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OPTIMAL TIME-VARIANT SYSTEMS AND FACTORIZATION OF OPERATORS. II: FACTORIZATION

D.Z. AROV, M.A. KAASHOEK, D.R. PIK

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ABSTRACT. For a block lower triangular contraction T the maximal block lower triangular outer solutions F and G of the operator inequalities $I - T^*T \geqslant F^*F$, and $I - TT^* \geqslant GG^*$ are identified in terms of optimal and star-optimal time-variant realizations of T, respectively. Special attention is given to the case when the inequality $I - T^*T \geqslant F^*F$ is satisfied for $F = 0$ only, and to the case when equality can be obtained. As a byproduct a characterization is derived of optimality of a time-variant system in terms of the input coefficients of the systems only. The existence of maximal block lower triangular solutions F of the operator inequality $I - T^*T \geqslant F^*F$ is also used to derive an optimal realization of T.

KEYWORDS: *Block lower triangular contractions, contractive systems, optimal systems, outer operators, factorization of operators.*

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0. INTRODUCTION

Throughout this paper $T = (t_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ is a block lower triangular contraction. Here $\mathcal{K}$ stands for the sequence of Hilbert spaces $(K_n)_{n\in\mathbb{Z}}$, and $\ell^2(\mathcal{K})$ denotes the Hilbert space of all square summable sequences $(k_n)_{n\in\mathbb{Z}}$ with $k_n \in K_n$ ($n \in \mathbb{Z}$). In other words $\ell^2(\mathcal{K})$ stands for the doubly infinite Hilbert space direct sum $\bigoplus\limits_{j\in\mathbb{Z}} K_j$. The space $\ell^2(\mathcal{L})$ is defined in a similar way. The operator $t_{i,j}$, which maps K_j into L_i, is the (i,j)-entry in the operator matrix representation of T relative to the natural direct sum decompositions of $\ell^2(\mathcal{K})$ and $\ell^2(\mathcal{L})$, and the requirement that T is block lower triangular means that $t_{i,j} = 0$ for $j > i$.

Given T as above, it is known (see for example [14], page 128) that there exists a block lower triangular operator $F = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$, where $\mathcal{N} = (N_j)_{j\in\mathbb{Z}}$ is some doubly infinite sequence of Hilbert spaces, such that

$$I_{\ell^2(\mathcal{K})} - T^*T \geqslant F^*F, \tag{0.1}$$

and F satisfies an additional maximality condition, namely, the operator F is outer, and if G is another block lower triangular contraction satisfying $I - T^*T \geqslant G^*G$, then $G = QF$ for some block lower triangular contraction Q. We shall refer to such an F as a *maximal outer solution* of the operator inequality (0.1). The property that the block lower triangular operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ is *outer* means (see [5]) that

$$\overline{F\Big[\bigoplus_{j\geqslant n} K_j\Big]} = \bigoplus_{j\geqslant n} N_j, \quad n \in \mathbb{Z}. \tag{0.2}$$

In this paper we show that such an F may be constructed by using techniques from system theory.

To explain the above in more detail, we first recall that a block lower triangular contraction $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ is the input-output operator of some causal contractive linear time-variant input-output system. Such a system is called a contractive realization of T. Among all contractive realizations of T there are ones with additional properties of minimality and optimality (see Section 1 for the terminology from system theory used in this introduction).

In this paper we associate with each contractive system Σ a natural complementary system Φ_Σ, and we show that if the original system Σ is minimal and optimal, then the input-output operator of the associated complementary system Φ_Σ is precisely a maximal outer solution of the operator inequality (0.1). We also show that, conversely, if the original system Σ is minimal, and the input-output operator of the complementary system Φ_Σ is a maximal outer solution of (0.1), then the system Σ must be optimal. Corresponding results are also proved for the operator inequality

$$I_{\ell^2(\mathcal{L})} - TT^* \geqslant GG^*. \tag{0.3}$$

Special attention will be given to two extremal cases in the operator inequalities (0.1) and (0.3), namely when the zero operator is the only block lower triangular solution to (0.1) (or to (0.3)), and when equality can be obtained in (0.1) (or in (0.3)). These extremal cases are identified in terms of special properties of optimal realizations of T. For example, if the realization Σ of T is minimal and optimal, then equality occurs in (0.1) if and only if the system Σ is pointwise stable (see Section 2).

As a by-product of our analysis we obtain a new characterization of optimality of time-variant systems in terms of the input coefficients of the systems only. The analogous characterizations for time invariant systems appears in [6] as a remark.

The paper consists of seven sections. The first section has a preliminary character. Here we briefly review the main facts concerning time-variant systems from [8] that are used in the present paper. In the second section we state the main theorems. In the third section semi-outer operators are introduced and such operators are characterized in three different ways. In Sections 4 and 5 the proofs of the main theorems are given. In Section 6 we give an alternative characterization of optimal systems in terms of the input coefficients only. In the final section an optimal realization of a block lower triangular contraction is obtained by using a maximal outer solution F to the inequality $I - T^*T \geqslant F^*F$.

1. PRELIMINARIES ABOUT SYSTEMS

In this section we review the basic facts about contractive time-variant systems that are used throughout this paper. We first recall some general facts concerning arbitrary discrete time-variant systems. For a more extensive treatment we refer to [8] (see also [14]). Consider the time-variant system with discrete time n:

$$\Sigma \begin{cases} x_{n+1} = A(n)x_n + B(n)u_n, \\ y_n = C(n)x_n + D(n)u_n, \end{cases} \qquad n \in \mathbb{Z}. \tag{1.1}$$

Here $A(n) : H_n \to H_{n+1}$, $B(n) : K_n \to H_{n+1}$, $C(n) : H_n \to L_n$ and $D(n) : K_n \to L_n$ are bounded linear operators acting between Hilbert spaces. We refer to H_n as the *state space* at time n, K_n as the *input space* at time n, and L_n as the *output space* at time n. By $M_\Sigma(n)$ we denote the *system matrix at time* n, i.e.:

$$M_\Sigma(n) = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix} : H_n \oplus K_n \to H_{n+1} \oplus L_n.$$

We will use the notation $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ instead of (1.1). The system Σ is called *contractive* (*isometric, co-isometric,* or *unitary*) if the system matrix $M_\Sigma(n)$ is a contraction (isometry, co-isometry, or unitary operator) for each integer n.

Given a time-variant system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ and unitary operators $U_n : H_n \to \widetilde{H}_n$, $n \in \mathbb{Z}$, we can form a new system:

$$\widetilde{\Sigma} = (U_{n+1}A(n)U_n^{-1}, U_{n+1}B(n), C(n)U_n^{-1}, D(n); \widetilde{H}_n, K_n, L_n). \tag{1.2}$$

In this case Σ and $\widetilde{\Sigma}$ are called *unitarily equivalent*.

Next, we introduce the notions of observability and controllability. We first consider these concepts at each time instant. Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a time-variant system. The *unobservable subspace at time* $n \in \mathbb{Z}$ is defined by

$$\mathrm{Ker}(C|A;n) = \bigcap_{j \geqslant n} \mathrm{Ker}\, C(j)\tau_{\mathcal{A}}(j,n). \tag{1.3}$$

Here $\tau_{\mathcal{A}}(j,n)$ is the *state transition operator* from H_n to H_j associated with the sequence of operators $A(\nu) : H_\nu \to H_{\nu+1}$, $\nu \in \mathbb{Z}$. In other words,

$$\tau_{\mathcal{A}}(k,l) = \begin{cases} A(k-1)A(k-2)\cdots A(l+1)A(l), & k > l, \\ I_{H_l}, & k = l, \\ 0, & k < l. \end{cases} \tag{1.4}$$

For $n, m \in \mathbb{Z}$, $n \leqslant m$, the *controllability operator* $\Lambda_{n,m}(\Sigma) : \bigoplus_{j=n}^{m} K_j \to H_{m+1}$ is defined by

$$\begin{aligned} \Lambda_{n,m}(\Sigma)\vec{v} &= \sum_{j=n}^{m} \tau_{\mathcal{A}}(m+1, j+1)B(j)v_j, \\ \vec{v} &= (v_n, v_{n+1}, \ldots, v_{m-1}, v_m)^{\mathrm{tr}} \in \bigoplus_{j=n}^{m} K_j. \end{aligned} \tag{1.5}$$

The *controllable subspace* at time n is by definition the closure of the linear manifold $\mathrm{Im}(A|B;n) = \operatorname*{span}_{p \leqslant n-1} \mathrm{Im}\, \Lambda_{p,n-1}(\Sigma)$. In other words,

$$\overline{\mathrm{Im}(A|B;n)} = \bigvee_{j \leqslant n-1} \mathrm{Im}\, \tau_{\mathcal{A}}(n, j+1)B(j), \tag{1.6}$$

where $\bigvee$ denotes the closed linear hull. Note that both $\mathrm{Ker}(C|A;n)$ and $\overline{\mathrm{Im}(A|B;n)}$ are subspaces of H_n.

A time-variant system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is called *observable at time n* if $\mathrm{Ker}(C|A;n) = \{0\}$ and *controllable at time n* if $\overline{\mathrm{Im}(A|B;n)} = H_n$. The system Σ will be called (*completely*) *observable* if Σ is observable at each time n, and (*completely*) *controllable* if Σ is controllable at each time n. Finally, the system Σ will be called *simple* if for each $n \in \mathbb{Z}$ we have

$$H_n = \overline{\mathrm{Ker}(C|A;n)^{\perp} + \mathrm{Im}(A|B;n)}.$$

Consider the systems $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ and $\widetilde{\Sigma} = (\widetilde{A}(n), \widetilde{B}(n), \widetilde{C}(n), \widetilde{D}(n); \widetilde{H}_n, K_n, L_n)$. The system $\widetilde{\Sigma}$ is called a *dilation* of Σ if for each n we have $\widetilde{D}(n) = D(n)$, and for each n the state space $\widetilde{H}_n$ admits an orthogonal sum decomposition $\widetilde{H}_n = E_n \oplus H_n \oplus F_n$ such that corresponding to this decomposition we have

$$(1.7)\quad A(n) = P_{H_{n+1}}\widetilde{A}(n)|H_n, \quad B(n) = P_{H_{n+1}}\widetilde{B}(n), \quad C(n) = \widetilde{C}(n)|H_n,$$

$$(1.8)\quad \widetilde{A}(n)E_n \subset E_{n+1},\ \widetilde{A}(n)^*F_{n+1} \subset F_n,\ \widetilde{C}(n)E_n = \{0\},\ \widetilde{B}(n)^*F_{n+1} = \{0\}$$

for each $n \in \mathbb{Z}$. A time-variant system is called *minimal* if it is not a dilation of any other (different) time-variant system. In this paper we shall often use the fact that a time-variant system is minimal if and only if it is controllable and observable (see [8], Proposition 2.1).

With a contractive system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ we associate the operator matrix $T_\Sigma = (t_{i,j})_{i,j=-\infty}^{\infty}$, where

$$(1.9)\qquad t_{i,j} = \begin{cases} 0, & i < j, \\ D(n), & i = j, \\ C(i)\tau_{\mathcal{A}}(i, j+1)B(j), & i > j. \end{cases}$$

Since Σ is contractive, T_Σ induces a contractive linear operator from $\ell^2(\mathcal{K})$ into $\ell^2(\mathcal{L})$, also denoted by T_Σ, which is referred to as the *input-output map* of Σ (see Theorem 4.1 from [8]).

A contractive system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is said to be a *realization* of a block lower triangular operator $T = (t_{i,j})_{i,j=-\infty}^{\infty}$ if $T = T_\Sigma$, i.e., if (1.9) is satisfied for each $i, j \in \mathbb{Z}$. Each block lower triangular contraction admits a simple unitary time-variant realization, which is uniquely determined by T up to unitary equivalence. If two time-variant systems Σ_1 and Σ_2 are unitarily equivalent, then the input-output maps T_{Σ_1} and T_{Σ_2} are equal. For proofs of these facts we refer to [8] and [14].

Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, and let

$$\Sigma_\circ = (A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n)$$

be a contractive realization of T. The system $\Sigma_\circ$ is called *optimal* (cf. [8]) if for each contractive realization $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ of T, for each $n \in \mathbb{Z}$, and each input sequence $u_n, u_{n+1}, u_{n+2}, \ldots$, where $u_j \in K_j$, we have

$$(1.10)\qquad \|x_{\circ,n+k}\| \leqslant \|x_{n+k}\|, \quad k \geqslant 0.$$

Here $x_{\circ,n+k}$ and x_{n+k} denote the states at time $n+k$ of the systems $\Sigma_\circ$ and Σ, respectively, corresponding to the given input sequence $u_n, u_{n+1}, u_{n+2}, \ldots$, and with initial states $x_{\circ,n}$ and x_n at time n being equal to zero. Each block lower triangular contraction has a minimal and optimal time-variant realization which is determined by T up to unitary equivalence (see [8], Theorem 6.1).

Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, and let

$$\Sigma_\bullet = (A_\bullet(n), B_\bullet(n), C_\bullet(n), D(n); H_{\bullet,n}, K_n, L_n)$$

be an observable contractive realization of T. The system $\Sigma_\bullet$ will be called *star-optimal* if for each observable contractive realization $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ of T, and for each input sequence $u_n, u_{n+1}, u_{n+2}, \ldots$, with $u_j \in K_j$, we have

$$\|x_{\bullet,n+k}\| \geqslant \|x_{n+k}\|, \quad k \geqslant 0. \tag{1.11}$$

Here $x_{\bullet,n+k}$ and x_{n+k} denote the states at time $n+k$ of the systems $\Sigma_\bullet$ and Σ, respectively, corresponding to the given input sequence $u_n, u_{n+1}, u_{n+2}, \ldots$, and with initial states $x_{\bullet,n}$ and x_n at time n being equal to 0. Each block lower triangular contraction has a minimal and star-optimal time-variant realization which is determined by T up to unitary equivalence (see [8], Theorem 7.1).

Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a contractive time-variant system with input-output map T_Σ. Fix $n \in \mathbb{Z}$. Since the system matrix $M_\Sigma(n)$ is a contraction, we may define the defect operator

$$D_{M_\Sigma(n)} = (I - M_\Sigma(n)^* M_\Sigma(n))^{1/2} : H_n \oplus K_n \to H_n \oplus K_n,$$

and the defect space $\mathcal{D}_{M_\Sigma(n)} = \overline{\operatorname{Im} D_{M_\Sigma(n)}}$. Define the operators

$$\begin{aligned} Y(n) &= D_{M_\Sigma(n)} \tau_{H_n} : H_n \to \mathcal{D}_{M_\Sigma(n)}, \\ Z(n) &= D_{M_\Sigma(n)} \tau_{K_n} : K_n \to \mathcal{D}_{M_\Sigma(n)}, \end{aligned} \tag{1.12}$$

where τ_{H_n} and τ_{K_n} are the canonical embeddings of H_n and K_n into $H_n \oplus K_n$, respectively. Then the operator matrix

$$\begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \\ Y(n) & Z(n) \end{bmatrix} : H_n \oplus K_n \to H_{n+1} \oplus L_n \oplus \mathcal{D}_{M_\Sigma(n)} \tag{1.13}$$

is an isometry. The latter holds for each $n \in \mathbb{Z}$, and hence the system

$$\Phi = \Phi_\Sigma = (A(n), B(n), Y(n), Z(n); H_n, K_n, \mathcal{D}_{M_\Sigma(n)}) \tag{1.14}$$

is a contractive time-variant system. We shall refer to Φ in (1.14) as the *first complementary system associated with* Σ. Its input-output map $T_\Phi : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{D})$, where $\mathcal{D}$ is the sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)})_{n\in\mathbb{Z}}$, satisfies

$$I - T_\Sigma^* T_\Sigma \geqslant T_\Phi^* T_\Phi, \tag{1.15}$$

as will be shown in Proposition 4.1 (ii).

The second complementary system associated with Σ is defined as follows. Fix $n \in \mathbb{Z}$. Since the system matrix $M_\Sigma(n) : H_n \oplus K_n \to H_{n+1} \oplus L_n$ is a contraction, we may define the defect operator

$$D_{M_\Sigma(n)^*} = (I - M_\Sigma(n) M_\Sigma(n)^*)^{1/2} : H_{n+1} \oplus L_n \to H_{n+1} \oplus L_n,$$

and the defect space $\mathcal{D}_{M_\Sigma(n)^*} = \overline{\operatorname{Im} D_{M_\Sigma(n)^*}}$. Put

$$\begin{aligned} V(n) &= \widetilde{\tau}^*_{H_{n+1}} D_{M_\Sigma(n)^*} : \mathcal{D}_{M_\Sigma(n)^*} \to H_{n+1}, \\ W(n) &= \widetilde{\tau}^*_{L_n} D_{M_\Sigma(n)^*} : \mathcal{D}_{M_\Sigma(n)^*} \to L_n, \end{aligned} \tag{1.16}$$

where $\widetilde{\tau}_{H_{n+1}}$ and $\widetilde{\tau}_{L_n}$ are the canonical embeddings of H_{n+1} and L_n into $H_{n+1} \oplus L_n$, respectively. Then the operator matrix

$$\begin{bmatrix} A(n) & B(n) & V(n) \\ C(n) & D(n) & W(n) \end{bmatrix} : H_n \oplus K_n \oplus \mathcal{D}_{M_\Sigma(n)^*} \to H_{n+1} \oplus L_n \tag{1.17}$$

is a co-isometry. The latter holds for each $n \in \mathbb{Z}$, and hence

$$\Psi = \Psi_\Sigma = (A(n), V(n), C(n), W(n); H_n, \mathcal{D}_{M_\Sigma(n)^*}, L_n) \tag{1.18}$$

is a contractive time-variant system. We shall refer to Ψ in (1.18) as the *second complementary system associated with* Σ. Its input-output map $T_\Psi : \ell^2(\mathcal{D}_*) \to \ell^2(\mathcal{L})$, where $\mathcal{D}_*$ denotes the sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)^*})_{n\in\mathbb{Z}}$, satisfies

$$I - T_\Sigma T_\Sigma^* \geqslant T_\Psi T_\Psi^*. \tag{1.19}$$

2. MAIN THEOREMS

In this section we state the main theorems of this paper. The first two will be proved in Section 4, the other ones in Section 5. The first theorem characterizes the optimality of a system Σ in properties of the input-output map of the first complementary system associated with Σ.

Theorem 2.1. *Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, and let Σ be a contractive realization of T. Let Φ be the first complementary system associated with Σ, and let its input-output map be given by $T_\Phi : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{D})$. Then Σ is optimal if and only if for any block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ satisfying $I_{\ell^2(\mathcal{K})} - T^*T \geqslant G^*G$ there exists a block lower triangular contraction $Q : \ell^2(\mathcal{D}) \to \ell^2(\mathcal{N})$ such that $G = QT_\Phi$. If, in addition, Σ is controllable then T_Φ is outer.*

To characterize when equality can be achieved in the factorization of $I - T^*T \geqslant F^*F$ with F a block lower triangular operator we need the notion of pointwise stability. A sequence of operators $(A(n) : H_n \to H_{n+1})_{n\in\mathbb{Z}}$ is called *pointwise stable* if for each $n \in \mathbb{Z}$ and for all $x \in H_n$ we have

$$\lim_{p\to\infty} \|\tau_{\mathcal{A}}(n+p, n)x\| = 0.$$

If a block lower triangular contraction $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ admits a contractive realization $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ with its sequence of main operators $(A(n) : H_n \to H_{n+1})_{n\in\mathbb{Z}}$ being pointwise stable, then the sequence of main operators of any controllable optimal realization of T is also pointwise stable. To see this, let $\Sigma_\circ = (A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n)$ be a controllable and optimal realization of T. Fix an integer m. By the optimality of the system $\Sigma_\circ$ we have

$$\begin{aligned}\|\tau_{A_\circ}(m+p, m+1)x\| &= \|\tau_{A_\circ}(m+p, m+1)\Lambda_{n,m}(\Sigma_\circ)\vec{u}\| \\ &\leqslant \|\tau_{A}(m+p, m+1)\Lambda_{n,m}(\Sigma)\vec{u}\|\end{aligned}$$

for each vector $x = \Lambda_{n,m}(\Sigma_\circ)\vec{u}$ with $\vec{u} \in \bigoplus_{j=n}^{m} K_j$, $n \leqslant m$ and $p \geqslant 1$. The sequence of main operators of the system Σ is pointwise stable, so $\|\tau_{A_\circ}(m+p, m+1)x\|$ tends to 0 for $p \to \infty$. Since the system $\Sigma_\circ$ is controllable, the linear manifold $\mathcal{M}_{m+1} = \operatorname{span}_{n\leqslant m} \Lambda_{n,m}(\Sigma_\circ)$ is dense in $H_{\circ,m+1}$. Using the fact that all the operators $A_\circ(n)$ are contractions and the fact that $\mathcal{M}_{m+1}$ is dense in $H_{\circ,m+1}$, a straightforward argument yields the pointwise stability of the sequence $(A_\circ(n) : H_n \to H_{n+1})_{n\in\mathbb{Z}}$.

Theorem 2.2. *Let* $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ *be a block lower triangular contraction, and let* $\Sigma_\circ = (A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n)$ *be a controllable and optimal realization of* T. *Then*

(i) *there exists a block lower triangular contraction* $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ *such that* $I - T^*T = F^*F$ *if and only if the sequence* $(A_\circ(n) : H_{\circ,n} \to H_{\circ,n+1})_{n\in\mathbb{Z}}$ *is pointwise stable;*

(ii) $F = 0$ *is the only block lower triangular operator satisfying the inequality* $I - T^*T \geqslant F^*F$ *if and only if* $\Sigma_\circ$ *is isometric.*

From [8], Corollary 5.5 (i) and Theorem 2.2 above it follows that $F = 0$ is the only block lower triangular operator satisfying the inequality $I - T^*T \geqslant F^*F$ if and only if any simple unitary realization of T is observable.

Theorem 2.1 has a dual version, for which we need the notion of a star-outer operator. A block lower triangular operator $R : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ is called *star-outer* if for each $n \in \mathbb{Z}$ we have $\overline{R^* \bigoplus\limits_{j\leqslant n} L_j} = \bigoplus\limits_{j\leqslant n} N_j$.

Theorem 2.3. *Let* $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ *be a block lower triangular contraction, and let* Σ *be a minimal and contractive realization of* T. *Let* Ψ *be the second complementary system associated with* Σ *with input-output map* $T_\Psi : \ell^2(\mathcal{D}_*) \to \ell^2(\mathcal{L})$. *Then* Σ *is star-optimal if and only if for any block lower triangular contraction* $G : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ *satisfying* $I_{\ell^2(\mathcal{L})} - TT^* \geqslant GG^*$ *there exists a block lower triangular contraction* $Q : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{D}_*)$ *such that* $G = T_\Psi Q$. *In this case* T_Ψ *is star-outer.*

To formulate the dual version of Theorem 2.2 we need the notion of pointwise star-stability. A sequence $(A(n) : H_n \to H_{n+1})_{n\in\mathbb{Z}}$ is called *pointwise star-stable* if for each $n \in \mathbb{Z}$ and for all $x \in H_n$ we have

$$\lim_{p\to\infty} \|\tau_A(n, n-p)^* x\| = 0.$$

In other words, the sequence $(A(n) : H_n \to H_{n+1})_{n\in\mathbb{Z}}$ is pointwise star-stable if and only if the sequence of operators $(A(n)^* : H_{n+1} \to H_n)_{n\in\mathbb{Z}}$ is pointwise stable in backwards time.

Theorem 2.4. *Let* $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ *be a block lower triangular contraction, and let* $\Sigma_\bullet = (A_\bullet(n), B_\bullet(n), C_\bullet(n), D(n); H_{\bullet,n}, K_n, L_n)$ *be a minimal and star-optimal realization of* T. *Then*

(i) *there exists a block lower triangular contraction* $R : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ *such that* $I - TT^* = RR^*$ *if and only if the sequence* $(A_\bullet(n) : H_{\bullet,n} \to H_{\bullet,n+1})_{n\in\mathbb{Z}}$ *is pointwise star-stable;*

(ii) $R = 0$ *is the only block lower triangular operator* R *satisfying the inequality* $I - TT^* \geqslant RR^*$ *if and only if* $\Sigma_\bullet$ *is co-isometric.*

From [8], Corollary 5.5 (ii) and Theorem 2.4 it follows that $R = 0$ is the only block lower triangular operator R satisfying the inequality $I - TT^* \geqslant RR^*$ if and only if any simple unitary realization of T is controllable.

Theorems 2.2 and 2.4 have the following corollary.

Corollary 2.5. *Let* $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ *be a block lower triangular contraction. A simple unitary realization of* T *is minimal if and only if the inequalities* $I - T^*T \geqslant F^*F$ *and* $I - TT^* \geqslant RR^*$*, with* F *and* R *block lower triangular, imply* $F = 0$ *and* $R = 0$.

Proof. Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a simple unitary realization of T.

Suppose first that the inequalities $I - T^*T \geqslant F^*F$ and $I - TT^* \geqslant RR^*$, where both F and R are assumed to be block lower triangular operators, imply $F = 0$ and $R = 0$. By [8], Theorem 6.1 the first minimal restriction $(\Sigma)_{\mathrm{res},1}$ is a minimal and optimal realization of T. By [8], Theorem 7.1 the second minimal restriction $(\Sigma)_{\mathrm{res},2}$ is a minimal and star-optimal realization of T. From Theorem 2.2 it follows that $(\Sigma)_{\mathrm{res},1}$ is isometric. From Theorem 2.4 it follows that $(\Sigma)_{\mathrm{res},2}$ is co-isometric. By [8], Corollary 5.5 it follows that Σ is minimal.

To prove the reverse implication, assume that Σ is minimal. So $\Sigma = \Sigma_{\mathrm{res},1} = \Sigma_{\mathrm{res},2}$. In particular, $\Sigma_{\mathrm{res},1}$ is an isometric system, and $\Sigma_{\mathrm{res},2}$ is a co-isometric system. By the construction of the operator F in Theorem 2.1 it follows that $F = 0$ is the only operator satisfying $I - T^*T \geqslant F^*F$, and by the construction of the operator R in Theorem 2.3 it follows that $R = 0$ is the only operator satisfying $I - TT^* \geqslant RR^*$. ∎

Notice that if a simple unitary realization of T is minimal then it is optimal and star-optimal. In that case all contractive and minimal realizations of T are unitarily equivalent, and they are unitary systems.

The existence of a block lower triangular outer operator F such that $F^*F \leqslant I - T^*T$ and satisfying the property that for any block lower triangular operator G satisfying $I - T^*T \geqslant G^*G$ there exists a block lower triangular contraction Q such that $G = QF$, may also be derived from [14]. Also in the time-invariant case when block lower triangular contractions are replaced by operator-valued functions from the Schur class, the existence of such an F is known (see [26]) and referred to (see [6]) as a best possible minorant. In [6] these best possible minorants are used to construct a time-invariant optimal realization of a Schur class function. We will give such a construction for the time-variant case in Section 7.

3. SEMI-OUTER OPERATORS

A block lower triangular (bounded) operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ is called *semi-outer* if for each $j \in \mathbb{Z}$ there exists a subspace $M_j \subset L_j$ such that we have

$$\overline{F\Big[\bigoplus_{j\geqslant n} K_j\Big]} = \bigoplus_{j\geqslant n} M_j, \quad n \in \mathbb{Z}. \tag{3.1}$$

Notice that the operator F is outer if $M_j = L_j$ for each $j \in \mathbb{Z}$. Throughout this paper, outer and semi-outer operators will be assumed to be bounded.

In Proposition 3.1 below we present three alternative characterizations of semi-outer operators. We first introduce the necessary notation. Let a block lower triangular operator $T = (t_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be given. We will frequently use the operator

$$H_p(T) = \begin{bmatrix} \cdots & t_{p,p-3} & t_{p,p-2} & t_{p,p-1} \\ \cdots & t_{p+1,p-3} & t_{p+1,p-2} & t_{p+1,p-1} \\ \cdots & t_{p+2,p-3} & t_{p+2,p-2} & t_{p+2,p-1} \\ & \vdots & \vdots & \vdots \end{bmatrix} : \bigoplus_{j=-\infty}^{p-1} K_j \to \bigoplus_{j=p}^{\infty} L_j. \tag{3.2}$$

Notice that for the case when T is a bounded block Laurent-operator with symbol $\theta(\lambda) = \sum_{j\geqslant 0} \theta_j \lambda^j$, so $t_{i,j} = \theta_{i-j}$, then the operator $H_p(T)$ is the block Hankel operator associated with the function θ.

Given a bounded operator $G = (g_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ for each $-\infty \leqslant n \leqslant m \leqslant \infty$, with n, m not both equal to $-\infty$ or ∞, we denote by $G(n,m)$ the operator

$$\tau^*_{\mathcal{L},n,m} G \tau_{\mathcal{K},n,m} : \bigoplus_{j=n}^{m} K_j \to \bigoplus_{j=n}^{m} L_j. \tag{3.3}$$

Here $\tau_{\mathcal{K},n,m} : \bigoplus_{j=n}^{m} K_j \to \ell^2(\mathcal{K})$ and $\tau_{\mathcal{L},n,m} : \bigoplus_{j=n}^{m} L_j \to \ell^2(\mathcal{L})$ are canonical embeddings. In particular, for $n, m \in \mathbb{Z}$, $n \leqslant m$, the operator $G(n,m)$ is defined by

(3.4)

$$G(n,m) = \begin{bmatrix} g_{n,n} & g_{n,n+1} & \cdots & g_{n,m-1} & g_{n,m} \\ g_{n+1,n} & g_{n+1,n+1} & \cdots & g_{n+1,m-1} & g_{n+1,m} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{m-1,n} & g_{m-1,n+1} & \cdots & g_{m-1,m-1} & g_{m-1,m} \\ g_{m,n} & g_{m,n+1} & \cdots & g_{m,m-1} & g_{m,m} \end{bmatrix} : \bigoplus_{j=n}^{m} K_j \to \bigoplus_{j=n}^{m} L_j.$$

Proposition 3.1. *For a block lower triangular operator* $F = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ *the following statements are equivalent:*

(i) *the operator* F *is semi-outer;*

(ii) *if* $G = (g_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M})$ *is a block lower triangular operator satisfying* $G^*G \leqslant F^*F$, *then* $G(n,N)^*G(n,N) \leqslant F(n,N)^*F(n,N)$ *for each* $n, N \in \mathbb{Z}$, $n \leqslant N$;

(iii) *if* $G = (g_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M})$ *is a block lower triangular operator satisfying* $G^*G \leqslant F^*F$, *then* $g_{n,n}^*g_{n,n} \leqslant f_{n,n}^*f_{n,n}$ *for each integer* n;

(iv) *for each* $n \in \mathbb{Z}$ *we have* $\operatorname{Im} H_n(F) \subset \overline{\operatorname{Im} F(n,\infty)}$.

Furthermore, if F *is semi-outer, then* (3.1) *holds for* $M_j = \overline{\operatorname{Im} f_{j,j}}$, $j \in \mathbb{Z}$.

Proof. We split the proof into three parts. In the first part we show that (iv) implies (i). In the second part we show the implications from (i) to (ii), and from (ii) to (iii). In the third part we prove the remaining implication.

Part (a). Suppose statement (iv) holds. We shall prove that F is semi-outer. For each $j \in \mathbb{Z}$ consider the subspace $N_j = \overline{\operatorname{Im} f_{j,j}}$ of L_j. Take $n \in \mathbb{Z}$. Let us partition the operator $F(n,\infty)$ as follows.

$$(3.5)\quad F(n,\infty) = \begin{bmatrix} f_{n,n} & 0 \\ h_n & F(n+1,\infty) \end{bmatrix} : K_n \oplus \Big(\bigoplus_{j \geqslant n+1} K_j \Big) \to L_n \oplus \Big(\bigoplus_{j \geqslant n+1} L_j \Big).$$

By assumption we have $\operatorname{Im} H_{n+1}(F) \subset \overline{\operatorname{Im} F(n+1,\infty)}$. This implies

$$(3.6)\qquad \operatorname{Im} h_n \subset \overline{\operatorname{Im} F(n+1,\infty)},$$

and hence

$$(3.7)\qquad \begin{aligned} \overline{\operatorname{Im} F(n,\infty)} &\subset \left(\operatorname{Im} \begin{bmatrix} f_{n,n} \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ F(n+1,\infty) \end{bmatrix} \right)^{-} \\ &= \overline{\operatorname{Im} f_{n,n}} \oplus \overline{\operatorname{Im} F(n+1,\infty)} = N_n \oplus \overline{\operatorname{Im} F(n+1,\infty)}. \end{aligned}$$

We shall prove that the inclusion in (3.7) is actually an equality. To do this, let $x = f_{n,n}v$ for some vector $v \in K_n$. From (3.6) it follows that there exist vectors $g_k \in \bigoplus_{j \geqslant n+1} K_j$, $k \in \mathbb{N}$, such that

$$-h_n v = \lim_{k \to \infty} F(n+1,\infty) g_k.$$

Put $w_k = F(n+1,\infty)g_k$ for each $k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} F(n,\infty) \begin{bmatrix} v \\ g_k \end{bmatrix} = \lim_{k\to\infty} \left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ h_n v \end{bmatrix} + \begin{bmatrix} 0 \\ w_k \end{bmatrix} \right) = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

It follows that $\operatorname{Im} f_{n,n} \subset \overline{\operatorname{Im} F(n,\infty)}$, and hence $N_n \subset \overline{\operatorname{Im} F(n,\infty)}$. Since $\overline{\operatorname{Im} F(n+1,\infty)}$ is contained in $\overline{\operatorname{Im} F(n,\infty)}$, it follows that $\overline{\operatorname{Im} F(n,\infty)} = N_n \oplus \overline{\operatorname{Im} F(n+1,\infty)}$. Since n is arbitrary, we conclude that

$$\overline{\operatorname{Im} F(n,\infty)} = \Big(\bigoplus_{j=n}^{m} N_j\Big) \oplus \overline{\operatorname{Im} F(m+1,\infty)} \tag{3.8}$$

for each $m \in \mathbb{Z}$, $m \geqslant n$.

From (3.8) it follows that $\overline{\operatorname{Im} F(n,\infty)} = \bigoplus\limits_{j=n}^{\infty} N_j$ for each integer n. To see this, fix $n \in \mathbb{Z}$. From (3.8) we see that $\bigoplus\limits_{j=n}^{m} N_j \subset \overline{\operatorname{Im} F(n,\infty)}$ for each $m \geqslant n$ and hence $\bigoplus\limits_{j=n}^{\infty} N_j \subset \overline{\operatorname{Im} F(n,\infty)}$. On the other hand, suppose that $x = (x_n, x_{n+1}, \ldots) \in \overline{\operatorname{Im} F(n,\infty)}$ and $x \perp \bigoplus\limits_{j=n}^{\infty} N_j$. By (3.8) this yields $x \in \overline{\operatorname{Im} F(m+1,\infty)}$ for each $m \geqslant n$. Notice that $x \in \overline{\operatorname{Im} F(m+1,\infty)}$ implies that the j-th coordinate x_j is zero for each $j \leqslant m$. Thus the latter holds for each $m \geqslant n$. Hence $x = 0$. So $\overline{\operatorname{Im} F(n,\infty)} = \bigoplus\limits_{j=n}^{\infty} N_j$, and F is semi-outer.

Part (b). In this part we show that (i) implies (ii). Suppose the operator F is semi-outer. Take an arbitrary block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M})$ satisfying $G^*G \leqslant F^*F$. Define the map $J : \operatorname{Im} F \to \ell^2(\mathcal{M})$ by $JFx = Gx$. Then J extends to a contraction $J : \overline{\operatorname{Im} F} \to \ell^2(\mathcal{M})$.

Since the operator F is semi-outer, there exists for each $j \in \mathbb{Z}$ a subspace $N_j \subset L_j$, such that $\overline{\operatorname{Im} F(n,\infty)} = \bigoplus\limits_{j \geqslant n} N_j$ for each $n \in \mathbb{Z}$. We will show that $\overline{\operatorname{Im} F} = \bigoplus\limits_{n \in \mathbb{Z}} N_j$. The fact that F is block lower triangular implies that $\overline{\operatorname{Im} F} \supset \overline{\operatorname{Im} F(n,\infty)} = \bigoplus\limits_{j \geqslant n} N_j$, and hence $\overline{\operatorname{Im} F} \supset \bigoplus\limits_{j \in \mathbb{Z}} N_j$. Next take $x \in \overline{\operatorname{Im} F}$, and assume $x \perp \bigoplus\limits_{j \in \mathbb{Z}} N_j$. We have to show that $x = 0$. Our hypothesis implies that $x \perp \operatorname{Im} F(n,\infty)$ for each integer $n \in \mathbb{Z}$. Since F is block lower triangular, it follows that $x \perp Fv$ for each $v \in \ell^2(\mathcal{K})$ with finite negative support. But vectors of the latter type are dense in $\ell^2(\mathcal{K})$, and therefore $x \perp \operatorname{Im} F$. Hence $x = 0$.

Now we will show that the operator $J : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{M})$ is block lower triangular. For arbitrary $n \in \mathbb{Z}$ we have

$$J\Big[\bigoplus_{j \geqslant n} N_j\Big] = J\left[\overline{\operatorname{Im} F(n,\infty)}\right] \subset \overline{\operatorname{Im} JF(n,\infty)} = \overline{\operatorname{Im} G(n,\infty)} \subset \bigoplus_{j \geqslant n} M_j.$$

Each of the operators F, G, and J is block lower triangular relative to the appropriate Hilbert space direct sums. So for each $n, N \in \mathbb{Z}$, $n \leqslant N$, we have

$$G(n,N)^*G(n,N) = F(n,N)^*J(n,N)^*J(n,N)F(n,N) \leqslant F(n,N)^*F(n,N).$$

We have shown that (i) implies (ii). Statement (iii) follows from (ii) trivially.

Part (c). In this part we show that (iii) implies (iv). So in what follows we assume that (iii) holds.

Take $n \in \mathbb{Z}$. Let us partition the operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ into

$$F = \begin{bmatrix} F_{1,1} & 0 & 0 \\ F_{2,1} & F_{2,2} & 0 \\ F_{3,1} & F_{3,2} & F_{3,3} \end{bmatrix} : X_1 \oplus X_2 \oplus X_3 \to Y_1 \oplus Y_2 \oplus Y_3, \tag{3.9}$$

where

$$X_1 = \bigoplus_{j \leqslant n-1} K_j, \quad X_2 = K_n, \quad X_3 = \bigoplus_{j \geqslant n+1} K_j, \tag{3.10}$$

$$Y_1 = \bigoplus_{j \leqslant n-1} L_j, \quad Y_2 = L_n, \quad Y_3 = \bigoplus_{j \geqslant n+1} L_j. \tag{3.11}$$

In particular, $F_{2,2} = f_{n,n}$.

First we will show that $\operatorname{Im} F_{3,2} \subset \overline{\operatorname{Im} F_{3,3}}$. Let P denote the orthogonal projection of Y_3 onto $\overline{\operatorname{Im} F_{3,3}}$. Introduce the spaces

$$Z_3 = Y_3 \ominus \overline{\operatorname{Im} F_{3,3}}, \quad Z_4 = \overline{\operatorname{Im} F_{3,3}},$$

and consider the operator

$$G := \begin{bmatrix} F_{1,1} & 0 & 0 \\ F_{2,1} & F_{2,2} & 0 \\ (I-P)F_{3,1} & (I-P)F_{3,2} & 0 \\ 0 & 0 & 0 \\ PF_{3,1} & PF_{3,2} & F_{3,3} \end{bmatrix} : X_1 \oplus X_2 \oplus X_3 \to Y_1 \oplus Y_2 \oplus Z_3 \oplus Z_3 \oplus Z_4.$$

Clearly, $G^*G = F^*F$. Put $\widetilde{Y}_2 = Y_2 \oplus Z_3$. Since $Z_3 \oplus Z_4$ is just Y_3, we see that G admits the following operator matrix representation

$$G = \begin{bmatrix} G_{1,1} & 0 & 0 \\ G_{2,1} & G_{2,2} & 0 \\ G_{3,1} & G_{3,2} & G_{3,3} \end{bmatrix} : X_1 \oplus X_2 \oplus X_3 \to Y_1 \oplus \widetilde{Y}_2 \oplus Y_3,$$

where $G_{1,1} = F_{1,1}$ and $G_{3,3} = F_{3,3}$. Recall that $F_{1,1}$ and $F_{3,3}$ are block lower triangular. So, we may view G as a block lower triangular operator form $\ell^2(\mathcal{K})$ into $\ell^2(\mathcal{M})$, where

$$M_j = \begin{cases} L_j, & j \neq n, \\ \widetilde{Y}_2, & j = n. \end{cases}$$

Because $G^*G = F^*F$, our assumption implies that

$$F_{2,2}^* F_{2,2} = f_{n,n}^* f_{n,n} \geqslant g_{n,n}^* g_{n,n} = G_{2,2}^* G_{2,2} = F_{2,2}^* F_{2,2} + F_{3,2}^*(I-P)F_{3,2}. \tag{3.12}$$

So, $(I-P)F_{3,2} = 0$, and hence $\operatorname{Im} F_{3,2} \subset \overline{\operatorname{Im} F_{3,3}}$.

Next we show that $\operatorname{Im} H_n(F) \subset \overline{\operatorname{Im} F(n,\infty)}$ for each $n \in \mathbb{Z}$. Fix $n \in \mathbb{Z}$. Let us partition the operator $H_n(F)$ in the following way:

$$H_n(F) = \begin{bmatrix} \cdots & h_{n-3} & h_{n-2} & h_{n-1} \end{bmatrix} : \bigoplus_{j=-\infty}^{n-1} K_j \to \bigoplus_{j=n}^{\infty} L_j,$$

where for each $j \geqslant 1$ we have

$$h_{n-j} = \begin{bmatrix} f_{n,n-j} \\ f_{n+1,n-j} \\ f_{n+2,n-j} \\ \vdots \end{bmatrix} : K_{n-j} \to \bigoplus_{j=n}^{\infty} L_j.$$

Take $j \geqslant 1$. Denote by $\tau : \bigoplus_{i=n}^{\infty} L_i \to \bigoplus_{i=n-j+1}^{\infty} L_i$ the canonical embedding. By the previous paragraph we conclude that

$$\begin{aligned} \operatorname{Im} h_{n-j} = \operatorname{Im} \tau^* \begin{bmatrix} f_{n-j+1,n-j} \\ f_{n-j+2,n-j} \\ \vdots \end{bmatrix} &\subset \tau^* \overline{\operatorname{Im} F(n-j,\infty)} \subset \overline{\operatorname{Im} \tau^* F(n-j,\infty)} \\ &\subset \overline{\operatorname{Im} F(n,\infty)}. \end{aligned}$$

It follows that $\operatorname{Im} H_n(F)$ is contained in $\overline{\operatorname{Im} F(n,\infty)}$. ∎

One can show directly that (iii) implies (i) by using a time-variant version of inner-outer factorization which appears in [14]. Indeed, assume that $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ is a block lower triangular operator satisfying property (iii) in Proposition 3.1. By [14], Proposition 1.6 there exists a block lower triangular outer operator $F_{\text{out}} = (f_{\text{out},i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}_{\text{out}})$, where $\mathcal{L}_{\text{out}}$ is the sequence of Hilbert spaces $(L_{\text{out},n})_{n\in\mathbb{Z}}$ and a block lower triangular isometry $Q = (q_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{L}_{\text{out}}) \to \ell^2(\mathcal{L})$ such that $F = QF_{\text{out}}$. Since F_{out}

is outer, we have $\overline{\operatorname{Im} f_{\mathrm{out},n,n}} = L_{\mathrm{out},n}$ for each $n \in \mathbb{Z}$. From the assumption (iii) and $F^*F = F^*_{\mathrm{out}}F_{\mathrm{out}}$ we have $f^*_{\mathrm{out},n}f_{\mathrm{out},n} \leqslant f^*_n f_n$ for each $n \in \mathbb{Z}$. Since $q_{n,n}f_{\mathrm{out},n,n} = f_{n,n}$ it follows that

$$f^*_{\mathrm{out},n,n}(I - q^*_{n,n}q_{n,n})f_{\mathrm{out},n,n} \leqslant 0. \tag{3.13}$$

The operator Q is an isometry, so $q_{n,n}$ is a contraction for each $n \in \mathbb{Z}$. Since $\overline{\operatorname{Im} f_{\mathrm{out},n,n}} = L_{\mathrm{out},n}$, it follows from (3.13) that $q^*_{n,n}q_{n,n} \geqslant I$ for each $n \in \mathbb{Z}$. We conclude that $q^*_{n,n}q_{n,n} = I$ for each $n \in \mathbb{Z}$. We have $\sum_{i=-\infty}^{n} q^*_{i,n}q_{i,n} = I$ for each $n \in \mathbb{Z}$, because Q is an isometry. We obtain $\sum_{i=-\infty}^{n-1} q^*_{i,n}q_{i,n} = 0$ for each $n \in \mathbb{Z}$. It follows that the operator Q is a diagonal isometry, and hence $F = QF_{\mathrm{out}}$ is a semi-outer operator.

4. PROOF OF THE THEOREMS 2.1 AND 2.2

In this section we will give the proofs of Theorems 2.1 and 2.2. We will start with two introductory lemmas. Part (i) of the next lemma is a variant of Lemma 3.4 in [21], which was stated there for unitary systems. Here we need a version for isometric systems. The proof is essentially the same, and is given here for the sake of completeness.

LEMMA 4.1. (i) *Let Σ be an isometric time-variant system with input-output map $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$. Then we have*

$$I - T(n,m)^*T(n,m) = \Lambda_{n,m}(\Sigma)^*\Lambda_{n,m}(\Sigma), \quad n,m \in \mathbb{Z},\ n \leqslant m. \tag{4.1}$$

(ii) *Let*

$$\Sigma = \left(A(n), B(n), \begin{bmatrix} C(n) \\ Y(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ Z(n) \end{bmatrix}; H_n, K_n, L_n \oplus M_n\right) \tag{4.2}$$

be an isometric system. Let T_{Σ_1} be the input-output map of $\Sigma_1 = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$, and let T_{Σ_2} be the input-output map of $\Sigma_2 = (A(n), B(n), Y(n), Z(n); H_n, K_n, M_n)$. Then we have

$$I - T_{\Sigma_1}(n,k)^*T_{\Sigma_1}(n,k) - T_{\Sigma_2}(n,k)^*T_{\Sigma_2}(n,k) = \Lambda_{n,k}(\Sigma_1)^*\Lambda_{n,k}(\Sigma_1) \tag{4.3}$$

for each pair of integers n,k with $n \geqslant k$. Furthermore, $I - T^*_{\Sigma_1}T_{\Sigma_1} \geqslant T^*_{\Sigma_2}T_{\Sigma_2}$.

Proof. (i) Let $\Sigma = (\alpha(n), \beta(n), \gamma(n), \delta(n); H_n, K_n, L_n)$ be our isometric system. Consider

$$\begin{cases} x_{j+1} = \alpha(j)x_j + \beta(j)u_j, \\ y_j = \gamma(j)x_j + \delta(j)u_j, \\ x_n = 0, \end{cases}$$

where $j = 1, 2, \ldots, m-1, m$. Let $u = (u_n, u_{n+1}, \ldots, u_{m-1}, u_m) \in \bigoplus_{j=n}^{m} K_j$ be an input sequence, and let $y = (y_n, y_{n+1}, \ldots, y_{m-1}, y_m)$ be the corresponding output sequence. Then $y = T(n,m)u$. Since the system is isometric, we have:

$$\|x_{j+1}\|^2 - \|x_j\|^2 = \|u_j\|^2 - \|y_j\|^2, \quad j = n, n+1, \ldots, m-1, m. \tag{4.4}$$

Hence, using $x_n = 0$, we see that

$$\|x_{m+1}\|^2 = \|x_{m+1}\|^2 - \|x_n\|^2 = \sum_{j=n}^{m} \|u_j\|^2 - \sum_{j=n}^{m} \|y_j\|^2, \tag{4.5}$$

and therefore

$$\langle (I - T(n,m)^* T(n,m))u, u\rangle = \sum_{j=n}^{m} \|u_j\|^2 - \sum_{j=n}^{m} \|y_j\|^2 = \|x_{m+1}\|^2. \tag{4.6}$$

On the other hand, the state vector x_{m+1} at time $m+1$ is given by $x_{m+1} = \Lambda_{n,m}(\Sigma)u$. We conclude that

$$\langle (I - T(n,m)^* T(n,m))u, u\rangle = \langle \Lambda_{n,m}(\Sigma)^* \Lambda_{n,m}(\Sigma)u, u\rangle,$$

which yields (4.1).

(ii) Define the unitary operator $Z : \Big(\bigoplus_{j\in\mathbb{Z}} L_j\Big) \oplus \Big(\bigoplus_{j\in\mathbb{Z}} M_j\Big) \to \bigoplus_{j\in\mathbb{Z}} (L_j \oplus M_j)$ by $Z\big((l_j)_{j\in\mathbb{Z}}, (m_j)_{j\in\mathbb{Z}}\big) = \big((l_j, m_j)\big)_{j\in\mathbb{Z}}$. The block lower triangular operator

$$T = Z \begin{bmatrix} T_{\Sigma_1} \\ T_{\Sigma_2} \end{bmatrix} : \ell^2(\mathcal{K}) \to \bigoplus_{j\in\mathbb{Z}} (L_j \oplus M_j), \tag{4.7}$$

is the input-output map of the isometric system Σ. It is contractive by [8], Theorem 4.1, and hence we have $I - T_{\Sigma_1}^* T_{\Sigma_1} \geqslant T_{\Sigma_2}^* T_{\Sigma_2}$. Since Σ is an isometric system, it follows from Part (i) that

$$\begin{aligned} I - T_{\Sigma_1}(n,n+k)^* T_{\Sigma_1}(n,n+k) - T_{\Sigma_2}(n,n+k)^* T_{\Sigma_2}(n,n+k) \\ = I - T(n,n+k)^* T(n,n+k) = \Lambda_{n,n+k}(\Sigma)^* \Lambda_{n,n+k}(\Sigma) \\ = \Lambda_{n,n+k}(\Sigma_1)^* \Lambda_{n,n+k}(\Sigma_1) \end{aligned} \tag{4.8}$$

for each pair of integers n, k with $k \geqslant 0$. ∎

Lemma 4.2. *A controllable isometric system is pointwise stable if and only if its input-output map is an isometry.*

Proof. Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a controllable isometric system with input-output map T. Fix integers n, m with $n \leqslant m$, and take $k \in \bigoplus_{j=n}^{m} K_j$. Denote by $\widetilde{k}$ the embedding of k into $\ell^2(\mathcal{K})$. Let $p > 0$. From (4.1) it follows that

$$\begin{aligned} &\|\tau_{\mathcal{A}}(m+p+1, m+1)\Lambda_{n,m}(\Sigma)k\|^2 \\ &= \left\|\Lambda_{n,m+p}(\Sigma)\begin{bmatrix} k \\ 0 \end{bmatrix}\right\|^2 = \left\langle (I - T(n,m+p)^* T(n,m+p)\begin{bmatrix} k \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \end{bmatrix}\right\rangle \\ &= \|\widetilde{k}\|^2 - \|P_{n,m+p}(\mathcal{L})T\widetilde{k}\|^2. \end{aligned} \tag{4.9}$$

Here $P_{n,m+p}(\mathcal{L})$ denotes the orthogonal projection of $\ell^2(\mathcal{L})$ onto $\bigoplus_{j=n}^{m+p} L_j$, and $\widetilde{k} = (k, 0)^{\mathrm{tr}}$ is considered as a vector in $\bigoplus_{j=n}^{m+p} K_j$. Since $\widetilde{k}$ is an element of the subspace $\bigoplus_{j=n}^{\infty} K_j$ of $\ell^2(\mathcal{K})$, and T is a block lower triangular operator, it follows that

$$\lim_{p\to\infty} P_{n,m+p}(\mathcal{L})T\widetilde{k} = T\widetilde{k}.$$

Now, if Σ is pointwise stable, then the left hand side of (4.9) tends to 0 for $p \to \infty$, and hence $\|T\widetilde{k}\| = \|\widetilde{k}\|$ for each $\widetilde{k}$ of finite support. By continuity we conclude that T is an isometry.

On the other hand, if we assume T to be an isometry, then for each vector $x = \Lambda_{n,m}(\Sigma)k$ we have $\lim_{p\to\infty} \tau_{\mathcal{A}}(m+p+1, m+1)\Lambda_{n,m}(\Sigma)k = 0$. Vectors of this type are dense, because Σ is controllable. By a limiting argument, and using the fact that Σ is a contractive system, it follows that Σ is pointwise stable. ∎

Proof of Theorem 2.1. Consider a contractive realization

$$\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n) \tag{4.10}$$

of the block lower triangular contraction $T = (t_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$, and let

$$\Phi = (A(n), B(n), Y(n), Z(n); H_n, K_n, \mathcal{D}_{M_\Sigma(n)}) \tag{4.11}$$

be the first complementary system associated with Σ. Let $T_\Phi = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{D})$ be the input-output map of Φ. Here $\mathcal{D}$ is the sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)})_{n\in\mathbb{Z}}$. Recall that by construction of the first complementary system Φ, the system

$$\widehat{\Sigma} = \left(A(n), B(n), \begin{bmatrix} C(n) \\ Y(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ Z(n) \end{bmatrix}; H_n, K_n, L_n \oplus \mathcal{D}_{M_\Sigma(n)}\right) \tag{4.12}$$

is an isometric system.

The proof is split into five parts. In Parts (a), (b), and (c), we show that Σ is optimal implies that for any block lower triangular operator G satisfying $I - T^*T \geqslant G^*G$ there exists a block lower triangular contraction J such that $G = JT_\Phi$. In Part (d) we show the reverse implication. In Part (e) we prove that if Σ is a controllable optimal system, then T_Φ is an outer operator.

Part (a). Assume that Σ is optimal. Let $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ be a block lower triangular operator satisfying $I - T^*T \geqslant G^*G$. We shall use the fact that Σ is optimal to show that

$$G(n,m)^*G(n,m) \leqslant T_\Phi(n,m)^*T_\Phi(n,m) \tag{4.13}$$

for each $n, m \in \mathbb{Z}, n \leqslant m$. Here $G(n,m)$ and $T_\Phi(n,m)$ are the (n,m)-th block sections of G and T_Φ, respectively (see formula (3.4)). Let us introduce the unitary operator $W : \Big(\bigoplus_{j\in\mathbb{Z}} L_j\Big) \oplus \Big(\bigoplus_{j\in\mathbb{Z}} N_j\Big) \to \bigoplus_{j\in\mathbb{Z}}(L_j \oplus N_j)$ by $W((l_j)_{j\in\mathbb{Z}}, (n_j)_{j\in\mathbb{Z}}) = (l_j, n_j)_{j\in\mathbb{Z}}$. The operator

$$\widetilde{T} = W \begin{bmatrix} T \\ G \end{bmatrix} : \ell^2(\mathcal{K}) \to \bigoplus_{j\in\mathbb{Z}}(L_j \oplus N_j)$$

is a block lower triangular contractive map, so by [8], Theorem 4.3 there exists a unitary realization

$$\widetilde{\Sigma} = \left(\widetilde{A}(n), \widetilde{B}(n), \begin{bmatrix} \widetilde{C}(n) \\ \widetilde{Y}(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ \widetilde{Z}(n) \end{bmatrix}; \widetilde{H}_n, K_n, L_n \oplus N_n\right) \tag{4.14}$$

of $\widetilde{T}$. It follows that the systems $\widetilde{\Sigma}_1 = (\widetilde{A}(n), \widetilde{B}(n), \widetilde{C}(n), D(n); \widetilde{H}_n, K_n, L_n)$ and $\widetilde{\Sigma}_2 = (\widetilde{A}(n), \widetilde{B}(n), \widetilde{Y}(n), \widetilde{Z}(n); \widetilde{H}_n, K_n, N_n)$ are contractive realizations of the operators T and G, respectively.

Since $\widetilde{\Sigma}$ is an isometric system, T is the input-output map of $\widetilde{\Sigma}_1$, and G is the input-output map of $\widetilde{\Sigma}_2$, formula (4.3) in Proposition 4.1 yields

$$I - T(n,m)^*T(n,m) - G(n,m)^*G(n,m) = \Lambda_{n,m}(\widetilde{\Sigma}_1)^*\Lambda_{n,m}(\widetilde{\Sigma}_1), \tag{4.15}$$

for all integers n and m, $n \leqslant m$. By the same lemma we have

$$(4.16) \qquad I - T(n,m)^* T(n,m) - T_\Phi(n,m)^* T_\Phi(n,m) = \Lambda_{n,m}(\Sigma)^* \Lambda_{n,m}(\Sigma)$$

for all integers n and m, $n \leqslant m$, because $\widehat{\Sigma}$ in (4.12) is an isometric system, T is the input-output map of Σ, and T_Φ is the input-output map of Φ.

Let us take an arbitrary input sequence $u = (u_n, u_{n+1}, \ldots, u_{m-1}, u_m)$, where $u_j \in K_j$. From (4.15), (4.16), and the optimality of the system Σ, we get

$$0 \leqslant \|\Lambda_{n,m}(\widetilde{\Sigma}_1)u\|^2 - \|\Lambda_{n,m}(\Sigma)u\|^2 = \langle (T_\Phi(n,m)^* T_\Phi(n,m) - G(n,m)^* G(n,m))u, u\rangle.$$

Since $\vec{u}$ was arbitrary chosen, inequality (4.13) follows.

Part (b). Assume that Σ is optimal. In this part $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ is a block lower triangular operator satisfying $I - T^*T \geqslant G^*G$, and we use the result of the previous part to show that $G^*G \leqslant T_\Phi^* T_\Phi$.

For each pair of integers n, m with $n \leqslant m$, let $P_{n,m}(\mathcal{K})$, $P_{n,m}(\mathcal{D})$, $P_{n,m}(\mathcal{N})$ denote the orthogonal projections of $\ell^2(\mathcal{K})$ onto $\bigoplus_{j=n}^{m} K_j$, of $\ell^2(\mathcal{D})$ onto $\bigoplus_{j=n}^{m} \mathcal{D}_{M_\Sigma(j)}$, and of $\ell^2(\mathcal{N})$ onto $\bigoplus_{j=n}^{m} N_j$, respectively. In Part (a) inequality (4.13) was shown for arbitrary integers n, m, $n \leqslant m$. Hence for each $x \in \ell^2(\mathcal{K})$ and n, m with $n \leqslant m$ we have

$$(4.17) \qquad \|P_{n,m}(\mathcal{N}) G P_{n,m}(\mathcal{K}) x\| \leqslant \|P_{n,m}(\mathcal{D}) T_\Phi P_{n,m}(\mathcal{K}) x\|.$$

Since for each vector $x \in \ell^2(\mathcal{K})$, we have

$$P_{n,m}(\mathcal{N}) G P_{n,m}(\mathcal{K}) x \to Gx, \quad P_{n,m}(\mathcal{D}) T_\Phi P_{n,m}(\mathcal{K}) x \to T_\Phi x,$$

when $n \to -\infty$ and $m \to \infty$, we conclude from (4.17) that $G^*G \leqslant T_\Phi^* T_\Phi$.

Part (c). Assume that Σ is optimal. In this part we show that for any block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ satisfying $I - T^*T \geqslant G^*G$ there exists a block lower triangular contraction J such that $G = JT_\Phi$.

If a block lower triangular operator $\widetilde{G}$ satisfies $\widetilde{G}^*\widetilde{G} \leqslant T_\Phi^* T_\Phi$, then also $I - T^*T \geqslant \widetilde{G}^*\widetilde{G}$ holds by the second part of Lemma 4.1, so from Part (a) it follows that

$$(4.18) \qquad \widetilde{G}(n,m)^* \widetilde{G}(n,m) \leqslant T_\Phi(n,m)^* T_\Phi(n,m)$$

for each $n, m \in \mathbb{Z}$, $n \leqslant m$. But then we can apply Proposition 3.1 to show that the operator T_Φ is semi-outer. So there exist subspaces $M_j \subset \mathcal{D}_{M_\Sigma(j)}$, $j \in \mathbb{Z}$, such that for each $n \in \mathbb{Z}$ we have $\overline{\operatorname{Im} T_\Phi(n,\infty)} = \bigoplus_{j \geqslant n} M_j$.

Let $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ be a block lower triangular operator satisfying $I - T^*T \geqslant G^*G$. From Part (b) it follows that $G^*G \leqslant T_\Phi^* T_\Phi$. Hence we can define the contractive operator

$$\widetilde{J} : \operatorname{Im} T_\Phi \to \ell^2(\mathcal{N}), \quad \widetilde{J}(T_\Phi x) = Gx, \tag{4.19}$$

and extend it by continuity to $\overline{\operatorname{Im} T_\Phi}$. Define then the contraction $J : \ell^2(\mathcal{D}) \to \ell^2(\mathcal{N})$ by

$$J(v + w) = \widetilde{J} v \quad (v \in \overline{\operatorname{Im} T_\Phi},\ w \perp \operatorname{Im} T_\Phi).$$

Notice that $JT_\Phi = G$. Since $\overline{\operatorname{Im} T_\Phi(n,\infty)} = \bigoplus_{j \geqslant n} M_j$, we conclude that

$$\begin{aligned} J\Big[\bigoplus_{j \geqslant n} \mathcal{D}_{M_\Sigma(j)}\Big] = J\Big[\bigoplus_{j \geqslant n} M_j\Big] &= J\left[\overline{\operatorname{Im} T_\Phi(n,\infty)}\right] \subset \overline{\operatorname{Im} JT_\Phi(n,\infty)} \\ &= \overline{\operatorname{Im} G(n,\infty)} \subset \bigoplus_{j \geqslant n} N_j, \quad n \in \mathbb{Z}. \end{aligned}$$

It follows that J is a block lower triangular contraction as desired.

Part (d). This part concerns the reverse implication. Let $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ be an arbitrary contractive realization of T, and let $\Phi_\Upsilon = (\alpha(n), \beta(n), \eta(n), \zeta(n); X_n, K_n, \mathcal{D}_{M_\Upsilon(n)})$ be the first complementary system associated with Υ. Let T_{Φ_Υ} be the input-output map of Φ_Υ. From the second part of Lemma 4.1 it follows that $I - T^*T \geqslant T_{\Phi_\Upsilon}^* T_{\Phi_\Upsilon}$. By assumption, there exists a block lower triangular contraction J such that $T_{\Phi_\Upsilon} = JT_\Phi$, and therefore for each $n, N \in \mathbb{Z}$ with $n \leqslant N$ we have $T_{\Phi_\Upsilon}(n,N)^* T_{\Phi_\Upsilon}(n,N) \leqslant T_\Phi(n,N)^* T_\Phi(n,N)$. The system

$$\widetilde{\Upsilon} = \left(\alpha(n), \beta(n), \begin{bmatrix} \gamma(n) \\ \eta(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ \zeta(n) \end{bmatrix}; X_n, K_n, L_n \oplus E_n\right)$$

is by construction an isometric system. First using formula (4.3) for the isometric system $\widehat{\Sigma}$ given by (4.12) and thereafter for the isometric system $\widetilde{\Upsilon}$, we obtain

$$\begin{aligned} \Lambda_{n,N}(\Sigma)^* \Lambda_{n,N}(\Sigma) &= I - T(n,N)^* T(n,N) - T_\Phi(n,N)^* T_\Phi(n,N) \\ &\leqslant I - T(n,N)^* T(n,N) - T_{\Phi_\Upsilon}(n,N)^* T_{\Phi_\Upsilon}(n,N) \\ &= \Lambda_{n,N}(\Upsilon)^* \Lambda_{n,N}(\Upsilon) \end{aligned}$$

for each pair of integers $n \geqslant N$. It follows that the system Σ is optimal.

Part (e). Assume now that Σ is an optimal and controllable system. We already have shown that in that case the operator T_Φ is semi-outer. Thus there exist subspaces $N_j \subset \mathcal{D}_{M_\Sigma(j)}$, $j \in \mathbb{Z}$, such that for each integer n we have $\overline{T_\Phi \bigoplus_{j\geqslant n} K_j} = \bigoplus_{j\geqslant n} N_j$.

Fix $n \in \mathbb{Z}$. Since T_Φ is semi-outer, it follows from Proposition 3.1 that $\operatorname{Im} H_n(T_\Phi) \subset \overline{\operatorname{Im} T_\Phi(n,\infty)}$. Hence we have $\operatorname{Im} f_{n,k} \subset \overline{\operatorname{Im} f_{n,n}}$ for each $k \leqslant n$. Now, use that Φ in (4.11) is a realization of T_Φ, so $Y(n)\tau_A(n,k+1)B(k) = f_{n,k}$ for each $k < n$. The system Σ is controllable, so the system Φ is controllable, and hence

$$\overline{\operatorname{Im} Y(n)} = \bigvee_{k<n} \operatorname{Im} Y(n)\tau_A(n,k+1)B(k) = \bigvee_{k<n} \operatorname{Im} f_{n,k} \subset \overline{\operatorname{Im} f_{n,n}}.$$

On the other hand, $Z(n) = f_{n,n}$. We conclude that

$$\mathcal{D}_{M_\Sigma(n)} = \overline{\operatorname{Im}\,[\,Y(n) \quad Z(n)\,]} \subset \overline{\operatorname{Im} Z(n)} \subset N_n \subset \mathcal{D}_{M_\Sigma(n)}. \tag{4.20}$$

Hence the operator T_Φ is outer. ∎

Proof of Theorem 2.2. Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, and let $\Sigma_\circ = (A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n)$ be a controllable and optimal realization of T.

Part (i). In this part we show when equality can be obtained in the factorization $I - T^*T = F^*F$. Let

$$\Phi = (A_\circ(n), B_\circ(n), Y_\circ(n), Z_\circ(n); H_{\circ,n}, K_n, \mathcal{D}_{M_\Sigma(n)}) \tag{4.21}$$

be the complementary system associated with $\Sigma_\circ$, and let T_Φ be its input-output map. By construction, the system

$$\widehat{\Sigma} = \left(A_\circ(n), B_\circ(n), \begin{bmatrix} C_\circ(n) \\ Y_\circ(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ Z_\circ(n) \end{bmatrix}; H_{\circ,n}, K_n, L_n \oplus \mathcal{D}_{M_\Sigma(n)} \right) \tag{4.22}$$

is isometric. It is controllable because the system $\Sigma_\circ$ is controllable. By Lemma 4.2, the sequence $A_\circ(n) : H_{\circ,n} \to H_{\circ,n+1}$ is pointwise stable if and only if the input-output map $\widehat{T}$ of the system $\widehat{\Sigma}$ is an isometry, that is, if and only if $T_\Phi^* T_\Phi + T^*T = I$. In particular, this proves the "if"-part of statement (i) in Theorem 2.2.

Suppose that for some block lower triangular operator F we have $I - T^*T = F^*F$. Then we conclude from Theorem 2.1 that

$$F^*F \leqslant T_\Phi^* T_\Phi \leqslant I - T^*T = F^*F.$$

It follows that $T_\Phi^* T_\Phi = I - T^*T$, and hence the input-output map of $\widehat{\Sigma}$ is an isometry. From Lemma 4.2 we conclude that the sequence $A_\circ(n) : H_{\circ,n} \to H_{\circ,n+1}$ is pointwise stable.

Part (ii). First suppose that the controllable and optimal system $\Sigma_\circ$ is isometric. Then the input-output map of the first complementary system Φ equals 0. From Theorem 2.1 it follows that for each block lower triangular operator F satisfying $I - T^*T \geqslant F^*F$ there exists a block lower triangular contraction Q such that $F = QT_\Phi$. Hence $F = 0$ is the only block lower triangular operator satisfying the inequality $I - T^*T \geqslant F^*F$.

Suppose now that $F = 0$ is the only block lower triangular operator satisfying the inequality $I - T^*T \geqslant F^*F$. Then, in particular, the input-output map T_Φ of the first complementary system associated with the system $\Sigma_\circ$ equals 0. Since $\Sigma_\circ$ is controllable, we have

$$\overline{\operatorname{Im}(A_\circ|B_\circ; n)} = H_{\circ,n}, \quad n \in \mathbb{Z}.$$

Since $T_\Phi = 0$, the operator $Z_\circ(n)$ appearing in (4.21) equals 0 for each $n \in \mathbb{Z}$, and for each $n, k \in \mathbb{Z}$, $n \geqslant k$, we have

$$Y_\circ(n)\tau_{A_\circ}(n, k+1)B_\circ(k) = 0.$$

So, for each $n \in \mathbb{Z}$ and for all vectors $x = \tau_{A_\circ}(n, k+1)B_\circ(k)u$, we have $Y_\circ(n)x = 0$. Vectors of this type are dense in $H_{\circ,n}$, so $Y_\circ(n) = 0$ for each $n \in \mathbb{Z}$. Hence the defect operator associated with the system matrix $M_\Sigma(n)$ satisfies

$$D_{M_{\Sigma_\circ}(n)} = D_{M_{\Sigma_\circ}(n)} \begin{bmatrix} \tau_{H_{0,n}} & \tau_{K_n} \end{bmatrix} = \begin{bmatrix} Y_\circ(n) & Z_\circ(n) \end{bmatrix} = 0.$$

We conclude that for each $n \in \mathbb{Z}$ the operator $M_{\Sigma_\circ}(n)$ is an isometry. ∎

Notice that in the last part of the above proof the optimality of $\Sigma_\circ$ is not used. In fact, by the same line of arguments one shows that if Σ is a controllable and contractive realization of a block lower triangular operator T, then the fact that $F = 0$ is the only block lower triangular operator satisfying the inequality $I - T^*T \geqslant F^*F$ implies the realization Σ to be isometric.

5. PROOFS OF THEOREMS 2.3 AND 2.4

In this section we prove Theorems 2.3 and 2.4. Furthermore, we will give an example to show that in Theorem 2.3 the condition that the realization Σ of T is minimal cannot be replaced by the weaker condition that Σ is just observable.

We shall prove Theorem 2.3 and Theorem 2.4 by using Theorem 2.1 and 2.2 and duality arguments. For this purpose we need the following notions. Let $\mathcal{X} = (X_i)_{i\in\mathbb{Z}}$ be a sequence of Hilbert-spaces. By $\ell^2(\mathcal{X}_-)$ we denote the Hilbert space direct sum $\bigoplus_{j=-\infty}^{\infty} X_{-j}$. For each sequence $\mathcal{X} = (X_j)_{j\in\mathbb{Z}}$ of Hilbert spaces we define the *flip-over operator* $J_{\mathcal{X}}$ to be

(5.1)
$$J_{\mathcal{X}} : \ell^2(\mathcal{X}) \to \ell^2(\mathcal{X}_-), \quad J_{\mathcal{X}}(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, x_1, \boxed{x_0}, x_{-1}, \ldots).$$

The flip-over operator is unitary.

Let $T = (t_{i,j})$ be a bounded operator acting from the Hilbert space $\ell^2(\mathcal{K})$ into $\ell^2(\mathcal{L})$, and consider the flip-over operators $J_{\mathcal{K}} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}_-)$, $J_{\mathcal{L}} : \ell^2(\mathcal{L}) \to \ell^2(\mathcal{L}_-)$. The operator $J_{\mathcal{K}}T^*J_{\mathcal{L}}^*$ is called the *reversed adjoint* of T and is denoted by $T^{\sim}$.

Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a contractive system. The system $\Sigma^* = (A_*(n), C_*(n), B_*(n), D_*(n); H_{*,n}, L_{*,n}, K_{*,n})$, where

$$H_{*,n} = H_{-n+1}, \quad K_{*,n} = K_{-n}, \quad L_{*,n} = L_{-n}, \tag{5.2}$$

and

$$\begin{aligned} A_*(n) &= A(-n)^* : H_{*,n} \to H_{*,n+1}, \quad & B_*(n) &= B(-n)^* : H_{*,n} \to K_{*,n}, \\ C_*(n) &= C(-n)^* : L_{*,n} \to H_{*,n+1}, \quad & D_*(n) &= D(-n)^* : L_{*,n} \to K_{*,n}, \end{aligned}$$

is called the *adjoint system* associated with Σ (see [8], Section 1).

If T is the input-output map of Σ, then the reversed adjoint $T^{\sim}$ is the input-output map of the adjoint system. Furthermore, by [8], Corollary 7.5, if Σ is minimal and star-optimal, then Σ^* is minimal and optimal.

Take arbitrary $n \in \mathbb{Z}$ and $x \in H_{*,n}$. Then, for each $p \geqslant 0$

$$\tau_{A_*}(n+p, n)x = \tau_A(-n+1, -n-p+1)^* x. \tag{5.3}$$

In particular, the sequence of main operators of Σ^* is pointwise stable if and only if the sequence of main operators of Σ is pointwise star-stable.

Proof of Theorem 2.3. Let Σ be a contractive realization of $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$. Its reversed adjoint is given by $T^{\sim} = J_{\mathcal{K}}T^*J_{\mathcal{L}}^* : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{K}_-)$. The adjoint system Σ^* is a contractive realization of $T^\sim$. Let Ψ be the second complementary system associated with Σ with input-output map $T_\Psi : \ell^2(\mathcal{D}_*) \to \ell^2(\mathcal{L})$. Here $\ell^2(\mathcal{D}_*) = \bigoplus_{j=-\infty}^{\infty} \mathcal{D}_{M_\Sigma(j)^*}$. Its reversed adjoint $(T_\Psi)^\sim = J_\mathcal{E} T_\Psi^* J_\mathcal{L}^* : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{D}_{*-})$ is the input-output map of the adjoint system Ψ^*, or in formula $(T_\Psi)^\sim = T_{\Psi^*}$.

The system Σ is minimal and star-optimal if and only if the adjoint system Σ^* is minimal and optimal by [8], Corollary 7.5. By Theorem 2.1, the system Σ^* is a minimal and optimal realization of $T^\sim : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{K}_-)$ if and only if for any block lower triangular operator $F : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{N})$ satisfying $I - (T^\sim)^* T^\sim \geqslant F^*F$ there exists a block lower triangular contraction $V : \ell^2(\mathcal{D}_{*-}) \to \ell^2(\mathcal{N})$ such that

$$F = V(T_\Psi)^\sim. \tag{5.4}$$

The latter is equivalent with the requirement that for any block lower triangular operator $G : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ satisfying $I - TT^* \geqslant GG^*$ there exists a block lower triangular contraction $Q : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{D}_*)$ such that $G = T_\Psi Q$.

Suppose now that Σ is minimal and star-optimal. Then Σ^* is minimal and optimal by [8], Corollary 7.5, and by Theorem 2.1 the input-output map $T_{\Psi^*} : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{D}_{*-})$ of the first complementary system associated with Σ^* is outer.

Fix $n \in \mathbb{Z}$. Then

$$\begin{aligned} T_\Psi^*\Big[\bigoplus_{j\leqslant -n} L_j\Big] &= T_\Psi^* J_\mathcal{L}^*\Big[\bigoplus_{j\geqslant n} L_{-n}\Big] = J_{\mathcal{D}_*}^*(T_\Psi)^\sim\Big[\bigoplus_{j\geqslant n} L_{-n}\Big] \\ &= J_{\mathcal{D}_*}^*(T_{\Psi^*})\Big[\bigoplus_{j\geqslant n} L_{-n}\Big] = J_{\mathcal{D}_*}^*\Big[\bigoplus_{j\geqslant n} \mathcal{D}_{M_\Sigma(-j)^*}\Big] = \bigoplus_{j\leqslant -n} \mathcal{D}_{M_\Sigma(j)^*}. \end{aligned}$$

The operator T_Ψ is star-outer. ∎

Proof of Theorem 2.4. We split the proof into two parts. Throughout $T = (t_{i,j})_{i,j=-\infty}^\infty : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ is a block lower triangular contraction, and $\Sigma_\bullet = (A_\bullet(n),\ B_\bullet(n),\ C_\bullet(n),\ D(n);\ H_{\bullet,n}, K_n, L_n)$ is a minimal and star-optimal realization of T.

Part (a). We will show that there exists a block lower triangular contraction $R : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ such that $I - TT^* = RR^*$ if and only if the sequence $A_\bullet(n) : H_{\bullet,n} \to H_{\bullet,n+1}$ is pointwise star-stable.

Suppose first that there exists a block lower triangular operator $R : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L})$ such that $I - TT^* = RR^*$. Then

$$I - (T^\sim)^* T^\sim = (R^\sim)^* R^\sim, \tag{5.5}$$

where $R^\sim$ is the reversed adjoint of R. By Theorem 2.2 and Equation (5.5), it follows that the sequence of main operators of the minimal and optimal system $(\Sigma_\bullet)^*$ is pointwise stable. Hence the sequence of main operators of $\Sigma_\bullet$ is pointwise star-stable.

To prove the converse, suppose that the sequence of main operators of $\Sigma_\bullet$ is pointwise star-stable. By Theorem 2.2 there exists a block lower triangular operator $S : \ell^2(\mathcal{L}_-) \to \ell^2(\mathcal{X})$, where $\mathcal{X}$ is a sequence of Hilbert spaces, such that $I - (T^\sim)^*T^\sim = S^*S$. It follows that

$$I - TT^* = J_{\mathcal{L}}^*(I - (T^\sim)^*T^\sim)J_{\mathcal{L}} = J_{\mathcal{L}}^*S^*SJ_{\mathcal{L}} = J_{\mathcal{L}}^*S^*J_{\mathcal{X}}J_{\mathcal{X}}^*SJ_{\mathcal{L}}.$$

Finally, notice that $J_{\mathcal{X}}^*SJ_{\mathcal{L}}$ is block lower triangular.

Part (b). Let $R : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ be a block lower triangular operator, and let $R^\sim = J_{\mathcal{M}}R^*J_{\mathcal{L}}^*$ be its reversed adjoint. Let $T^\sim = J_{\mathcal{K}}T^*J_{\mathcal{L}}^*$ be the reversed adjoint of T. It is straightforward that $R = 0$ is the only operator satisfying $I - TT^* \geqslant RR^*$ if and only if $R^\sim = 0$ is the only operator satisfying $I - (T^\sim)^*T^\sim \geqslant (R^\sim)^*R^\sim$. From Theorem 2.2 we see that this is equivalent with the requirement that there exists a minimal and optimal realization of $T^\sim$ which is isometric. This in turn is equivalent with the requirement that there exists a minimal and star-optimal realization of T which is co-isometric. ∎

We conclude this section with an example showing that in Theorem 2.3 the condition that the realization Σ of T is minimal cannot be replaced by the weaker condition of observability. Consider the system

$$\Sigma = \left(\begin{bmatrix} -1/8 & -1/6\sqrt{3} \\ 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/8\sqrt{3} & 1/2 \end{bmatrix}, 1/2; \mathbb{C}^2, \mathbb{C}, \mathbb{C} \right),$$

with input-output map T. In [3], Section 4, it was shown that Σ is a star-optimal system. Notice that Σ is observable, but not minimal. The restriction

$$\Sigma_1 = (-1/8, 1/2\sqrt{3}, 1/8\sqrt{3}, 1/2; \mathbb{C}, \mathbb{C}, \mathbb{C})$$

of Σ is a minimal and star-optimal realization of T. Define for each $n \in \mathbb{Z}$ the spaces

$$\mathcal{D}_{M_\Sigma(n)^*} = \overline{\mathrm{Im}(I - M_\Sigma(n)M_\Sigma(n)^*)^{1/2}},$$
$$\mathcal{D}_{M_{\Sigma_1}(n)^*} = \overline{\mathrm{Im}(I - M_{\Sigma_1}(n)M_{\Sigma_1}(n)^*)^{1/2}}.$$

Let

$$\Psi = \left(\begin{bmatrix} -1/8 & -1/6\sqrt{3} \\ 0 & 1/2 \end{bmatrix}, V, \begin{bmatrix} 1/8\sqrt{3} & 1/2 \end{bmatrix}, W; \mathbb{C}^2, \mathcal{D}_{M_\Sigma(n)^*}, \mathbb{C} \right)$$

be the second complementary system associated with Σ, and

$$\Psi_1 = (-1/8, V_1, 1/8\sqrt{3}, W_1; \mathbb{C}, \mathcal{D}_{M_{\Sigma_1}(n)^*}, \mathbb{C})$$

be the second complementary system associated with Σ_1. Denote the input-output map of Ψ and Ψ_1 by T_Ψ and T_{Ψ_1}, respectively. By construction

$$WW^* = 1 - [\,1/8\sqrt{3} \quad 1/2\,] \begin{bmatrix} 1/8\sqrt{3} \\ 1/2 \end{bmatrix} - 1/4 < 1 - (1/8\sqrt{3})^2 - 1/4 = W_1 W_1^*. \tag{5.6}$$

Suppose now that Theorem 2.3 holds for the system Σ. Since $I - TT^* \geqslant T_{\Psi_1} T_{\Psi_1}^*$, it follows that there exists a block lower triangular operator Q such that $T_{\Psi_1} = T_\Psi Q$. Let $Q_{0,0}$ be the $(0,0)$-entry of Q. Then $W_1 = WQ_{0,0}$. But this is impossible, since

$$WW^* < W_1 W_1^* = WQ_{0,0}Q_{0,0}^* W^* \leqslant WW^*.$$

6. A characterization of optimality

Theorems 2.1 and 2.3 together yield the following characterization of optimality and star-optimality.

Theorem 6.1. *Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, and let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a minimal and contractive realization of T. Then*

(i) *the system Σ is optimal if and only if for each minimal and contractive realization $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ of T we have*

$$B(n)^* B(n) \leqslant \beta(n)^* \beta(n), \quad n \in \mathbb{Z}; \tag{6.1}$$

(ii) *the system Σ is star-optimal if and only if for each minimal and contractive realization $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ of T we have*

$$C(n)C(n)^* \leqslant \gamma(n)\gamma(n)^*, \quad n \in \mathbb{Z}. \tag{6.2}$$

Proof. Part (a). The necessity of condition (6.1) is clear from the definition of optimality. It suffices to prove the sufficiency of this condition.

Assume that Σ is a contractive realization of T, such that for each minimal and contractive realization $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ of T, the inequality (6.1) holds for each integer n.

Let $\Sigma_\circ = (A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n)$ be a minimal and optimal realization of T, and let $\Phi_\circ = (A_\circ(n), B_\circ(n), Y_\circ(n), Z_\circ(n); H_{\circ,n}, K_n, \mathcal{D}_{M_{\Sigma_\circ}(n)})$ be the first complementary system associated with $\Sigma_\circ$ with input-output map $T_{\Phi_\circ} = (f^\circ_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{D}_\circ)$, where $\mathcal{D}_\circ$ is the sequence of defect spaces $\mathcal{D}_{M_{\Sigma_\circ}(n)}$. Let $\Phi = (A(n), B(n), Y(n), Z(n); H_n, K_n, \mathcal{D}_{M_\Sigma(n)})$ be the first complementary system associated with Σ with input-output map $T_\Phi = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{D})$, where $\mathcal{D}$ is the sequence of defect spaces $\mathcal{D}_{M_\Sigma(n)}$.

By assumption, we have $B(n)^*B(n) \leqslant B_\circ(n)^*B_\circ(n)$ for each integer n. Since $\Sigma_\circ$ is an optimal realization of T, it follows that $B(n)^*B(n) = B_\circ(n)^*B(n)$ for each integer n. Hence

$$(f^\circ_{n,n})^* f^\circ_{n,n} = I - D(n)^*D(n) - B_\circ(n)^*B_\circ(n) \\ = I - D(n)^*D(n) - B(n)^*B(n) = f^*_{n,n} f_{n,n}. \tag{6.3}$$

Since $\Sigma_\circ$ is an optimal system, and $T_\Phi^* T_\Phi \geqslant I - T^*T$, it follows from Theorem 2.1 that $T_\Phi = QT_{\Phi_\circ}$, where $Q = (q_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{D}_\circ) \to \ell^2(\mathcal{D})$ is a block lower triangular contraction. Since each of the operators T_Φ, Q, and $T_{\Phi_\circ}$ are block lower triangular, we have $f_{n,n} = q_{n,n} f^\circ_{n,n}$ for each $n \in \mathbb{Z}$. Since the optimal system $\Sigma_\circ$ is minimal, by Theorem 2.1 the operator $T_{\Phi_\circ}$ is outer. Hence we have $\overline{\operatorname{Im} f^\circ_{n,n}} = \mathcal{D}_{M_{\Sigma_\circ}(n)}$.

From (6.3) it follows that $q_{n,n} : \mathcal{D}_{M_{\Sigma_\circ}(n)} \to \mathcal{D}_{M_\Sigma(n)}$ is an isometry for each integer n. Consequently

$$\sum_{j=1}^{\infty} q^*_{n+j,n} q_{n+j,n} = I - q^*_{n,n} q_{n,n} = 0,$$

so the operator Q is block diagonal.

Let now $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ be a block lower triangular contraction such that $I - T^*T \geqslant G^*G$. From Theorem 2.1 it follows that there exists a block lower triangular contraction Q_G such that $G = Q_G T_{\Phi_\circ}$. It follows that $G = Q_G Q^* T_\Phi$. The product $Q_G Q^*$ is a block lower triangular contraction, and therefore, by Theorem 2.1, the system Σ is optimal.

Part (b). In [8], Corollary 7.5 it was proved that the system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is minimal and star-optimal if and only if its adjoint $\Sigma^* = (A_*(n), C_*(n), B_*(n), D_*(n); H_{*,n}, L_{*,n}, K_{*,n})$ is minimal and star-optimal. Here we have in particular $A_*(n) = A(-n)^*$ and $C_*(n) = C(-n)^*$. Let $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ be a contractive realization of T. Then its adjoint $\Upsilon^* = (\alpha_*(n), \gamma_*(n), \beta_*(n), D_*(n); X_{*,n}, K_{*,n}, L_{*,n})$ is a contractive realization of $T^\sim = J_\mathcal{K} T^* J^*_\mathcal{L}$. Here we have in particular $\alpha_*(n) = \alpha(-n)^*$, and $\gamma_*(n) = \gamma(-n)^*$.

If Σ^* is optimal, then it follows that $C_*(n)^*C_*(n) \leqslant \gamma_*(n)^*\gamma_*(n)$ for each integer n. Hence we have $C(-n)C(-n)^* \leqslant \gamma(-n)\gamma(-n)^*$ for each integer n.

Assume now that for each minimal and contractive realization $\Upsilon = (\alpha(n), \beta(n), \gamma(n), D(n); X_n, K_n, L_n)$ we have $C(n)C(n)^* \leqslant \gamma(n)\gamma(n)^*$ for each $n \in \mathbb{Z}$. So for each contractive realization $\Upsilon^* = (\alpha_*(n), \beta_*(n), \gamma_*(n), D_*(n); X_{*,n}, L_{*,n}, K_{*,n})$ of $T^\sim$ we have

$$C_*(n)^*C_*(n) = C(-n)C(-n)^* \leqslant \gamma(-n)\gamma(-n)^* = \gamma_*(n)^*\gamma_*(n).$$

But then the system Σ^* is optimal. Since Σ is assumed to be minimal, it follows from [8], Corollary 7.5 that Σ is star-optimal. ∎

7. A third construction of an optimal realization

Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction. So far we have used that T has a minimal and optimal realization (by [8], Theorem 6.1), and we have shown how such realizations can be employed to construct maximal outer solutions F of the operator inequality

$$I - T^*T \geqslant F^*F. \tag{7.4}$$

In this section we reverse the direction, and we show, conversely, how a maximal outer solution of (7.4) can be used to construct an optimal realization of T. In the time-invariant case, when block lower triangular contractions are replaced by Schur-class function, the latter approach has been used in [6]. In fact, it was the first method for constructing optimal time invariant realizations.

For the time variant case two different constructions of minimal and optimal realizations have been given in [8]. In the first approach a minimal and optimal realization appears as a first minimal restriction of a unitary realization of T. The second method is a generalization of a restricted shift realization where each state space is endowed with an appropriate de Branges-Rovnyak norm. The next theorem gives a third way to construct a minimal and optimal realization which is based on the existence of a maximal outer factor and the fact that a block lower triangular contraction admits a time-variant isometric realization.

Theorem 7.1. *Let $T = (t_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction. Let F be a maximal outer solution of the inequality* (7.4)*, i.e., $F = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M})$ satisfies $I - T^*T \geqslant F^*F$ and for any block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ satisfying $I_{\ell^2(\mathcal{K})} - T^*T \geqslant G^*G$ there exists a block lower triangular contraction $Q : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ such that $G = QF$. Let*

$$\widetilde{\Sigma} = \left(A(n), B(n), \begin{bmatrix} C(n) \\ Y(n) \end{bmatrix} \begin{bmatrix} D(n) \\ Z(n) \end{bmatrix}, H_n, K_n, L_n \oplus M_n\right) \tag{7.5}$$

be an isometric realization of the block lower triangular contraction

$$\widetilde{T} = \left(\begin{bmatrix} t_{i,j} \\ f_{i,j} \end{bmatrix}\right)_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \bigoplus_{j=-\infty}^{\infty} (L_j \oplus M_j). \tag{7.6}$$

Then $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is an optimal realization of T.

Furthermore, if $H_{\circ,n} = \overline{\operatorname{Im}(A|B;n)}$, then

$$\Sigma_\circ = (P_{H_{\circ,n+1}} A(n)|H_{\circ,n}, P_{H_{\circ,n+1}} B(n), C(n)|H_{\circ,n}, D(n); H_{\circ,n}, K_n, L_n)$$

is a minimal and optimal realization of T.

Proof. Let $\Phi = (A(n), B(n), \widetilde{Y}(n), \widetilde{Z}(n); H_n, K_n, \mathcal{D}_{M_{\Sigma_1}(n)})$ be the first complementary system associated with Σ_1, and let its input-output map be given by T_Φ. Since $I - T^*T \geqslant T_\Phi^* T_\Phi$, by assumption there exists a block lower triangular contraction $\widetilde{Q}$ such that $T_\Phi = \widetilde{Q}F$.

Fix $n \in \mathbb{Z}$. Using the fact that Σ is an isometric realization of $\widetilde{T}$, and using the construction of the first complementary system, it follows that

$$\begin{aligned} f_{n,n}^* f_{n,n} &= Z(n)^* Z(n) = I - B(n)^* B(n) - D(n)^* D(n) \\ &= \widetilde{Z}(n)^* \widetilde{Z}(n) = T_\Phi(n,n)^* T_\Phi(n,n). \end{aligned} \tag{7.7}$$

The operator F is outer, so $\overline{\operatorname{Im} f_{n,n}} = M_n$. Since $T_\Phi(n,n) = \widetilde{Q}(n,n) f_{n,n}$, it follows from (7.7) that $\widetilde{Q}(n,n)$ is an isometry on M_n. Hence the operator $\widetilde{Q}$ is a block diagonal isometry on $\ell^2(\mathcal{M})$.

For each block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ satisfying $I - T^*T \geqslant G^*G$ there exists (by assumption) a block lower triangular operator Q such that $G = QF$. It follows that

$$G = QF = Q\widetilde{Q}^* T_\Phi,$$

where $Q\widetilde{Q}^*$ is block lower triangular. By Theorem 2.1 the system Σ is optimal.

Since the system Σ is optimal, by [8], Proposition 6.4 (iv) we have $\operatorname{Im}(A|B;n) \subset \operatorname{Ker}(C|A;n)^\perp$. Hence the system $\Sigma_\circ$ is precisely the first minimal restriction of the optimal system Σ. By the same proposition, item (ii), it follows that $\Sigma_\circ$ is minimal. ∎

Let us remark that any block lower triangular contraction admits a controllable isometric realization. Indeed, by [8], Theorem 4.3 of each block lower triangular contraction $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ admits a (simple) unitary realization $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$. For each n, we can make an orthogonal decomposition $H_n = H_{1,n} \oplus H_{2,n}$, where $H_{1,n} = \overline{\operatorname{Im}(A|B;n)}$ and $H_{2,n} = \operatorname{Im}(A|B;n)^{\perp}$. Since for each $n \in \mathbb{Z}$ the inclusions $A(n)H_{1,n} \subset H_{1,n+1}$ and $\operatorname{Im} B(n) \subset H_{1,n+1}$ hold, the system Σ decomposes as

$$\Sigma = \left(\begin{bmatrix} A_{11}(n) & A_{12}(n) \\ 0 & A_{22}(n) \end{bmatrix}, \begin{bmatrix} B_1(n) \\ 0 \end{bmatrix}, \begin{bmatrix} C_1(n) & C_2(n) \end{bmatrix}, D(n); H_{1,n} \oplus H_{2,n}, K_n, L_n \right).$$

The system $\Sigma_1 = (A_{1,1}(n), B_1(n), C_1(n), D(n); H_{1,n}, K_n, L_n)$ is controllable by construction. Since the system Σ is unitary, it follows that Σ_1 is isometric.

Using the remark in the previous paragraph, we see that the following alternative version of Theorem 7.1 is also of interest.

THEOREM 7.2. *Let $T = (t_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction. Let F be a maximal outer solution of the inequality* (7.4)*, i.e., $F = (f_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M})$ satisfies $I - T^*T \geqslant F^*F$ and for any block lower triangular operator $G : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ satisfying $I_{\ell^2(\mathcal{K})} - T^*T \geqslant G^*G$ there exists a block lower triangular contraction $Q : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ such that $G = QF$. Let*

$$\widetilde{\Sigma} = \left(A(n), B(n), \begin{bmatrix} C(n) \\ Y(n) \end{bmatrix} \begin{bmatrix} D(n) \\ Z(n) \end{bmatrix}, H_n, K_n, L_n \oplus M_n \right) \tag{7.8}$$

be a controllable isometric realization of the block lower triangular contraction

$$\widetilde{T} = \left(\begin{bmatrix} t_{i,j} \\ f_{i,j} \end{bmatrix} \right)_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{K}) \to \bigoplus_{j=-\infty}^{\infty} (L_j \oplus M_j). \tag{7.9}$$

Then $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is a minimal and optimal realization of T.

Proof. The system Σ is optimal by Theorem 7.1. It is controllable since the system (7.8) is controllable. It follows from the last statement in Theorem 7.1 that the system Σ is minimal. ∎

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D.Z. AROV
Department of Mathematics
South-Ukrainian Pedagogical University
270020 Odessa
UKRAINE
E-mail: aspect@farlep.net

M.A. KAASHOEK
Faculteit der Wiskunde en Informatica
De Boelelaan 1081 a
1081 HV Amsterdam
THE NETHERLANDS
E-mail: kaash@cs.vu.nl

D.R. PIK
Faculteit der Wiskunde en Informatica
De Boelelaan 1081 a
1081 HV Amsterdam
THE NETHERLANDS
E-mail: drpik@cs.vu.nl

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