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PSEUDOSPECTRA OF DIFFERENTIAL OPERATORS

E. BRIAN DAVIES

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ABSTRACT. We study the pseudospectral theory of a variety of non-self-adjoint constant coefficient and variable coefficient differential operators, showing that the phenomenon of non-trivial pseudospectra is typical rather than exceptional. We prove that the pseudospectra provide more stable information about the operators under various limiting procedures than does the spectrum.

KEYWORDS: *Spectrum, pseudospectrum, norms of inverses, resolvent operators, differential operators.*

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1. INTRODUCTION

It is well established that the spectrum of a self-adjoint operator is of crucial importance in understanding its action in various applied contexts. For highly non-self-adjoint operators, on the other hand, there is increasing evidence that the spectrum is often not very helpful, and that the pseudospectra are of more importance. We refer to [20], [21] for references to the increasing literature on this concept, and for a series of examples in which the pseudospectra have been computed and displayed graphically.

The goal of the first part of this paper is to determine the stability (in the sense of [1], [2], [18]) of certain sequences of ordinary differential operators acting over an increasing sequence of bounded intervals. In particular, we prove that the pseudospectra converge to a specified limit as the widths of the intervals become infinite. We always impose Dirichlet boundary conditions (DBCs), but discuss the effect of making other choices of boundary conditions at the end of the paper. The operators concerned are not self-adjoint, and our analysis covers the case of the one-dimensional convection-diffusion operator studied in detail in [15], [21].

In Section 5 we treat elliptic partial differential operators with variable coefficients acting on $L^2(\mathbb{R}^N)$, including variable coefficient convection-diffusion operators of the form

$$Af(x) := -\delta\Delta f(x) + a(x)\cdot\nabla f(x).$$

The new feature of these operators is that they are compactly supported perturbations of self-adjoint operators, but this does not prevent the instability of the spectrum under the limit $\delta \to 0$. Our results once again support the view that the pseudospectra have substantial importance when investigating non-self-adjoint operators; see Section 5 for further details.

Our theorems extend the work of [1], [2], [14], [15], [17], [21] by providing a further range of operators for which one cannot compute the spectrum by the standard process of restricting the operator to a bounded interval, computing the spectrum there and then taking the limit as the interval expands. It now appears that these features are fairly typical for differential operators of the type which occur throughout applied mathematics.

Our analysis will be described in the language of pseudospectral theory; this could have been avoided but we find that the geometrical pictures it affords provide very useful motivation. Let A be a closed densely defined operator on a Hilbert space $\mathcal{H}$ and suppose that $\mathrm{Spec}(A)$ is not equal to the entire complex plane $\mathbb{C}$. Given $\varepsilon > 0$ the pseudospectrum $\mathrm{Spec}_\varepsilon(A)$ is defined to be the union of its spectrum with the set of $z \in \mathbb{C}$ such that

$$\|(z-A)^{-1}\| > \varepsilon^{-1}.$$

Equivalently (see [18], Proposition 4.15)

$$\mathrm{Spec}_\varepsilon(A) = \bigcup\{\mathrm{Spec}(A+D) : \|D\| < \varepsilon\}.$$

The proof of this is elementary because we impose a strict inequality in the definition of the pseudospectrum, instead of the more conventional non-strict inequality, thus making the pseudospectrum an open set.

The pseudospectrum of any operator contains the ε-neighbourhood of its spectrum, and is contained in the ε-neighbourhood of its numerical range under mild conditions; see Section 3. If A is normal, the pseudospectrum of A is equal to the ε-neighbourhood of the spectrum, but in general it may be much bigger.

See [18], [21] for references to the proofs of basic properties of the pseudospectra and [19] for a discussion of methods of computing the pseudospectra of matrices.

The first part of this paper is devoted to the analysis of the spectrum and pseudospectra of an elliptic ordinary differential operator of the form

$$Af(x) := \sum_{r=0}^{2n} a_r \frac{\mathrm{d}^r f}{\mathrm{d}x^r},$$

where a_r are complex constants and $a_{2n} \neq 0$. We assume that A acts in $L^2(I)$ where $I = (-b, b)$, $(0, \infty)$ or $\mathbb{R}$. We assume Dirichlet boundary conditions in the sense that $f(c) = f'(c) = \cdots = f^{(n-1)}(c) = 0$ at any finite boundary point c.

As an illustration of the peculiar nature of such operators we mention the convection-diffusion operator

$$Af(x) := -f''(x) - 2f'(x).$$

When acting on $L^2(\mathbb{R})$, this operator is normal with spectrum the parabolic curve

$$\{x + \mathrm{i}y : y^2 = 2x\}.$$

However, for any finite b, the operator acting in $L^2(-b, b)$ has discrete real spectrum which converges as $b \to \infty$ to the real interval $[1, \infty)$. The lack of connection between the spectra of these operators is remedied if one considers the pseudospectra instead ([15]). Similar conclusions follow for general even order constant coefficient ODEs from our general theory in Section 4, which extends and completes earlier results of Reddy ([14]).

2. THE SPECTRUM

The precise domain of the ordinary differential operator A on $L^2(I)$ is described as follows. We may write $A = A_0 + A_1$ where $A_0 := a_{2n}\mathrm{d}^{2n}/\mathrm{d}x^{2n}$ has domain $W_0^{n,2}(I) \cap W^{2n,2}(I)$. The perturbation A_1 is relatively bounded with relative bound 0, so the domain of A equals that of A_0; the adjoint operator has the same domain. One may also define A by quadratic form methods. In this case one considers the sectorial form

$$Q(f, g) := \langle Af, g\rangle$$

initially defined on $C_c^\infty(I)$. The operator A is then that operator associated with the closure of the form Q as described in [7]; the domain of the closure is precisely $W_0^{n,2}(I)$.

The spectrum of the operator A acting on $L^2(\mathbb{R})$ may be determined by Fourier analysis ([5], Chapter 3). If

$$p(\xi) := \sum_{r=0}^{2n} a_r \xi^r,$$

then the spectrum of A, which is normal, is

$$\mathrm{Spec}(A) = \{p(\mathrm{i}\xi) : \xi \in \mathbb{R}\}.$$

There are several definitions of the essential spectrum for closed operators, and we follow [3], Chapter 3, Section 2 which defines it via the theory of Fredholm operators. Since the essential spectrum is invariant under relatively compact perturbations, the essential spectrum of an elliptic differential operator is not changed by local perturbations of the coefficients ([7], [3]). However, in the non-normal case the non-essential spectrum may be quite different from the set of isolated eigenvalues of finite multiplicity. For the operator A on $L^2(\mathbb{R})$ its spectrum coincides with its essential spectrum.

Let A_b denote the restriction of A to $L^2(-b,b)$ subject to DBCs. A_b has compact resolvent so its spectrum is discrete. The eigenvalues may be computed by standard methods. We do not provide a systematic description of the spectrum, but note that the case of the convection-diffusion operator, Example 2.3 below, shows that it does not generally converge in any sense to the spectrum of A as $b \to \infty$. The link between the two involves the pseudospectra and is mediated by the operator acting on $L^2(0,\infty)$, to which we now turn.

Let A, B, C denote our operator acting on $L^2(\mathbb{R})$, $L^2(0,\infty)$ and $L^2(-\infty,0)$ respectively. Among these operators only A is normal.

Lemma 2.1. *The spectrum of B is the union of the spectrum of A, the set of L^2 eigenvalues of B and the complex conjugate of the set of L^2 eigenvalues of B^*.*

Proof. If $\lambda \in \mathrm{Spec}(A)$ then there exists $\xi \in \mathbb{R}$ such that $p(\mathrm{i}\xi) = \lambda$. Let $\varphi \in C_c^\infty(\mathbb{R})$ satisfy $\varphi(x) = 0$ if $x \leqslant 1$ or $x \geqslant 4$, and $\varphi(x) = 1$ if $2 \leqslant x \leqslant 3$. Then define $f_n \in C_c^\infty(0,\infty)$ for positive n by

$$f_n(x) := \mathrm{e}^{\mathrm{i}\xi x}\varphi(x/n).$$

A direct computation establishes that

$$\lim_{n\to\infty} \|Bf_n - \lambda f_n\|/\|f_n\| = 0$$

and this proves that $\lambda \in \mathrm{Spec}(B)$.

Since $B+C$ differs from A only by imposing DBCs at 0, the difference of the resolvents is of finite rank. If $\lambda \notin \mathrm{Spec}(A)$ then it follows that $B - \lambda I$ and $C - \lambda I$ are Fredholm operators. Thus $\lambda \in \mathrm{Spec}(B)$ if and only if λ is an eigenvalue of B or $\overline{\lambda}$ is an eigenvalue of B^*. ∎

The computation of the eigenvalues of B and B^* involves studying the location of the solutions of $p(z) = \lambda$. Given λ, let R be large enough so that all solutions of $p(z) = \lambda$ lie inside the circle $\{|z| = R\}$. Let γ_0 be the anticlockwise closed curve composed of the straight line from $-\mathrm{i}R$ to $\mathrm{i}R$ and the semicircle $\{R\mathrm{e}^{\mathrm{i}\theta} : \pi/2 \leqslant \theta \leqslant 3\pi/2\}$. Then the number of roots of $p(z) = \lambda$ which have negative real parts equals the winding number of λ with respect to the closed curve $\gamma_1(t) := p(\gamma_0(t))$. Note that for large enough R the semicircular part of γ_0 corresponds to a part of γ_1 which winds approximately n times anticlockwise around the origin.

The following theorem is due to Reddy ([14]); analogues for Toeplitz and Wiener-Hopf operators have a long ancestry ([1], [2], [9], [15]).

THEOREM 2.2. *We have* $\lambda \in \mathrm{Spec}(B) \setminus \mathrm{Spec}(A)$ *if and only if the winding number of the curve* γ_1 *around* λ *is not equal to* n.

Proof. We assume for simplicity that the solutions of $p(z) = \lambda$ are distinct; the proof may easily be modified to deal with the general case. Assuming that $\lambda \notin \mathrm{Spec}(A)$, every solution of $p(z) = \lambda$ has non-zero real part. If there are $n + 1$ (or more) solutions $z_1, \ldots, z_{n+1}$ with $\mathrm{Re}(z_r) < 0$ then there exists a linear combination

$$f(x) := \sum_{r=1}^{n+1} \alpha_r \mathrm{e}^{z_r x}$$

which lies in $L^2(0, \infty)$ and satisfies the boundary conditions at 0. One may directly check that λ is an eigenvalue of B with eigenvector f. If there are n or fewer solutions which satisfy $\mathrm{Re}(z) < 0$ then λ is not an eigenvalue of B.

If there are $n+1$ (or more) solutions $z_1, \ldots, z_{n+1}$ of $p(z) = \lambda$ with $\mathrm{Re}(z_r) > 0$ then there exists a linear combination

$$f(x) := \sum_{r=1}^{n+1} \alpha_r \mathrm{e}^{-\overline{z}_r x}$$

which lies in $L^2(0, \infty)$ and satisfies the boundary conditions at 0. One may directly check that $\overline{\lambda}$ is an eigenvalue of B^* with eigenvector f. If there are n or fewer solutions which satisfy $\mathrm{Re}(z) > 0$ then $\overline{\lambda}$ is not an eigenvalue of B^*. ∎

EXAMPLE 2.3. Let

$$Af(x) := -f''(x) - 2f'(x)$$

acting on $L^2(\mathbb{R})$. The spectrum of A is the parabola $y^2 = 4x$. Every point inside the parabola has winding number 2 with respect to γ_1 while every point outside has winding number 1 (for large enough R). Therefore the operator B on $L^2(0,\infty)$ has spectrum consisting of the parabola together with its interior. The operator A_b on $L^2(-b,b)$ has spectrum consisting of the real eigenvalues $1+\pi^2n^2/4b^2$ where $n=1,2,\ldots$, which does not converge to the spectrum of A or B as $b\to\infty$. See [15] for further discussion of this case.

Example 2.4. Let A be the differential operator associated with the polynomial

$$p(\mathrm{i}\xi) := 2\xi^6 - 6\xi^2 + \xi^{10} + \mathrm{i}\xi + \mathrm{i}c\,\xi(\xi^2-1/4)(\xi^2-1)$$

where c is a real constant. For small $c>0$ the complement of the spectrum of the operator B on $L^2(0,\infty)$ subject to DBCs is connected but as c increases two lobes come together and overlap leaving a hole in the spectrum. See Figure 1, which has $c=9$.

Figure 1. Polynomial spectrum.

3. THE NUMERICAL RANGE

The material in this section is all standard, but is included for completeness. If A is a closed linear operator with dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$, its numerical range

$$\mathrm{Num}(A) := \{\langle Af, f\rangle : \|f\| = 1\}$$

is a convex subset of $\mathbb{C}$. The ε-neighbourhood of any subset S of $\mathbb{C}$ is defined by

$$N_\varepsilon(S) := \{s + x : s \in S \text{ and } |x| < \varepsilon\}.$$

Lemma 3.1. *If A is a closed linear operator on $\mathcal{H}$ then*

$$N_\varepsilon\left(\mathrm{Spec}(A)\right) \subseteq \mathrm{Spec}_\varepsilon(A)$$

for all $\varepsilon > 0$. If the complement of $\mathrm{Num}(A)$ *is a connected set containing at least one point not in* $\mathrm{Spec}(A)$ *then one also has*

$$\mathrm{Spec}_\varepsilon(A) \subseteq N_\varepsilon\left(\mathrm{Num}(A)\right).$$

Proof. The first statement follows from standard resolvent estimates; see [3], Corollary 2.3. If $\mathrm{Num}(A) \subseteq \{\mathrm{Re}(z) \geqslant 0\}$ then either every z with $\mathrm{Re}(z) < 0$ lies in $\mathrm{Spec}(A)$ or no such z does; see Theorem 2.24 of [3]. In the latter case $-A$ is the generator of a one-parameter contraction semigroup $T_t = \mathrm{e}^{-At}$ and if $\mathrm{Re}(z) < 0$ we have

$$\|(A - z)^{-1}\| \leqslant \int_0^\infty \|T_t\| \mathrm{e}^{t\mathrm{Re}(z)}\, \mathrm{d}t \leqslant \frac{1}{|\mathrm{Re}(z)|}.$$

The second statement of the theorem follows by applying the above argument to $\alpha A + \beta$ where α and β depend upon the choice of $z \notin \mathrm{Num}(A)$. ∎

Corollary 3.2. *If A is a bounded linear operator on $\mathcal{H}$ then for any $\varepsilon > 0$,*

$$N_\varepsilon\left(\mathrm{Spec}(A)\right) \subseteq \mathrm{Spec}_\varepsilon(A) \subseteq N_\varepsilon\left(\mathrm{Num}(A)\right).$$

We comment that one may also prove this without any use of semigroup theory.

Lemma 3.3. *Let $\mathcal{H}_0$ be a closed linear subspace of $\mathcal{H}$ and let A be a closed sectorial normal linear operator on $\mathcal{H}$. Let A_0 be the restriction of A to $\mathcal{H}_0$ in the sense of quadratic forms. Then*

$$\mathrm{Num}(A_0) \subseteq \overline{\mathrm{conv}}(\mathrm{Spec}A).$$

Proof. Let Q be the closed sectorial form associated with A. Then A_0 is by definition the closed operator associated with the restriction Q_0 of the form Q to $\mathcal{H}_0$. Since A is normal we have

$$\begin{aligned}\mathrm{Num}(A_0) &= \{Q_0(f,f) : \|f\| = 1,\ f \in \mathcal{H}_0\}\\ &\subseteq \{Q(f,f) : \|f\| = 1,\ f \in \mathcal{H}\}\\ &\subseteq \overline{\mathrm{conv}}(\mathrm{Spec}A). \quad \blacksquare\end{aligned}$$

Corollary 3.4. *Let A_I be a constant coefficient even order differential operator acting on $L^2(I)$ subject to* DBCs*, where I is any interval in $\mathbb{R}$. Then*

$$\mathrm{Num}(A_I) \subseteq \overline{\mathrm{conv}}\{p(\mathrm{i}\xi) : \xi \in \mathbb{R}\}.$$

Proof. If A is the same operator acting on $L^2(\mathbb{R})$ then A is normal with spectrum $\{p(\mathrm{i}\xi) : \xi \in \mathbb{R}\}$. The restriction of A to $L^2(I)$ in the sense of quadratic forms is precisely the restriction which satisfies DBCs at the endpoints of I. $\blacksquare$

4. THE PSEUDOSPECTRUM

Although there is no relationship between the spectrum of A_b, the operator acting on $L^2(-b,b)$, and the spectra of A or B, the pseudospectra of these operators are closely related. The conclusion we reach can be summarised in the statement that

$$\lim_{b\to\infty} \mathrm{Spec}_\varepsilon(A_b) = \mathrm{Spec}_\varepsilon(B) \tag{4.1}$$

for all $\varepsilon > 0$. Such a formula is well known for Toeplitz and Wiener-Hopf operators. It was in principle already contained in Henry Landau's papers [10], [11], [12], it was rediscovered and explicitly stated in [15], and a new approach to the problem along with generalizations of former results is in [1], [2]. The formula was proved by Reddy ([14]) for constant coefficient differential operators subject to “triangular” boundary conditions, but it is new for Dirichlet boundary conditions. It will be apparent that our method actually applies to much more general boundary conditions as well. We break the proof down into several small lemmas.

LEMMA 4.1. *If $\varepsilon > 0$ then any compact subset K of $N_\varepsilon(\mathrm{Spec}(A))$ lies in $\mathrm{Spec}_\varepsilon(A_b)$ provided b is large enough.*

Proof. Using the normality of A one can prove that if $\lambda \in N_\varepsilon(\mathrm{Spec}(A))$ then there exists $f \in \mathrm{Dom}(A)$ such that

$$\|Af - \lambda f\| < \varepsilon \|f\|.$$

Since $C_c^\infty(\mathbb{R})$ is dense in $\mathrm{Dom}(A)$ for the graph norm, there exists $g \in C_c^\infty(\mathbb{R})$ such that

$$\|Ag - \lambda g\| < \varepsilon \|g\|.$$

For large enough b we have $g \in \mathrm{Dom}(A_b)$ and can deduce that $\lambda \in \mathrm{Spec}_\varepsilon(A_b)$. For a compact subset K of $\mathrm{Spec}_\varepsilon(A)$ we need only observe that every step may be carried out uniformly with respect to $\lambda \in K$. ∎

LEMMA 4.2. *Let K be any compact subset of $\mathrm{Spec}(B) \setminus \mathrm{Spec}(A)$ and let $\varepsilon > 0$. Then $K \subset \mathrm{Spec}_\varepsilon(A_b)$ provided b is large enough.*

Proof. We first observe that K can be written as a finite union of compact subsets, each of which is contained in a single component of $\mathrm{Spec}(B) \setminus \mathrm{Spec}(A)$. We need only consider one such component, within which the winding number m is constant. There are two cases depending upon whether $m > n$ or $m < n$, and we consider only the first for brevity. For each $\lambda \in K$ there exists an eigenvector $f \in \mathrm{Dom}(B)$ for λ of the form

$$f(x) := \sum_{r=1}^{n+1} \alpha_r \mathrm{e}^{z_r x}$$

where $\mathrm{Re}(z_r) < 0$ for all $1 \leqslant r \leqslant n+1$.

Let φ be a smooth function on $[0, \infty)$ which satisfies $\varphi(x) = 1$ if $x \leqslant 1$ and $\varphi(x) = 0$ if $x \geqslant 2$. Then for $n > 0$ put

$$f_n(x) := f(x+b)\varphi((x+b)/n).$$

One may check that $f_n \in \mathrm{Dom}(A_b)$ for all large enough b. Moreover

$$\lim_{n\to\infty} \lim_{b\to\infty} \|A_b f_n - \lambda f_n\| / \|f_n\| = 0.$$

Given $\varepsilon > 0$ this implies that $\lambda \in \mathrm{Spec}_\varepsilon(A_b)$ for all large enough b. The corresponding result for K depends upon observing that the estimates are uniform with respect to $\lambda \in K$. ∎

It follows from the proof of the next theorem that if K is a compact subset of $\mathrm{Spec}_\varepsilon(B) \setminus \mathrm{Spec}(B)$ then $K \subseteq \mathrm{Spec}_\varepsilon(A_b)$ for all large enough b. By combining this with Lemmas 4.1 and 4.2 we finally deduce that if K is a compact subset of $\mathrm{Spec}_\varepsilon(B)$ then $K \subseteq \mathrm{Spec}_\varepsilon(A_b)$ for all large enough b. In the reverse direction it also follows from the following theorem that if $K \cap \overline{\mathrm{Spec}_\varepsilon(B)} = \emptyset$ then $K \cap \overline{\mathrm{Spec}_\varepsilon(A_b)} = \emptyset$ for all large enough b. These two statements complete the proof of (4.1).

Theorem 4.3. *We have*

$$\lim_{b\to\infty} \|(\lambda - A_b)^{-1}\| = \|(\lambda - B)^{-1}\|$$

uniformly for all λ in any compact subset of $\mathbb{C} \setminus \mathrm{Spec}(B)$.

Proof. We use a twisting idea taken from [5], Section 8.6 or [4], [6]. Let H_1 be the operator on $\mathcal{H}_1 := L^2(\mathbb{R}) \oplus L^2(-b, b)$ defined by

$$H_1 = \begin{pmatrix} A & 0 \\ 0 & A_b \end{pmatrix}$$

and let H_2 be the operator on $\mathcal{H}_2 := L^2(-b, \infty) \oplus L^2(-\infty, b)$ defined by

$$H_2 = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}$$

where B_+ and B_- are given by the same formula as A and satisfy DBCs at $-b$ and b, respectively. The spectra of B_+, B_- and H_2 are all equal to the spectrum of B already determined. Indeed there exist unitary equivalences which enable us to see that

$$\|(\lambda - B)^{-1}\| = \|(\lambda - B_\pm)^{-1}\| = \|(\lambda - H_2)^{-1}\| \tag{4.2}$$

for all $\lambda \notin \mathrm{Spec}(B)$. Since A is normal and $\mathrm{Spec}(A) \subseteq \mathrm{Spec}(B)$ we also have

$$\|(\lambda - A)^{-1}\| = \mathrm{dist}(\lambda, \mathrm{Spec}(A))^{-1} \leqslant \mathrm{dist}(\lambda, \mathrm{Spec}(B))^{-1} \leqslant \|(\lambda - B)^{-1}\| \tag{4.3}$$

for all $\lambda \notin \mathrm{Spec}(B)$, by Lemma 3.1.

We prove the existence of a b-dependent unitary operator $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\lim_{b\to\infty} \|(V H_1 V^{-1} - \lambda)^{-1} - (H_2 - \lambda)^{-1}\| = 0. \tag{4.4}$$

We follow the construction of Theorem 8.6.1 of [5] replacing r by b and replacing σ by a smooth function $\sigma : \mathbb{R} \to [0, \pi/2]$ such that $\sigma(s) = 0$ if $s \leqslant -1/3$ and $\sigma(s) = \pi/2$ if $s \geqslant 1/3$. This leads to the formula

$$VH_1V^{-1} = H_2 + P$$

where

$$\lim_{b\to\infty} \|P(H_2 - \lambda)^{-1}\| = 0 \tag{4.5}$$

for all $\lambda \notin \mathrm{Spec}(H_2)$. Here P is a differential operator of order at most $2n - 1$ whose coefficients have support in $(-b/3, b/3)$ and vanish uniformly as $b \to \infty$. By combining (4.5) with

$$(VH_1V^{-1} - \lambda)^{-1} = (H_2 - \lambda)^{-1} \left(1 + P(H_2 - \lambda)^{-1}\right)^{-1}$$

we can now deduce (4.4) and hence

$$\lim_{b\to\infty} \|(H_1 - \lambda)^{-1}\| = \|(H_2 - \lambda)^{-1}\|$$

for all $\lambda \notin \mathrm{Spec}(B)$, the RHS being independent of b. The theorem follows by combining this with (4.2) and (4.3) and observing that the above estimates are uniform on any compact set K disjoint from $\mathrm{Spec}(B)$. ∎

5. VARIABLE COEFFICIENT CONVECTION-DIFFUSION OPERATORS

In this section we consider the operator

$$Af(x) := -\delta\Delta f(x) + a(x) \cdot \nabla f(x)$$

acting on $L^2(\mathbb{R}^N)$, where a is a real, bounded, possibly discontinuous function and $\delta > 0$. This operator is a relatively bounded perturbation with relative bound zero of the operator $B := -\delta\Delta$, so it has the same domain $W^{2,2}(\mathbb{R}^N)$ and its spectrum is contained in $\{z : \mathrm{Re}(z) \geqslant c\}$ for some real c. The operator is said to become stiff as $\delta \to 0$ and we wish to investigate the limit of the spectrum and pseudospectra in this limit. The limit may be problematical if a is not a Lipschitz continuous function, because then the limiting operator

$$A_0 f(x) := a(x) \cdot \nabla f(x) \tag{5.1}$$

need not generate a flow on $\mathbb{R}^N$. The corresponding problem for higher order operators with discontinuous lower order coefficients is, however, not so clear.

Under the following condition one has no problems about the limit $\delta \to 0$.

THEOREM 5.1. *Suppose that* $a \in W^{1,\infty}$ *and that* $\nabla \cdot a(x) \leqslant 0$ *for all* $x \in \mathbb{R}^N$. *Then* $\operatorname{Spec}(A) \subseteq \{\lambda : \operatorname{Re}(\lambda) \geqslant 0\}$ *and* $\operatorname{Spec}_\varepsilon(A) \subseteq \{\lambda : \operatorname{Re}(\lambda) > -\varepsilon\}$ *for all* $\varepsilon > 0$.

Proof. An integration by parts shows that the hypotheses imply that

$$\operatorname{Re}\langle Af, f\rangle \geqslant 0$$

for all $f \in \operatorname{Dom}(A)$. This implies that e^{-At} is a contraction semigroup for $t \geqslant 0$, from which the statements of the theorem follow. ∎

The conditions on a in the above theorem can be relaxed if we are only interested in the spectrum, but we shall see later that this is misleading.

THEOREM 5.2. *If* $a \in L^\infty$ *and* $Af = \lambda f$ *for some* $f \in \operatorname{Dom}(A)$ *and some* $\lambda \in \mathbb{C}$, *then* $\operatorname{Re}(\lambda) \geqslant 0$. *If moreover* $\lim_{|x|\to\infty} a(x) = 0$ *then* $\operatorname{Spec}(A) \subseteq \{\lambda : \operatorname{Re}(\lambda) \geqslant 0\}$.

Proof. Elliptic regularity theorems imply that f, which a priori only belongs to $W^{2,2}$, is actually bounded and continuous. We need to use the fact that e^{-At} is a contraction semigroup on L^∞. This is intuitively obvious on probabilistic grounds, but we include a proof for completeness; the most general results of this genre which we know are [8] and [13] — several other abstract approaches assume that the semigroup on L^2 is contractive, which is not the case for our operators. Let C_0 denote the space of bounded, continuous functions on $\mathbb{R}^N$ which vanish at ∞. We have $A = B + A_0$ where B and A_0 are defined above. The operator $-B$ generates a positivity preserving holomorphic contraction semigroup on L^2 and on C_0. The operator A_0 is a class $\mathcal{P}$ perturbation in both spaces in the sense of [3], Theorem 3.5, provided a is a bounded continuous function. Therefore $-A$ generates a strongly continuous semigroup on both spaces. Now A is dissipative in the sense of [3], Theorem 2.25 on C_0, so the semigroup on C_0 is a contraction semigroup. That is

$$\|\mathrm{e}^{-At} f\|_\infty \leqslant \|f\|_\infty \tag{5.2}$$

for all $f \in L^2 \cap C_0$ and all $t \geqslant 0$ provided a is bounded and continuous. If a is not continuous then we construct a sequence a_n of bounded continuous functions which converges to a boundedly and pointwise, and let A_n be the associated operators. The operators A_n converge to A in the strong resolvent sense ([3], Corollary 3.18), so the semigroups converge strongly in L^2. We observe that (5.2) is preserved under L^2 norm limits. The validity of (5.2) for all $f \in L^2 \cap L^\infty$ is deduced by constructing a sequence $f_n \in L^2 \cap C_0$ which converges to f in L^2 norm and satisfies $\|f_n\|_\infty \leqslant \|f\|_\infty$ for all n. Applying (5.2) to the eigenfunction f we conclude that $\operatorname{Re}(\lambda) \geqslant 0$. ∎

If a vanishes at infinity then A_0 is a relatively compact perturbation of B and so A has the same essential spectrum as B, namely $[0, \infty)$. All other points of the spectrum are isolated eigenvalues of finite multiplicity by Weyl's essential spectrum theorem ([16], Theorem 13.14), and were dealt with in the first part of the proof. Note that [16] uses a different definition of essential spectrum, so the text must be read carefully.

In order to construct counterexamples to the corresponding statement about the limit of the pseudospectra as $\delta \to 0$, we need to make some hypotheses about the vector field $a(x)$ on $\mathbb{R}^N$. We assume that S is a compact subset of $\mathbb{R}^N$ with C^2 boundary and that a vanishes outside S. We assume that a is C^1 inside S and that there is a C^2 function φ on $\mathbb{R}^N$ such that $a(x) \cdot \nabla\varphi(x) = 1$ for all $x \in S$.

The significance of this assumption may be seen by supposing that γ is a flow line of the vector field, more precisely

$$\gamma'(t) = a(\gamma(t)) \in S$$

for all $t_0 \leqslant t \leqslant t_1$. It is easy to prove under the above assumption on φ that

$$\varphi(\gamma(t_1)) = \varphi(\gamma(t_0)) + t_1 - t_0$$

so φ measures the passage along the flow lines. The existence of φ implies that the flow lines cannot be closed loops.

THEOREM 5.3. *Let the operator A on $L^2(\mathbb{R}^N)$ be given by*

$$Af(x) := -\delta\Delta f(x) + a(x) \cdot \nabla f(x)$$

where a satisfies the above conditions and $\delta > 0$. Let K be any compact subset of $\mathbb{C}$ and let $\varepsilon > 0$. Then

$$K \subseteq \mathrm{Spec}_\varepsilon(A)$$

for all small enough $\delta > 0$. If A_0 is any extension of the operator initially defined on $C_c^2(\mathbb{R}^N)$ by (5.1) *then* $\mathrm{Spec}(A_0) = \mathbb{C}$.

Proof. Given $\lambda \in \mathbb{C}$ we construct a function $f \in \mathrm{Dom}(A)$ such that

$$\|Af - \lambda f\| < \varepsilon \|f\|$$

provided δ is small enough. It will be apparent that the construction is uniform for λ in any compact subset of $\mathbb{C}$.

We put

$$f(x) := \mathrm{e}^{\lambda\varphi(x)}\psi(x)$$

where ψ is a C^2 function on $\mathbb{R}^N$ satisfying the following conditions for a parameter $\beta \in (0,1)$ to be determined. We assume that $0 \leqslant \psi \leqslant 1$, that $\psi = 1$ on S, that $|\mathrm{supp}(\psi) \setminus S| \leqslant c_1\beta$, that $|\nabla\psi| \leqslant c_2\beta^{-1}$ and finally that $|\Delta\psi| \leqslant c_3\beta^{-2}$.

If $x \in S$ then a direct calculation shows that

$$Af(x) = -\delta\Delta \mathrm{e}^{\lambda\varphi(x)} + \lambda f(x)$$

so

$$\|(Af - \lambda f)|_S\| \leqslant c_4\delta.$$

If $x \notin S$ then $|\Delta f(x)| \leqslant c_5\beta^{-2}$, $a(x) \cdot \nabla f(x) = 0$ and $|f(x)| \leqslant c_6$. Since we neeed only integrate over a set of measure $c_1\beta$ we deduce that

$$\|(Af - \lambda f)|\mathbb{R}^n \setminus S\| \leqslant c_7\delta\beta^{-3/2} + c_8\beta^{1/2}.$$

Putting $\beta := \delta^{1/2}$ we conclude that

$$\lim_{\delta\to 0} \|Af - \lambda f\| = 0.$$

On the other hand we also have

$$\lim_{\delta\to 0} \|f\|^2 = \int_S \mathrm{e}^{2\mathrm{Re}(\lambda)\varphi(x)}\, \mathrm{d}^N x > 0$$

so the proof of the statements concerning A is completed.

Applying the same proof to A_0 we deduce that $\mathrm{Spec}_\varepsilon(A_0) = \mathbb{C}$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ the final statement of the theorem follows. ∎

Corollary 5.4. *Given $\delta > 0$, let*

$$Af(x) := -\delta\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a(x)\frac{\mathrm{d}f}{\mathrm{d}x}$$

where a is a positive C^1 function on $[-b,b]$ and $a(x) = 0$ for all $|x| > b$. Let K be any compact subset of $\mathbb{C}$ and let $\varepsilon > 0$. Then

$$K \subseteq \mathrm{Spec}_\varepsilon(A)$$

for all small enough $\delta > 0$.

Proof. We define

$$\varphi(x) := \int_0^x \frac{\mathrm{d}s}{a(s)}$$

for all $x \in [-b,b]$, and extend it to a C^2 function on $\mathbb{R}$ by any convenient procedure. ∎

EXAMPLE 5.5. We mention another example for which detailed calculations can be carried out by the methods of this paper, namely that for which $a(x) = \alpha_-$ if $x < 0$ and $a(x) = \alpha_+$ if $x > 0$. This example has different asymptotics on the left and right as did the examples in [6]. The spectral and pseudospectral behaviour of this operator as $\delta \to 0$ depends upon the signs and relative sizes of $\alpha_\pm$. The essential spectrum is always the union of the essential spectra of the two constant coefficient operators

$$A_\pm f(x) := -\delta \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \alpha_\pm \frac{\mathrm{d}f}{\mathrm{d}x}$$

on $L^2(\mathbb{R})$.

EXAMPLE 5.6. Let $A : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the operator

$$Af(x) := \delta \Delta^2 f(x) + a(x) \cdot \nabla f(x)$$

where $\delta > 0$. It is easy to prove analogues of Theorems 5.1 and 5.3 by similar methods. However we are not able to prove a version of Theorem 5.2 because e^{-At} is not a contraction semigroup on L^∞. The determination of the limit of the spectrum of A as $\delta \to 0$ is an open question, but the behaviour of the pseudospectra suggests that its solution is not of great importance.

We next turn to another class of operators acting on $L^2(\mathbb{R}^N)$, given by

$$Hf(x) := -(\delta + a(x))\Delta f(x)$$

where $\delta > 0$ and a is a non-negative bounded function. It may be seen that H is a relatively bounded perturbation of $H_0 := -(\delta + \|a\|_\infty)\Delta f(x)$ with relative bound less than 1, and we take H to have the same domain, namely $W^{2,2}(\mathbb{R}^N)$.

LEMMA 5.7. *For any $\delta > 0$, H has real non-negative spectrum and*

$$\|\mathrm{e}^{-Ht}\| \leqslant (1 + \delta^{-1}\|a\|_\infty)^{1/2}$$

for all $t \geqslant 0$.

Proof. Let us define the new inner product

$$\langle f, g \rangle_\delta := \int_{\mathbb{R}^N} (\delta + a(x))^{-1} f(x)\overline{g(x)}\, \mathrm{d}^N x$$

on $L^2(\mathbb{R}^N)$, the two norms being obviously equivalent. We have

$$\langle Hf, g \rangle_\delta = -\int_{\mathbb{R}^N} (\Delta f(x))\overline{g(x)}\, \mathrm{d}^N x = \int_{\mathbb{R}^N} \nabla f(x) \cdot \overline{\nabla g(x)}\, \mathrm{d}^N x$$

for all $f, g \in \mathrm{Dom}(H)$, so H is self-adjoint and non-negative with respect to this new inner product. Thus $\mathrm{Spec}(H) \subseteq [0, \infty)$ and

$$\begin{aligned}\|\mathrm{e}^{-Ht} f\| &\leqslant (\delta + \|a\|_\infty)^{1/2} \|\mathrm{e}^{-Ht} f\|_\delta \leqslant (\delta + \|a\|_\infty)^{1/2} \|f\|_\delta \\ &\leqslant (1 + \delta^{-1} \|a\|_\infty)^{1/2} \|f\|\end{aligned}$$

for all $f \in L^2$. ∎

If $\lim_{|x|\to\infty} a(x) = 0$ then the invertible operator relating the two norms on L^2 has condition number increasing to $+\infty$ as $\delta \to 0$ so the above spectral results do not tell the full story. We consider only one simple example to illustrate what presumably happens in much greater generality.

Theorem 5.8. *Let S be a compact subset of $\mathbb{R}^N$ with C^2 boundary and let*

$$a(x) := \begin{cases} 1, & \text{if } x \in S; \\ 0, & \text{otherwise.} \end{cases}$$

If $\varepsilon > 0$ and K is any compact subset of $\mathbb{C}$, then

$$K \subseteq \mathrm{Spec}_\varepsilon(H)$$

for all small enough $\delta > 0$.

Proof. The proof is very similar to that of Theorem 5.3, except that we put $\varphi(x) := x_1$. ∎

Our final example is simpler than those above, and illustrates the problems involved in taking a different type of limit. Consider the operator

$$Af(x) := -f''(x) - 2f'(x) - V(x)f(x)$$

acting on $L^2(\mathbb{R})$ where V is a non-negative potential of compact support. As a relatively compact perturbation of the same operator with zero potential, its spectrum is

$$S := \{\xi^2 - 2\mathrm{i}\xi : \xi \in \mathbb{R}\}$$

together with some possible discrete eigenvalues. By examining the asymptotics of the eigenfunctions as $x \to \pm\infty$, one finds that any eigenvalues must lie outside S. The computation of the pseudospectra is non-trivial, and we consider a special degenerate example to illustrate the surprising possibilities.

We replace V by $c\delta$ where δ is a delta potential concentrated at the origin and $c > 0$. That is we consider the operator

$$Af(x) := -f''(x) - 2f'(x)$$

acting on $L^2(\mathbb{R})$ where we impose the internal boundary condition

$$f'(0+) - f'(0-) = -cf(0)$$

and require f to be continuous at $x = 0$, as well as everywhere else. It is easy to show that if $c > 2$ then

$$\mathrm{Spec}(A) = S \cup \{\gamma\}$$

where $\gamma := 1 - c^2/4$ is the eigenvalue associated with the eigenfunction

$$f(x) := \begin{cases} \mathrm{e}^{-(1+c/2)x}, & \text{if } x > 0; \\ \mathrm{e}^{-(1-c/2)x}, & \text{if } x < 0. \end{cases}$$

It might be expected that $\|(A - \lambda)^{-1}\|$ would be the same order of magnitude as $\mathrm{dist}(\lambda, S)^{-1}$ if γ is far enough away from S and λ. This hope is dashed by the following theorem. An alternative statement of the theorem is that the pseudospectrum of this operator expands to fill the interior of S as c increases.

THEOREM 5.9. *If λ is inside S then*

$$\lim_{c\to\infty} \|(A - \lambda)^{-1}\| = \infty.$$

Proof. It suffices to construct a function $f \in \mathrm{Dom}(A)$ such that

$$\lim_{c\to\infty} \|Af - \lambda f\|/\|f\| = 0.$$

We do not display the dependence of all of the quantites on c. Let λ be inside S and let z_1, z_2 be the two solutions of $-z^2 - 2z = \lambda$, so that both z_i have negative real parts. Then put

$$f(x) := \begin{cases} 0, & \text{if } x < -\delta, \\ -(x + \delta)^2, & \text{if } -\delta \leqslant x \leqslant 0, \\ u\mathrm{e}^{z_1 x} + v\mathrm{e}^{z_2 x}, & \text{if } x > 0; \end{cases}$$

where $\delta > 0$ and $u, v \in \mathbb{C}$ are to be determined. We choose u, v so that $f'(0+) = 1$ and $f(0+) = -\delta^2$. This ensures the continuity of f at $x = 0$. The boundary condition at $x = 0$ is satisfied if $c\delta^2 - 2\delta - 1 = 0$, or equivalently

$$\delta := (1 + \{1 + c\}^{1/2})/c.$$

Direct computations show that $\delta \sim c^{-1/2}$, $u \to (z_1 - z_2)^{-1}$ and $v \to (z_2 - z_1)^{-1}$ as $c \to \infty$. It then follows that $\|Af - \lambda f\| \to 0$ and $\|f\| \to k \neq 0$ as $c \to \infty$, and this completes the proof. ∎

6. DISCUSSION

We start by discussing our results for constant coefficient ordinary differential operators. It might be suspected that the lack of relationship between $\mathrm{Spec}(A_b)$ and $\mathrm{Spec}(A)$ is related to the choice of Dirichlet boundary conditions at the ends of $(-b,b)$. In one sense this is correct — if we were to choose periodic boundary conditions instead, then it is easy to prove that

$$\lim_{b\to\infty} \mathrm{Spec}(A_b) = \mathrm{Spec}(A).$$

However, we know of no theorem which states that the choice of periodic boundary conditions implies the above limiting formula if A has variable coefficients. If one passes to similar problems in higher dimensions the problem reasserts itself in a form which is even harder to resolve. For example suppose that the convection-diffusion operator

$$Af(x) := -\Delta f(x) + a(x)\cdot\nabla f(x)$$

acting on $L^2(\mathbb{R}^n)$ is rotationally invariant, so that $a(x) = \widetilde{a}(|x|)x$ for all $x\in\mathbb{R}^n$. Then the natural space cut-off is two balls with centres at the origin and radii which increase to infinity. One has to impose boundary conditions before one can start to investigate spectral behaviour, and the natural boundary conditions for such an operator are again rotationally invariant. Dirichlet boundary conditions are an obvious choice, and there is no analogue of periodic boundary conditions.

Even if the spectral behaviour of a particular operator acting on $L^2(\mathbb{R}^n)$ can be determined by the choice of an appropriate sequence of cut-offs and boundary conditions, this does not really help the numerical analyst unless one can specify a general procedure to be followed for a wide class of operators. The developing view is that even if such a procedure exists, the use of pseudospectra is both more stable and also in many contexts more relevant for applications to highly non-normal operators ([21]).

While the results above arise from restricting the operators in question to smaller subspaces, Section 5 concerns taking singular limits within a fixed Hilbert space. The theorems which we have proved concerning the limiting behaviour as $\delta\to 0$ of the operators

$$Af(x) := -\delta\Delta f(x) + a(x)\cdot\nabla f(x)$$

are not entirely surprising in view of the fact that the limit operator is of order 1 and one can prove directly that such operators are problematical if the coefficients are not Lipschitz continuous. Our point is rather that if one relied upon the spectrum this pathology as $\delta \to 0$ would be entirely invisible. On the other hand if one examines the pseudospectra one learns that the limit is singular from the fact that the pseudospectra expands to fill the entire complex plane as $\delta \to 0$. In other problems for which the behaviour of the singular limit is not so clear, it seems that one may well learn more by examining the limiting behaviour of the pseudospectra than by relying upon the spectrum.

Similar comments apply to the operators

$$Hf(x) := -(\delta + a(x))\Delta f(x)$$

acting on $L^2(\mathbb{R}^N)$ studied in the second half of Section 5. Here, however, we have the further feature that each operator is non-negative and self-adjoint with respect to a norm equivalent to the usual L^2 norm. The pseudospectra may still expand to fill the entire complex plane as $\delta \to 0$ because the invertible operator relating the two norms has condition number increasing to $+\infty$ as $\delta \to 0$.

The final example studied in Section 5, namely

$$Af(x) := -f''(x) - 2f'(x) - c\delta(x)f(x),$$

where $c > 0$ and δ is a delta function at the origin, is exactly soluble. It again illustrates the disadvantages of relying upon the spectrum for information about the behaviour of the operator as c increases.

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E. BRIAN DAVIES
Department of Mathematics
King's College
Strand, London WC2R 2LS
ENGLAND

E-mail: E.Brian.Davies@kcl.ac.uk

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