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COMPLETELY COMPLEMENTED SUBSPACE PROBLEM

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Abstract. We will prove that, if every finite dimensional subspace of an *infinite dimensional* operator space E is 1-completely complemented in it, E is 1-Hilbertian and 1-homogeneous. However, this is not true for finite dimensional operator spaces: we give an example of an n-dimensional operator space E, such that all of its subspaces are 1-completely complemented in E, but which is not 1-homogeneous. Moreover, we will show that, if E is an operator space such that both E and E^* are c-exact and every subspace of E is λ-completely complemented in it, then E is $f(c, \lambda)$-completely isomorphic either to row or column operator space.

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1. INTRODUCTION

The problem of characterizing the Banach spaces for which all subspaces are complemented goes back at least to Kakutani. It is known that if E is a Banach space of dimension more than 2, $2 \leqslant k < \dim E$ and every k-dimensional subspace of E is 1-complemented, then E is linearly isometric to a Hilbert space. This was first stated (without proof) for $k = 2$ by Kakutani in [6]; for general k it was proved by Papini in [12]. Both of these facts can be found in [1] (Theorems 12.4 and 12.8, respectively) in the real case. In the case of complex scalars the same results can be proved by using a similar argument. In [9] it was shown that if E is an infinite dimensional Banach space and every finite dimensional subspace of it is λ-complemented in E, then E is $f(\lambda)$-isomorphic to a Hilbert space, where we can take $f(\lambda) = K\lambda^4$ for some numerical constant K. In [4] a similar result was proved for finite dimensional Banach spaces.

Below we will consider the operator space version of this problem. For general information on operator spaces we refer the reader to [2], [3] or [17]. As usual, M_n will stand for the space of $n \times n$ matrices, and R_n and C_n will stand for $\mathrm{span}[e_{1j} \mid j = 1, \dots, n]$ and $\mathrm{span}[e_{i1} \mid i = 1, \dots, n]$, respectively. More generally, if H is a Hilbert space, we may equip it with row (column) operator space structures by identifying it with $H_{\mathrm{r}} = B(H^*, \mathbb{C})$ ($H_{\mathrm{c}} = B(\mathbb{C}, H)$). In particular, $R_n = (\ell_2^n)_{\mathrm{r}}$, $C_n = (\ell_2^n)_{\mathrm{c}}$. We will say that an operator space E is *λ-Hilbertian* if it is λ-isomorphic to a Hilbert space (in the Banach space sense); if E is λ-Hilbertian for some λ, we will call it *Hilbertian*. We will say that E is *λ-homogeneous* if every bounded operator $T : E \to E$ is completely bounded, and $\|T\|_{\mathrm{cb}} \leqslant \lambda \|T\|$. If E is λ-homogeneous for some λ, it will be called *homogeneous*. These definitions were introduced in [14].

Two different settings of the "completely complemented subspace problem" will be considered:

(1) Every subspace of E is 1-completely complemented (an operator space analog of the problem solved by Kakutani). This case will be considered in Section 2. Clearly, in this situation E is 1-Hilbertian. If E is infinite dimensional, it is also 1-homogeneous (Theorem 2.1). On the other hand, if E is finite dimensional, it need not be 1-homogeneous (Proposition 2.3).

(2) Every subspace of E is λ-completely complemented (an operator space analog of the problem solved by Lindenstrauss and Tzafriri). This case will be considered in Section 3. In this case, if we assume that both E and E^* are exact (see below for the definition of exactness), we can prove that E is completely isomorphic either to row or column operator space (Theorem 3.1, Corollary 3.2).

2. ALL SUBSPACES OF E ARE 1-COMPLETELY COMPLEMENTED

The main result of this section is

THEOREM 2.1. (i) *If E is an n-dimensional operator space ($n > 1$) such that all of its k-dimensional subspaces are 1-completely complemented, then E is 1-Hilbertian and n/k-homogeneous.*

(ii) *If E is an infinite dimensional operator space such that all of its finite dimensional subspaces are 1-completely complemented, then E is 1-Hilbertian and 1-homogeneous.*

Proof. By the result of Kakutani quoted above, E is 1-Hilbertian. Assume first that $\dim E = n < \infty$. To prove that E is n/k-homogeneous, it suffices to show that, if $u : E \to E$ is a linear operator and $\|u\| < 1$, then $\|u\|_{\mathrm{cb}} \leqslant n/k$. By the results of Kuo and Wu (see [7], [8]), if $v : E \to E$ satisfies $\operatorname{rank} v < n$ and $\|v\| < 1$, then there exist orthogonal projections $P_1, \dots, P_m$ such that $v = P_1 \cdots P_m$. Moreover, the projections can be chosen in such a way that $\operatorname{rank} P_i = \operatorname{rank} v$ ($1 \leqslant i \leqslant m$). By assumption, all orthogonal projections of rank k (or less) have c.b. norm 1. Therefore, if $v : E \to E$ is a linear operator, $\|v\| < 1$ and $\operatorname{rank} v \leqslant k$, then $\|v\|_{\mathrm{cb}} \leqslant 1$.

Now consider an orthonormal basis $e_1, \dots, e_n$ in E, and let P_i ($1 \leqslant i \leqslant n$) be the orthogonal projection onto $\operatorname{span}[e_1, \dots, e_{i-1}, e_{i+n-k}, \dots, e_n]$. Then, if $u : E \to E$ is a linear operator, $u = \sum_{i=1}^{n} P_i u/k$. Therefore,

$$\|u\|_{\mathrm{cb}} \leqslant \frac{1}{k} \sum_{i=1}^{n} \|P_i u\|_{\mathrm{cb}} = \frac{1}{k} \sum_{i=1}^{n} \|P_i u\| \leqslant \frac{n}{k} \|u\|.$$

This proves part (i) of the theorem. To prove part (ii), we must show that if F is a finite dimensional subspace of E and $v : E \to E$ satisfies $\|v\| < 1$, then $\|v|_F\|_{\mathrm{cb}} \leqslant 1$. To this end, define $w : E \to E$ as $w = vP_F$, where P_F is the orthogonal projection onto F. Then $\|w\| \leqslant \|v\| < 1$ and, since w is finite rank, it can be represented as a product of finite rank orthogonal projections (here we are using the results of Kuo and Wu again). Since all orthogonal projections of finite rank are completely contractive, $\|v|_F\|_{\mathrm{cb}} \leqslant \|w\|_{\mathrm{cb}} \leqslant 1$. This proves (ii). ∎

Corollary 2.2. *If E is an n-dimensional operator space ($n > 1$) such that all of its subspaces of dimension k ($1 \leqslant k < n$) are 1-completely complemented, then all k-dimensional subspaces of E are 1-homogeneous and completely isometric to each other, and all subspaces of dimension less than k are 1-completely complemented.*

The estimate for the maximal homogeneity constant of an n-dimensional 1-Hilbertian operator space in which all rank k orthogonal projections have c.b. norm 1, obtained in Theorem 2.1, is, in a sense, optimal, as is demonstrated by the following example.

Proposition 2.3. *For every $n > 2$ and $2 \leqslant k < n$ there exists an n-dimensional 1-Hilbertian operator space E, all of whose subspaces of dimension k or less are 1-completely complemented in it and a unitary $U : E \to E$ with $\|U\|_{\mathrm{cb}} > 1 + (n-k)/(8n)$, if $n > k > n/2$, and $\|U\|_{\mathrm{cb}} \geqslant n/(5k)$, if $2 \leqslant k \leqslant n/2$.*

Proof. First consider the case $n > k > n/2$. We will use an idea of Zhang (see [19], Example 8). Fix an orthogonal basis $e_1, \dots, e_n$ in ℓ_2^n. We will denote by J the formal identity map from ℓ_2^n into C_n. To describe the operator space

E which will provide us with a counterexample, define a norm on $\ell_2^n \otimes B(H)$ as follows: for $a_1, \dots, a_n \in B(H)$ set

$$\Big\| \sum_{i=1}^n e_i \otimes a_i \Big\|_{E \otimes B(H)}$$
$$\stackrel{\text{def}}{=} \max\Big\{ \Big\| \sum_{i=1}^k a_i^* a_i + \sum_{i=k+1}^n \frac{a_i^* a_i}{2} \Big\|^{\frac12}, \sup \Big\| (J \otimes \mathrm{id}_{B(H)}) \Big(\sum_{i=1}^n P e_i \otimes a_i \Big) \Big\|_{C_n \otimes B(H)} \Big\},$$

where the supremum is taken over all rank k orthogonal projections P. One can easily verify that the norm on $E \otimes B(H)$ satisfies Ruan's conditions (see, e.g., [2] or [3]), i.e. E is indeed an operator space, and the underlying Banach space of E is ℓ_2^n. Since every orthogonal projection of rank $\leqslant k$ acting on E can be represented as a product of rank k orthogonal projections, it has c.b. norm 1. However, consider an operator $U : E \to E$, defined as follows:

$$U e_i \stackrel{\text{def}}{=} \begin{cases} e_{n+1-i} & 1 \leqslant i \leqslant n-k \text{ or } k+1 \leqslant i \leqslant n; \\ e_i & n-k+1 \leqslant i \leqslant k. \end{cases}$$

U is unitary; we will show that $\|U\|_{\mathrm{cb}} > 1 + (n-k)/(8n)$. To this end, consider

$$x = \sum_{i=n-k+1}^n e_i \otimes e_{i1} + \sum_{i=1}^{n-k} \frac{1}{\sqrt{2}} e_i \otimes e_{i1} \in E \otimes M_n.$$

Then, $\|(U \otimes \mathrm{id}_{B(H)})(x)\| = \sqrt{k + (n-k)/4}$. We will show that $\|x\| = \sqrt{k}$; this will imply the inequality

$$\|U\|_{\mathrm{cb}} \geqslant \sqrt{1 + \frac{n-k}{4k}} > 1 + \frac{n-k}{8n}.$$

To prove that $\|x\| = \sqrt{k}$, note that for every rank k orthogonal projection P,

$$\Big\| \sum_{i=1}^{n-k} JPe_i \otimes \frac{e_{i1}}{\sqrt{2}} + \sum_{i=n-k+1}^n JPe_i \otimes e_{i1} \Big\|_{C_n \otimes M_n} = \left\| \begin{pmatrix} JPe_1/\sqrt{2} \\ \vdots \\ JPe_{n-k}/\sqrt{2} \\ JPe_{n-k+1} \\ \vdots \\ JPe_n \end{pmatrix} \right\|$$
$$\leqslant \Big(\sum_{i=1}^{n-k} \frac{\|JPe_i\|^2}{2} + \sum_{i=n-k+1}^n \|JPe_2\|^2 \Big)^{\frac12} \leqslant \Big(\sum_{i=1}^n \|JPe_i\|^2 \Big)^{\frac12} = \|P\|_{HS} = \sqrt{k}.$$

Thus,

$$\|x\| = \max\left\{\left\|\frac{1}{2}\Big(\sum_{i=1}^{n-k} e_{i1}^* e_{i1} + \sum_{i=k+1}^{n} e_{i1}^* e_{i1}\Big) + \sum_{i=n-k+1}^{n-k} e_{i1}^* e_{i1}\right\|^{\frac{1}{2}}, \sqrt{k}\right\} = \sqrt{k}.$$

This implies the lower estimate for $\|U\|_{\mathrm{cb}}$.

Now consider the case $k \leqslant n/2$. Once again, we use the ideas of Zhang. In an n-dimensional Hilbert space E, fix an orthonormal basis $e_1, \ldots, e_n$, let $m = k\lfloor n/(2k)\rfloor$, and define the norm on $E \otimes B(H)$:

$$\left\|\sum_{i=1}^{n} e_i \otimes a_i\right\| \stackrel{\mathrm{def}}{=} \max\left\{\left\|\sum_{i=1}^{m} e_i \otimes a_i\right\|_{\max(\ell_2^n)\otimes B(H)}, \sup\left\|\sum_{i=1}^{n} Pe_i \otimes a_i\right\|_{\max(\ell_2^k)\otimes B(H)}\right\},$$

where the supremum is taken over all rank k orthogonal projections P (the range of such a projection can be identified with ℓ_2^k) and $\max(Z)$ denotes the Banach space Z endowed with its maximal operator space structure (see [13], [17]). The above norm satisfies Ruan's conditions, and hence, E is an operator space; E is 1-Hilbertian. Moreover, every orthogonal projection $P : E \to E$ of rank k or less has c.b. norm 1. Consider an operator $U : E \to E$ defined by

$$Ue_i \stackrel{\mathrm{def}}{=} \begin{cases} e_{i+m} & 1 \leqslant i \leqslant m; \\ e_{i-m} & m+1 \leqslant i \leqslant 2m; \\ e_i & \text{otherwise.} \end{cases}$$

We will show that $\|U\|_{\mathrm{cb}} \geqslant m/(2k)$. By Theorem 2.16 of [13], there exist operators $A_1, \ldots, A_m \in B(H)$ such that

$$\left\|\sum_{i=1}^{m} e_i \otimes A_i\right\|_{\max(\ell_2^n)\otimes B(H)} \geqslant \frac{m}{2},$$

but

$$\sup_{\sum |\lambda_i|^2 \leqslant 1} \left\|\sum_{i=1}^{m} \lambda_i A_i\right\| \leqslant 1.$$

Let $x = \sum\limits_{i=m+1}^{2m} e_i \otimes A_{i-m}$. Clearly, $\|(U \otimes \mathrm{id}_{B(H)})(x)\| \geqslant m/2$. We will show that $\|x\| \leqslant k$, i.e. that for every orthogonal projection P of rank k, $\left\|\sum\limits_{i=1}^{m} Pe_i \otimes A_i\right\| \leqslant k$. Indeed, if $f_1, \ldots, f_k$ form an orthonormal basis in the range of P,

$$\begin{aligned}\left\|\sum_{i=1}^{m} Pe_i \otimes A_i\right\| &= \left\|\sum_{j=1}^{k}\sum_{i=1}^{m} \langle Pe_i, f_j\rangle f_j \otimes A_i\right\| \\ &\leqslant \sum_{j=1}^{k}\left\|\sum_{i=1}^{m} \langle Pe_i, f_j\rangle f_j \otimes A_i\right\| \leqslant \sum_{j=1}^{k}\left(\sum_{i=1}^{m} |\langle Pe_i, f_j\rangle|^2\right)^{\frac{1}{2}} \leqslant k.\end{aligned}$$

Therefore,

$$\|U\|_{\mathrm{cb}} \geqslant \frac{m/2}{k} = \frac{m}{2k} \geqslant \frac{n}{5k}.$$

This completes the construction. ∎

REMARK 2.4. Similarly, for every positive integer k we can construct an infinite dimensional 1-Hilbertian operator space E such that all orthogonal projections of rank k, acting on E, have c.b. norm 1, but E is not homogeneous.

3. ALL SUBSPACES OF E ARE λ-COMPLETELY COMPLEMENTED

Below, we will need the notion of *exactness* of operator spaces introduced in [15]. An operator space E is called *exact* if there exists $c > 0$ such that for every finite dimensional subspace F of E, there exists a subspace $F' \hookrightarrow M_N$ with $d_{\mathrm{cb}}(F, F') \leqslant c$. The infimum of all such c's is called the *exactness constant* of E, and is denoted by $\mathrm{ex}(E)$. For more information on exactness of operator spaces see [15] or [17].

In [16] it was proved that if E is an *infinite dimensional* operator space, and for every finite dimensional subspace $F \hookrightarrow E$ there exists a projection $P : E \to F$ with $\|P\|_{\mathrm{cb}} \leqslant \lambda$, then E is λ'-homogeneous and λ'-Hilbertian, where $\lambda' = K\lambda^2$ (K is a constant). Unfortunately, Pisier's proof cannot be applied to finite dimensional spaces. However, if we assume that both E and E^* are exact and every subspace of E is completely complemented in E, we can not only show that E is homogeneous and Hilbertian, but in fact, we can show something stronger. Namely, we have:

THEOREM 3.1. *If E is an n-dimensional operator space such that $\mathrm{ex}(E)\mathrm{ex}(E^*) \leqslant c$ and for every subspace $F \hookrightarrow E$ there exists a projection $P : E \to F$ with $\|P\|_{\mathrm{cb}} \leqslant \lambda$, then $\min\{d_{\mathrm{cb}}(E, R_n), d_{\mathrm{cb}}(E, C_n)\} \leqslant \kappa c^{121}\lambda^{60}$ (κ is a constant, independent of c and λ).*

COROLLARY 3.2. *If E is an operator space such that both E and E^* are exact and for every finite dimensional subspace $F \hookrightarrow E$ there exists a projection $P : E \to F$ with $\|P\|_{\mathrm{cb}} \leqslant \lambda$, then E is completely isomorphic to either row or column operator space.*

We will say that an operator space X is λ-*injective* if it is λ-completely complemented in every operator space containing it. We will say that X is *injective* if it is λ-injective for some λ. Corollary 6.4.13 of [17] states that if E is a λ-injective finite dimensional operator space, then E is λ-exact. This implies

COROLLARY 3.3. *If E is an injective homogeneous Hilbertian operator space and E^* is exact, then E is completely isomorphic to either row or column operator space. In particular, if both E and E^* are injective homogeneous Hilbertian operator spaces, then E is completely isomorphic to either row or column operator space.*

To prove Theorem 3.1, we need the following lemmas:

LEMMA 3.4. *If E is a 1-Hilbertian operator space of dimension n and $\mathrm{ex}(E)\,\mathrm{ex}(E^*) \leqslant c$, then we can find orthonormal vectors $u_1, \ldots, u_m \in E$ ($m \geqslant n/50c^2$), such that either*

$$\Big\| \sum u_i \otimes a_i \Big\| \geqslant \frac{1}{10c} \Big\| \sum a_i^* a_i \Big\|^{\frac{1}{2}}$$

or

$$\Big\| \sum u_i \otimes a_i \Big\| \geqslant \frac{1}{10c} \Big\| \sum a_i a_i^* \Big\|^{\frac{1}{2}}$$

holds for any $a_1, \ldots, a_m \in B(K)$.

Proof. This proof is due to Alvaro Arias; the original one was more complicated and yielded a worse constant. Let (e_i) be an orthonormal basis in $E \hookrightarrow B(H)$, and let (f_i) be the dual basis in E^*. By Theorem 1.4 of [5],

$$n \leqslant 4c \max\Big\{\Big\| \sum e_i^* e_i \Big\|^{\frac{1}{2}}, \Big\| \sum e_i e_i^* \Big\|^{\frac{1}{2}}\Big\} \max\Big\{\Big\| \sum f_i^* f_i \Big\|^{\frac{1}{2}}, \Big\| \sum f_i f_i^* \Big\|^{\frac{1}{2}}\Big\}.$$

However, $\max\big\{\big\| \sum f_i^* f_i \big\|^{1/2}, \big\| \sum f_i f_i^* \big\|^{1/2}\big\} \leqslant \sqrt{n}$, hence

$$\sqrt{n} \leqslant 4c \max\Big\{\Big\| \sum e_i^* e_i \Big\|^{\frac{1}{2}}, \Big\| \sum e_i e_i^* \Big\|^{\frac{1}{2}}\Big\}.$$

Assume without loss of generality that $\Big\| \sum e_i^* e_i \Big\|^{1/2} \geqslant \sqrt{n}/4c$. Then there exists $\xi \in H$, $\|\xi\| = 1$ such that $\big(\sum \|e_i \xi\|^2\big)^{1/2} > \sqrt{n}/5c$. Consider an operator $T : E \to B(\mathbb{C}, H) = H_{\mathrm{c}} : e \mapsto e\xi$. Clearly, $\|T\| \leqslant \|T\|_{\mathrm{cb}} \leqslant 1$ and $\mathrm{rank}(T) \leqslant n$. On the other hand, $\|T\|_{HS} = \big(\sum \|Te_i\|^2\big)^{1/2} > \sqrt{n}/5c$. Hence, T has $m \geqslant n/50c^2$ singular values $\lambda_i > 1/10c$. Denote the corresponding orthonormal vectors by $u_1, \ldots, u_m$. Then $\langle Tu_i, Tu_j \rangle = \lambda_i \lambda_j \delta_{ij}$, where δ_{ij} is Kronecker's delta. Thus,

$$\Big\| \sum u_i \otimes a_i \Big\|_{E \otimes B(K)} \geqslant \Big\| \sum Tu_i \otimes a_i \Big\|_{C \otimes B(K)} > \frac{1}{10c} \Big\| \sum a_i^* a_i \Big\|^{1/2}$$

for any $a_1, \ldots, a_m \in B(K)$. This completes the proof. ∎

LEMMA 3.5. *Suppose E is an n-dimensional 1-Hilbertian operator space with $\mathrm{ex}(E)\mathrm{ex}(E^*) \leqslant c$, and for every subspace $G \hookrightarrow E^*$ there exists a subspace $F \hookrightarrow G$ such that $\dim F \geqslant a \dim G$ and a projection $P : E^* \to F$ with $\|P\|_{\mathrm{cb}} \leqslant \lambda$. Then there exists a subspace $E_1 \hookrightarrow E$ of dimension $m \geqslant c_1 n$ such that*

$$\min\{d_{\mathrm{cb}}(E_1, C_m), d_{\mathrm{cb}}(E_1, R_m)\} \leqslant c_2;$$

and furthermore, if $n > 2.5 \cdot 10^7 c^8 \lambda^4/a$, we can take $c_1 = a/(2500c^4\lambda^2)$ and $c_2 = 100c^2\lambda^2$.

Proof. If E_1 and E_2 are operator spaces which share the same underlying Banach space, we can consider a formal identity operator $\mathrm{id} : E_1 \to E_2$. By Lemma 3.4, we can assume without loss of generality that there exists a subspace $F \hookrightarrow E^*$ of dimension $k \geqslant an/(50c^2)$ such that $\|\mathrm{id} : F \to C_k\|_{\mathrm{cb}} \leqslant 10c$ and F is λ-completely complemented in E^*. Therefore, F^* is λ-completely isomorphic to a subspace of E, and $\mathrm{ex}(F)\mathrm{ex}(F^*) \leqslant c\lambda$. Hence, by applying Lemma 3.4 again, we can find an m-dimensional subspace $G \hookrightarrow F^*$ ($m \geqslant k/(50c^2\lambda^2) \geqslant an/(2500c^4\lambda^2)$) such that either

$$\|\mathrm{id} : G \to R_m\|_{\mathrm{cb}} \leqslant 10c\lambda \tag{3.1}$$

or

$$\|\mathrm{id} : G \to C_m\|_{\mathrm{cb}} \leqslant 10c\lambda. \tag{3.1$'$}$$

However,

$$\|\mathrm{id} : R_m \to G\|_{\mathrm{cb}} \leqslant \|\mathrm{id} : R_k \to F^*\|_{\mathrm{cb}} = \|\mathrm{id} : F \to C_k\|_{\mathrm{cb}} \leqslant 10c. \tag{3.2}$$

If (3.1$'$) holds,

$$\sqrt{m} = \|\mathrm{id} : R_m \to C_m\|_{\mathrm{cb}} \leqslant \|\mathrm{id} : R_m \to G\|_{\mathrm{cb}} \|\mathrm{id} : G \to C_m\|_{\mathrm{cb}} \leqslant 100c^2\lambda,$$

which contradicts our assumptions about n. Then, by (3.1) and (3.2), $\|\mathrm{id} : G \to R_m\|_{\mathrm{cb}} \leqslant 10c\lambda$ and $\|\mathrm{id} : R_m \to G\|_{\mathrm{cb}} \leqslant 10c$, i.e. $d_{\mathrm{cb}}(G, R_m) \leqslant 100c^2\lambda$. To complete the proof, we need only to recall that G is λ-completely isomorphic to a subspace of E. ∎

LEMMA 3.6. *Suppose E is an n-dimensional 1-Hilbertian operator space such that* $\mathrm{ex}(E)\mathrm{ex}(E^*) \leqslant c$ *and every subspace of E is λ-completely complemented. Then*

$$\min\{d_{\mathrm{cb}}(E, C_n), d_{\mathrm{cb}}(E, R_n)\} \leqslant f(c, \lambda)$$

(if $n > (10^{48}c^{20}\lambda^{60})^2$, we can take $f(c, \lambda) = 10^{48}c^{20}\lambda^{60}$).

Proof. Below, we will heavily use the ideas from [9]. Consider an arbitrary k-dimensional subspace G of E, with $k = \lceil n/(2 \cdot 10^4 c^4 \lambda^6)\rceil$. Then, there exists a projection $Q : E \to G$ with $\|Q\|_{\mathrm{cb}} \leqslant \lambda$. Let $Y = \ker Q$. Since Y is λ-completely complemented in E, $\mathrm{ex}(Y)\mathrm{ex}(Y^*) \leqslant c\lambda$. Since every subspace of Y is λ-completely complemented in Y, by Lemma 3.5 we may assume without loss of generality that Y contains a k-dimensional subspace F such that $d_{\mathrm{cb}}(F, C_k) \leqslant K = 400c^2\lambda^4$. Let $A = d_{\mathrm{cb}}(G, F)^{1/2}$. We will show first that $A \leqslant 8\sqrt{2}\lambda^2 K^2$. Indeed, there exists an operator $T : G \to F$ such that $\|T\|_{\mathrm{cb}} = \|T^{-1}\|_{\mathrm{cb}} = A$. Consider the space

$$S = \{(x, Tx) | x \in G\} \hookrightarrow G \oplus_\infty F \hookrightarrow G \oplus_\infty Y.$$

The space $G \oplus_\infty Y$ is $2(\lambda + 1)$-completely isomorphic to E; hence, there exists a projection $P : G \oplus_\infty F \to S$ with $\|P\|_{\mathrm{cb}} \leqslant 4\lambda^2$. Then there exist operators $\alpha : G \to G$ and $\beta : F \to G$ for which

$$P(x, y) = \big(\alpha(x) + \beta(y), T(\alpha(x) + \beta(y))\big).$$

Then,

$$x = \alpha(x) + \beta T(x) = T^{-1}(T\alpha)(x) + \beta T(x). \tag{3.3}$$

One can see that $\|\beta\|_{\mathrm{cb}} \leqslant 4\lambda^2$ and $\|T\alpha\|_{\mathrm{cb}} \leqslant 4\lambda^2$. Set $s = 4\lambda^2/A$ and consider the space

$$\widetilde{G} = \{(T\alpha(x), sT(x)) | x \in G\} \hookrightarrow F \oplus_\infty F.$$

Consider an operator $u : G \to \widetilde{G} : x \mapsto (T\alpha(x), sT(x))$. Then $\|u\|_{\mathrm{cb}} \leqslant \max\{\|\beta\|_{\mathrm{cb}}, \|T\alpha\|_{\mathrm{cb}}\} \leqslant 4\lambda^2$. Moreover, by (3.3), $u^{-1}(y, z) = T^{-1}y + s^{-1}\beta z$; hence, $\|u^{-1}\|_{\mathrm{cb}} \leqslant \|T^{-1}\|_{\mathrm{cb}} + \|\beta\|_{\mathrm{cb}}/s \leqslant 2A$. Therefore, $d_{\mathrm{cb}}(G, \widetilde{G}) \leqslant \|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}} \leqslant 8\lambda^2 A$. However,

$$d_{\mathrm{cb}}(F, \widetilde{G}) \leqslant d_{\mathrm{cb}}(C_k, \widetilde{G}) d_{\mathrm{cb}}(C_k, F) \leqslant d_{\mathrm{cb}}(F \oplus_\infty F, C_{2k}) d_{\mathrm{cb}}(C_k, F) \leqslant$$

$$d_{\mathrm{cb}}(C_k, F) d_{\mathrm{cb}}(C_k \oplus_\infty C_k, C_{2k}) d_{\mathrm{cb}}(C_k, F) \leqslant \sqrt{2}K^2.$$

Hence, $d_{\mathrm{cb}}(G, F) = A^2 \leqslant 8\sqrt{2}A\lambda^2 K^2$, which implies $A \leqslant 8\sqrt{2}\lambda^2 K^2$. Thus, every k-dimensional subspace of E is $128\lambda^4 K^5$-completely isomorphic to either C_k or R_k. Since we assume that $n > (10^{48}c^{20}\lambda^{60})^2$, $k > (128\lambda^4 K^5)^2$.

Our next step is to show that either every k-dimensional subspace of E is $128\lambda^4K^5$-completely isomorphic to C_k, or every k-dimensional subspace of E is $128\lambda^4K^5$-completely isomorphic to R_k. If G and G' are k-dimensional subspaces of E, set $d_H(G,G') = \|P_G - P_{G'}\|$, where P_G and $P_{G'}$ are the orthogonal projections onto G and G', respectively. The set $G_{n,k}(E)$ of k-dimensional subspaces of E, equipped with the metric $d_H(\,\cdot\,,\cdot\,)$, can be identified with the Grassman manifold $G_{n,k}$. Note that for every $0 < \varepsilon < 1/2$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if G and G' are k-dimensional subspaces of E and $d_H(G,G') < \delta$, then $d_{\mathrm{cb}}(G,G') < 1+\varepsilon$. Indeed, for any such ε we can find a $\delta > 0$ such that if $d_H(G,G') < \delta$, then there exists a linear isometry $U : G \to G'$ such that $\|J_{G'}U - J_G\| < \varepsilon/(4n)$, where $J_G : G \hookrightarrow E$ and $J_{G'} : G' \hookrightarrow E$ are the injection maps. Therefore, $\|J_{G'}U - J_G\|_{\mathrm{cb}} \leqslant n\|J_{G'}U - J_G\| < \varepsilon/4$. Similarly, $\|J_{G'} - J_G U^{-1}\|_{\mathrm{cb}} < \varepsilon/4$. Thus, $\|U\|_{\mathrm{cb}} < 1+\varepsilon/4$ and $\|U^{-1}\|_{\mathrm{cb}} < 1+\varepsilon/4$, which implies that $d_{\mathrm{cb}}(G,G') \leqslant \|U\|_{\mathrm{cb}}\|U^{-1}\|_{\mathrm{cb}} < (1+\varepsilon/4)^2 < 1+\varepsilon$.

Denote the set of k-dimensional subspaces of E which are $128\lambda^4K^5$-completely isomorphic to C_k (respectively R_k) by $\mathcal{E}_C$ (respectively $\mathcal{E}_R$). By the above, both $\mathcal{E}_C$ and $\mathcal{E}_R$ are closed in the metric $d_H(\cdot,\cdot)$. Since the Grassman manifold $G_{n,k}$ is connected (see e.g. Chapter 1 of [10] or pp. 41–42 of [18]), either $\mathcal{E}_C$ and $\mathcal{E}_R$ intersect or one of these two sets is empty. In the former case, there exists a k-dimensional subspace of E which is $128\lambda^4K^5$-completely isomorphic to both C_k and R_k. Therefore, $d_{\mathrm{cb}}(C_k,R_k) \leqslant (128\lambda^4K^5)^2$. On the other hand, by [14], $d_{\mathrm{cb}}(C_k,R_k) = k > (128\lambda^4K^5)^2$. This shows that either $\mathcal{E}_C$ or $\mathcal{E}_R$ is empty, i.e. either every k-dimensional subspace of E is $128\lambda^4K^5$-completely isomorphic to R_k, or every k-dimensional subspace of E is $128\lambda^4K^5$-completely isomorphic to C_k.

Pick a k-dimensional subspace G of E. By the reasoning above, there exists an operator $u : G \to C_k$, such that $\|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}} \leqslant 128\lambda^4K^5 \leqslant 2^{17}\cdot 10^{10}\cdot c^{10}\lambda^{24}$. Since G is 1-Hilbertian, it makes sense to consider a formal identity operator $\mathrm{id} : G \to C_k$. Then $\|\mathrm{id} : G \to C_k\|_{\mathrm{cb}} \leqslant \|u\|_{\mathrm{cb}}\|u^{-1}\| \leqslant 2^{17}\cdot 10^{10}\cdot c^{10}\lambda^{24}$, and similarly, $\|\mathrm{id}^{-1}\|_{\mathrm{cb}} \leqslant 2^{17}\cdot 10^{10}\cdot c^{10}\lambda^{24}$. Hence, if $(e_i)_1^n$ is an orthonormal basis in E and σ is a subset of $\{1,\dots,n\}$ of cardinality k,

$$\frac{1}{2^{17}\cdot 10^{10}c^{10}\lambda^{24}}\Big\|\sum_{i\in\sigma} a_i^*a_i\Big\| \leqslant \Big\|\sum_{i\in\sigma} e_i\otimes a_i\Big\| \leqslant 2^{17}\cdot 10^{10}c^{10}\lambda^{24}\Big\|\sum_{i\in\sigma} a_i^*a_i\Big\|$$

for every $a_1,\dots,a_n \in B(K)$. Thus,

$$\Big\|\sum_{i=1}^n e_i\otimes a_i\Big\| \leqslant 2^{18}\cdot 10^{14}c^{10}\lambda^{30}\Big\|\sum_{i=1}^n a_i^*a_i\Big\|. \tag{3.4}$$

Since every subspace of E^* is $(\lambda+1)$-completely complemented, we can prove, in a similar fashion, that for every orthonormal basis $(f_i)_1^n$ in E^* and every $b_1, \ldots, b_k \in B(K)$, either

$$\Big\| \sum_{i=1}^{n} f_i \otimes b_i \Big\| \leqslant 10^{28} c^{10} \lambda^{30} \Big\| \sum_{i=1}^{n} b_i b_i^* \Big\| \tag{3.5}$$

or

$$\Big\| \sum_{i=1}^{n} f_i \otimes b_i \Big\| \leqslant 10^{28} c^{10} \lambda^{30} \Big\| \sum_{i=1}^{n} b_i^* b_i \Big\|. \tag{3.5$'$}$$

By duality, (3.5$'$) cannot be true if n satisfies our conditions, hence (3.5) holds. Together, (3.4) and (3.5) imply

$$\frac{1}{10^{28} c^{10} \lambda^{30}} \Big\| \sum_{i=1}^{n} a_i^* a_i \Big\| \leqslant \Big\| \sum_{i=1}^{n} e_i \otimes a_i \Big\| \leqslant 10^{20} c^{10} \lambda^{30} \Big\| \sum_{i=1}^{n} a_i^* a_i \Big\|,$$

which completes the proof of this lemma. ∎

Proof of Theorem 3.1. Suppose $\mathrm{ex}(E)\mathrm{ex}(E^*) \leqslant c$. Then, by Corollary 1.6 of [5], there exists an operator $u : E \to \ell_2^n$ such that $\|u\| \leqslant 4c$ and $\|u^{-1}\| \leqslant 1$. Then, we equip ℓ_2^n with an operator space structure (the resulting operator space will be called F) as follows: for $x \in \ell_2^n \otimes B(H)$, define

$$\|x\|_{F \otimes_{\min} B(H)} = \max\{\|(u^{-1} \otimes \mathrm{id}_{B(H)})(x)\|_{E \otimes_{\min} B(H)}, \|x\|_{\min(\ell_2^n) \otimes_{\min} B(H)}\}$$

(see [13] or [17] for the definition of $\min(\ell_2^n)$). Clearly, this is an operator space norm (i.e. it satisfies the axioms of Ruan), $\|u : E \to F\|_{\mathrm{cb}} \leqslant 4c$ and $\|u^{-1} : F \to E\|_{\mathrm{cb}} \leqslant 1$. Thus, F is an n-dimensional 1-Hilbertian operator space, every subspace of F is $4c\lambda$-complemented in F, and $\mathrm{ex}(F)\mathrm{ex}(F^*) \leqslant 16c^3$. The statement of the theorem now follows from Lemma 3.6 since $d_{\mathrm{cb}}(E, C_n) \leqslant d_{\mathrm{cb}}(E, F) d_{\mathrm{cb}}(F, C_n)$. ∎

Remark 3.7. A conjecture of G. Pisier and M. Junge (implicitly contained in [5]) states that if $u : E \to F$ is a completely bounded map and E, F^* are exact operator spaces, then u completely factors through $H_{\mathrm{r}} \oplus_\infty H_{\mathrm{c}}$ for some Hilbert space H. In particular, if both E and E^* are exact, E is completely isomorphic to a completely complemented subspace of $H_{\mathrm{r}} \oplus_\infty H_{\mathrm{c}}$; according to [11], this implies that E is completely isomorphic to $X_{\mathrm{r}} \oplus_\infty Y_{\mathrm{c}}$, where X and Y are Hilbert spaces. Corollary 3.2 can be regarded as a proof of a particular case of this conjecture — namely, the case of all finite dimensional subspaces of E being λ-completely complemented. In [11] we were able to prove another particular case:

if E is a coordinate subspace of the space $\mathcal{K}$ of compact linear operators on ℓ_2, i.e. $E = \mathrm{span}[E_{ij} | (i,j) \in \sigma]$, (here σ is a subset of $\mathbb{N}^2$ and E_{ij} are matrix units) and $\mathrm{ex}(E^*) \leqslant c$, then $d_{\mathrm{cb}}(E, R_k \oplus_\infty C_m) < 8c^2$ for some k and m (finite or infinite).

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