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ON THE AH ALGEBRAS WITH THE IDEAL PROPERTY

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ABSTRACT. A C^*-algebra has the *ideal property* if any ideal (closed, two-sided) is generated (as an ideal) by its projections. We prove a theorem which implies, in particular, that an AH algebra (AH stands for "approximately homogeneous") stably isomorphic to a C^*-algebra with the ideal property has the ideal property. It is shown that, for any AH algebra A with the ideal property and slow dimension growth, the projections in $M_\infty(A)$ satisfy the Riesz decomposition and interpolation properties and $\mathrm{K}_0(A)$ is a Riesz group. We prove a theorem which describes the partially ordered set of all the ideals generated by projections of an AH algebra A; the special case when the projections in $M_\infty(A)$ satisfy the Riesz decomposition property is also considered. This theorem generalizes a result of G.A. Elliott which gives the ideal structure of an AF algebra. We answer — jointly with M. Dadarlat — a question of G.K. Pedersen, constructing extensions of C^*-algebras with the ideal property which do not have the ideal property.

KEYWORDS: *C^*-algebra,* AH *algebra, ideal property, stable isomorphism, Riesz decomposition and interpolation property, Riesz group, ideal generated by projections, extension of two C^*-algebras.*

MSC (2000): 46L05, 46L99.

1. INTRODUCTION

A C^*-algebra has the *ideal property* if any ideal (closed, two-sided) is generated (as an ideal) by its projections. An AH algebra is a C^*-algebra which is the inductive limit of a sequence of finite direct sums of C^*-algebras of the form $PC(X, M_n)P$, where X is a finite, connected CW-complex and P is a projection of $C(X, M_n)$. In this paper we continue our study of the AH algebras with the ideal property (see [25], [26]). This class of C^*-algebras is a common generalization of the two most important classes of AH algebras: the simple AH algebras and the real rank zero AH algebras (see [4]), for which a lot of interesting results have been proved in the last years. The ideal property is very important since, in general, any real rank zero C^*-algebra or any simple, unital C^*-algebra has the ideal property ([4]). The study of the AH algebras with the ideal property is part of an important problem of E.G. Effros ([10]) (namely: Find suitable topological invariants for AH algebras) and it is also related to G.A. Elliott's remarkable Project of the classification of the separable, amenable C^*-algebras by invariants containing K-theory ([16]), for which the AH algebras and their extensions are important tools. Therefore, the properties and the structure of the AH algebras with the ideal property have a special importance.

In this paper we show that the ideal property is preserved by stable isomorphism in the class of AH algebras. In fact, we prove a more general result (Theorem 2.1), which implies that an AH algebra stably isomorphic to a C^*-algebra with the ideal property has itself the ideal property (see Corollary 2.7).

It is shown that, for any AH algebra A with the ideal property and with slow dimension growth, the projections in $M_\infty(A)$ satisfy the Riesz decomposition and interpolation properties and $\mathrm{K}_0(A)$ is a Riesz group (Theorem 3.1). We used this result to deduce that for such kind of C^*-algebras A, $\mathrm{K}_0(A)/\mathrm{tor}\,\mathrm{K}_0(A)$ is an unperforated Riesz group (Theorem 3.4).

We have been also interested in the study of the partially ordered set of ideals generated by projections of an AH algebra A. We proved that there is an order isomorphism from this partially ordered set of ideals to the partially ordered set of ideals of $D(A \otimes \mathcal{K})$ — the abelian local semigroup of Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$ — and that if, moreover, the projections in $M_\infty(A)$ satisfy the Riesz decomposition property (e.g. this happens when A has the ideal property and slow dimension growth (see Theorem 3.1)) then the same partially ordered set of ideals of A is also order isomorphic with the partially ordered set of ideals of $D(A)$ and also, with the partially ordered set of ideals of $\mathrm{K}_0(A)$ (Theorem 4.1). Actually, the above mentioned partially odered sets are proved to be lattices (see also Theorem 4.1). Note that recently, the ideals generated by projections in an AH algebra played a role in the proof of the surprising fact that there are inductive limits of AH algebras which are not AH algebras (see [6]). Our above result (Theorem 4.1) gives in particular a description of the ideal structure of an AH algebra with the ideal property and generalizes a result of G.A. Elliott ([13]) giving the ideal structure of an AF algebra. We prove also that the partially ordered set of ideals generated by projections of a separable C^*-algebra is shape invariant (Proposition 4.13). (M. Dadarlat obtained independently the same result.) We use this fact and other results of us to prove that two AH algebras with the ideal property and with slow dimension growth for which there is a graded, ordered, scaled isomorphism between their total K-theory groups which commutes with the Bockstein operations have isomorphic ideal lattices (Proposition 4.14).

The extension problem for AH algebras is difficult and important. While it is well-known that extensions of AH algebras are not AH algebras in general (the Toeplitz extension gives a trivial example), we proved in [26] that the class of the AH algebras with the ideal property has a "good behavior" with respect to extensions (e.g., we showed there that if $0 \to I \to A \to B \to 0$ is an exact sequence of AH algebras, then A has the ideal property if and only if I and B have the ideal property). In this paper we prove that in fact this behavior is "not very good". To be more precise, we settle — jointly with M. Dadarlat — a problem of G.K. Pedersen, constructing extensions of C^*-algebras with the ideal property which *do not have* the ideal property. In fact, we prove that there are extensions of simple A$\mathbb{T}$ algebras (i.e. AH algebras over the torus $\mathbb{T}$) with real rank zero by simple A$\mathbb{T}$ algebras with real rank zero which do not have the ideal property (Theorem 5.1). (Note that any C^*-algebra with real rank zero has the ideal property, by a result in [4].)

In this paper we shall use the version of the *slow dimension growth* condition defined in [18].

Let A be a C^*-algebra. By an ideal in A we shall mean a closed, two-sided ideal of A; the fact that I is an ideal of A will be denoted by $I \triangleleft A$. By the ideal of A generated by a family of projections of A we shall mean the closed, two-sided ideal of A generated by that family of projections. $D(A)$ will denote the abelian local semigroup of Murray-von Neumann equivalence classes of projections in A, the addition of two classes being defined when they have orthogonal representatives. If p is a projection in A, we shall denote by $[p]$ its class in $D(A)$. By an ideal in $D(A)$ we shall understand a nonempty hereditary subset which is closed under addition, where defined. The projections of A will be denoted by $\mathcal{P}(A)$. If $p, q \in \mathcal{P}(A)$ we shall write $p \sim q$ if p and q are Murray-von Neumann equivalent (i.e. there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$) and we shall write $p \precsim q$ if $p \sim q' \leqslant q$ for some $q' \in \mathcal{P}(A)$. We shall denote by $M_\infty(A)$ the algebraic inductive limit of the matrix algebras $M_n(A)$ (where $n \in \mathbb{N}$) under the embeddings:

$$M_n(A) \ni a \mapsto a \oplus 0 \in M_{n+1}(A).$$

If $p \in M_\infty(A)$ is a projection, its class in $\mathrm{K}_0(A)$ will be denoted by $[p]$; sometimes, in order to avoid confusion, we shall write $[p]_{\mathrm{K}_0(A)}$. Also, if $p \in \mathcal{P}(\mathcal{A})$, the difference between $[p] \in D(A)$ and $[p] \in \mathrm{K}_0(A)$ will follow from the context. By an ideal of $\mathrm{K}_0(A)$ we shall mean a subgroup H of $\mathrm{K}_0(A)$ such that $H^+ := H \cap \mathrm{K}_0(A)^+$ is hereditary (i.e., if $0 \leqslant g \leqslant h$ for some $g \in \mathrm{K}_0(A)$ and $h \in H^+$, then $g \in H$) and $H = H^+ - H^+$. The real rank of A ([4]) will be denoted by $\mathrm{RR}(A)$ and the topological stable rank of A ([28]) will be denoted by $\mathrm{tsr}(A)$.

We shall denote by $\mathcal{K}$ the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$.

A C^*-algebra is called an $\mathrm{A}\mathbb{T}$ *algebra* if it is isomorphic to the inductive limit of a sequence of finite direct sums of C^*-algebras of the form $C(\mathbb{T}, M_n)$, where n may vary (called circle algebras).

2. STABLE ISOMORPHISM

In this section we shall give the answer to the following natural

QUESTION. Let A be an AH algebra. Suppose that A is stably isomorphic to a C^*-algebra B which has the ideal property. Is it true that A has also the ideal property?

The answer is positive, and it follows from the following more general result:

THEOREM 2.1. *Let A and E be* AH *algebras such that* $\mathrm{RR}(E) = 0$ *and E has a non-zero, minimal projection. Let B be a C^*-algebra and let F be an* AF *algebra. Suppose that:*

$$A \otimes E \cong B \otimes F.$$

If B has the ideal property, then A has the ideal property.

The proof of the above theorem will need the following five propositions.

PROPOSITION 2.2. *Let A be a C^*-algebra with the ideal property. Then $M_n(A)$ has the ideal property for every n.*

Proof. Let I be an ideal of $M_n(A)$. It is known that there is an ideal J of A such that $I = M_n(J)$. Since A has the ideal property, J is generated by its projections and hence $I = J \otimes M_n(\mathbb{C})$ is generated by the projections $e \otimes 1 \in I$, where e is a projection of J. Hence I is generated by its projections. ∎

PROPOSITION 2.3. *If a C^*-algebra A is the inductive limit of a net $(A_\lambda)_{\lambda \in \Lambda}$ of C^*-algebras with the ideal property, then A has the ideal property.*

Proof. Let $A = \varinjlim(A_\lambda, \Phi_\lambda, \gamma)$ and let I be an ideal of A. For each $\lambda \in \Lambda$, let $\Phi_{\lambda,\infty} : A_\lambda \to A$ be the canonical $*$-homomorphsim. Then

$$I = \varinjlim(I_\lambda, \Phi_\lambda, \gamma | I_\lambda)$$

where $I_\lambda = \{x \in A_\lambda : \Phi_{\lambda,\infty}(x) \in I\}$ ([2]). Now let $x \in I$ and let $\varepsilon > 0$. Then, there is $\lambda_0 \in \Lambda$ and $a \in I_{\lambda_0}$ such that:

$$\|x - \Phi_{\lambda_0,\infty}(a)\| < \frac{\varepsilon}{2}.$$

Since I_{λ_0} is an ideal of A_{λ_0} and A_{λ_0} has the ideal property, it follows that there are $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A_{\lambda_0}$ and projections $e_1, e_2, \ldots, e_n \in I_{\lambda_0}$ (for some positive integer n) such that:

$$\Big\|a - \sum_{i=1}^{n} a_i e_i b_i\Big\| < \frac{\varepsilon}{2}.$$

This implies:

$$\Big\|\Phi_{\lambda_0,\infty}(a) - \sum_{i=1}^{n} \Phi_{\lambda_0,\infty}(a_i)\Phi_{\lambda_0,\infty}(e_i)\Phi_{\lambda_0,\infty}(b_i)\Big\| < \frac{\varepsilon}{2}.$$

Hence:

$$\Big\|x - \sum_{i=1}^{n} \Phi_{\lambda_0,\infty}(a_i)\Phi_{\lambda_0,\infty}(e_i)\Phi_{\lambda_0,\infty}(b_i)\Big\| < \varepsilon$$

which ends the proof since all the $\Phi_{\lambda_0,\infty}(e_i)$'s are projections in I. ∎

PROPOSITION 2.4. *Let A be a C^*-algebra with the ideal property and let E be an* AF *algebra. Then $A \otimes E$ has the ideal property.*

Proof. Since, obviously, a finite direct sum of C^*-algebras with the ideal property is a C^*-algebra which has the ideal property, the proof follows using Proposition 2.2 and Proposition 2.3. ∎

PROPOSITION 2.5. *Let A be an* AH *algebra with the ideal property, and let p be a projection in A. Then pAp is an* AH *algebra with the ideal property.*

Proof. This follows essentially from the equivalence (a) $\Leftrightarrow$ (b) in [25], Theorem 3.1. ∎

PROPOSITION 2.6. *Let A and B be* AH *algebras such that* $\mathrm{RR}(B) = 0$ *and B has a non-zero, minimal projection. If $A \otimes B$ has the ideal property, then A has the ideal property.*

Proof. Since A is an AH algebra, there is an increasing sequence $(p_n)_{n=1}^{\infty}$ of projections in A such that:

$$A = \overline{\bigcup_{n=1}^{\infty} p_n A p_n}. \tag{2.1}$$

(If $A = \varinjlim(A_n, \Phi_{n,m})$ and $A_n = \bigoplus_{i=1}^{k_n} P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i}$ with $X_{n,i}$ finite, connected CW-complexes and $P_{n,i}$ projections in $C(X_{n,i}, M_{[n,i]})$, then for any n one may take p_n to be the canonical image in A of the identity of A_n.) Let $e \neq 0$ be a minimal projection of B. By [4], $\mathrm{RR}(eBe) = 0$. In particular, eBe is the closed linear span of its projections. But, since $0 \neq e$ is a minimal projection of B, the projections of eBe are only 0 and e. Hence:

$$eBe = \mathbb{C}e \cong \mathbb{C}. \tag{2.2}$$

For any n, define $q_n = p_n \otimes e \in A \otimes B$. Obviously, $(q_n)_{n=1}^{\infty}$ is an increasing sequence of projections in $A \otimes B$. We have:

$$\overline{\bigcup_{n=1}^{\infty} q_n(A \otimes B)q_n} = \overline{\bigcup_{n=1}^{\infty} (p_nAp_n \otimes eBe)} \overset{(2.2)}{\cong} \overline{\bigcup_{n=1}^{\infty} p_nAp_n} \overset{(2.1)}{\cong} A.$$

Hence:

$$A \cong \varinjlim q_n(A \otimes B)q_n.$$

But, since each q_n is a projection in the AH algebra with the ideal property $A \otimes B$, by Proposition 2.5 it follows that each $q_n(A \otimes B)q_n$ has the ideal property. Now the proof follows from Proposition 2.3. ∎

Proof of Theorem 2.1. By Proposition 2.4 it follows that $B \otimes F$ has the ideal property. Hence $A \otimes E$ has the ideal property. Now the proof ends applying Proposition 2.6. ∎

PROPOSITION 2.7. *An* AH *algebra which is strongly Morita equivalent to a σ-unital C^*-algebra with the ideal property has the ideal property.*

Proof. By [3], strong Morita equivalence is the same as stable ismorphism when both C^*-algebras are σ-unital. Now the result follows from Theorem 2.1. ∎

3. THE RIESZ DECOMPOSITION PROPERTY

The main result of this section is Theorem 3.1 which says that for "many" AH algebras A with the ideal property, the projections in $M_\infty(A)$ satisfy the Riesz decomposition and interpolation properties and $\mathrm{K}_0(A)$ is a Riesz group. Some consequences of this theorem are also discussed.

Theorem 3.1. *Let A be an* AH *algebra with the ideal property and with slow dimension growth. Then the projections in $M_\infty(A)$ satisfy the Riesz interpolation and decomposition properties and* $\mathrm{K}_0(A)$ *is a Riesz group.*

Proof. By [25], Theorem 2.6, A is shape equivalent with an AH algebra B with real rank zero and with slow dimension growth. Now, since the projections in $M_\infty(B)$ satisfy cancellation (as well as those from $M_\infty(A)$) — because B and A have stable rank one by [25], Theorem 4.1 — it follows, by a result of S. Zhang ([30], Corollary 1.6), that $\mathrm{K}_0(B)$ is a Riesz group. Since the above mentioned shape equivalence obviously implies that:

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+) \cong (\mathrm{K}_0(B), \mathrm{K}_0(B)^+)$$

as ordered groups, it follows that $\mathrm{K}_0(A)$ is a Riesz group. Since, as observed before, the projections in $M_\infty(A)$ satisfy cancellation, it follows that the projections in $M_\infty(A)$ satisfy the Riesz interpolation and decomposition properties (note that in $\mathrm{K}_0(A)$ these properties are equivalent ([19], 2.1)). ∎

Remark 3.2. Observe that one can prove the above theorem without using S. Zhang's result [30], Corollary 1.6 or [25], Theorem 2.6. Indeed, since the C^*-algebra A has cancellation (by [25], Theorem 4.1), it suffices to prove that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Let p, q_1, q_2 be projections in $M_\infty(A)$. If $A = \varinjlim(A_n, \Phi_{n,m})$, where $A_n = \bigoplus_{i=1}^{k_n} A_n^i$ and $(A_n, \Phi_{n,m})$ has slow dimension growth, then we may suppose that $p, q_1, q_2 \in M_\infty(A_1^i)$ for some i. Then, using [25], Lemma 2.11, [22], 2.9 and [17], Lemma 2.13 and working as in the proof of [19], Theorem 2.7 or as in the proof of [22], Theorem 3.7) (see also the proof of [24], Theorem 3.1) — and using that for m large enough and any j either the partial homomorphism $\Phi_{1,m}^{i,j} : A_1^i \to A_m^j$ satisfies $\mathrm{rank}(\Phi_{1,m}^{i,j}(1_{A_1^i})) \geqslant (\dim(X_{m,j}){+}2)\mathrm{rank}(1_{A_1^i})$ or we may suppose that $\Phi_{1,m}^{i,j}(p_1), \Phi_{1,m}^{i,j}(q_1)$ and $\Phi_{1,m}^{i,j}(q_2)$ belong to a finite dimensional C^*-algebra of A_m^j — it follows that the projections in $M_\infty(A)$ have the Riesz decomposition property. The first proof is shorter and more "elegant" while this proof is more "elementary". ∎

THEOREM 3.3. *Let A be a unital* AH *algebra with the ideal property and with slow dimension growth. Then the state space of* $(\mathrm{K}_0(A), [1_A])$ *is a Choquet simplex.*

Proof. It follows from Theorem 3.1 and [19], Theorem 1.2. ∎

THEOREM 3.4. *Let A be an* AH *algebra with the ideal property and with slow dimension growth. Then* $\mathrm{K}_0(A)/\mathrm{tor}\,\mathrm{K}_0(A)$ *is an unperforated Riesz group.*

Proof. By Theorem 3.1 and [25], Theorem 5.1 (a), $\mathrm{K}_0(A)$ is a weakly unperforated Riesz group. By [15], 4.5, it follows that $\mathrm{K}_0(A)/\mathrm{tor}\,\mathrm{K}_0(A)$ is a Riesz group. The fact that $\mathrm{K}_0(A)/\mathrm{tor}\,\mathrm{K}_0(A)$ is unperforated follows from [20], 8.1. ∎

4. THE IDEAL STRUCTURE

The main result of this section is Theorem 4.1 which describes the partially ordered set of all the ideals generated by projections of an AH algebra. In particular, this theorem gives the ideal structure of an AH algebra with the ideal property. Other special cases and consequences are also discussed.

THEOREM 4.1. *Let A be an* AH *algebra. Then there is a lattice isomorphism:*

$$\{I \lhd A \mid I \textit{ is generated by projections}\} \xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A\otimes\mathcal{K})\}.$$

If moreover the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, then there are lattice isomorphisms:

$$\begin{aligned}\{I \lhd A \mid I \textit{ is generated by projections}\} &\xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A)\}\\ &\xrightarrow{\sim} \{L \mid L \textit{ is an ideal of } \mathrm{K}_0(A)\}.\end{aligned}$$

Before starting the proof, observe that for each of the above mentioned partially ordered sets it's far from obvious that it is a lattice.

To prove the above theorem we shall need several preliminary results. The first one is a lemma which was pointed out to us by G.A. Elliott, who remarked that its proof could be obtained using the argument on page 227 in [14]. We shall give a different proof, inspired by the proof of [30], Theorem 2.3, which seems to be simpler.

Lemma 4.2. *Let A be a C^*-algebra. Then there is an order isomorphism:*

$$\{I \lhd A \otimes \mathcal{K} \mid I \textit{ is generated by projections}\} \xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

Proof. Let $D(\cdot)$ be a map from the set of the ideals of $A \otimes \mathcal{K}$ generated by projections to the set of ideals of $D(A \otimes \mathcal{K})$, where for any ideal I of $A \otimes \mathcal{K}$ generated by projections, $D(I)$ is the set of equivalence classes of projections in I. Consider also $E(\cdot)$ the map from the set of ideals of $D(A \otimes \mathcal{K})$ to the set of ideals of $A \otimes \mathcal{K}$ generated by projections, where $E(J)$ is defined to be the ideal of $A \otimes \mathcal{K}$ generated by the projections $P \in A \otimes \mathcal{K}$ with $[P] \in J$. We shall prove that $D(\cdot)$ and $E(\cdot)$ are mutual inverses.

Note that for any ideal I of $A \otimes \mathcal{K}$ which is generated by its projections we have that $I = E(D(I))$. Since the inclusion $J \subseteq D(E(J))$ is obvious, we need only to prove that $D(E(J)) \subseteq J$, since $D(\cdot)$ and $E(\cdot)$ are obviously inclusion-preserving.

Let $[P] \in D(E(J))$. Then there are projections P_i and elements X_i and Y_i in $A \otimes \mathcal{K}$ such that:

$$\Big\| \sum_{i=1}^{n} X_i P_i Y_i - P \Big\| < 1$$

and $[P_i] \in J$, $1 \leqslant i \leqslant n$. By [20], 10.7 (see also [12]), it follows that $P \lesssim P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Now, using the fact that $A \otimes \mathcal{K} = \varinjlim (A \otimes M_n)$ and a simple argument, it follows that there are projections $P_i' \in A \otimes \mathcal{K}$ such that $P_i' \sim P_i$, $i = 1, 2, \ldots, n$, $P_i' P_j' = 0$ if $i \neq j$, $i, j = 1, 2, \ldots, n$ and:

$$P \lesssim \sum_{i=1}^{n} P_i' \quad \text{in } A \otimes \mathcal{K}.$$

Since $[P_i'] = [P_i] \in J$, $i = 1, 2, \ldots, n$ and J is an ideal in $D(A \otimes \mathcal{K})$, it follows that $[P] \in J$. ∎

Remark 4.3. Let $\Phi : E \to F$ be an order isomorphism of partially ordered sets, where any two elements in E have a supremum and any two elements in F have an infimum. Then E and F are lattices and Φ is a lattice isomorphism.

Using this remark obtained in a conversation with Mircea Martin, it easily follows that $\{I \lhd A \otimes \mathcal{K} \mid I$ is generated by projections$\}$ and $\{J \mid J$ is an ideal in $D(A \otimes \mathcal{K})\}$ in the above lemma are in fact *lattices*.

Lemma 4.4. ([6], [27]; the implication (a) $\Rightarrow$ (b) below is contained in the proof of [26], Theorem 3.1) *Let $A = \varinjlim (A_n, \Phi_{n,m})$ be an* AH *algebra, where $A_n = \bigoplus_{i=1}^{k_n} A_n^i$, $A_n^i = P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}$, $X_{n,i}$ are finite, connected* CW*-complexes and $P_{n,i} \in C(X_{n,i} M_{[n,i]})$ are projections. Then, the following are equivalent:*

(i) *I is generated by its projections;*

(ii) $I = \lim\limits_{\to}(I_n, \Phi_{n,m}|I_n)$, *with each I_n consisting of a direct sum of full blocks of A_n.*

The following result was obtained in a conversation with M. Dadarlat.

LEMMA 4.5. *Let A be an* AH *algebra. Then there is a lattice isomorphism:*

$$\begin{aligned}&\{I \lhd A \otimes \mathcal{K} \mid I \textit{ is generated by projections}\}\\ &\xrightarrow{\sim} \{J \lhd A \mid J \textit{ is generated by projections}\}.\end{aligned}$$

Proof. Let $A = \lim\limits_{\to}(A_n, \Phi_{n,m})$, where

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i,\, A_n^i = P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i},$$

$X_{n,i}$ are finite, connected CW-complexes and $P_{n,i}$ are projections in $C(X_{n,i}M_{[n,i]})$. Let I be an ideal of $A \otimes \mathcal{K}$ generated by projections. Then, it is known that there is a *unique* ideal J of A such that:

$$(*) \qquad I = J \otimes \mathcal{K}.$$

We shall prove that J is also generated by projections. Let $\mathcal{K} = \lim\limits_{\to}(M_n, \Psi_{n,m})$. Since I is an ideal of the AH algebra $A \otimes \mathcal{K}$ generated by projections, Lemma 4.4 implies that for any n there is $F_n \subseteq \{1, 2, \ldots, k_n\}$ such that:

$$\begin{aligned} I &= \lim_{\to}\Big(\Big(\bigoplus_{i\in F_n} A_n^i\Big) \otimes M_n, \Big(\Phi_{n,m}\Big| \bigoplus_{i\in F_n} A_n^i\Big) \otimes \Psi_{n,m}\Big)\\ &= \Big(\lim_{\to}\Big(\bigoplus_{i\in F_n} A_n^i, \Phi_{n,m}\Big| \bigoplus_{i\in F_n} A_n^i\Big)\Big) \otimes \Big(\lim_{\to}(M_n, \Psi_{n,m})\Big)\\ &= \Big(\lim_{\to}\Big(\bigoplus_{i\in F_n} A_n^i, \Phi_{n,m}\Big| \bigoplus_{i\in F_n} A_n^i\Big)\Big) \otimes \mathcal{K}. \end{aligned}$$

By the uniqueness of the decomposition $(*)$, it follows that:

$$J = \lim_{\to}\Big(\bigoplus_{i\in F_n} A_n^i, \Phi_{n,m}\Big| \bigoplus_{i\in F_n} A_n^i\Big)$$

which, applying again Lemma 4.4, implies that J is an ideal of A generated by its projections.

Hence we can define a map:

$$\begin{aligned}\Lambda : &\{I \lhd A \otimes \mathcal{K} \mid I \text{ is generated by projections}\}\\ &\to \{J \lhd A \mid J \text{ is generated by projections}\}\end{aligned}$$

by:

$$I \mapsto J$$

where $I = J \otimes \mathcal{K}$ and $J \lhd A$.

Since $\mathcal{K}$ is generated by projections (as an ideal of $\mathcal{K}$), we can also define a map:

$$\begin{aligned}\Gamma : \{J \lhd A \mid J \text{ is generated by projections}\} \\ \to \{I \lhd A \otimes \mathcal{K} \mid I \text{ is generated by projections}\}\end{aligned}$$

by:

$$J \mapsto J \otimes \mathcal{K}.$$

It is clear that Λ and Γ are mutual inverses and that they are inclusion preserving, and this together with Remark 4.3 end the proof. ∎

The following lemma is inspired by the proof of [30], Theorem 2.3 (see also [14]).

Lemma 4.6. *Let A be a C^*-algebra such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then there is a lattice isomorphism:*

$$\{I \lhd A \mid I \textit{ is generated by projections}\} \xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A)\}.$$

Proof. Let $D : \{I \lhd A \mid I \text{ is generated by projections}\} \to \{J \mid J \text{ is an ideal of } D(A)\}$ be the map given by:

$$D(I) = \text{the set of equivalence classes of projections in } I$$

for any ideal I of A generated by projections, and let $E : \{J \mid J \text{ is an ideal of } D(A)\} \to \{I \lhd A \mid I \text{ is generated by projections}\}$ be the map given by:

$$E(J) = \text{the ideal of } A \text{ generated by the projections } P \in A \text{ with } [P] \in J$$

for any ideal J of $D(A)$.

Note that, for any ideal I of A which is generated by its projections, we have that $I = E(D(I))$. Since the inclusion $J \subseteq D(E(J))$ is obvious, we need only to prove that $D(E(J)) \subseteq J$, since $D(\cdot)$ and $E(\cdot)$ are obviously inclusion-preserving maps (see also Remark 4.3).

Let $[P] \in D(E(J))$. Then there are projections P_i and elements X_i and Y_i in A such that:

$$\Big\| \sum_{i=1}^{n} X_i P_i Y_i - P \Big\| < 1$$

and $[P_i] \in J$, $1 \leqslant i \leqslant n$. By [20], 10.7 (see also [12]), it follows that $P \lesssim P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Since the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, it follows that there are projections R_i in $M_\infty(A)$, $i = 1, 2, \ldots, n$ such that:

$$P \sim \bigoplus_{i=1}^{n} R_i \quad \text{in } M_\infty(A)$$

and $R_i \leqslant P_i, 1 \leqslant i \leqslant n$. But then, by a standard general argument, it follows that there are pairwise orthogonal projections Q_i in A such that $Q_i \sim R_i$ in $M_\infty(A)$ $(i = 1, 2, \ldots, n)$ and:

$$P = \sum_{i=1}^{n} Q_i$$

(see e.g. [23], Lemma 7.2.3). Since also $Q_i \lesssim P_i$ in A $(i = 1, 2, \ldots, n)$ and J is an ideal of $D(A)$, it follows that $[Q_i] \in J$, $i = 1, 2, \ldots, n$ and hence $[P] = \sum_{i=1}^{n} [Q_i]$ (in $D(A)$) belongs to J. ∎

The following four results have been inspired by results and proofs in Section 10 of [20].

LEMMA 4.7. *Let A be a C^*-algebra with an approximate unit consisting of projections and such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then any projection in a matrix algebra $M_n(A)$ is equivalent to an orthogonal sum of n projections from A.*

Proof. This follows as in the proof of [20], Lemma 10.4. ∎

LEMMA 4.8. *Let A be an* AH *algebra. Let I be an ideal of A generated by projections. Denote by $i : I \to A$ the canonical inclusion and let $H = \mathrm{K}_0(i)(\mathrm{K}_0(I))$. Then H is an ideal of $\mathrm{K}_0(A)$ and I equals the ideal generated by the set:*

$$\{p \in A \mid p \text{ is a projection and } [p] \in H\}.$$

Proof. By the six-term exact sequence in K-theory associated with the exact sequence of C^*-algebras $0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0$ (where $\pi : A \to A/I$ is the canonical map) it follows that $H = \ker \mathrm{K}_0(\pi)$. But $\mathrm{K}_0(A/I)$ is a partially ordered group, since A/I being an AH algebra is stably finite. Hence $H^+ (:= H \cap \mathrm{K}_0(A)^+)$ is hereditary. Now, since I has an approximate unit of projections ([27] or the proof of [26], Theorem 3.1), it follows that $\mathrm{K}_0(I) = \mathrm{K}_0(I)^+ - \mathrm{K}_0(I)^+$. Hence $H = \mathrm{K}_0(i)(\mathrm{K}_0(I)) = \{[p]_{\mathrm{K}_0(A)} - [q]_{\mathrm{K}_0(A)} \mid p \text{ and } q \text{ are projections in } M_\infty(I)\}$. This, together with the fact that $\{[p]_{\mathrm{K}_0(A)} \mid p \text{ is a projection in } M_\infty(I)\} \subset H^+$, imply that $H = H^+ - H^+$.

Since I is generated by its projections, in order to prove the last part of the lemma, it is enough to prove that:

$$\mathcal{P}(I) = \{p \in A \mid p \text{ is a projection and } [p] \in H\}.$$

Since the inclusion $\mathcal{P}(I) \subseteq \{p \in \mathcal{P}(A) \mid [p] \in H\}$ follows from the fact — proved above — that $H = \{[p]_{\mathrm{K}_0(A)} - [q]_{\mathrm{K}_0(A)} \mid p \text{ and } q \text{ are projections in } M_\infty(I)\}$, we have to prove only that $\{p \in \mathcal{P}(A) \mid [p] \in H\} \subseteq \mathcal{P}(I)$. For this purpose, let $p \in \mathcal{P}(A)$ be such that $[p] \in H = \ker \mathrm{K}_0(\pi)$. Hence $[\pi(p)] = 0$ in $\mathrm{K}_0(A/I)$, and so there is a projection $q \in M_\infty(A/I)$ such that $\pi(p) \oplus q \sim q$. Since A/I is stably finite (being an AH algebra) it follows that $\pi(p) = 0$, that is $p \in I$ as desired. ∎

Lemma 4.9. *Let A be an* AH *algebra such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, let H be an ideal of* $\mathrm{K}_0(A)$ *and let I be the ideal of A generated by the set:*

$$\{p \in A \mid p \text{ is a projection and } [p] \in H\}.$$

Then:

(i) *Let p be a projection in $M_\infty(A)$. Then $[p] \in H$ if and only if $p \in M_\infty(I)$.*

(ii) *H equals the subgroup of* $\mathrm{K}_0(A)$ *generated by the set:*

$$\{[p] \mid p \text{ is a projection in } M_\infty(I)\}.$$

Proof. Observe that since I is an ideal of A, the hypothesis implies easily that the projections in $M_\infty(I)$ satisfy also the Riesz decomposition property. Also — as remarked before — by [27] or by the proof of [26], Theorem 3.1 it follows that I has an approximate unit of projections. The proof goes now as in the proof of [20], Lemma 10.8 (b) and (c) and using Lemma 4.7. ∎

Lemma 4.10. *Let A be an* AH *algebra such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then there is a lattice isomorphism:*

$$\{I \triangleleft A \mid I \text{ is generated by projections}\} \xrightarrow{\sim} \{H \mid H \text{ is an ideal of } \mathrm{K}_0(A)\}.$$

More precisely, there are inclusion-preserving inverse maps sending each ideal I of A generated by projections to $\mathrm{K}_0(i_I)(\mathrm{K}_0(I))$ *(where $i_I : I \to A$ is the canonical inclusion), and sending each ideal H of* $\mathrm{K}_0(A)$ *to the ideal of A generated by those projections $p \in A$ for which $[p] \in H$.*

Proof. By the previous two lemmas, the maps described in the lemma are inclusion-preserving inverse bijections. Also, by Lemma 4.5 or by Lemma 4.6, $\{I \triangleleft A \mid I \text{ is generated by projections}\}$ is a lattice. ∎

Proof of Theorem 4.1. Combining Lemma 4.2 with Remark 4.3 and Lemma 4.5 we obtain a lattice isomorphism:

$$\{I \triangleleft A \mid I \text{ is generated by projections}\} \xrightarrow{\sim} \{J \mid J \text{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

Suppose now that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. The fact that there is a lattice isomorphism:

$$\{I \triangleleft A \mid I \text{ is generated by projections}\} \xrightarrow{\sim} \{J \mid J \text{ is an ideal of } D(A)\}$$

follows from Lemma 4.6, which says that this is true in a much more general setting. Finally, the fact that there is a lattice isomorphism:

$$\{I \triangleleft A \mid I \text{ is generated by projections}\} \xrightarrow{\sim} \{L \mid L \text{ is an ideal of } \mathrm{K}_0(A)\}$$

follows from Lemma 4.10. ∎

REMARKS 4.11. (i) Theorem 4.1 gives, in particular, the ideal structure of an AH algebra A with the ideal property (with a special case when the projections of $M_\infty(A)$ satisfy the Riesz decomposition property).

(ii) Theorem 4.1 generalizes a result of G.A. Elliott in [13] concerning the ideal structure of an AF algebra. ∎

COROLLARY 4.12. *Let A be an* AH *algebra with the ideal property and with slow dimension grouwth. Then there are lattice isomorphisms:*

$$\begin{aligned}\{I \mid I \textit{ is an ideal of } A\} &\xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A \otimes \mathcal{K})\}\\ &\xrightarrow{\sim} \{K \mid K \textit{ is an ideal of } D(A)\}\\ &\xrightarrow{\sim} \{L \mid L \textit{ is an ideal of } \mathrm{K}_0(A)\}.\end{aligned}$$

Proof. It follows from Theorem 4.1 and Theorem 3.1. ∎

Our next result has been also obtained, independently, by M. Dadarlat.

PROPOSITION 4.13. *Let A and B be separable C^*-algebras. If A and B are shape equivalent, then there is an order isomorphism:*

$$\{I \triangleleft A \mid I \textit{ is generated by projections}\} \xrightarrow{\sim} \{J \triangleleft B \mid J \textit{ is generated by projections}\}.$$

Proof. Let

$$\begin{tikzcd}[column sep=small]
A_1 \arrow[rr, "\Phi_1"] \arrow[dr, "\alpha_1"'] & & A_2 \arrow[rr, "\Phi_2"] \arrow[dr, "\alpha_2"'] & & A_3 \arrow[rr, "\Phi_3"] \arrow[dr, "\alpha_3"'] & & A_4 \arrow[rr, "\Phi_4"] & & \cdots \\
& B_1 \arrow[ur, "\beta_1"'] \arrow[rr, "\Psi_1"] & & B_2 \arrow[ur, "\beta_2"'] \arrow[rr, "\Psi_2"] & & B_3 \arrow[ur, "\beta_3"'] \arrow[rr, "\Psi_3"] & & \cdots &
\end{tikzcd}$$

be a diagram of C^*-algebras and homomorphisms which is commutative within homotopy and suppose that $A = \varinjlim(A_n, \Phi_n), B = \varinjlim(B_n, \Psi_n)$.

To simplify the notation, we shall denote in the same way any element in some A_n and its canonical image in A. We shall make the same convention for the elements of the B_n's and their canonical images in B.

We shall define now a map:

$$\begin{aligned}\Lambda :& \{I \triangleleft A \mid I \text{ is generated by projections}\} \\ &\to \{J \triangleleft B \mid J \text{ is generated by projections}\}.\end{aligned}$$

Let I be an ideal of A generated by a family $(p_i)_i$ of projections. Approximating these projections in the A_n's (in fact, in the canonical images of the A_n's in A) we may suppose that each p_i belongs to some A_n (close enough projections are unitarily equivalent). So, for each i let n be a natural number (depending on i) such that $p_i \in A_n$. Let $q_i = \alpha_n(p_i) \in B_n$. Define $\Lambda(I)$ to be the ideal of B generated by the family of projections $(q_i)_i$. Observe that Λ is well defined (the above diagram is commutative within homotopy and homotopic projections are unitarily equivalent).

Now, define a map:

$$\begin{aligned}\Gamma :& \{J \triangleleft B \mid J \text{ is generated by projections}\} \\ &\to \{I \triangleleft A \mid I \text{ is generated by projections}\}\end{aligned}$$

in a similar way, but using the maps β_n instead of α_n. Observe that Λ and Γ are inverse bijections, since the above diagram is commutative within homotopy and homotopic projections are unitarily equivalent ($\beta_n(\alpha_n(p_i))$ is homotopic to $\Phi_n(p_i)$). Also, we used again the fact that if in a C^*-algebra D we have a family of projections $(e_k)_k$ and if $(u_k)_k$ is a family of unitaries in $\widetilde{D} = D + \mathbb{C} \cdot 1$, then the ideal generated in D by $(e_k)_k$ is clearly equal to the ideal generated in D by $(u_k e_k u_k^*)_k$. Obviously, Λ and Γ are inclusion-preserving. ∎

Proposition 4.14. *Let A and B be* AH *algebras with the ideal property and with slow dimension growth. If there is a graded isomorphism of ordered, scaled groups:*

$$(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(A)^+, \textstyle\sum(A)) \xrightarrow{\sim} (\underline{\mathrm{K}}(B), \underline{\mathrm{K}}(B)^+, \textstyle\sum(B))$$

which commutes with the Bockstein operations ([9], [7]*; see also* [8], [11])*, then there is a lattice isomorphism:*

$$\{I \mid I \textit{ is an ideal of } A\} \xrightarrow{\sim} \to \{J \mid J \textit{ is an ideal of } B\}.$$

Proof. By [25], Theorem 2.15, it follows that A and B are shape equivalent. Now, use the above proposition and Lemma 4.5. ∎

5. EXTENSIONS AND THE IDEAL PROPERTY

In this section we shall give the answer to the following

QUESTION. (G.K. Pedersen) Is it true that if

$$0 \to I \to A \to B \to 0$$

is an extension of C^*-algebras such that I and B have the ideal property, then A has the ideal property?

In [26], Theorem 3.1, we proved that if I, A and B above are all AH *algebras*, then the answer is "yes". Even though this might have been thought as an encouraging evidence for a positive answer to the above question of G.K. Pedersen, *the answer* however — *obtained jointly with M. Dadarlat* — turns out to be negative:

THEOREM 5.1. (joint with M. Dadarlat) *There are extensions of* C^**-algebras*

$$0 \to I \to A \to B \to 0$$

such that:

(i) I *and* B *are simple* $\mathrm{A}\mathbb{T}$ *algebras with* $\mathrm{RR}(I) = \mathrm{RR}(B) = 0$*;*

(ii) $\mathrm{RR}(A) = \mathrm{tsr}(A) = 1$*;*

and A *does not have the ideal property.*

In order to prove the above theorem we shall need the following result:

LEMMA 5.2. (joint with M. Dadarlat) *Let* A *be a* C^**-algebra generated (as an ideal of* A*) by its projections. Let* I *be an ideal of* A *such that* $I \neq A$*.*

Then, either A/I *is not stably finite or the index map* $\delta_0 : \mathrm{K}_0(A/I) \to \mathrm{K}_1(I)$ *is not injective.*

Proof. By contradiction. Suppose that A/I is stably finite and that δ_0 is injective. By the six-term exact sequence in K-theory associated with the exact sequence $0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0$ (where $i : I \to A$ is the canonical inclusion and $\pi : A \to A/I$ is the canonical surjection), we get that $\delta_0 \circ \mathrm{K}_0(\pi) = 0$. Now, let p be an arbitrary projection of A. It follows that $(\delta_0 \circ \mathrm{K}_0(\pi))([p]) = 0$, i.e. $\delta_0([\pi(p)]) = 0$. Since δ_0 is injective, we get that $[\pi(p)] = 0$ in $\mathrm{K}_0(A)/I)$, which implies — since A/I is stably finite — that $\pi(p) = 0$, i.e. $p \in I$. Hence any projection of A is also a projection of I. Since A is generated (as an ideal of A) by its projections and I is an ideal of A, it follows that $A \subseteq I$, and hence $A = I$, which is a contradiction. This ends the proof. ∎

Proof of Theorem 5.1. Let $B \in \mathcal{N}$ (where $\mathcal{N}$ is the "bootstrap" category defined in [29]) such that B has the ideal property, is stably finite and $\mathrm{K}_0(B) = \mathbb{Z}$, and let I be a stable, σ-unital C^*-algebra with the ideal property and such that $\mathrm{K}_1(I)$ is a non-zero, torsion free group. By the Universal Coefficient Theorem ([29], Theorem 1.17) there is an extension of C^*-algebras:

$$0 \to I \to A \to B \to 0$$

such that the index map $\delta_0 : \mathrm{K}_0(B) = \mathbb{Z} \to \mathrm{K}_1(I) \neq \{0\}$ is injective (i.e. $\delta_0(1) \neq 0$). Then, by Lemma 5.2, it follows that A is not generated (as an ideal of A) by its projections, and hence A does not have the ideal property.

We can take $B = \mathbb{C}$ and $I = D \otimes \mathcal{K}$, where D is a Bunce-Deddens algebra ([5]). Since D is a simple $\mathrm{A}\mathbb{T}$ algebra ([5]), it follows that I is also a simple $\mathrm{A}\mathbb{T}$ algebra ($\mathcal{K}$ is simple). On the other hand, by [1], D has real rank zero and hence, by [4], $\mathrm{RR}(I) = 0$. Also, note that $\mathrm{K}_1(I) = \mathbb{Z}$, since it is well-known that $\mathrm{K}_1(D) = \mathbb{Z}$. We want to prove now that $\mathrm{tsr}(A) = \mathrm{RR}(A) = 1$. Indeed, $\mathrm{tsr}(I) = 1$ (I is an inductive limit of circle algebras which have stable rank one and now use [28]), and obviously, $\mathrm{tsr}(\mathbb{C}) = 1$ and the index map $\delta_1 : \mathrm{K}_1(B) = \{0\} \to \mathrm{K}_0(I)$ is zero. These things imply that $\mathrm{tsr}(A) = 1$ (see e.g. [21]). By [4], Proposition 1.2 it follows that $\mathrm{RR}(A)$ is zero or one. But since A does not have the ideal property, it follows that $\mathrm{RR}(A) \neq 0$ and hence $\mathrm{RR}(A) = 1$. This ends the proof since, obviously, $B = \mathbb{C}$ is simple, stably finite, $\mathrm{RR}(\mathbb{C}) = 0$ and $\mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$. ∎

Remark 5.3. The above proof of Theorem 5.1 shows in fact how one could construct a lot of examples of extensions A of C^*-algebras I and B with the ideal property which do not have the ideal property (with I and B not necessarily simple $\mathrm{A}\mathbb{T}$ algebras of real rank zero).

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REFERENCES

1. B. Blackadar, A. Kumjian, Skew products of relations and structure of simple C^*-algebras, *Math. Z.* **189**(1985), 55–63.
2. O. Bratteli, Inductive limits of finite-dimensional C^*-algebras, *Trans. Amer. Math. Soc.* **171**(1972), 195–234.
3. L.G. Brown, P. Green, M.A. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pacific J. Math.* **71**(1977), 349–363.
4. L.G. Brown, G.K. Pedersen, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.
5. J. Bunce, J. Deddens, A family of simple C^*-algebras related to weighted shift operators, *J. Funct. Anal.* **19**(1975), 13–24.

6. M. DADARLAT, S. EILERS, Approximate homogeneity is not a local property, preprint.
7. M. DADARLAT, G. GONG, A classification result for approximately homogeneous C^*-algebras of real rank zero, *Geom. Funct. Anal.* **7**(1997), 646–711.
8. M. DADARLAT, T.A. LORING, Classifying C^*-algebras via ordered mod-p K-theory, *Math. Ann.* **305**(1996), 601–616.
9. M. DADARLAT, T.A. LORING, A universal multi-coefficient theorem for the Kasparov groups, *Duke Math. J.* **84**(1996), 355–377.
10. E.G. EFFROS, *Dimensions and C^*-Algebras*, CBMS Regional Conf. Ser. in Math., vol. 46, Amer. Math. Soc., Providence, RI 1981.
11. S. EILERS, A complete invariant for AD algebras with real rank zero and bounded torsion in K_1, *J. Funct. Anal.* **139**(1996), 325–348.
12. G.A. ELLIOTT, Derivations of matroid C^*-algebras. II, *Ann. of Math.* **100**(1974), 407–422.
13. G.A. ELLIOTT, On totally ordered groups, and K_0, in *Ring Theory Waterloo*, edited by D. Handelman and J. Lawrence, Lecture Notes in Math., vol. 734, Springer-Verlag, New York 1978, pp. 1-49.
14. G.A. ELLIOTT, The ideal structure of the multiplier algebra of an AF algebra, *C.R. Math. Rep. Acad. Sci. Canada* **IX**(1987), 225–230.
15. G.A. ELLIOTT, Dimension groups with torsion, *Internat. J. Math. Sci.* **1**(1990), 361–380.
16. G.A. ELLIOTT, The classification problem for amenable C^*-algebras, *Proc. I.C.M.* 1994, Zürich, Switzerland, Birkhäuser Verlag, Basel, 1995, 922–932.
17. G.A. ELLIOTT, G. GONG, On the classification of C^*-algebras of real rank zero. II, *Ann. of Math.* **144**(1996), 497–610.
18. G. GONG, On inductive limits of matrix algebras over higher dimensional spaces. II, *Math. Scand.* **80**(1997), 56–100.
19. K.R. GOODEARL, Riesz decomposition in inductive limit C^*-algebras, *Rocky Mountain J. Math.* **24**(1994), 1405–1430.
20. K.R. GOODEARL, K_0 of multiplier algebras of C^*-algebras with real rank zero, *K-Theory* **10**(1996), 419–489.
21. H. LIN, M. RØRDAM, Extensions of inductive limits of circle algebras, *J. London Math. Soc. (2)* **51**(1995), 603–613.
22. M. MARTIN, C. PASNICU, Some comparability results in inductive limits of C^*-algebras, *J. Operator Theory* **30**(1994), 137–147.
23. G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Inc., 1990.
24. C. PASNICU, Nonstable K-theory results for some AH-algebras, *Math. Scand.* **78** (1996), 112–120.
25. C. PASNICU, Shape equivalence, nonstable K-theory and AH algebras, *Pacific J. Math.*, to appear.
26. C. PASNICU, Extensions of AH algebras with the ideal propery, *Proc. Edinburgh Math. Soc.* **42**(1999), 65–76.
27. C. PASNICU, Ideals generated by projections and inductive limit C^*-algebras, preprint.
28. M. RIEFFEL, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc.* **46** (1983), 301–333.
29. J. ROSENBERG, C. SCHOCHET, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized functor, *Duke Math. J.* **55**(1987), 431–474.

30. S. ZHANG, A Riesz decomposition property and ideal structure of multiplier algebras, *J. Operator Theory* **24**(1990), 209–225.

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