each point is in the span of $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ ([8]).

Recall that G acts on M_c from the left. The vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ can be recovered from this action. To see this consider the two one-parameter subgroups of G of the form $x = r$, $y = z = 0$ and $y = s$, $x = z = 0$; the generators of their actions on M_c are the two desired vector fields. That is, flowing along the vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ produces the actions α_r and β_s on M_c defined by

$$\alpha_r(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot A, \qquad \beta_s(A) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot A$$

$(A \in M_c)$. We will use this observation to define an analogous construction for quantum Heisenberg manifolds.

2. QUANTUM HEISENBERG MANIFOLDS

We recall the definition of the quantum Heisenberg C^*-algebras, and define corresponding von Neumann algebras. Recall that c is a fixed positive integer, $\hbar$, μ, and ν are fixed real numbers, and $H = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$.

DEFINITION 2.1. ([12], Theorem 5.5) Let S^c denote the space of C^∞ functions Φ on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ which satisfy

(i) $\Phi(x + k, y, p) = e^{ickpy}\Phi(x, y, p)$ for all $k \in 2\pi\mathbb{Z}$; and

(ii) for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\widetilde{X} = \partial^{m+n}/\partial x^m \partial y^n$ on $\mathbb{R} \times \mathbb{T}$ the function $P(p) \cdot (\widetilde{X}\Phi)(x, y, p)$ is bounded on $C \times \mathbb{Z}$ for any compact subset C of $\mathbb{R} \times \mathbb{T}$.

Define an action of $\Phi \in S^c$ on H by

$$(\Phi\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q - 2p)\mu, y - \hbar(q - 2p)\nu, q)\xi(x, y, p - q)$$

(recall that $H = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$). Then let $D_\hbar = D_{\hbar,c}^{\mu,\nu}$ be the norm closure of $S^c \subset B(H)$ and let $N_\hbar = N_{\hbar,c}^{\mu,\nu}$ be its weak operator closure.

It is shown in [12] that $D_\hbar$ is a C^*-algebra, and it follows that $N_\hbar$ is a von Neumann algebra. Note that our conventions differ from [12] by a factor of 2π in the x variable.

The C^*-algebras $D_\hbar$ are classified up to isomorphism in [2] and [4].

We require alternative characterizations of $D_\hbar$ and $N_\hbar$. The results are analogous to, but a bit more complicated than, the corresponding facts about noncommutative tori ([15]). (Another characterization of $D_\hbar$ is given in [3].) Our main tool is a kind of Fourier expansion of elements of $N_\hbar$, given in the next definition. We record its basic properties in the subsequent lemmas.

DEFINITION 2.2. For $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ define unitary operators U_t and Y_n on H by

$$(U_t\xi)(x, y, p) = e^{ipt}\xi(x, y, p) \qquad \text{and} \qquad (Y_n\xi)(x, y, p) = \xi(x, y, p + n).$$

For any $T \in B(H)$ and $n \in \mathbb{Z}$ define $a_n(T) \in B(H)$ by

$$a_n(T) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} U_t T U_t^{-1} e^{-int} \, dt.$$

This and all other operator integrals are taken in the weak operator sense, i.e.

$$\langle a_n(T)\xi, \eta \rangle = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \langle U_t T U_t^{-1} \xi, \eta \rangle \mathrm{e}^{-int} \, \mathrm{d}t$$

for all $\xi, \eta \in H$.

We regard $a_n(T)$ as a sort of Fourier coefficient of T; similarly, for $N \in \mathbb{N}$ we define the Cesaro mean $\sigma_N(T)$ by

$$\sigma_N(T) = \sum_{-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) a_n(T) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} U_t T U_t^{-1} K_N(t) \, \mathrm{d}t,$$

where K_N is the Fejér kernel

$$K_N(t) = \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) \mathrm{e}^{-int} = \frac{1}{N+1} \left(\frac{\sin((N+1)t/2)}{\sin(t/2)} \right)^2.$$

LEMMA 2.3. *For any $T \in B(H)$ we have $\sigma_N(T) \to T$ (weak operator) as $N \to \infty$. If the map $t \mapsto U_t T U_t^{-1}$ is continuous for the norm topology on $B(H)$ then $\sigma_N(T) \to T$ in norm as $N \to \infty$.*

Proof. For the weak operator statement, pick $\xi, \eta \in H$ and observe that the map

$$t \mapsto \langle (U_t T U_t^{-1} - T)\xi, \eta \rangle$$

is continuous and vanishes at $t = 0$. Therefore its integral against K_N goes to zero as $N \to \infty$ (e.g. see [11]), hence

$$\langle (\sigma_N(T) - T)\xi, \eta \rangle = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \langle (U_t^{-1} T U_t - T)\xi, \eta \rangle K_N(t) \, \mathrm{d}t \to 0.$$

This shows that $\sigma_N(T) \to T$ (weak operator).

For the norm statement, note that the function $t \mapsto \|U_t^{-1} T U_t^{-1} - T\|$ is continuous and vanishes at zero. Therefore

$$\|\sigma_N(T) - T\| = \frac{1}{2\pi} \left\| \int\limits_{-\pi}^{\pi} (U_t T U_t^{-1} - T) K_N(t) \, \mathrm{d}t \right\|$$

$$\leqslant \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \|U_t T U_t^{-1} - T\| \, K_N(t) \, \mathrm{d}t \to 0$$

as $N \to \infty$. ∎

LEMMA 2.4. *For any $T \in B(H)$, the operator $Y_n a_n(T)$ preserves constant p subspaces of $H = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ $(p \in \mathbb{Z})$.*

Proof. Note that $Y_n a_n(T)$ commutes with U_s for all s:

$$Y_n a_n(T) U_s = \frac{1}{2\pi} Y_n \int_{-\pi}^{\pi} U_t T U_t^{-1} U_s e^{-int} \, dt = \frac{1}{2\pi} Y_n U_s \int_{-\pi}^{\pi} U_{t-s} T U_{t-s}^{-1} e^{-int} \, dt$$

$$= \frac{1}{2\pi} e^{ins} U_s Y_n \int_{-\pi}^{\pi} U_t T U_t^{-1} e^{-in(t+s)} \, dt = U_s Y_n a_n(T).$$

But the operators U_s generate the von Neumann algebra $l^\infty(\mathbb{Z}) \subset B(H)$, so we conclude that $Y_n a_n(T)$ preserves the constant p subspaces of H. ∎

LEMMA 2.5. *Let $T \in B(H)$. Suppose T commutes with the operators V_f, W_k, and X_r defined by*

$$(V_f \xi)(x, y, p) = f(x, y)\xi(x, y, p)$$
$$(W_k \xi)(x, y, p) = e^{-ick(p^2\hbar\nu + py)}\xi(x + k, y, p)$$
$$(X_r \xi)(x, y, p) = \xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r)$$

for all $f \in L^\infty(\mathbb{R} \times \mathbb{T})$, $k \in 2\pi\mathbb{Z}$, and $r \in \mathbb{Z}$. Then $a_n(T)$ satisfies

$$(a_n(T)\xi)(x, y, p) = g(x, y, p)\xi(x, y, p - n)$$

for some $g \in L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$, and the function g satisfies

$$(*) \qquad\qquad g(x + k, y, p) = e^{-ick((n^2 - 2np)\hbar\nu - ny)} g(x, y, p)$$

$(k \in 2\pi\mathbb{Z})$ and

$$(\dagger) \qquad\qquad g(x, y, p) = g(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r)$$

$(r \in \mathbb{Z})$.

The function $\Phi \in L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ defined by $\Phi(x, y, p) = 0$ for $p \neq n$ and

$$\Phi(x, y, n) = g(x - \hbar n\mu, y - \hbar n\nu, n)$$

satisfies condition (i) *of Definition 2.1, and $a_n(T) = \Phi$ where Φ acts on H as in Definition 2.1.*

Proof. Every V_f commutes with T by hypothesis and with both U_t and Y_n by easy computations. It follows that V_f also commutes with $Y_n a_n(T)$. Thus $Y_n a_n(T)$, which preserves constant p subspaces by Lemma 2.4, must be a multiplication operator on each constant p subspace of H. So $a_n(T)$ has the form

$$(a_n(T)\xi)(x, y, p) = g(x, y, p)\xi(x, y, p - n)$$

for some $g \in L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$.

Next observe that every W_k commutes with both T and U_t, hence W_k commutes with $a_n(T)$. So

$$(a_n(T)W_k\xi)(x, y, p) = g(x, y, p)\mathrm{e}^{-\mathrm{i}ck((p-n)^2\hbar\nu + (p-n)y)}\xi(x + k, y, p - n)$$

equals

$$(W_k a_n(T)\xi)(x, y, p) = \mathrm{e}^{-\mathrm{i}ck(p^2\hbar\nu + py)}g(x + k, y, p)\xi(x + k, y, p - n),$$

which implies $(*)$.

Similarly, we have $X_r U_t = \mathrm{e}^{\mathrm{i}rt}U_t X_r$, hence X_r commutes with $U_t T U_t^{-1}$ and therefore with $a_n(T)$, which implies that

$$(a_n(T)X_r\xi)(x, y, p) = g(x, y, p)\xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r - n)$$

equals

$$(X_r a_n(T)\xi)(x, y, p) = g(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r)\xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r - n).$$

From this we obtain $(\dagger)$.

Finally, define Φ as in the statement of the lemma. It follows more or less immediately from $(*)$ that Φ satisfies condition (i) of Definition 2.1. Furthermore it follows from $(\dagger)$ that the action of Φ on H given in Definition 2.1 agrees with the action of $a_n(T)$, that is, taking $r = n - p$,

$$\begin{aligned}
(a_n(T)\xi)(x, y, p) &= g(x, y, p)\xi(x, y, p - n) \\
&= g(x - 2\hbar(n - p)\mu, y - 2\hbar(n - p)\nu, n)\xi(x, y, p - n) \\
&= \Phi(x - \hbar(n - 2p)\mu, y - \hbar(n - 2p)\nu, n)\xi(x, y, p - n) \\
&= \sum_q \Phi(x - \hbar(q - 2p)\mu, y - \hbar(q - 2p)\nu, q)\xi(x, y, p - q).
\end{aligned}$$

(Note that even if Φ does not satisfy condition (ii) of Definition 2.1, it still acts as a bounded operator on H. In fact $Y_n\Phi$ is a multiplication operator and so the operator norm of Φ equals $\|\Phi\|_\infty$.) ∎

THEOREM 2.6. *Let $T \in B(H)$. Then $T \in N_\hbar$ if and only if T commutes with the operators V_f, W_k, and X_r defined in Lemma 2.5 for all $f \in L^\infty(\mathbb{R} \times \mathbb{T})$, $k \in 2\pi\mathbb{Z}$, and $r \in \mathbb{Z}$.*

Proof. The forward direction can be proved by checking that every $\Phi \in S^c \subset B(H)$ commutes with V_f, W_k, and X_r. This is an elementary calculation and we omit it.

Suppose T commutes with V_f, W_k, and X_r; we must show that $T \in N_\hbar$. Since $\sigma_N(T) \to T$ (weak operator) by Lemma 2.3, it will suffice to show that $\sigma_N(T) \in N_\hbar$ for all $N \in \mathbb{N}$; and since $\sigma_N(T)$ is a linear combination of the Fourier coefficients $a_n(T)$ it will suffice to show that $a_n(T) \in N_\hbar$ for all $n \in \mathbb{N}$.

Let g and Φ be the functions defined in Lemma 2.5 and let (h_m) be a sequence of functions in $C^\infty(\mathbb{R} \times \mathbb{T})$ with $\text{supp}(h_m) \subset [-1/m, 1/m] \times [-1/m, 1/m]$ and $\|h_m\|_1 = 1$. Define the smoothings Φ_m of Φ by twisted convolution:

$$\Phi_m(x, y, p) = \int\limits_{\mathbb{R} \times \mathbb{T}} h_m(r, s)\Phi(x - r, y - s, p)\mathrm{e}^{\mathrm{i}cpxs}\, \mathrm{d}r\, \mathrm{d}s.$$

Then each Φ_m satisfies condition (i) of Definition 2.1 because

$$\begin{aligned}
\Phi_m(x + k, y, p) &= \int h_m(r, s)\Phi(x + k - r, y - s, p)\mathrm{e}^{\mathrm{i}cp(x+k)s}\, \mathrm{d}r\, \mathrm{d}s \\
&= \int h_m(r, s)\mathrm{e}^{\mathrm{i}ckp(y-s)}\Phi(x - r, y - s, p)\mathrm{e}^{\mathrm{i}cp(x+k)s}\, \mathrm{d}r\, \mathrm{d}s \\
&= \mathrm{e}^{\mathrm{i}ckpy}\Phi_m(x, y, p),
\end{aligned}$$

and each Φ_m satisfies condition (ii) of Definition 2.1 because it is C^∞ and supported on $p = n$. So $\Phi_m \in S^c$, and a standard application of Fubini's theorem and dominated convergence shows that $\Phi_m \to \Phi$ weak* in $L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$. Thus the multiplication operators $Y_n\Phi_m$ weak operator converge to $Y_n\Phi$, so that $\Phi_m \to \Phi$ (weak operator). This shows that $a_n(T) = \Phi$ belongs to $N_\hbar$. ∎

The corresponding characterization of the C^*-algebra $D_\hbar$ can be stated most naturally in terms of the action of the Heisenberg group G on the $D_\hbar$ given in [12].

DEFINITION 2.7. ([12], p. 557) For $r, s, t \in \mathbb{R}$, let $U_{(r,s,t)}$ be the unitary operator on H defined by

$$(U_{(r,s,t)}\xi)(x, y, p) = \mathrm{e}^{\mathrm{i}p(t+cs(x+\hbar p\mu - r))}\xi(x - r, y - s, p).$$

Then $L_{(r,s,t)}(T) = U_{(r,s,t)}TU_{(r,s,t)}^{-1}$ defines an action L of G on $B(H)$, where we take $(r, s, t) \in \mathbb{R}^3 \cong G$. A short computation shows that this action preserves S^c, hence it preserves $D_\hbar$ and $N_\hbar$.

By specializing to the three coordinate axes in $G \cong \mathbb{R}^3$ we get three actions α, β, γ of $\mathbb{R}$, defined by

$$\alpha_r(T) = U_{(r,0,0)} T U^*_{(r,0,0)}$$
$$\beta_s(T) = U_{(0,s,0)} T U^*_{(0,s,0)}$$
$$\gamma_t(T) = U_{(0,0,t)} T U^*_{(0,0,t)}.$$

(Note that $U_{(0,0,t)}$ equals the unitary U_t of Definition 2.2. Thus, for instance, $a_n(T) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \gamma_t(T) \mathrm{e}^{-int} \, \mathrm{d}t$.)

For $\Phi \in S^c \subset B(H)$ we have

$$\alpha_r(\Phi)(x, y, p) = \Phi(x - r, y, p)$$
$$\beta_s(\Phi)(x, y, p) = \mathrm{e}^{ipscx} \Phi(x, y - s, p)$$
$$\gamma_t(\Phi)(x, y, p) = \mathrm{e}^{ipt} \Phi(x, y, p).$$

LEMMA 2.8. *Let $\Phi \in S^c$. Then there exists $K > 0$ such that*

$$\|\alpha_r(\Phi) - \Phi\| \leqslant Kr \qquad and \qquad \|\beta_s(\Phi) - \Phi\| \leqslant Ks$$

for all $r, s > 0$.

Proof. By condition (ii) of Definition 2.1, together with the fact that $|\partial \Phi / \partial x|$ is periodic in x, there exists a positive function $f \in L^1(\mathbb{Z})$ such that

$$\left| \frac{\partial \Phi}{\partial x}(x, y, p) \right| \leqslant f(p)$$

for all $p \in \mathbb{Z}$. So for $r > 0$ we have

$$|(\alpha_r(\Phi) - \Phi)(x, y, p)| = |\Phi(x - r, y, p) - \Phi(x, y, p)| \leqslant r \cdot f(p).$$

It follows that $\|\alpha_r(\Phi) - \Phi\| \leqslant r \cdot \|f\|_1$.

Now use condition (ii) of Definition 2.1 to find a positive function f_1 such that $pf_1(p) \in L^1(\mathbb{Z})$ and $|\Phi(x, y, p)| \leqslant f_1(p)$ for all $p \in \mathbb{Z}$ and a positive function $f_2 \in L^1(\mathbb{Z})$ such that

$$\left| \frac{\partial \Phi}{\partial y}(x, y, p) \right| \leqslant f_2(p)$$

for all $p \in \mathbb{Z}$. Also note that

$$\sup_{0 \leqslant x < 2\pi} |\mathrm{e}^{ipcsx} - 1| \leqslant 2\pi cps.$$

Then

$$|(\beta_s(\Phi) - \Phi)(x, y, p)| = |\mathrm{e}^{\mathrm{i}pcsx}\Phi(x, y - s, p) - \Phi(x, y, p)|$$
$$\leqslant |\mathrm{e}^{\mathrm{i}pcsx} - 1|f_1(p) + |\Phi(x, y - s, p) - \Phi(x, y, p)|$$
$$\leqslant (2\pi cp f_1(p) + f_2(p))s$$

for all $x \in [0, 2\pi)$. But for any $x \in [0, 2\pi)$ and $k \in 2\pi\mathbb{Z}$ we have

$$(\beta_s(\Phi) - \Phi)(x + k, y, p) = \mathrm{e}^{\mathrm{i}pcs(x+k)}\Phi(x + k, y - s, p) - \Phi(x + k, y, p)$$
$$= \mathrm{e}^{\mathrm{i}pcs(x+k)}\mathrm{e}^{\mathrm{i}ckp(y-s)}\Phi(x, y - s, p) - \mathrm{e}^{\mathrm{i}ckpy}\Phi(x, y, p)$$
$$= \mathrm{e}^{\mathrm{i}cpky}(\beta_s(\Phi) - \Phi)(x, y, p),$$

so that the previous estimate holds for all $x \in \mathbb{R}$. It follows that $\|\beta_s(\Phi) - \Phi\| \leqslant \left(2\pi c\|pf_1\|_1 + \|f_2\|_1\right)s$. ∎

THEOREM 2.9. *Let $T \in N_\hbar$. Then $T \in D_\hbar$ if and only if the maps $r \mapsto \alpha_r(T)$ and $s \mapsto \beta_s(T)$ are continuous for the norm topology on $N_\hbar$.*

Proof. "$\Rightarrow$" The set of operators T for which α and β are norm-continuous is easily seen to be norm-closed. Thus it suffices to show that every $\Phi \in S^c \subset B(H)$ has this property. This was shown in Lemma 2.8.

"$\Leftarrow$" Let $T \in N_\hbar$ and suppose α and β are norm-continuous for T. It follows that γ is also norm-continuous for T by the identity

$$\gamma_t = \beta_{t'}^{-1}\alpha_{t'}^{-1}\beta_{t'}\alpha_{t'}$$

where $t' = \sqrt{t/c}$. Thus $\sigma_N(T) \to T$ in norm as $N \to \infty$ by Lemma 2.3. Therefore, to prove that $T \in D_\hbar$ it will suffice to show that each $\sigma_N(T) \in D_\hbar$, or indeed that each $a_n(T) \in D_\hbar$.

Now α and γ commute, so

$$\|\alpha_r(a_n(T)) - a_n(T)\| = \frac{1}{2\pi}\left\|\int\limits_{-\pi}^{\pi}(\alpha_r(\gamma_t(T)) - \gamma_t(T))\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t\right\|$$

$$\leqslant \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\|\alpha_r(\gamma_t(T)) - \gamma_t(T)\|\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\|\alpha_r(T) - T\|\,\mathrm{d}t = \|\alpha_r(T) - T\|.$$

Since T is norm-continuous for α, this shows that $a_n(T)$ is as well; the same argument shows that $a_n(T)$ is norm-continuous for β.

Using the fact that $\alpha_r(Y_n a_n(T)) = Y_n \alpha_r(a_n(T))$, we get that $\alpha_r(Y_n a_n(T))$ is continuous in norm as a function of r. Similarly, a short computation shows that $\beta_s(Y_n)$ is continuous in norm as a function of s, hence $\beta_s(Y_n a_n(T)) = \beta_s(Y_n)\beta_s(a_n(T))$ is also continuous in norm. It is then standard that the functions g and Φ defined in Lemma 2.5 must be uniformly continuous. We let Φ act on H by the formula given in Definition 2.1, so that $a_n(T) = \Phi$ as operators.

Now, just as in the proof of Theorem 2.6, we can smooth Φ by taking a twisted convolution with a C^∞ approximate unit of L^1-norm one, to get a sequence (Φ_m) in S^c. But since Φ is continuous, $\Phi_m \to \Phi$ in sup norm ($=$ operator norm), hence $a_n(T) = \Phi \in D_\hbar$. ∎

COROLLARY 2.10. *$D_\hbar$ consists of precisely the elements of $N_\hbar$ for which the action of G is norm-continuous.*

Proof. Norm-continuity for G implies norm-continuity for α and β, so one direction follows immediately from Theorem 2.9. For the other direction, we also know from Theorem 2.9 that every element of $D_\hbar$ is norm-continuous for α and β, and that this implies norm-continuity for γ as well (as in the proof of Theorem 2.9). But α, β, and γ generate G, so this is enough. ∎

3. THE NONCOMMUTATIVE SUB-RIEMANNIAN METRIC

There is a natural noncommutative sub-Riemannian metric on $N_\hbar$, and it can be presented in both local and global forms. The local version is a sort of noncommutative exterior derivative, while the global version is the noncommutative Lipschitz algebra that is the former's domain.

DEFINITION 3.1. Let $E = N_\hbar \oplus N_\hbar$; we regard it as a Hilbert bimodule over $N_\hbar$ with left and right $N_\hbar$-valued inner products given by

$$\langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle_l = x_1 y_1^* + x_2 y_2^*$$

and

$$\langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle_r = x_1^* y_1 + x_2^* y_2.$$

Let δ_1 and δ_2 be the generators of the actions α and β defined in the last section, i.e.

$$\delta_1(x) = \lim_{r \to 0} \frac{\alpha_r(x) - x}{r} \qquad \text{and} \qquad \delta_2(x) = \lim_{s \to 0} \frac{\beta_s(x) - x}{s}$$

for all $x \in N_\hbar$ for which the limits exist in the weak operator sense. Define $\mathcal{L}_\hbar = \mathrm{dom}(\delta_1) \cap \mathrm{dom}(\delta_2)$ and define $d : \mathcal{L}_\hbar \to E$ by $d(x) = \delta_1(x) \oplus \delta_2(x)$. Give $\mathcal{L}_\hbar$ the norm

$$\|x\|_L = \max(\|x\|, \|d(x)\|_1, \|d(x)\|_r)$$

where $\| \cdot \|_1$ and $\| \cdot \|_r$ are the left and right Hilbert module norms on E.

In the case $\hbar = 0$, δ_1 and δ_2 are genuine partial derivatives, and $d(x)$ is the projection of the exterior derivative of x onto the cotangent subbundle dual to B.

The following alternative characterization of $\mathcal{L}_\hbar$ is useful. It follows immediately from [7], Proposition 3.1.6.

LEMMA 3.2. *Let $x \in N_\hbar$. Then $x \in \mathcal{L}_\hbar$ if and only if $\sup\limits_{r>0} \|\alpha_r(x) - x\|/r$ and $\sup\limits_{s>0} \|\beta_s(x) - x\|/s$ are finite.*

THEOREM 3.3. *The map d and its domain $\mathcal{L}_\hbar$ have the following properties:*
  (i) *d is an unbounded derivation with (weak*)-closed graph.*
  (ii) *$\mathcal{L}_\hbar$ is a dual Banach algebra. It contains S^c and is densely contained in $D_\hbar$.*
  (iii) *If $\hbar = 0$ then $\mathcal{L}_\hbar$ is naturally identified with the algebra of functions on M_c which are Lipschitz for the sub-Riemannian metric defined in Section 1, and $\|d(x)\|_1 = \|d(x)\|_r$ equals the Lipschitz number of x, for any $x \in \mathcal{L}_0 \cong \mathrm{Lip}\,(M_c)$.*

Proof. (i) The fact that d is a derivation, i.e. is linear and satisfies the Liebniz formula $d(xy) = xd(y) + d(x)y$, is an elementary calculation. Weak*-closure of the graph follows from [7], Proposition 3.1.6.

(ii) The norm $\| \cdot \|_L$ equals the graph norm when $\mathcal{L}_\hbar$ is identified with the graph of d by the map $x \mapsto x \oplus d(x)$ and E is given the max of its left and right Hilbert norms (which is equivalent to the von Neumann algebra norm on E). Thus $\mathcal{L}_\hbar$ is isometric to a weak*-closed subspace of a dual Banach space, hence $\mathcal{L}_\hbar$ is a dual space. It is an algebra because $\mathrm{dom}(\delta_1)$ and $\mathrm{dom}(\delta_2)$ are both algebras (being domains of derivations).

$S^c \subset \mathcal{L}_\hbar$ follows from Lemma 2.8 and Lemma 3.2. $\mathcal{L}_\hbar \subset D_\hbar$ follows from Theorem 2.9 and Lemma 3.2, using the fact that continuity of an $\mathbb{R}$-action is equivalent to continuity at 0. Also, $\mathcal{L}_\hbar$ is dense in $D_\hbar$ because it contains S^c.

(iii) Let $\hbar = 0$. It is straightforward to check that D_0 and N_0 are, respectively, naturally isomorphic to $C(M_c)$ and $L^\infty(M_c)$; this simply involves taking the Fourier transform in the p variable. Now if $f \in L^\infty(M_c)$ is Lipschitz for the sub-Riemannian metric then it satisfies

$$\|f \circ \varphi - f\|_\infty \leqslant L(f) \cdot h$$

for any isometry φ of M_c such that $d_B(\varphi(\rho), \rho) = h$ for all $\rho \in M_c$, where $L(f)$ is the Lipschitz number of f. Taking

$$\varphi(r, s, t) = (r - h\cos\theta, s - h\sin\theta, t - hr\sin\theta),$$

this shows that $\mathrm{Lip}\,(M_c) \subset \mathcal{L}_0$ by Lemma 3.2; and

$$\|d(f)\|_1^2(\rho) = |\delta_1(f)(\rho)|^2 + |\delta_2(f)(\rho)|^2 \leqslant L(f)^2$$

for almost every $\rho \in M_c$, so $\|d(f)\| \leqslant L(f)$.

Conversely, let f be any function in $\mathcal{L}_0$ and let $\rho, \sigma \in M_c$. By [5], Theorem 2.7 there exists a constant velocity geodesic $p : [0,1] \to M_c$ which is everywhere tangent to the subbundle B of $\mathrm{T}M_c$ defined in Section 1 and which satisfies $p(0) = \rho$, $p(1) = \sigma$, and $l(p) = d_B(\rho, \sigma)$. Then the function $g = f \circ p : [0,1] \to \mathbb{C}$ satisfies $g(0) = f(\rho)$, $g(1) = f(\sigma)$, and $g'(t) = \langle d(f)(t), dp(t) \rangle$ for almost every $t \in [0,1]$. This implies that g is Lipschitz with

$$L(g) \leqslant \|dg\| = \|d(f)\| \cdot l(p),$$

so

$$\frac{|f(\rho) - f(\sigma)|}{d_B(\rho, \sigma)} = \frac{g(0) - g(1)}{l(p)} \leqslant L(g)/l(p) \leqslant \|d(f)\|.$$

Taking the supremum over all ρ and σ shows that f is Lipschitz and $L(f) \leqslant \|d(f)\|$. ∎

We conclude this section with a proof that the unit ball of $\mathcal{L}_\hbar$ is compact in operator norm. In the commutative case, this is true for $\mathrm{Lip}\,(X)$ precisely when X is compact. In addition it was proved for noncommutative Lipschitz algebras associated to noncommutative tori in [15] and [16], and our proof here uses basically the same method. These results also follow from an unpublished theorem of Rieffel which deals with the general situation of a Lie group acting on a Banach space ([13]). I do not know whether that line of reasoning implies our current result (it is not obvious because $\mathbb{R}^2$, the Lie group that appears here, is not compact).

LEMMA 3.4. *For any $\varepsilon > 0$ there exists N large enough such that $\|x - \sigma_n(x)\| \leqslant \varepsilon$ for all $x \in \mathrm{ball}\,(\mathcal{L}_\hbar)$ and $n \geqslant N$.*

Proof. Recall that $\gamma_t = \beta_{t'}^{-1}\alpha_{t'}^{-1}\beta_{t'}\alpha_{t'}$ where $t' = \sqrt{t/c}$. Now for any $y \in \mathrm{ball}(\mathcal{L}_\hbar)$ we have

$$\|\alpha_{t'}(y) - y\|, \ \|\beta_{t'}(y) - y\| \leqslant t' \cdot \|d(y)\| \leqslant t',$$

so that $\|\gamma_t(x) - x\| \leqslant 4t'$ for any $x \in \text{ball}(\mathcal{L}_\hbar)$. So we have

$$\|x - \sigma_n(x)\| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|x - \gamma_t(x)\| K_n(t)\, dt \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} 4\sqrt{t/c} K_n(t)\, dt,$$

and the last formula goes to zero as $n \to \infty$. This is what we needed to show. ∎

THEOREM 3.5. *The unit ball of $\mathcal{L}_\hbar$ is compact in operator norm.*

Proof. Let (x_k) be any sequence in $\text{ball}(\mathcal{L}_\hbar)$; we will find a convergent subsequence.

As in Lemma 2.5, $Y_n a_n(x_k)$ is the operator of multiplication by some function $g_n^k \in L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$. Now

$$\|\alpha_r(Y_n a_n(x_k)) - Y_n a_n(x_k)\| = \|\alpha_r(a_n(x_k)) - a_n(x_k)\|$$

$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|(\alpha_r(\gamma_t(x_k)) - \gamma_t(x_k))\mathrm{e}^{-int}\|\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\alpha_r(x_k) - x_k\|\, dt \leqslant r\|\delta_1(x_k)\| \leqslant r.$$

Similarly, $\|\beta_s(Y_n a_n(x_k)) - Y_n a_n(x_k)\| \leqslant s$, and this implies that the function g_n^k is Lipschitz with Lipschitz number at most 2. Since $[0, 2\pi] \times \mathbb{T} \times \{0\}$ is compact, we may choose a subsequence $g_n^{k_j}$ which converges in sup norm on this set; by $(*)$ and $(\dagger)$ of Lemma 2.5 this implies that $g_n^{k_j}$ converges in sup norm on all of $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$. Allowing n to vary, finding successive subsequences for which $g_n^{k_j}$ converges, and diagonalizing, we get a subsequence (x_{k_j}) of (x_k) such that $(g_n^{k_j})$ converges in sup norm for all n.

Let x be a weak operator cluster point of (x_{k_j}) and let $Y_n a_n(x_k)$ be multiplication by g_n. Then g_n is a weak*-cluster point of $(g_n^{k_j})$ in $L^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$, but since $(g_n^{k_j})$ already converges in sup norm we have $g_n^{k_j} \to g_n$.

Given $\varepsilon > 0$, by Lemma 3.4 we can find a positive integer N such that $\|x - \sigma_N(x)\| \leqslant \varepsilon$ and $\|x_{k_j} - \sigma_N(x_{k_j})\| \leqslant \varepsilon$ for all j. Taking M large enough so that $j \geqslant M$ implies

$$\|g_n - g_n^{k_j}\|_\infty \leqslant \frac{\varepsilon}{(2N+1)}$$

for all $|n| \leqslant N$, we get

$$\|\sigma_N(x) - \sigma_N(x_{k_j})\| \leqslant \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \|a_n(x) - a_n(x_{k_j})\|$$

$$= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \|g_n - g_n^{k_j}\| \leqslant \varepsilon.$$

Thus

$$\|x - x_{k_j}\| \leqslant \|x - \sigma_N(x)\| + \|\sigma_N(x) - \sigma_N(x_{k_j})\| + \|\sigma_N(x_{k_j}) - x_{k_j}\| \leqslant 3\varepsilon$$

for $j \geqslant M$. So $x_{k_j} \to x$ in operator norm. ∎

4. FURTHER PROPERTIES

In this section we first identify the sub-Riemannian noncommutative Lipschitz algebra $\mathcal{L}_\hbar$ defined in Section 3 with a noncommutative Hölder algebra. This generalizes the classical fact (see [10]) that the sub-Riemannian metric on the Heisenberg manifolds is comparable to the square root of the Riemannian metric in the z direction. Then we use work of Sauvageot ([14]) to establish the existence of a heat semigroup on $N_\hbar$ and identify its generator with a noncommutative Laplacian.

DEFINITION 4.1. Let $A, B, C \in (0, 1]$ and define $\mathcal{L}_\hbar^{A,B,C}$ to be the set of $x \in N_\hbar$ for which there exists a constant $K \geqslant 0$ such that

$$\|x - \alpha_r(x)\| \leqslant Kr^A, \qquad \|x - \beta_s(x)\| \leqslant Ks^B, \qquad \|x - \gamma_t(x)\| \leqslant Kt^C$$

for all $r, s, t > 0$. Let $L(x) = L^{A,B,C}(x)$ be the smallest possible value of K and norm $\mathcal{L}_\hbar^{A,B,C}$ by

$$\|x\|_{A,B,C} = \max(\|x\|, L(x)).$$

Note that $A' \leqslant A, B' \leqslant B, C' \leqslant C$ implies $\mathcal{L}_\hbar^{A,B,C} \subset \mathcal{L}_\hbar^{A',B',C'}$. Indeed, if $K = \max(L^{A,B,C}(x), 2\|x\|)$ then

$$\|x - \alpha_r(x)\| \leqslant \begin{cases} Kr^A \leqslant Kr^{A'} & \text{for } r \leqslant 1 \\ 2\|x\| \leqslant Kr^{A'} & \text{for } r \geqslant 1, \end{cases}$$

and similarly for β and γ, so that $L^{A',B',C'}(x) \leqslant \max(L^{A,B,C}(x), 2\|x\|)$.

Note also that $\mathcal{L}_\hbar^{1,1,1} = \text{dom}(\alpha') \cap \text{dom}(\beta') \cap \text{dom}(\gamma')$ by an obvious extension of Lemma 3.2. For this reason we can realize $\mathcal{L}_\hbar^{1,1,1}$ as the domain of a derivation into the Hilbert module $N_\hbar \oplus N_\hbar \oplus N_\hbar$ in the same way that we treated $\mathcal{L}_\hbar$ in Definition 3.1. However, for non-unit values of A, B, or C, Hilbert modules are not appropriate. We instead use a construction from [16] to handle this case.

DEFINITION 4.2. Let

$$F = \bigoplus_{t>0}^{\infty}(N_\hbar \oplus N_\hbar \oplus N_\hbar)$$

be the l^∞ direct sum of von Neumann algebras. Regard it as a dual operator $N_\hbar$-bimodule with left action given by the diagonal embedding of $N_\hbar$ in F and right action given by the embedding

$$x \mapsto \bigoplus_{t>0}(\alpha_t(x) \oplus \beta_t(x) \oplus \gamma_t(x)).$$

Define a map $d : \mathcal{L}_\hbar^{A,B,C} \to F$ by $d(x) = \bigoplus d_t(x)$ with

$$d_t(x) = \frac{x - \alpha_t(x)}{t^A} \oplus \frac{x - \beta_t(x)}{t^B} \oplus \frac{x - \gamma_t(x)}{t^C}.$$

THEOREM 4.3. *The map d and its domain $\mathcal{L}_\hbar^{A,B,C}$ have the following properties:*

(i) *d is an unbounded derivation with weak*-closed graph.*

(ii) *$\mathcal{L}_\hbar^{A,B,C}$ is a dual Banach algebra. It is densely contained in $D_\hbar$ and, if $C < 1$, it contains S^c.*

(iii) *The unit ball of $\mathcal{L}_\hbar^{A,B,C}$ is compact in operator norm.*

Proof. (i) It is routine to check that d is a derivation. To verify weak*-closedness of the graph, let $(x_\lambda) \subset \mathcal{L}_\hbar^{A,B,C}$ be a net which weak operator converges to $x \in N_\hbar$ and suppose that $d(x_\lambda)$ weak operator converges to $\bigoplus y_t^1 \oplus y_t^2 \oplus y_t^3 \in F$. By the Krien-Smulian theorem we may assume that (x_λ) is bounded in norm. Then, restricting attention to the tth summand of F, we have $d_t(x_\lambda) \to y_t^1 \oplus y_t^2 \oplus y_t^3$. Thus

$$\frac{x_\lambda - \alpha_t(x_\lambda)}{t^A} \to y_t^1$$

(weak operator), but the left side also converges to $(x - \alpha_t(x))/t^A$. An identical argument applies to the other two summands of $d_t(x)$ and so we conclude that $d_t(x_\lambda) \to d_t(x)$. Boundedness of the net then implies that $x \in \mathcal{L}_\hbar^{A,B,C}$ and $d(x_\lambda) \to d(x)$.

(ii) $\mathcal{L}_\hbar^{A,B,C}$ is a dual Banach algebra by the same easy argument used in Theorem 3.3 (ii). It is contained in $D_\hbar$ by Theorem 2.9. Also, as remarked above, it contains $L_\hbar^{1,1,1}$. We claim that $\mathcal{L}_\hbar^{1,1,1}$ contains all $\Phi \in S^c$ which are zero for all but a finite number of values of p. This implies density in $D_\hbar$ by Lemma 2.3.

To prove the claim it suffices to consider only those $\Phi \in S^c$ for which $\Phi(x, y, p) = 0$ unless $p = n$, for some fixed $n \in \mathbb{Z}$. For such Φ the operator norm equals the L^∞ norm, and so

$$\|\gamma_t(\Phi) - \Phi\| = \|(e^{int} - 1)\Phi\|_\infty \leqslant nt\|\Phi\|_\infty.$$

Together with Lemma 2.8, this implies $\Phi \in \mathcal{L}_\hbar^{1,1,1}$, as claimed.

Now suppose $C < 1$ and let Φ be any function in S^c. Choose $N \geqslant 1 + 2/(1-C)$. Then by part (ii) of Definition 2.1, there exists a constant K such that $|p^N \Phi(x, y, p)| \leqslant K$ for $x \in [0, 2\pi]$ and all y and p. By Part (i) of Definition 2.1, this implies that $p^N \Phi$ is bounded on all of $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$. Now, for any $t > 0$ define $\Phi_q(x, y, p) = \delta_{p,q}\Phi(x, y, p)$ and $\Phi' = \Phi - \sum\limits_{|p| \leqslant t^{-1/(N-1)}+1} \Phi_q$. Then $\|\Phi_q\|_\infty \leqslant q^{-N}K$ and so we can bound the operator norm of Φ' by

$$\|\Phi'\| \leqslant \sum_{|q| > t^{-1/(N-1)}+1} q^{-N}K \leqslant 2 \int\limits_{t^{-1/(N-1)}}^{\infty} q^{-N}K \, dq$$
$$= 2\frac{(t^{-1/(N-1)})^{1-N}}{N-1}K = \frac{2tK}{N-1}.$$

Thus $\|\gamma_t(\Phi') - \Phi'\| \leqslant 4tK/(N-1)$, so that $\|\gamma_t(\Phi') - \Phi'\|/t^C$ is bounded for $t \leqslant 1$. Moreover, we have

$$|(\gamma_t(\Phi_q) - \Phi_q)(x, y, p)| = |(e^{iqt} - 1)\Phi_q(x, y, p)| \leqslant |qt|\|\Phi\|_\infty \leqslant (t + t^{1-1/(N-1)})\|\Phi\|_\infty$$

for $|q| \leqslant t^{-1/(N-1)} + 1$, so that for $t \leqslant 1$ and $|q| \leqslant t^{-1/(N-1)} + 1$ we have

$$\|\gamma_t(\Phi_q) - \Phi_q\| \leqslant 2t^{1-1/(N-1)}\|\Phi\|_\infty.$$

Hence

$$\|\gamma_t(\Phi - \Phi') - (\Phi - \Phi')\| \leqslant (2t^{-1/(N-1)} + 3) \cdot 2t^{1-1/(N-1)}\|\Phi\|_\infty \leqslant 12t^C\|\Phi\|_\infty.$$

We conclude that $\|\gamma_t(\Phi) - \Phi\|/t^C$ is bounded for $t \leqslant 1$. But for $t \geqslant 1$ we have

$$\|\gamma_t(\Phi) - \Phi\| \leqslant 2\|\Phi\| \leqslant 2t^C\|\Phi\|,$$

so that $\|\gamma_t(\Phi) - \Phi\|/t^C$ is bounded for all $t > 0$. Thus $\Phi \in \mathcal{L}_\hbar^{A,B,C}$.

(iii) This is proved in the same way as Theorem 3.5. ∎

THEOREM 4.4. $\mathcal{L}_\hbar$ and $\mathcal{L}_\hbar^{1,1,1/2}$ are identical as sets and have isomorphic norms.

Proof. $\mathcal{L}_\hbar^{1,1,1/2} \subset \mathcal{L}_\hbar$ is clear from the definitions. Conversely, it was noted in the proof of Lemma 3.4 that if $x \in \mathrm{ball}\,(\mathcal{L}_\hbar)$, then $\|\gamma_t(x) - x\| \leqslant 4\sqrt{t/c}$. Hence $\|\gamma_t(x) - x\|/\sqrt{t}$ is bounded and we get $x \in \mathcal{L}_\hbar^{1,1,1/2}$. So $\mathcal{L}_\hbar = \mathcal{L}_\hbar^{1,1,1/2}$ as sets.

Isomorphism follows from the estimate

$$\|x\|_L \leqslant \max(\|x\|, \|\delta_1(x)\| + \|\delta_2(x)\|) \leqslant \max(\|x\|, 2L(x)) \leqslant 2\|x\|_{A,B,C}$$

together with the open mapping theorem. ∎

As a final topic we consider a natural Laplacian and the heat semigroup it generates. The following tool is needed.

DEFINITION 4.5. Define $\tau : N_\hbar \to \mathbb{C}$ by

$$\tau(T) = \frac{1}{2\pi} \int\limits_0^1 \int\limits_0^{2\pi} \Phi(x, y, 0)\, \mathrm{d}x\, \mathrm{d}y$$

where Φ is the function associated to $a_0(T)$ in Lemma 2.5.

THEOREM 4.6. ([12], p. 558) *The map τ is a faithful normal finite trace. It is invariant for the action L given in Definition* 2.7.

(The fact that τ is normal follows from the fact that it is the composition of two normal maps: integration of $\gamma_t(T)$ and integration over $\mathbb{T}^2$.)

PROPOSITION 4.7. *The* GNS *representation of $N_\hbar$ associated to τ is unitarily equivalent to the restriction of its original representation on $H = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ to $H' = L^2([0, 2\pi] \times \mathbb{T} \times \mathbb{Z})$.*

Proof. Define a map $\Psi : N_\hbar \to H'$ by $\Psi(T) = T(\xi)$ where $\xi(x, y, p) = 1$ for $(x, y, p) \in [0, 2\pi] \times \mathbb{T} \times \{0\}$ and $\xi(x, y, p) = 0$ elsewhere. Then for any $T \in N_\hbar$ we have $\tau(T) = \langle T\xi, \xi \rangle$, so that Ψ extends to a unitary map from the Hilbert space of the GNS representation onto H'. It intertwines the action of $N_\hbar$ because

$$\Psi(TS) = TS(\xi) = T\Psi(S). \quad \blacksquare$$

Now we define a Laplacian on S^c, and apply a theorem of Sauvageot to establish the existence of a heat semigroup.

DEFINITION 4.8. For $\Phi \in S^c$ define $\Delta\Phi$ by $\Delta\Phi = \delta_1(\delta_1(\Phi)) + \delta_2(\delta_2(\Phi))$. Concretely, we have

$$\Delta\Phi = \Phi_{xx} - p^2 c^2 x^2 \Phi(x,y,p) - 2ipcx\Phi_y + \Phi_{yy}.$$

The fact that $\Delta\Phi \in S^c$ follows immediately from the first definition of $\Delta\Phi$, or is a routine computation from the second.

THEOREM 4.9. *The operators* $e^{-t\Delta}$, $t \geqslant 0$, *form a weak-operator continuous semigroup of completely positive normal contractions of* $N_\hbar$.

Proof. First we show that $\delta_1 \oplus \delta_2$ is closed when regarded as an operator from H' into $H' \oplus H'$.

Identify H' with the functions Φ on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ which satisfy

$$\Phi(x+k,y,p) = e^{ickpy}\Phi(x,y,p)$$

for all $k \in 2\pi\mathbb{Z}$ and whose restriction to $[0, 2\pi] \times \mathbb{T} \times \mathbb{Z}$ is square integrable. Then for all $r, s, t \in \mathbb{R}$ the formula

$$(U_{(r,s,t)}\xi)(x,y,p) = e^{ip(t+cs(x+\hbar p\mu - r))}\xi(x-r, y-s, p)$$

from Definition 2.7 defines a unitary operator on H'. Thus setting $\alpha'_r(\xi) = U_{(r,0,0)}\xi$ and $\beta'_s(\xi) = U_{(0,s,0)}\xi$ gives us two strongly continuous one-parameter unitary groups on H', whose generators D_1 and D_2 satisfy $D_i(\Phi) = \delta_i(\Phi)$, treating Φ respectively as an element of H' and of $N_\hbar$.

Since D_1 and D_2 are self-adjoint they are closed, hence

$$D = D_1 \oplus D_2 : H' \mapsto H' \oplus H'$$

is closed. Also $D^*(\xi \oplus 0) = D_1(\xi) = \delta_1(\xi)$ and $D^*(0 \oplus \xi) = D_2(\xi) = \delta_2(\xi)$, so

$$D^*D(\Phi) = D^*(\delta_1(\Phi) \oplus \delta_2(\Phi)) = \delta_1(\delta_1(\Phi)) + \delta_2(\delta_2(\Phi)) = \Delta\Phi$$

for all $\Phi \in S^c$. The desired conclusion now follows from ([14], Corollary 3.5).

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