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MULTIPARAMETRIC DISSIPATIVE LINEAR STATIONARY DYNAMICAL SCATTERING SYSTEMS: DISCRETE CASE

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Abstract. We propose the new generalization of linear stationary dynamical systems with discrete time $t \in \mathbb{Z}$ to the case $t \in \mathbb{Z}^N$. The dynamics of such a system can be reproduced by means of its associated multiparametric Lax-Phillips semigroup. We define multiparametric dissipative and conservative scattering systems and interpret them in terms of operator colligations, of the associated semigroup, and of "energy" relations for system data. We prove the Agler's type theorem describing the class of holomorphic operator-valued functions on the polydisc $\mathbb{D}^N$ that are the transfer functions of multiparametric conservative scattering systems.

Keywords: *Dissipative systems, multiparametric Lax-Phillips semigroup, generalized Schur class, conservative realizations.*

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0. INTRODUCTION AND PRELIMINARIES

In the present paper we introduce the concept of multiparametric linear stationary dynamical system (LSDS) and, in particular, the concept of dissipative (conservative) scattering LSDS, establish some properties of dissipative systems and give the description of the class of contractive operator-valued functions holomorphic on the open unit polydisc $\mathbb{D}^N$ which are the transfer functions of multiparametric conservative scattering LSDSs. We are based on the results of J. Agler on the characterization of the generalized Schur class (see [2]).

In our considerations, the multidimensional parameter $t \in \mathbb{Z}^N$ plays a role of "multidimensional time" since the introduced concepts generalize the concept of LSDS with (one-dimensional) discrete time and, in particular, the concept of dissipative (conservative) scattering LSDS. In this introductory section we shall recall the basic definitions and the most important facts of the theory of one-parametric dissipative scattering systems (see [6] or [7]) and give the motivation of such generalization.

Let us start with the standard definition of LSDS for the case $t \in \mathbb{Z}$ as the following system of equations:

$$\alpha : \begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t = 0, 1, 2, \ldots; \tag{0.1}$$

where $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{N}^- \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{N}^+$, $D : \mathcal{N}^- \to \mathcal{N}^+$ are bounded linear operators in separable Hilbert spaces, and the initial condition

$$x(0) = x_0, \tag{0.2}$$

with prescribed $x_0 \in \mathcal{X}$. The spaces $\mathcal{X}, \mathcal{N}^-, \mathcal{N}^+$ are called the *state space*, the *input space*, and the *output space*, respectively, for the LSDS α. The system α is called a *dissipative scattering* LSDS if the system matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+] \tag{0.3}$$

(here $[\mathcal{H}_1, \mathcal{H}_2]$ denotes the space of all bounded linear operators acting from a separable Hilbert space $\mathcal{H}_1$ into a separable Hilbert space $\mathcal{H}_2$) defines a contractive operator, i.e., $G^*G \leqslant I_{\mathcal{X}\oplus\mathcal{N}^-}$ as Hermitian operators. The system α is called a *conservative scattering* LSDS if G is unitary, i.e.,

$$G^*G = I_{\mathcal{X}\oplus\mathcal{N}^-}, \quad GG^* = I_{\mathcal{X}\oplus\mathcal{N}^+}. \tag{0.4}$$

Note that dissipativity (respective conservativity) condition has the physical meaning of the dissipation (respective of the full balance) of energy in a system. We hold the terminology of [14], [28]; however, some authors (see e.g. [6]) use the term "passive system" instead of the term "dissipative system". One may gather the data of a system using the notation $\alpha = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$. In the case of dissipative (respective conservative) scattering LSDS, the aggregate $\alpha = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ is called a *contractive* (respective *unitary*) *operator colligation.*

The operator-valued function

$$\theta_\alpha(z) = D + zC(I_{\mathcal{X}} - zA)^{-1}B \tag{0.5}$$

which has to be considered on some neighbourhood of $z = 0$ in $\mathbb{C}$ is called the *transfer function* of the system α. It is known (see [6] or [7]) that the transfer function θ_α of an arbitrary dissipative (in particular, conservative) scattering LSDS α belongs to the *Schur class* $S(\mathcal{N}^-, \mathcal{N}^+)$ consisting of all functions holomorphic on the open unit disc $\mathbb{D}$ which value contractive operators from $[\mathcal{N}^-, \mathcal{N}^+]$. The converse statement is also true (see [11], [26]): for an arbitrary $\theta \in S(\mathcal{N}^-, \mathcal{N}^+)$ there exists its conservative realization, that is, a conservative scattering LSDS (a unitary colligation) α of which θ is the transfer function (the characteristic function of the corresponding unitary colligation), i.e., $\theta = \theta_\alpha$. Moreover, the following result, fundamental for the theory of characteristic operator-valued functions (see [11]), is valid (we shall recall it here in the systems theory setting from [7]): for an arbitrary $\theta \in S(\mathcal{N}^-, \mathcal{N}^+)$ there exists a *closely connected* conservative realization $\alpha = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$, i.e., α is a conservative system and

$$\mathcal{X} = \bigvee_p p(A, A^*)(B\mathcal{N}^- + C^*\mathcal{N}^+) \tag{0.6}$$

(here the symbol "$\bigvee$" denotes the closure of the linear span of some subspaces and p runs the set of all monomials in two noncommuting variables); such a realization is uniquely determined by θ up to unitary similarity. The close connectedness of a conservative system is equivalent to the complete non-unitarity of its (contractive) *main operator* A (see [7]). Thus, there are close relations between the theory of dissipative (in particular, conservative) scattering systems, the theory of contractions by Sz.-Nagy–Foiaş ([26]), the theory of unitary colligations ([11]) and the theory of holomorphic functions of a complex variable, namely of Schur class functions.

The dynamics of LSDS (0.1)–(0.2) can be described by means of its associated Lax-Phillips semigroup in the corresponding scattering scheme (see [16], and also [1], [5], [21]). The *abstract Lax-Phillips scattering scheme* for the discrete case includes some separable Hilbert space $\mathcal{H}$, its subspaces $\mathcal{D}_+$ (the *outgoing space*) and $\mathcal{D}_-$ (the *ingoing space*), and some bounded linear operator W in $\mathcal{H}$ (the *generator* of the *Lax-Phillips semigroup* $\mathfrak{W} = \{W^t \mid t = 0, 1, 2, \dots\}$) such that:

(i) $\mathcal{D}_+ \perp \mathcal{D}_-$;
(ii) $W\mathcal{D}_+ \subset \mathcal{D}_+$, $W^*\mathcal{D}_- \subset \mathcal{D}_-$;
(iii) the operators $W|\mathcal{D}_+$ and $W^*|\mathcal{D}_-$ are isometric;
(iv) $\bigcap_{n=0}^{\infty} W^n\mathcal{D}_+ = \{0\} = \bigcap_{n=0}^{\infty} W^{*n}\mathcal{D}_-$.

We postpone the definition of the associated Lax-Phillips semigroup $\mathfrak{W}_\alpha$ for LSDS α till Subsection 1.1; we only notice here that the semigroup $\mathfrak{W}_\alpha$ is contractive (respective unitary) if and only if the system α is dissipative (respective conservative).

The powerful instrument in the investigation of the properties of systems and their main operators is the notion of dilation (see e.g. [6], [7]). The LSDS $\widetilde{\alpha} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathcal{X}}, \mathcal{N}^-, \mathcal{N}^+)$ is called a *dilation of the LSDS* $\alpha = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ if the following conditions are fulfilled:

$$\widetilde{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_*, \quad A = P_{\mathcal{X}}\widetilde{A}|\mathcal{X}, \quad B = P_{\mathcal{X}}\widetilde{B}, \quad C = \widetilde{C}|\mathcal{X},$$
$$\widetilde{A}\mathcal{D} \subset \mathcal{D}, \quad \widetilde{C}\mathcal{D} = \{0\}, \quad \widetilde{A}^*\mathcal{D}_* \subset \mathcal{D}_*, \quad \widetilde{B}^*\mathcal{D}_* = \{0\}$$

(here $P_{\mathcal{X}}$ denotes the orthogonal projector onto $\mathcal{X}$). In particular, it means that $A^t = P_{\mathcal{X}}\widetilde{A}^t|\mathcal{X}$ $(t = 1, 2, \ldots)$, i.e., the operator $\widetilde{A}$ is a *dilation of the operator* A. The LSDS α is called *minimal* if it is not a dilation of any other system, different from it. The notion of minimal system plays an important role in control theory; in the case of finite-dimensional spaces $\mathcal{N}^-$ and $\mathcal{N}^+$ and a rational matrix-valued function $\theta(\cdot)$ there exists a minimal system among all systems with the transfer function θ, and its state space has the definite finite dimension which is minimal of all state space dimensions of such systems (see [4]). An arbitrary LSDS α is a dilation of some minimal LSDS $\alpha_{\min}$ with the same transfer function; moreover, if α is a dissipative scattering LSDS then $\alpha_{\min}$ is also dissipative (see [6]). Together with the theorem on the conservative realization of a Schur class function this implies that any Schur class function has a minimal dissipative realization. On the other hand (see [6]), each dissipative scattering LSDS α has a conservative dilation $\widetilde{\alpha}$, that is, the system analogue of the classical B. Sz.-Nagy theorem (see [26]) on the existence of a unitary dilation for an arbitrary contraction in a Hilbert space.

Summarizing all the foregoing, let us extract the following aspects of the theory of dissipative scattering LSDSs with (one-dimensional) discrete time:

(a) the connection with the theory of holomorphic functions of one complex variable (namely, Schur class functions on $\mathbb{D}^N$);

(b) the connection with operator theory (namely, with the theory of contractive and unitary operators and operator colligations in Hilbert spaces);

(c) the connection with the Lax-Phillips scattering theory;

(d) the theory of dilations of systems as a useful method in control theory and in the theory of operators (colligations) in Hilbert spaces.

When constructing the theory of multiparametric dissipative scattering LS-DSs it seems important that the above-mentioned aspects of systems theory have meaningful generalizations. In our approach this is realized in the following way.

(a′) By means of the notion of the transfer function of a system the connection with the theory of holomorphic functions of several complex variables (namely, of contractive operator-valued holomorphic functions on $\mathbb{D}^N$) is established.

(b′) If $\zeta = (\zeta_1, \dots, \zeta_N)$ is a varying point on the unit torus $\mathbb{T}^N$, $\mathbf{A} = (A_1, \dots, A_N)$ and $\mathbf{G} = (G_1, \dots, G_N)$ are N-tuples of operators from $[\mathcal{X}, \mathcal{X}]$ and $[\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]$ respectively, then the pencil of operators $\zeta\mathbf{A} := \sum_{k=1}^{N} \zeta_k A_k$ is an analogue of the main operator A of a one-parametric LSDS, and the pencil of block matrices $\zeta\mathbf{G} := \sum_{k=1}^{N} \zeta_k G_k$ is an analogue of the system matrix G (see (0.3)). Moreover, the pencil of contractions $\zeta\mathbf{A}$ (respective $\zeta\mathbf{G}$) is an analogue of the contraction A (respective G), and the pencil of unitary operators $\zeta\mathbf{G}$ (of unitary colligations) is an analogue of the unitary operator G (of the unitary colligation) ($\zeta \in \mathbb{T}^N$).

(c′) The multiparametric semigroup of operators which reproduces the dynamics of a system and serves as an analogue of the Lax-Phillips semigroup associated with a one-parametric system is constructed.

(d′) The notion of dilation for multiparametric systems is introduced, and by means of it some multidimensional analogues of results in control theory as well as the criterion of the existense of conservative dilations for multiparametric dissipative scattering LSDSs (in particular, of unitary dilations for linear pencils of contractive operators in a Hilbert space that are considered on $\mathbb{T}^N$) are obtained.

Let us notice that generally speaking there is an extensive bibliography on multiparametric or, as one usually says, multidimensional systems theory (see e.g. the survey [10]). We shall indicate here only those approaches where metric properties of systems (dissipativity, conservativity and their analogues) are essential and where some aspects extracted above from systems theory are reflected. First of all, it is a system approach to the investigation of N-tuples of commuting nonselfadjoint operators in a Hilbert space having its origin in works of M.S. Livšic and his co-workers (see [17], [19], and also [18] and references indicated there). In this approach systems with continuous "multidimensional time" are overdetermined that brings to the necessity of additional relations which have to be imposed upon the N-tuple of the main operators of such systems. As a consequence, this induces connections with the theory of functions on Riemann surfaces (in fact, not with functions of several complex variables). In the discrete case, as V. Vinnikov told to the author, in his joint work with J.A. Ball an analogous systems theory is constructed, that is connected with the investigation of N-tuples of commuting contractions in a Hilbert space and also brings to functions on Riemann surfaces (these results are unpublished yet). Multidimensional analogues of dissipative resistance systems (see [6]) and their connections with the theory of functions of several complex variables were considered in [9]. The different multidimensional generalizations of the Lax-Phillips scattering scheme in the continuous case were constructed in [24] and [29]. In the discrete case, the multidimensional analogue of the abstract scattering scheme (including the Lax-Phillips scheme ([16]), and also the Adamjan-Arov scheme ([1])) has appeared in [12]. The structure of a multiparametric semigroup of isometries was investigated in [25] and [13]. Finally, in the paper by J.A. Ball and T.T. Trent ([8]) systems known in literature on the multidimensional systems theory as the "Roesser model" (see e.g. [15]), with imposed metric constraints, namely conservative (in particular, unitary) systems are investigated (in the terminology of the authors such a system is called conservative if it satisfies one of "energy" equalities, and unitary if it satisfies both of these equalities analogous to (0.4)). Using the result of [2] on the realization of operator-valued functions from the generalized Schur class (we shall recall its definition in Section 3) by unitary systems of the above-mentioned type, in [8] various functional models of those systems are constructed, and also interpolation problems for holomorphic functions on $\mathbb{D}^N$ are solved.

Our paper has the following structure. In Section 1, the appropriate for the further generalization reformulation of the definition of LSDS with (one-dimensional) discrete time and corresponding reformulations of main results on such systems are given. Then, the definition of a multiparametric LSDS and related definitions of the conjugate system, the transfer function, and also of the associated Lax-Phillips semigroup and of the associated one-parametric LSDS that reproduce the dynamics of such a system are given. In Section 2, multiparametric dissipative and conservative scattering LSDSs are defined and their characterizations in terms of the associated Lax-Phillips semigroup and of the associated one-parametric system are obtained. In Section 3, the class of transfer functions of multiparametric conservative scattering LSDSs is described as the subclass of functions from the generalized Schur class that are equal to zero at $z = 0$. Then the refinement (in one direction) of this result is obtained: the theorem on the closely connected conservative realization of such a function is proved.

Having in mind the restriction on the size of the paper, we plan to present the definition of a dilation for multiparametric systems and results on conservative dilations of multiparametric dissipative scattering LSDSs in our next paper.

1. MULTIPARAMETRIC LSDS AND ITS TRANSFER FUNCTION

1.1. Some remarks on LSDSs with (one-dimensional) discrete time. For the further generalization of the notion of LSDS, we need to make some renotations in (0.1). Namely, for all $t \in \mathbb{Z}_+ := \{t \in \mathbb{Z} : t \geqslant 0\}$ we set $\varphi^-(t) := u(t)$, $\varphi^+(t+1) := y(t)$. Then one can rewrite (0.1) as follows:

$$\alpha^0 : \begin{cases} x(t) = Ax(t-1) + B\varphi^-(t-1), \\ \varphi^+(t) = Cx(t-1) + D\varphi^-(t-1), \end{cases} \quad t \in \mathbb{Z}_+ \setminus \{0\}; \tag{1.1}$$

and the initial condition (0.2) remains. We call $x(t)$ $(\in \mathcal{X})$, $\varphi^-(t)$ $(\in \mathcal{N}^-)$, $\varphi^+(t+1)$ $(\in \mathcal{N}^+)$ for $t \in \mathbb{Z}_+$ *states, input data and output data* of the system (1.1) and (0.2), respectively. For α^0 of the form (1.1), we shall use also the short notation $\alpha^0 = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$. The dissipativity condition for α^0 can be rewritten in the following way:

$$\begin{gathered} \forall x(0) \in \mathcal{X}, \quad \forall t \in \mathbb{Z}_+ \setminus \{0\}, \quad \forall \{\varphi^-(\tau) \mid 0 \leqslant \tau < t\} \subset \mathcal{N}^- \\ \|\varphi^-(t-1)\|^2 - \|\varphi^+(t)\|^2 \geqslant \|x(t)\|^2 - \|x(t-1)\|^2, \end{gathered} \tag{1.2}$$

and the conservativity conditions for α^0 are obtained if we change the symbol "$\geqslant$" in (1.2) by "$=$" and require the analogous equalities for the *conjugate system* $(\alpha^0)^* := (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{N}^+, \mathcal{N}^-)$. If one interprets input data $\varphi^-(t)$ and output data $\varphi^+(t)$ as the "amplitudes of the incident and reflected waves", $\|\varphi^-(t)\|^2$ and $\|\varphi^+(t)\|^2$ as their "powers", and $\|x(t)\|^2$ as the "energy" of inner states $x(t)$, then the condition (1.2) for α^0 mean the dissipation of energy, and the conservativity conditions for α^0 means the full balance of energy, i.e. its conservation both for α^0 and for $(\alpha^0)^*$. This last system may be interpreted as one "with inverse time and the inverse direction of waves propagation".

Substitute $x(t-1)$ by $Ax(t-2) + B\varphi^-(t-2)$ in the right-hand side of (1.1); after this $x(t-2)$ by $Ax(t-3)+B\varphi^-(t-3)$ and so on. Then we get for $t \in \mathbb{Z}_+ \setminus \{0\}$

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1} B\varphi^-(\tau), \tag{1.3}$$

$$\varphi^+(t) = CA^{t-1}x(0) + D\varphi^-(t-1) + \sum_{\tau=0}^{t-2} CA^{t-\tau-2} B\varphi^-(\tau) \tag{1.4}$$

(for $t = 1$ only two summands remain in (1.4)). If we specify the zero initial condition in (0.2) (i.e., take $x_0 = 0$), then

$$x(t) = \sum_{\tau=0}^{t-1} A^{t-\tau-1} B\varphi^-(\tau), \quad t \in \mathbb{Z}_+ \setminus \{0\}; \tag{1.5}$$

$$\varphi^{+}(t)=D\varphi^{-}(t-1)+\sum_{\tau=0}^{t-2}CA^{t-\tau-2}B\varphi^{-}(\tau),\quad t\in\mathbb{Z}_{+}\setminus\{0\}. \tag{1.6}$$

Consider the formal power series

$$\widehat{x}(z)=\sum_{t\in\mathbb{Z}_{+}\setminus\{0\}}x(t)z^{t},\ \ \widehat{\varphi}^{-}(z)=\sum_{t\in\mathbb{Z}_{+}}\varphi^{-}(t)z^{t},\ \ \widehat{\varphi}^{+}(z)=\sum_{t\in\mathbb{Z}_{+}\setminus\{0\}}\varphi^{+}(t)z^{t}. \tag{1.7}$$

Then it is easy to obtain (formally) from (1.5) and (1.6)

$$\widehat{x}(z)=\left(\sum_{n=0}^{\infty}(zA)^{n}zB\right)\widehat{\varphi}^{-}(z), \tag{1.8}$$

$$\widehat{\varphi}^{+}(z)=\left(zD+\sum_{n=0}^{\infty}zC(zA)^{n}zB\right)\widehat{\varphi}^{-}(z). \tag{1.9}$$

Since the operator A is bounded, for $z\in\mathbb{C}$ from a small neighbourhood of zero, the series in (1.8) and (1.9) are convergent in the operator norm (moreover, this covergence is uniform on compact sets in this neighbourhood). Thus the operator-valued functions

$$(I_{\mathcal{X}}-zA)^{-1}zB=\sum_{n=0}^{\infty}(zA)^{n}zB,$$

$$zD+zC(I_{\mathcal{X}}-zA)^{-1}zB=zD+\sum_{n=0}^{\infty}zC(zA)^{n}zB$$

turn out to be holomorphic on this neighbourhood. We shall call the operator-valued function

$$\theta_{\alpha^{0}}(z):=zD+zC(I_{\mathcal{X}}-zA)^{-1}zB \tag{1.10}$$

the *transfer function* of the system α^{0} of the form (1.1).

From (1.8) and (1.9), one can deduce the formal relations

$$\widehat{\alpha^{0}}:\begin{cases}\widehat{x}(z)=zA\widehat{x}(z)+zB\widehat{\varphi}^{-}(z),\\ \widehat{\varphi}^{+}(z)=zC\widehat{x}(z)+zD\widehat{\varphi}^{-}(z).\end{cases} \tag{1.11}$$

As in [7] for systems of the form (0.1), we shall call $\widehat{\alpha^{0}}$ the Z-*transform* of α^{0}. If the $\mathcal{N}^{-}$-valued function $\widehat{\varphi}^{-}(z)$ from (1.7) is holomorphic on some neighbourhood of zero in $\mathbb{C}$, then by (1.8) and (1.9), $\widehat{x}(z)$ and $\widehat{\varphi}^{+}(z)$ from (1.7) are $\mathcal{X}$-valued (respective $\mathcal{N}^{+}$-valued) functions holomorphic on a neighbourhood of $z=0$; thus

(1.11) turns out to be a system of equations with holomorphic functions, and (1.8) and (1.9) turn into equations

$$\widehat{x}(z) = (I_{\mathcal{X}} - zA)^{-1} zB\widehat{\varphi}^{-}(z), \tag{1.12}$$

$$\widehat{\varphi}^{+}(z) = \theta_{\alpha^0}(z)\widehat{\varphi}^{-}(z), \tag{1.13}$$

that are equivalent to (1.11) in this case.

Comparing (1.10) with (0.5), we obtain

$$\theta_{\alpha^0}(z) = z\theta_{\alpha}(z). \tag{1.14}$$

Let us define the class $S^0(\mathcal{N}^-,\mathcal{N}^+)$ as the subclass of all operator-valued functions from $S(\mathcal{N}^-,\mathcal{N}^+)$ that are equal to zero at $z = 0$. If $\theta \in S(\mathcal{N}^-,\mathcal{N}^+)$ then $z\theta \in S^0(\mathcal{N}^-,\mathcal{N}^+)$. Conversely, if $\psi \in S^0(\mathcal{N}^-,\mathcal{N}^+)$ then by the Schwarz lemma for Banach space-valued functions (see e.g. Section 8.1.2 in [22]) $\psi = z\theta$ with $\theta \in S(\mathcal{N}^-,\mathcal{N}^+)$. So, we have the canonical bijection between the classes $S(\mathcal{N}^-,\mathcal{N}^+)$ and $S^0(\mathcal{N}^-,\mathcal{N}^+)$:

$$S^0(\mathcal{N}^-,\mathcal{N}^+) = zS(\mathcal{N}^-,\mathcal{N}^+). \tag{1.15}$$

From (1.14) and (1.15) we obtain the following reformulations of the results quoted in the previous section.

Theorem 1.1. *The transfer function θ_{α^0} of an arbitrary dissipative scattering* LSDS α^0 *of the form* (1.1) *belongs to the class $S^0(\mathcal{N}^-,\mathcal{N}^+)$.*

Theorem 1.2. *Any function $\theta \in S^0(\mathcal{N}^-,\mathcal{N}^+)$ has a closely connected conservative realization of the form* (1.1). *This realization is unique up to a unitary similarity.*

Remark 1.3. The definition of unitary similarity for systems (1.1) is the same as for systems (0.1) (see e.g. [7]).

Theorem 1.1 and Theorem 1.2 imply the following.

Theorem 1.4. *The class of transfer functions of conservative scattering* LSDSs *with the input space $\mathcal{N}^-$ and the output space $\mathcal{N}^+$ coincides with $S^0(\mathcal{N}^-,\mathcal{N}^+)$.*

One can associate to a system α^0 of the form (1.1) (as well as to a system α of the form (0.1)) its *Lax-Phillips semigroup* (see [1], [5], [21]) $\mathfrak{W}_{\alpha^0} = \{W^t_{\alpha^0} \mid$

$t \in \mathbb{Z}_+\}$ where $W_{\alpha^0} \in [\mathcal{H}_{\alpha^0}, \mathcal{H}_{\alpha^0}]$ is its *generator*, and $\mathcal{H}_{\alpha^0} := \mathcal{D}_+ \oplus \mathcal{X} \oplus \mathcal{D}_-$, $\mathcal{D}_+ := \cdots \oplus \mathcal{N}^+ \oplus \mathcal{N}^+$, $\mathcal{D}_- := \mathcal{N}^- \oplus \mathcal{N}^- \oplus \cdots$. If $h \in \mathcal{H}_{\alpha^0}$, i.e.,

$$h = \operatorname{col}(\ldots, v_{-1}, v_0; \boxed{x_0}; u_0, u_1, \ldots) \tag{1.16}$$

(here the element x_0 of the subspace $\mathcal{X}$ in $\mathcal{H}_{\alpha^0}$ is distinguished by a frame), then

$$W_{\alpha^0} h := \operatorname{col}(\ldots, v_0, Cx_0 + Du_0; \boxed{Ax_0 + Bu_0}; u_1, u_2, \ldots).$$

Evidently, this semigroup fits in the abstract scattering scheme of Lax-Phillips which was described in Introduction. For the conjugate system $(\alpha^0)^*$, the associated semigroup $\mathfrak{W}_{(\alpha^0)^*}$ is the "conjugate semigroup with inverse time", i.e., for $h \in \mathcal{H}_{\alpha^0}$ from (1.16) one defines the transform

$$\gamma h := \operatorname{col}(\ldots, u_1, u_0; \boxed{x_0}; v_0, v_{-1}, \ldots)$$

mapping the space $\mathcal{H}_{\alpha^0}$ isometrically onto $\mathcal{H}_{(\alpha^0)^*}$, and then

$$W^t_{(\alpha^0)^*} = \gamma (W^*_{\alpha^0})^t \gamma^{-1}, \quad t \in \mathbb{Z}_+.$$

The semigroup $\mathfrak{W}_{\alpha^0}$ is contractive (respective unitary) if and only if α^0 (as well as α) is a dissipative (respective conservative) scattering LSDS. The semigroup $\mathfrak{W}_{\alpha^0}$ reproduces the dynamics of the system α^0 in the following sense: if the sequence $\{u_\tau \mid \tau = 0, 1, \ldots\}$ from (1.16) is given to the input of α^0, i.e., $\varphi^-(\tau) = u_\tau$ ($\tau = 0, 1, \ldots$), and x_0 from (1.16) is substituted into the initial condition (0.2), then for h from (1.16) we have:

$$W^t_{\alpha^0} h = \operatorname{col}(\ldots, v_{-1}, v_0, \varphi^+(1), \varphi^+(2), \ldots, \varphi^+(t); \boxed{x(t)}; \varphi^-(t), \varphi^-(t+1), \ldots)$$

where $\varphi^+(1), \varphi^+(2), \ldots, \varphi^+(t)$, $x(t)$ turn out to be the output data and states of the system α^0 at corresponding moments of time $t \in \mathbb{Z}_+ \setminus \{0\}$.

1.2. THE DEFINITION OF A MULTIPARAMETRIC LSDS. Let us introduce the notion of multiparametric LSDS generalizing the notion of LSDS (1.1) and (0.2). For $t \in \mathbb{Z}^N$, we set $|t| := \sum_{k=1}^N t_k$, $\widetilde{\mathbb{Z}}^N_+ := \{t \in \mathbb{Z}^N : |t| \geqslant 0\}$, $\widetilde{\mathbb{Z}}^N_0 := \{t \in \mathbb{Z}^N : |t| = 0\}$, and for $k \in \{1, \ldots, N\}$ let $e_k := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^N$ (here 1 is on the k-th place and zeros are otherwise). We define a *multiparametric LSDS* as a system of equalities:

$$\alpha : \begin{cases} x(t) = \sum_{k=1}^N (A_k x(t - e_k) + B_k \varphi^-(t - e_k)), \\ \varphi^+(t) = \sum_{k=1}^N (C_k x(t - e_k) + D_k \varphi^-(t - e_k)), \end{cases} \quad t \in \widetilde{\mathbb{Z}}^N_+ \setminus \widetilde{\mathbb{Z}}^N_0. \tag{1.17}$$

Here for all $t \in \widetilde{\mathbb{Z}}_+^N$, $x(t)$ $(\in \mathcal{X})$, $\varphi^-(t)$ $(\in \mathcal{N}^-)$, and for all $t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N$, $\varphi^+(t)$ $(\in \mathcal{N}^+)$ are respectively *states*, *input data* and *output data* of α; $\mathcal{X}, \mathcal{N}^-, \mathcal{N}^+$ are separable Hilbert spaces that are called the *state space*, the *input space* and the *output space* respectively; for all $k \in \{1, \ldots, N\}$, $A_k \in [\mathcal{X}, \mathcal{X}]$, $B_k \in [\mathcal{N}^-, \mathcal{X}]$, $C_k \in [\mathcal{X}, \mathcal{N}^+]$, $D_k \in [\mathcal{N}^-, \mathcal{N}^+]$; moreover the following analogue of the initial condition (0.2) is given:

$$x(t) = x_0(t), \quad t \in \widetilde{\mathbb{Z}}_0^N, \tag{1.18}$$

where $x_0(\cdot) : \widetilde{\mathbb{Z}}_0^N \to \mathcal{X}$ is a prescribed function. If one denotes the N-tuple of operators T_k $(k = 1, \ldots, N)$ by $\mathbf{T} := (T_1, \ldots, T_N)$, then for such a system one may use the short notation $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$. It is clear that $(1; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+) = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$.

Remark 1.5. A system α can be considered on the positive octant $\mathbb{Z}_+^N := \{t \in \mathbb{Z}^N : t_k \geqslant 0, \, k = 1, \ldots, N\}$ only, i.e., we can take an input function $\varphi^-(\cdot)$ with the support in $\mathbb{Z}_+^N$ and choose the following initial data in (1.18):

$$x_0(t) = \begin{cases} 0 & \text{for } t \in \widetilde{\mathbb{Z}}_0^N \setminus \{0\}, \\ x_0 & \text{for } t = 0, \end{cases}$$

with some prescribed $x_0 \in \mathcal{X}$, and then, according to (1.17), $\operatorname{supp} x(\cdot) \subset \mathbb{Z}_+^N$, $\operatorname{supp} \varphi^+(\cdot) \subset \mathbb{Z}_+^N$.

Substitute $x(t - e_k)$ by $\sum_{j=1}^{N} (A_j x(t - e_k - e_j) + B_j \varphi^-(t - e_k - e_j))$ in the right-hand side of (1.17), and after this the corresponding expression instead of $x(t - e_k - e_j)$, etc. Then we get for $t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N$

$$\begin{aligned}
x(t) = {} & \sum_{k_1=1}^{N} \cdots \sum_{k_{|t|}=1}^{N} A_{k_1} \cdots A_{k_{|t|}} x\Bigl(t - \sum_{j=1}^{|t|} e_{k_j}\Bigr) \\
& + \sum_{l=1}^{|t|} \sum_{k_1=1}^{N} \cdots \sum_{k_{l-1}=1}^{N} \sum_{k_l=1}^{N} A_{k_1} \cdots A_{k_{l-1}} B_{k_l} \varphi^-\Bigl(t - \sum_{j=1}^{l} e_{k_j}\Bigr), \\
\varphi^+(t) = {} & \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_{|t|}=1}^{N} C_{k_1} A_{k_2} \cdots A_{k_{|t|}} x\Bigl(t - \sum_{j=1}^{|t|} e_{k_j}\Bigr) + \sum_{k=1}^{N} D_k \varphi^-(t - e_k) \\
& + \sum_{l=2}^{|t|} \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_{l-1}=1}^{N} \sum_{k_l=1}^{N} C_{k_1} A_{k_2} \cdots A_{k_{l-1}} B_{k_l} \varphi^-\Bigl(t - \sum_{j=1}^{l} e_{k_j}\Bigr).
\end{aligned}$$

Let $\tau \leqslant t$ mean $t-\tau \in \mathbb{Z}_+^N$. We denote by

$$c_s := \frac{|s|!}{s_1! \cdots s_N!}, \quad s \in \mathbb{Z}_+^N,$$

the numbers of permutations with repetitions (the polynomial coefficients). We introduce also the following notations:

$$\mathbf{A}^s := c_s^{-1} \sum_\sigma A_{[\sigma(1)]} \cdots A_{[\sigma(|s|)]}, \qquad s \in \mathbb{Z}_+^N \tag{1.19}$$

$$(\mathbf{A}\sharp\mathbf{B})^s := c_s^{-1} \sum_\sigma A_{[\sigma(1)]} \cdots A_{[\sigma(|s|-1)]} B_{[\sigma(|s|)]}, \qquad s \in \mathbb{Z}_+^N \setminus \{0\} \tag{1.20}$$

$$(\mathbf{C}\flat\mathbf{A})^s := c_s^{-1} \sum_\sigma C_{[\sigma(1)]} A_{[\sigma(2)]} \cdots A_{[\sigma(|s|)]}, \qquad s \in \mathbb{Z}_+^N \setminus \{0\} \tag{1.21}$$

$$\begin{aligned}(\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^s &:= c_s^{-1} \sum_\sigma C_{[\sigma(1)]} A_{[\sigma(2)]} \cdots A_{[\sigma(|s|-1)]} B_{[\sigma(|s|)]} \\ & \qquad s \in \mathbb{Z}_+^N \setminus \{0, e_1, \dots, e_N\}\end{aligned} \tag{1.22}$$

for *symmetrized multipowers of the operator N-tuple* $\mathbf{A}$ (1.19), *of the N-tuple* $\mathbf{A}$ *bordered with the* N-tuple $\mathbf{B}$ *from the right* (1.20), *of the N-tuple* $\mathbf{A}$ *bordered with the* N-tuple $\mathbf{C}$ *from the left* (1.21), *of the* N-tuple $\mathbf{A}$ *bordered with the* N-tuple $\mathbf{C}$ *from the left and with the* N-tuple $\mathbf{B}$ *from the right* (1.22). In these formulas the summation index σ runs the set of all permutations of $|s|$ elements of N different types with repetitions (an element of the j-th type repeats itself s_j times, $[k]$ ($\in \{1, \dots, N\}$) denotes the type of the element k). Note that in the case of a commutative N-tuple $\mathbf{A}$, we have $\mathbf{A}^s = A_1^{s_1} \cdots A_N^{s_N}$, i.e., the usual multipower. With notations (1.19)–(1.22) we obtain for $t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N$,

$$x(t) = \sum_{\tau \leqslant t,\, |\tau|=0} c_{t-\tau} \mathbf{A}^{t-\tau} x(\tau) + \sum_{\tau \leqslant t,\, \tau \neq t} c_{t-\tau} (\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau), \tag{1.23}$$

$$\begin{aligned}\varphi^+(t) = &\sum_{\tau \leqslant t,\, |\tau|=0} c_{t-\tau} (\mathbf{C}\flat\mathbf{A})^{t-\tau} x(\tau) + \sum_{k=1}^N D_k \varphi^-(t-e_k) \\ &+ \sum_{\tau \leqslant t,\, |t-\tau| \geqslant 2} c_{t-\tau} (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau).\end{aligned} \tag{1.24}$$

It is easy to assure that, for $N=1$, (1.23) and (1.24) coincide with (1.3) and (1.4) respectively.

1.3. THE $\boldsymbol{Z}$-TRANSFORM AND THE TRANSFER FUNCTION. Let us specify the zero condition in (1.18), i.e., $x_0(t) = 0$ for all $t \in \widetilde{\mathbb{Z}}_0^N$. Then, from (1.23) and (1.24) we get for $t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N$

$$x(t) = \sum_{\tau \leqslant t,\, \tau \neq t} c_{t-\tau} (\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau), \tag{1.25}$$

$$\varphi^+(t) = \sum_{k=1}^N D_k \varphi^-(t-e_k) + \sum_{\tau \leqslant t,\, |t-\tau| \geqslant 2} c_{t-\tau} (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau). \tag{1.26}$$

Consider the formal power series

$$\widehat{x}(z) = \sum_{t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N} x(t) z^t, \quad \widehat{\varphi}^-(z) = \sum_{t \in \widetilde{\mathbb{Z}}_+^N} \varphi^-(t) z^t, \quad \widehat{\varphi}^+(z) = \sum_{t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N} \varphi^+(t) z^t \tag{1.27}$$

(here $z^t := z_1^{t_1} \cdots z_N^{t_N}$ for any N-tuple $z = (z_1, \ldots, z_N)$ of commuting variables and for any $t \in \mathbb{Z}_+^N$). Then from (1.25) we get (formally)

$$\begin{aligned} \widehat{x}(z) &= \sum_{t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N} z^t \sum_{\tau \leqslant t,\, \tau \neq t} c_{t-\tau} (\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau) \\ &= \sum_{s \in \mathbb{Z}_+^N \setminus \{0\}} c_s z^s (\mathbf{A}\sharp\mathbf{B})^s \sum_{\tau \in \widetilde{\mathbb{Z}}_+^N} z^\tau \varphi^-(\tau) \\ &= \sum_{n=0}^\infty \left(\sum_{k=1}^N z_k A_k \right)^n \sum_{j=1}^N z_j B_j \widehat{\varphi}^-(z). \end{aligned} \tag{1.28}$$

Using the notation $z\mathbf{T} := \sum_{k=1}^N z_k T_k$ for the N-tuples $z = (z_1, \ldots, z_N)$ and $\mathbf{T} = (T_1, \ldots, T_N)$ we get

$$\widehat{x}(z) = \left(\sum_{n=0}^\infty (z\mathbf{A})^n z\mathbf{B} \right) \widehat{\varphi}^-(z)$$

(cf. (1.8)). From (1.26) we get (formally)

$$\begin{aligned} \widehat{\varphi}^+(z) &= \sum_{t \in \widetilde{\mathbb{Z}}_+^N \setminus \widetilde{\mathbb{Z}}_0^N} z^t \left(\sum_{k=1}^N D_k \varphi^-(t-e_k) + \sum_{\tau \leqslant t,\, |t-\tau| \geqslant 2} c_{t-\tau} (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^{t-\tau} \varphi^-(\tau) \right) \\ &= \sum_{k=1}^N z_k D_k \sum_{\tau \in \widetilde{\mathbb{Z}}_+^N} z^\tau \varphi^-(\tau) + \sum_{s \in \mathbb{Z}_+^N \setminus \{0, e_1, \ldots, e_N\}} c_s z^s (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^s \sum_{\tau \in \widetilde{\mathbb{Z}}_+^N} z^\tau \varphi^-(\tau). \end{aligned}$$

Finally,

$$\widehat{\varphi}^+(z) = \left(z\mathbf{D} + \sum_{n=0}^\infty z\mathbf{C} (z\mathbf{A})^n z\mathbf{B} \right) \widehat{\varphi}^-(z) \tag{1.29}$$

(cf. (1.9)). Since $\mathbf{A}$ is a N-tuple of bounded operators, in a small neighbourhood of $z = 0$ in $\mathbb{C}^N$ the series in (1.28) and (1.29) are convergent in the operator norm (moreover, this convergence is uniform on compact sets in this neighbourhood). Thus, the operator-valued functions

$$(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} = \sum_{n=0}^{\infty} (z\mathbf{A})^n z\mathbf{B},$$
$$z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} = z\mathbf{D} + \sum_{n=0}^{\infty} z\mathbf{C}(z\mathbf{A})^n z\mathbf{B}$$

turn out to be holomorphic on this neighbourhood. We shall call the operator-valued function

$$\theta_\alpha(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} \tag{1.30}$$

the *transfer function* of the system α of the form (1.17) (cf. (1.10)). From (1.28) and (1.29) one can deduce the formal relations

$$\widehat{\alpha} : \begin{cases} \widehat{x}(z) = z\mathbf{A}\widehat{x}(z) + z\mathbf{B}\widehat{\varphi}^-(z), \\ \widehat{\varphi}^+(z) = z\mathbf{C}\widehat{x}(z) + z\mathbf{D}\widehat{\varphi}^-(z); \end{cases} \tag{1.31}$$

(cf. (1.11)). We shall call $\widehat{\alpha}$ the *Z-transform* of the system α of the form (1.17). If the $\mathcal{N}^-$-valued function $\widehat{\varphi}^-(z)$ from (1.27) is holomorphic on some neighbourhood of $z = 0$ in $\mathbb{C}^N$, then by (1.28) and (1.29) $\widehat{x}(z)$ and $\widehat{\varphi}^+(z)$ from (1.27) are $\mathcal{X}$-valued (respectively $\mathcal{N}^+$-valued) functions also holomorphic on a neighbourhood of $z = 0$, thus, (1.31) turns out to be a system of equations with holomorphic functions, and (1.28) and (1.29) turn into equations

$$\widehat{x}(z) = (I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B}\widehat{\varphi}^-(z), \tag{1.32}$$
$$\widehat{\varphi}^+(z) = \theta_\alpha(z)\widehat{\varphi}^-(z); \tag{1.33}$$

(cf. (1.12) and (1.13)), that are equivalent to (1.31) in this case.

1.4. THE CONJUGATE SYSTEM. Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ be a multiparametric LSDS. Then we call $\alpha^* = (N; \mathbf{A}^*, \mathbf{C}^*, \mathbf{B}^*, \mathbf{D}^*; \mathcal{X}, \mathcal{N}^+, \mathcal{N}^-)$ the *conjugate LSDS* for α, where for an N-tuple $\mathbf{T} = (T_1, \ldots, T_N)$ of operators from $[\mathcal{Y}, \mathcal{V}]$ we set $\mathbf{T}^* := (T_1^*, \ldots, T_N^*)$, that is an N-tuple of operators from $[\mathcal{V}, \mathcal{Y}]$. For an arbitrary function $\theta(z) = \theta(z_1, \ldots, z_N)$ with values in $[\mathcal{Y}, \mathcal{V}]$ we define the function $\theta^*(z) := \theta(\bar{z})^* = \theta(\bar{z_1}, \ldots, \bar{z_N})^*$ with values in $[\mathcal{V}, \mathcal{Y}]$.

PROPOSITION 1.6. *Let* $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ *be a multiparametric* LSDS, $\theta_\alpha(z)$ *be its transfer function which is defined and holomorphic on some open neighbourhood* Υ *of* $z = 0$ *in* $\mathbb{C}^N$. *Then the transfer function* $\theta_{\alpha^*}(z)$ *of the conjugate* LSDS α^* *is defined and holomorphic on* $\Upsilon^* := \{z \in \mathbb{C}^N : \bar{z} \in \Upsilon\}$ *and* $\theta_{\alpha^*}(z) = \theta_\alpha^*(z)$ *for all* $z \in \Upsilon^*$.

Proof. Evidently, if Υ is an open neighbourhood of $z = 0$ then Υ^* is also an open neighbourhood of $z = 0$. For $z \in \Upsilon^*$, we have $\bar{z} \in \Upsilon$, $\theta_\alpha^*(z) = \theta_\alpha(\bar{z})^*$ is holomorphic on Υ^* and

$$\begin{aligned}\theta_\alpha^*(z) = \theta_\alpha(\bar{z})^* &= (\bar{z}\mathbf{D} + \bar{z}\mathbf{C}(I_{\mathcal{X}} - \bar{z}\mathbf{A})^{-1}\bar{z}\mathbf{B})^* \\ &= z\mathbf{D}^* + z\mathbf{B}^*(I_{\mathcal{X}} - z\mathbf{A}^*)^{-1}z\mathbf{C}^* = \theta_{\alpha^*}(z)\end{aligned}$$

that completes the proof. ∎

1.5. THE ASSOCIATED SEMIGROUP AND THE ASSOCIATED ONE-PARAMETRIC LSDS.

Let us introduce for the LSDS $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ the multiparametric analogue of the associated Lax-Phillips semigroup (see Subsection 1.1). Set $\widetilde{\mathbb{Z}}_-^N := \{t \in \mathbb{Z}^N : |t| \leqslant 0\}$ and let $\widetilde{\mathcal{D}}_\pm := l^2(\widetilde{\mathbb{Z}}_\mp^N, \mathcal{N}^\pm)$, $\widetilde{\mathcal{X}} := l^2(\widetilde{\mathbb{Z}}_0^N, \mathcal{X})$ be Hilbert spaces of multisequences $\{u^\pm(t) \mid t \in \widetilde{\mathbb{Z}}_\mp^N\} \subset \mathcal{N}^\pm$, $\{y(t) \mid t \in \widetilde{\mathbb{Z}}_0^N\} \subset \mathcal{X}$ respectively, such that

$$\sum_{t \in \widetilde{\mathbb{Z}}_\mp^N} \|u^\pm(t)\|^2 < \infty, \qquad \sum_{t \in \widetilde{\mathbb{Z}}_0^N} \|y(t)\|^2 < \infty.$$

Set also $\mathcal{H}_\alpha := \widetilde{\mathcal{D}}_+ \oplus \widetilde{\mathcal{X}} \oplus \widetilde{\mathcal{D}}_-$. Then we shall define the operators $W_{\alpha,k} \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$ $(k = 1, \ldots, N)$ in the following way. For $h = (u^+, y, u^-) \in \widetilde{\mathcal{D}}_+ \oplus \widetilde{\mathcal{X}} \oplus \widetilde{\mathcal{D}}_- = \mathcal{H}_\alpha$ set

$$h_k := W_{\alpha,k} h = (u_k^+, y_k, u_k^-), \quad k = 1, \ldots, N; \tag{1.34}$$

where

$$u_k^+(t) = \begin{cases} u^+(t + e_k), & |t| \leqslant -1, \\ \sum\limits_{j=1}^N (C_j y(t + e_k - e_j) + D_j u^-(t + e_k - e_j)), & |t| = 0; \end{cases} \tag{1.35}$$

$$y_k(t) = \sum_{j=1}^N (A_j y(t + e_k - e_j) + B_j u^-(t + e_k - e_j)), \quad |t| = 0, \tag{1.36}$$

$$u_k^-(t) = u^-(t + e_k), \qquad |t| \geqslant 0. \tag{1.37}$$

It is not difficult to assure that, for $k \neq j$, $W_{\alpha,k} W_{\alpha,j} = W_{\alpha,j} W_{\alpha,k}$ and thus the semigroup $\mathfrak{W}_\alpha := \{\mathbf{W}_\alpha^t \mid t \in \mathbb{Z}_+^N\} \subset [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$ is well defined (here $\mathbf{W}_\alpha = (W_{\alpha,1}, \ldots, W_{\alpha,N})$ and for $t \in \mathbb{Z}_+^N$, $\mathbf{W}_\alpha^t = W_{\alpha,1}^{t_1} \cdots W_{\alpha,N}^{t_N}$).

REMARK 1.7. From (1.34)–(1.37) one can see that for $k \in \{1, \ldots, N\}$, $W_{\alpha,k}|\widetilde{\mathcal{D}}_+$ and $W^*_{\alpha,k}|\widetilde{\mathcal{D}}_-$ are forward shift operators with wandering generating subspaces

$$(1.38)\quad \widetilde{\mathcal{N}}^+ := \widetilde{\mathcal{D}}_+ \ominus W_{\alpha,k}\widetilde{\mathcal{D}}^+ = l^2(\widetilde{\mathbb{Z}}_0^N, \mathcal{N}^+),\ \widetilde{\mathcal{N}}^- := \widetilde{\mathcal{D}}_- \ominus W^*_{\alpha,k}\widetilde{\mathcal{D}}^- = l^2(\widetilde{\mathbb{Z}}_0^N, \mathcal{N}^-)$$

which means that $\mathfrak{W}_\alpha$ is a multiparametric analogue of Lax-Phillips semigroup, and its generators $W_{\alpha,k}$ $(k = 1, \ldots, N)$ is generators of some one-parametric Lax-Phillips semigroups.

REMARK 1.8. The semigroup $\mathfrak{W}_\alpha$ reproduces the dynamics of the system α in the following sense: if the multisequence $\{u^-(t) \mid t \in \widetilde{\mathbb{Z}}_+^N\}$ $(\subset \widetilde{\mathcal{D}}_-)$ is given as the input of α, i.e., $\varphi^-(t) = u^-(t)$ $(t \in \widetilde{\mathbb{Z}}_+^N)$, and $\{y(t) \mid t \in \widetilde{\mathbb{Z}}_0^N\}$ $(\subset \widetilde{\mathcal{X}})$ is substituted into (1.18) then (1.35)–(1.37) turn into

$$\begin{aligned} u_k^+(t) &= \begin{cases} u^+(t+e_k), & |t| \leqslant -1; \\ \varphi^+(t+e_k), & |t| = 0; \end{cases} \\ y_k(t) &= x(t+e_k), \qquad |t| = 0; \\ u_k^-(t) &= \varphi^-(t+e_k), \qquad |t| \geqslant 0; \end{aligned}$$

where $x(t+e_k)$ and $\varphi^+(t+e_k)$ are the states and the output signals of the system α at the moment $t+e_k$. Iterating these formulas, we obtain for $h_s = (u_s^+, y_s, u_s^-) := \mathbf{W}_\alpha^s h$ $(s \in \mathbb{Z}_+^N)$:

$$\begin{aligned} u_s^+(t) &= \begin{cases} u^+(t+s), & |t| \leqslant -s, \\ \varphi^+(t+s), & -s < |t| \leqslant 0; \end{cases} \\ y_s(t) &= x(t+s), \qquad |t| = 0; \\ u_s^-(t) &= \varphi^-(t+s), \qquad |t| \geqslant 0; \end{aligned}$$

where $x(t+s)$ and $\varphi^+(t+s)$ are the states and the output signals of α at the moment $t+s$.

We shall show now that for the conjugate system α^* of some system α, the associated semigroup $\mathfrak{W}_{\alpha^*}$ is a "conjugate semigroup with inverse time". More exactly, we define the isomorphism $\gamma : \mathcal{H}_\alpha \to \mathcal{H}_{\alpha^*}$ in the following way: for $h = (u^+, y, u^-) \in \mathcal{H}_\alpha$, we set $h^* = \gamma h := (u_*^+, y_*, u_*^-) \in \mathcal{H}_{\alpha^*}$, where

$$(1.39)\qquad \begin{cases} u_*^+(t) = u^-(-t), & t \in \widetilde{\mathbb{Z}}_-^N, \\ y_*(t) = y(-t), & t \in \widetilde{\mathbb{Z}}_0^N, \\ u_*^-(t) = u^+(-t), & t \in \widetilde{\mathbb{Z}}_+^N; \end{cases}$$

and then we obtain the following.

Proposition 1.19. *For every $t \in \mathbb{Z}_+^N$ we have $\mathbf{W}_{\alpha^*}^t = \gamma(\mathbf{W}_\alpha^*)^t\gamma^{-1}$.*

Proof. Evidently, it is sufficient to show that

$$W_{\alpha^*,k} = \gamma W_{\alpha,k}^*\gamma^{-1}, \quad k \in \{1,\dots,N\}. \tag{1.40}$$

It is not difficult to assure that for $h = (u^+, y, u^-) \in \mathcal{H}_\alpha$, $W_{\alpha,k}^* h$ are defined by the following formulas:

$$\begin{aligned}
&(P_{\widetilde{\mathcal{D}}_+}W_{\alpha,k}^*h)(t) = u^+(t-e_k), \quad |t| \leqslant 0;\\
&(P_{\widetilde{\mathcal{X}}}W_{\alpha,k}^*h)(t) = \sum_{j=1}^N (A_j^* y(t-e_k+e_j) + C_j^* u^+(t-e_k+e_j)), \quad |t| = 0;\\
&(P_{\widetilde{\mathcal{D}}_-}W_{\alpha,k}^*h)(t) = \begin{cases} \sum_{j=1}^N (B_j^* y(t-e_k+e_j) + D_j^* u^+(t-e_k+e_j)), & |t| = 0,\\ u^-(t-e_k), & |t| \geqslant 1;\end{cases}
\end{aligned}$$

hence for $h = (u^+, y, u^-) \in \mathcal{H}_\alpha$ and for $h^* = \gamma h = (u_*^+, y_*, u_*^-) \in \mathcal{H}_{\alpha^*}$, we have $\gamma W_{\alpha,k}^*\gamma^{-1}h_* = \gamma W_{\alpha,k}^* h =: (u_{*,k}^+, y_{*,k}, u_{*,k}^-) \in \mathcal{H}_{\alpha^*}$, where by (1.39)

$$\begin{aligned}
&u_{*,k}^+(t) = \begin{cases} u_*^+(t+e_k), & |t| \leqslant -1,\\ \sum_{j=1}^N (B_j^* y_*(t+e_k-e_j) + D_j^* u_*^-(t+e_k-e_j)), & |t| = 0;\end{cases}\\
&y_{*,k}(t) = \sum_{j=1}^N (A_j^* y_*(t+e_k-e_j) + C_j^* u_*^-(t+e_k-e_j)), \quad |t| = 0;\\
&u_{*,k}^-(t) = u_*^-(t+e_k), \quad |t| \geqslant 0.
\end{aligned}$$

By the definition of α^* and by (1.34)–(1.37), we get $W_{\alpha^*,k}h_* = (u_{*,k}^+, y_{*,k}, u_{*,k}^-)$. Taking into account the arbitrariness of h_* and the unitarity of the operator γ, (1.40) follows. ∎

Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ be a multiparametric LSDS and $\mathfrak{W}_\alpha$ be its associated semigroup. Consider for an arbitrary $k \in \{1,\dots,N\}$ the one-parametric semigroup $\mathfrak{W}_{\alpha,k} := \{W_{\alpha,k}^n \mid n \in \mathbb{Z}_+\}$. In accordance with Remark 1.7 it turns out to be the associated Lax-Phillips semigroup of some one-parametric LSDS, namely $\alpha_k := (\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k; \widetilde{\mathcal{X}}, \widetilde{\mathcal{N}}^-, \widetilde{\mathcal{N}}^+)$ where $\widetilde{\mathcal{N}}^+$ and $\widetilde{\mathcal{N}}^-$ are defined in (1.38), $\widetilde{A}_k := P_{\widetilde{\mathcal{X}}}W_{\alpha,k}|\widetilde{\mathcal{X}}$, $\widetilde{B}_k := P_{\widetilde{\mathcal{X}}}W_{\alpha,k}|\widetilde{\mathcal{N}}^-$, $\widetilde{C}_k := P_{\widetilde{\mathcal{N}}^+}W_{\alpha,k}|\widetilde{\mathcal{X}}$, and $\widetilde{D}_k := P_{\widetilde{\mathcal{N}}^+}W_{\alpha,k}|\widetilde{\mathcal{N}}^-$. Define the unitary operators $T_{jk} : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ and $S_{jk} : \widetilde{\mathcal{N}}^- \to \widetilde{\mathcal{N}}^-$ for all $j \neq k$ as follows:

$$\forall y \in \widetilde{\mathcal{X}} \qquad (T_{jk}y)(t) := y(t+e_k-e_j), \qquad t \in \mathbb{Z}_0^N, \tag{1.41}$$

$$\forall u^- \in \widetilde{\mathcal{N}}^- \quad (S_{jk}u^-)(t) := u^-(t+e_k-e_j), \quad t \in \mathbb{Z}_0^N. \tag{1.42}$$

Then, according to (1.35)–(1.37), the system matrix of α_k is

$$\widetilde{G}_k = \begin{pmatrix} \widetilde{A}_k = A_k I_{\widetilde{\mathcal{X}}} + \sum_{j\neq k} A_j T_{jk} & \widetilde{B}_k = B_k I_{\widetilde{\mathcal{N}}^-} + \sum_{j\neq k} B_j S_{jk} \\ \widetilde{C}_k = C_k I_{\widetilde{\mathcal{X}}} + \sum_{j\neq k} C_j T_{jk} & \widetilde{D}_k = D_k I_{\widetilde{\mathcal{N}}^-} + \sum_{j\neq k} D_j S_{jk} \end{pmatrix} \tag{1.43}$$

where $A_k I_{\widetilde{\mathcal{X}}}$, $B_k I_{\widetilde{\mathcal{N}}^-}$, $C_k I_{\widetilde{\mathcal{X}}}$, $D_k I_{\widetilde{\mathcal{N}}^-}$ are "block diagonal" operators, i.e., $(A_k I_{\widetilde{\mathcal{X}}} y)(t) = A_k y(t)$ for $t \in \widetilde{\mathbb{Z}}_0^N$, etc. We shall call α_k the *k-th associated one-parametric LSDS* for α. Each system α_k reproduces the dynamics of α: if an input sequence of α_k is given as the input of α, i.e., $\varphi^-(s+ne_k) = \varphi_n^-(s)$ for $s \in \widetilde{\mathbb{Z}}_0^N$, $n \in \mathbb{Z}_+$ where φ^- and φ_n^- are the input data of α and α_k respectively, and the initial condition for α_k is substituted into (1.18), then the states and the output signals of α_k at any moment $n > 0$ coincide with collections of the states and the output signals of α taken for all $t \in \mathbb{Z}^N$ such that $|t| = n$, i.e.,

$$x_n(s) = x(s+ne_k), \quad \varphi_n^+(s) = \varphi^+(s+ne_k), \quad s \in \mathbb{Z}_0^N. \tag{1.44}$$

Indeed, from (1.41)–(1.43), we get for $s \in \widetilde{\mathbb{Z}}_0^N$:

$$\begin{aligned} x_1(s) &= (\widetilde{A}_k x_0 + \widetilde{B}_k \varphi_0^-)(s) \\ &= \sum_{j=1}^N (A_j x_0(s+e_k-e_j) + B_j \varphi_0^-(s+e_k-e_j)) = x(s+e_k), \\ \varphi_1^+(s) &= (\widetilde{C}_k x_0 + \widetilde{D}_k \varphi_0^-)(s) \\ &= \sum_{j=1}^N (C_j x_0(s+e_k-e_j) + D_j \varphi_0^-(s+e_k-e_j)) = \varphi^+(s+e_k). \end{aligned}$$

Iterating this calculation n times, we obtain (1.44).

2. MULTIPARAMETRIC DISSIPATIVE SCATTERING LSDS

2.1. THE DEFINITION AND SOME PROPERTIES. We shall call $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ a *multiparametric dissipative scattering LSDS* if for any $\zeta \in \mathbb{T}^N$,

$$\zeta\mathbf{G} := \begin{pmatrix} \zeta\mathbf{A} & \zeta\mathbf{B} \\ \zeta\mathbf{C} & \zeta\mathbf{D} \end{pmatrix} \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]$$

is a contractive operator. Evidently, by virtue of the maximum principle for holomorphic operator-valued functions (see e.g. [23]), it is equivalent to require for the operator-valued linear function $L_{\mathbf{G}}(z) := z\mathbf{G}$ to be contractive on $\mathbb{D}^N$. For $N = 1$ this notion coincides with the notion of dissipative scattering LSDS with discrete time (see Introduction and Subsection 1.1).

PROPOSITION 2.1. *Let* $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ *be given,* $\mathfrak{W}_\alpha$ *and* α_k *(for some* $k \in \{1, \dots, N\}$*) be the associated semigroup and the associated one-parametric* LSDS *respectively. Then, the following statements are equivalent:*

(i) α *is a multiparametric dissipative scattering* LSDS*;*

(ii) $\mathfrak{W}_\alpha$ *is a semigroup of contractions in* $\mathcal{H}_\alpha$*;*

(iii) α_k *is a dissipative scattering* LSDS*;*

(iv) *if an input multisequence* $\{\varphi^-(t) \mid t \in \widetilde{\mathbb{Z}}_+^N\}$ *of* α *satisfies*

$$\sum_{|t|=n} \|\varphi^-(t)\|^2 < \infty, \quad \forall n \in \mathbb{Z}_+, \tag{2.1}$$

and its states $\{x_0(t) \mid t \in \widetilde{\mathbb{Z}}_0^N\}$ *from* (1.18) *satisfy*

$$\sum_{|t|=0} \|x_0(t)\|^2 < \infty; \tag{2.2}$$

then $\forall n \in \mathbb{Z}_+ \setminus \{0\}$, $\sum_{|t|=n} \|x(t)\|^2 < \infty$, $\sum_{|t|=n} \|\varphi^+(t)\|^2 < \infty$, *and*

$$\sum_{|t|=n-1} \|\varphi^-(t)\|^2 - \sum_{|t|=n} \|\varphi^+(t)\|^2 \geqslant \sum_{|t|=n} \|x(t)\|^2 - \sum_{|t|=n-1} \|x(t)\|^2. \tag{2.3}$$

Proof. (iii) $\Leftrightarrow$ (iv). If α_k is a dissipative scattering LSDS, then according to (1.2) for any $x_0 \in \widetilde{\mathcal{X}}$ in (1.18), for an arbitrary $n \in \mathbb{Z}_+ \setminus \{0\}$ and for any finite input sequence $\{\varphi_m^- \mid 0 \leqslant m < n\} \subset \widetilde{\mathcal{N}}^-$, we have $\|\varphi_{n-1}^-\|^2 - \|\varphi_n^+\|^2 \geqslant \|x_n\|^2 - \|x_{n-1}\|^2$, where $x_n \in \widetilde{\mathcal{X}}$ is the state and $\varphi_n^+ \in \widetilde{\mathcal{N}}^+$ is the output signal of α_k at the moment n. This means that

$$\sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|\varphi_{n-1}^-(s)\|^2 - \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|\varphi_n^+(s)\|^2 \geqslant \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|x_n(s)\|^2 - \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|x_{n-1}(s)\|^2. \tag{2.4}$$

Let $\{\varphi^-(t) \mid t \in \widetilde{\mathbb{Z}}_+^N\}$ and $\{x_0(t) \mid t \in \widetilde{\mathbb{Z}}_0^N\}$ satisfy (2.1) and (2.2), the initial state of α_k be equal to $x_0 = \{x_0(s) \mid s \in \widetilde{\mathbb{Z}}_0^N\}$ ($\in \widetilde{\mathcal{X}}$ according to (2.2)), and define the input sequence of α_k by $\varphi_n^-(s) := \varphi^-(s + ne_k)$ ($s \in \widetilde{\mathbb{Z}}_0^N$) for all $n \in \mathbb{Z}_+$. By (2.1), $\varphi_n^- \in \widetilde{\mathcal{N}}^-$. Then (1.44) is valid. Since for $n \in \mathbb{Z}_+$, $x_n \in \widetilde{\mathcal{X}}$, $\varphi_n^+ \in \widetilde{\mathcal{N}}^+$ we have

$$\sum_{|t|=n} \|x(t)\|^2 = \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|x(s + ne_k)\|^2 = \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|x_n(s)\|^2 = \|x_n\|^2 < \infty,$$
$$\sum_{|t|=n} \|\varphi^+(t)\|^2 = \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|\varphi^+(s + ne_k)\|^2 = \sum_{s \in \widetilde{\mathbb{Z}}_0^N} \|\varphi_n^+(s)\|^2 = \|\varphi_n^+\|^2 < \infty,$$

and (2.4) implies (2.3). Thus, (iii) $\Rightarrow$ (iv) holds. Reversing the argument we obtain (iv) $\Rightarrow$ (iii).

(iii) $\Leftrightarrow$ (i). Define for an arbitrary separable Hilbert space $\mathcal{Y}$ the operator $F_{\mathcal{Y}}: l^2(\widetilde{\mathbb{Z}}_0^N,\mathcal{Y}) \to \mathcal{L}^2(\mathbb{T}^N,\mathcal{Y})$ by $\widehat{y}(\zeta) = (F_{\mathcal{Y}}y)(\zeta) := \sum_{t\in\widetilde{\mathbb{Z}}_0^N} y(t)\zeta^t$ (a.e. $\zeta \in \mathbb{T}^N$) where $\mathcal{L}^2(\mathbb{T}^N,\mathcal{Y})$ denotes the subspace in the space $L^2(\mathbb{T}^N,\mathcal{Y})$ of $\mathcal{Y}$-valued functions square integrable on $\mathbb{T}^N$, having null Fourier coefficients for all multiindices $t \notin \widetilde{\mathbb{Z}}_0^N$. Evidently, $F_{\mathcal{Y}}$ is a unitary operator (it is the restriction of the discrete Fourier transform mapping $l^2(\mathbb{Z}^N,\mathcal{Y})$ onto $L^2(\mathbb{T}^N,\mathcal{Y})$). If α_k is a dissipative scattering LSDS then the operator $\widetilde{G}_k$ in (1.43) is contractive. Thus, for any $\widehat{x}_0 \in \mathcal{L}^2(\mathbb{T}^N,\mathcal{X})$ and $\widehat{\varphi}_0^- \in \mathcal{L}^2(\mathbb{T}^N,\mathcal{N}^-)$, we have for almost every $\zeta \in \mathbb{T}^N$

$$
\begin{aligned}
\begin{pmatrix} \widehat{x}_1(\zeta) \\ \widehat{\varphi}_1^+(\zeta) \end{pmatrix} &= \left\{ \begin{pmatrix} F_{\mathcal{X}} & O \\ 0 & F_{\mathcal{N}^+} \end{pmatrix} \widetilde{G}_k \begin{pmatrix} F_{\mathcal{X}} & O \\ 0 & F_{\mathcal{N}^-} \end{pmatrix}^{-1} \begin{pmatrix} \widehat{x}_0 \\ \widehat{\varphi}_0^- \end{pmatrix} \right\}(\zeta) \\
&= \begin{pmatrix} \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t \sum_{j=1}^N (A_j x_0(t+e_k-e_j) + B_j\varphi_0^-(t+e_k-e_j)) \\ \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t \sum_{j=1}^N (C_j x_0(t+e_k-e_j) + D_j\varphi_0^-(t+e_k-e_j)) \end{pmatrix} \\
&= \begin{pmatrix} \zeta_k^{-1} \sum_{j=1}^N \zeta_j A_j \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t x_0(t) + \zeta_k^{-1} \sum_{j=1}^N \zeta_j B_j \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t \varphi_0^-(t) \\ \zeta_k^{-1} \sum_{j=1}^N \zeta_j C_j \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t x_0(t) + \zeta_k^{-1} \sum_{j=1}^N \zeta_j D_j \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \zeta^t \varphi_0^-(t) \end{pmatrix} \\
&= \zeta_k^{-1} \begin{pmatrix} \zeta\mathbf{A} & \zeta\mathbf{B} \\ \zeta\mathbf{C} & \zeta\mathbf{D} \end{pmatrix} \begin{pmatrix} \widehat{x}_0(\zeta) \\ \widehat{\varphi}_0^-(\zeta) \end{pmatrix} = \zeta_k^{-1}\zeta\mathbf{G} \begin{pmatrix} \widehat{x}_0(\zeta) \\ \widehat{\varphi}_0^-(\zeta) \end{pmatrix}.
\end{aligned}
$$

Since the operators $F_{\mathcal{X}}$ and $F_{\mathcal{N}^+}$ are unitary, the operator of multiplication by the block matrix-valued function "$\cdot\,\zeta_k^{-1}\zeta\mathbf{G}$" : $\mathcal{L}^2(\mathbb{T}^N,\mathcal{X}) \oplus \mathcal{L}^2(\mathbb{T}^N,\mathcal{N}^-) \to \mathcal{L}^2(\mathbb{T}^N,\mathcal{X})$ $\oplus\mathcal{L}^2(\mathbb{T}^N,\mathcal{N}^+)$ is contractive. It is easy to show that in this case, for each $\zeta \in \mathbb{T}^N$, both $\zeta_k^{-1}\zeta\mathbf{G}$ and $\zeta\mathbf{G}$ are contractive operators from $[\mathcal{X}\oplus\mathcal{N}^-, \mathcal{X}\oplus\mathcal{N}^+]$, that means the dissipativity of α, and (iii) $\Rightarrow$ (i) holds. Reversing the argument, we obtain (i) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii). As it was shown earlier, (i) $\Rightarrow$ (iii), i.e., from the dissipativity of α the dissipativity of α_k follows for an arbitrary k. This means that the generators $W_{\alpha_k} = W_{\alpha,k}$ of the associated one-parametric semigroups are contractive operators for all $k \in \{1,\dots,N\}$ (see Subsection 1.1 and Introduction). Hence $\mathfrak{W}_\alpha$ is a semigroup of contractions.

(ii) $\Rightarrow$ (iii). If $\mathfrak{W}_\alpha$ is a semigroup of contractions in $\mathcal{H}_\alpha$, then each one-parametric semigroup $\mathfrak{W}_{\alpha_k} = \{W_{\alpha,k}^n \mid n \in \mathbb{Z}_+\}$ is also contractive and hence α_k is a dissipative scattering LSDS (see Subsection 1.1).

The proof of Proposition 2.1 is now complete. ∎

REMARK 2.2. The inequalities in (2.3) are multiparametric analogues of the inequalities in (1.2) characterizing dissipative scattering LSDS in the case $N = 1$; one may attach to (2.3) also the physical notion of dissipation of energy in a system, and in this case the relations for "powers of incident and reflected waves" and the "energy" of inner states of a system are considered on "wave fronts" $|t| = \text{const}$.

THEOREM 2.3. *The transfer function $\theta_\alpha(z)$ of a dissipative scattering* LSDS $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ *is a holomorphic contractive operator-valued function on $\mathbb{D}^N$.*

Proof. Since $L_{\mathbf{G}}(z) = z\mathbf{G}$ is a contractive $[\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]$-valued function on $\mathbb{D}^N$, $L_{\mathbf{A}}(z) := z\mathbf{A} = P_{\mathcal{X}}(z\mathbf{G})|\mathcal{X}$ is a contractive $[\mathcal{X}, \mathcal{X}]$-valued function on $\mathbb{D}^N$; moreover, by virtue of the maximum principle for holomorphic operator-valued functions (see [23]), $\|z\mathbf{A}\| < 1$ for $z \in \mathbb{D}^N$. This implies that $\theta_\alpha(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B}$ is well-defined and holomorphic on $\mathbb{D}^N$. Fix an arbitrary $z^0 \in \mathbb{D}^N$. Then the one-parameter system $\alpha_{z^0} = (z^0\mathbf{A}, z^0\mathbf{B}, z^0\mathbf{C}, z^0\mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ is a dissipative scattering LSDS, and by Theorem 1.1, $\theta_{\alpha_{z^0}}(\lambda) = \lambda z^0\mathbf{D} + \lambda z^0\mathbf{C}(I_{\mathcal{X}} - \lambda z^0\mathbf{A})^{-1}\lambda z^0\mathbf{B}$ is a contractive operator-valued function on $\mathbb{D}$. The function $\theta_\alpha(z)$ is holomorphic at z^0 and hence continuous at this point. Therefore

$$\|\theta_\alpha(z^0)\| = \lim_{\lambda \in \mathbb{D},\, \lambda \to 1} \|\theta_\alpha(\lambda z^0)\| = \lim_{\lambda \in \mathbb{D},\, \lambda \to 1} \|\theta_{\alpha_{z^0}}(\lambda)\| \leqslant 1,$$

that completes the proof. ∎

2.2. MULTIPARAMETRIC CONSERVATIVE SCATTERING LSDS. We shall call $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ a *multiparametric conservative scattering LSDS* if for any $\zeta \in \mathbb{T}^N$, $\zeta\mathbf{G} \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]$ is a unitary operator. It is evident that conservative scattering LSDS is a special case of dissipative one. For $N = 1$, this notion coincides with the notion of conservative scattering LSDS with discrete time (see Introduction and Subsection 1.1).

PROPOSITION 2.4. *Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ be given, $\mathfrak{W}_\alpha$ and α_k (for some $k \in \{1, \ldots, N\}$) be the associated semigroup and the associated one-parametric* LSDS *respectively. Then the following statements are equivalent:*

(i) *α is a multiparametric conservative scattering* LSDS*;*

(ii) *$\mathfrak{W}_\alpha$ is a semigroup of unitary operators in $\mathcal{H}_\alpha$;*

(iii) *α_k is a conservative scattering* LSDS*;*

(iv) *the N-tuple of matrices of the system α*

$$G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+], \quad k = 1, \dots, N,$$

satisfies the following conditions:

$$\sum_{k=1}^{N} G_k^* G_k = I_{\mathcal{X}\oplus\mathcal{N}^-}, \tag{2.5}$$

$$G_k^* G_j = 0, \quad k \neq j, \tag{2.6}$$

$$\sum_{k=1}^{N} G_k G_k^* = I_{\mathcal{X}\oplus\mathcal{N}^+}, \tag{2.7}$$

$$G_k G_j^* = 0, \quad k \neq j; \tag{2.8}$$

(v) *the spaces $\mathcal{H}^+ := \mathcal{X} \oplus \mathcal{N}^+$ and $\mathcal{H}^- := \mathcal{X} \oplus \mathcal{N}^-$ have the decompositions*

$$\mathcal{H}^+ = \bigoplus_{k=1}^{N} \mathcal{H}_k^+, \quad \mathcal{H}^- = \bigoplus_{k=1}^{N} \mathcal{H}_k^- \tag{2.9}$$

such that with respect to them, the matrices G_k have the block structure

$$(G_k)_{ij} = \begin{cases} G_k^0 & \text{if } (i,j) = (k,k), \\ 0 & \text{if } (i,j) \neq (k,k), \end{cases} \quad k = 1, \dots, N; \tag{2.10}$$

the operator $G^0 := \sum_{k=1}^{N} G_k$ is unitary and represented by the block-diagonal matrix $G^0 = diag(G_1^0, \dots, G_N^0)$;

(vi) *in the assumptions of* (iv) *from Proposition* 2.1 *for α, the following equalities are valid for any $n \in \mathbb{Z}_+ \setminus \{0\}$:*

$$\sum_{|t|=n-1} \|\varphi^-(t)\|^2 - \sum_{|t|=n} \|\varphi^+(t)\|^2 = \sum_{|t|=n} \|x(t)\|^2 - \sum_{|t|=n-1} \|x(t)\|^2 \tag{2.11}$$

and in the same assumptions for α^*, for any $n \in \mathbb{Z}_+ \setminus \{0\}$:

$$\sum_{|t|=n-1} \|\varphi_*^-(t)\|^2 - \sum_{|t|=n} \|\varphi_*^+(t)\|^2 = \sum_{|t|=n} \|x_*(t)\|^2 - \sum_{|t|=n-1} \|x_*(t)\|^2. \tag{2.12}$$

Proof. (i) $\Leftrightarrow$ (iv). Rewrite the conservativity conditions for α as follows:

$$(\zeta\mathbf{G})^*(\zeta\mathbf{G}) = \sum_{k=1}^{N}\sum_{j=1}^{N} \bar{\zeta}_k \zeta_j G_k^* G_j = I_{\mathcal{X}\oplus\mathcal{N}^-}, \quad \forall \zeta \in \mathbb{T}^N \tag{2.13}$$

$$(\zeta\mathbf{G})(\zeta\mathbf{G})^* = \sum_{k=1}^{N}\sum_{j=1}^{N} \zeta_k \bar{\zeta}_j G_k G_j^* = I_{\mathcal{X}\oplus\mathcal{N}^+}, \quad \forall \zeta \in \mathbb{T}^N. \tag{2.14}$$

Equating the coefficients under corresponding multipowers of $\zeta \in \mathbb{T}^N$ (here $\bar{\zeta}_k\zeta_j = \zeta^{e_j-e_k}$), we obtain that (2.13) is equivalent to (2.5) and (2.6), and (2.14) is equivalent to (2.7) and (2.8).

(iv) $\Leftrightarrow$ (v). Let (iv) be fulfilled. Denote by $\overline{\mathcal{Y}}$ the closure of $\mathcal{Y}$. Define $\mathcal{H}_k^- := \overline{G_k^*\mathcal{H}^+}$, $\mathcal{H}_k^+ := \overline{G_k\mathcal{H}^-}$ $(k = 1,\dots,N)$. Then $\mathcal{H}_k^- \perp \mathcal{H}_j^-$, $\mathcal{H}_k^+ \perp \mathcal{H}_j^+$ for $k \neq j$. Indeed, if $h_1^+, h_2^+ \in \mathcal{H}^+$ then by (2.8), $\langle G_k^*h_1^+, G_j^*h_2^+\rangle = \langle h_1^+, G_kG_j^*h_2^+\rangle = 0$ for $k \neq j$, hence by the continuity argument we get $\mathcal{H}_k^- \perp \mathcal{H}_j^-$ for $k \neq j$. Analogously, $\mathcal{H}_k^+ \perp \mathcal{H}_j^+$ for $k \neq j$. As it was shown earlier, (iv) $\Leftrightarrow$ (i), hence $G^0 = \sum_{j=1}^N G_j$ is a unitary operator. This implies $\mathcal{H}^+ = \sum_{j=1}^N G_j\mathcal{H}^- \subset \sum_{j=1}^N \overline{G_j\mathcal{H}^-} = \sum_{j=1}^N \mathcal{H}_j^+ \subset \mathcal{H}^+$. Thus $\mathcal{H}^+ = \sum_{j=1}^N \mathcal{H}_j^+$; moreover $\mathcal{H}^+ = \bigoplus_{j=1}^N \mathcal{H}_j^+$. Analogously, $\mathcal{H}^- = \bigoplus_{j=1}^N \mathcal{H}_j^-$, and (2.9) is valid. Since by (2.8) for any $h^+ \in \mathcal{H}^+$ we have $G_kG_j^*h^+ = 0$ for $k \neq j$, by the continuity argument we get $G_k\mathcal{H}_j^- = \{0\}$ for $k \neq j$. Then $\mathcal{H}_k^+ = \overline{G_k\mathcal{H}^-} = \overline{G_k \bigoplus_{j=1}^N \mathcal{H}_j^-} = \overline{G_k\mathcal{H}_k^-} = \overline{\sum_{j=1}^N G_j\mathcal{H}_k^-} = \overline{G^0\mathcal{H}_k^-} = G^0\mathcal{H}_k^- = \sum_{j=1}^N G_j\mathcal{H}_k^- = G_k\mathcal{H}_k^-$. Analogously, $\mathcal{H}_k^- = G_k^*\mathcal{H}_k^+$. Thus, with respect to (2.9), the operators G_k have the structure of the block matrices (2.10) where $G_k^0 = P_{\mathcal{H}_k^+}G_k|\mathcal{H}_k^-$ $(k = 1,\dots,N)$, and (iv) $\Rightarrow$ (v) is obtained. Conversely, if (v) is fulfilled, then one can verify (2.5)–(2.8) immediately, that gives (v) $\Rightarrow$ (iv).

(iii) $\Leftrightarrow$ (vi). The proof is analogous to the proof of (iii) $\Leftrightarrow$ (iv) in Proposition 2.1, but here it is necessary to carry out the same argument twice: both for α and for α^*, and both for the corresponding equalities (2.11) and (2.12).

(iii) $\Leftrightarrow$ (i), (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) can be established similarly to the corresponding parts of Proposition 2.1. ∎

REMARK 2.5. The relations (2.11) and (2.12) are multidimensional analogues of energy equalities characterizing conservative scattering LSDS in the case $N = 1$ and also mean the conservation of energy under the direct and the inverse directions of waves propagation; as for the general case of dissipative system (see Remark 2.2), energy relations are considered here on the "wave fronts" $|t| = \text{const}$.

REMARK 2.6. It follows from Proposition 2.4 that for a conservative scattering LSDS α, the semigroup $\mathfrak{W}_\alpha = \{\mathbf{W}_\alpha^t \mid t \in \mathbb{Z}_+^N\}$ can be extended to the group $\widetilde{\mathfrak{W}}_\alpha = \{\mathbf{W}_\alpha^t \mid t \in \mathbb{Z}^N\}$ of unitary operators in $\mathcal{H}_\alpha$, and one can consider the *associated Lax-Phillips group* $\widetilde{\mathfrak{W}}_\alpha$ for α. Indeed, it is sufficient to show that $W_{\alpha,k}^{-1}W_{\alpha,j} = W_{\alpha,j}W_{\alpha,k}^{-1}$ for $k \neq j$. But these relations are equivalent to commutativity relations for $\mathfrak{W}_\alpha$: $W_{\alpha,j}W_{\alpha,k} = W_{\alpha,k}W_{\alpha,j}$ $(k \neq j)$ since they follow after the multiplication of the last ones from the right and from the left by the unitary operator $W_{\alpha,k}^{-1}$.

3. THE CLASS OF TRANSFER FUNCTIONS OF MULTIPARAMETRIC CONSERVATIVE SCATTERING LSDSs

Recall that the *generalized Schur class* $S_N(\mathcal{N}^-, \mathcal{N}^+)$ (see [2]) is the class of all holomorphic functions $\theta(z)$ on $\mathbb{D}^N$ with values in $[\mathcal{N}^-, \mathcal{N}^+]$, where $\mathcal{N}^-$ and $\mathcal{N}^+$ are separable Hilbert spaces such that for any separable Hilbert space $\mathcal{Y}$, for any N-tuple $\mathbf{T} = (T_1, \dots, T_N)$ of commuting contractions in $\mathcal{Y}$ and for any positive $r < 1$

$$\|\theta(r\mathbf{T})\| \leqslant 1, \tag{3.1}$$

$$\theta(r\mathbf{T}) = \theta(rT_1, \dots, rT_N) := \sum_{t \in \mathbb{Z}_+^N} \widehat{\theta}_t \otimes (r\mathbf{T})^t \in [\mathcal{N}^- \otimes \mathcal{Y}, \mathcal{N}^+ \otimes \mathcal{Y}]; \tag{3.2}$$

$\widehat{\theta}_t$ are the Maclaurin's coefficients of $\theta(z)$, and the convergence of series in (3.2) is understood in the sense of norm in $[\mathcal{N}^- \otimes \mathcal{Y}, \mathcal{N}^+ \otimes \mathcal{Y}]$. Since one can choose in particular $\mathcal{Y} = H^2(\mathbb{D}^N)$ (the Hardy space on $\mathbb{D}^N$) and $T_k =$"$\cdot\, z_k$" (i.e., the operator of multiplication by the k-th independent variable in $\mathcal{Y} = H^2(\mathbb{D}^N)$) for $k = 1, \dots, N$, by virtue of the isomorphism of the Hilbert spaces $\mathcal{N}^{\pm} \otimes H^2(\mathbb{D}^N)$ and $H^2(\mathbb{D}^N, \mathcal{N}^{\pm})$ (the last ones are the Hardy spaces of $\mathcal{N}^{\pm}$-valued functions on $\mathbb{D}^N$) according to (3.1), we have

$$\|\theta\|_\infty = \sup_{0<r<1} \|\text{“}\cdot\theta(rz)\text{”}\|_{[H^2(\mathbb{D}^N, \mathcal{N}^-),\, H^2(\mathbb{D}^N, \mathcal{N}^+)]} = \sup_{0<r<1} \|\theta(r\mathbf{T})\| \leqslant 1$$

(here $\|\cdot\|_\infty$ is the norm in the space $H^\infty(\mathbb{D}^N, [\mathcal{N}^-, \mathcal{N}^+])$ of bounded holomorphic $[\mathcal{N}^-, \mathcal{N}^+]$-valued functions on $\mathbb{D}^N$), thus θ turns out to be an element of the unit ball $B_N(\mathcal{N}^-, \mathcal{N}^+)$ of the Banach space $H^\infty(\mathbb{D}^N, [\mathcal{N}^-, \mathcal{N}^+])$, and we obtain

$$S_N(\mathcal{N}^-, \mathcal{N}^+) \subseteq B_N(\mathcal{N}^-, \mathcal{N}^+). \tag{3.3}$$

If $N = 1$, then due to the von Neumann inequality (see [20]), $S_N(\mathcal{N}^-, \mathcal{N}^+) = S(\mathcal{N}^-, \mathcal{N}^+)$, i.e. the Schur class. It is known that for $N = 1$ (see [20]) and for $N = 2$ (see [3]) one has in fact the sign "=" in (3.3) for any $\mathcal{N}^-$ and $\mathcal{N}^+$, i.e., the classes $S_N(\mathcal{N}^-, \mathcal{N}^+)$ and $B_N(\mathcal{N}^-, \mathcal{N}^+)$ coincide. For $N > 2$, as it follows from [27], these classes do not coincide for any $\mathcal{N}^-$ and $\mathcal{N}^+$ different from $\{0\}$. As J. Agler showed in [2], $\theta \in S_N(\mathcal{N}^-, \mathcal{N}^+)$ if and only if there exist separable Hilbert spaces $\mathcal{M}_k$ and holomorphic functions $F_k : \mathbb{D}^N \to [\mathcal{N}^-, \mathcal{M}_k]$ $(k = 1, \ldots, N)$ such that

$$I_{\mathcal{N}^-} - \theta(\lambda)^* \theta(z) = \sum_{k=1}^{N} (1 - \overline{\lambda}_k z_k) F_k(\lambda)^* F_k(z) \quad \text{for all } \lambda, z \in \mathbb{D}^N. \tag{3.4}$$

We shall also use the following result of [2].

LEMMA 3.1. *Let $\mathcal{N}, \mathcal{K}, \mathcal{L}$ be separable Hilbert spaces, $g : \mathbb{D}^N \to [\mathcal{N}, \mathcal{K}]$ and $f : \mathbb{D}^N \to [\mathcal{N}, \mathcal{L}]$ be holomorphic functions, and*

$$\mathcal{G} := \bigvee_{\lambda \in \mathbb{D}^N} g(\lambda)\mathcal{N}, \quad \mathcal{F} := \bigvee_{\lambda \in \mathbb{D}^N} f(\lambda)\mathcal{N}. \tag{3.5}$$

If

$$g(\lambda)^* g(z) = f(\lambda)^* f(z) \quad \text{for all } \lambda, z \in \mathbb{D}^N, \tag{3.6}$$

then there exists a unique isomorphism $L : \mathcal{G} \to \mathcal{F}$ such that

$$f(\lambda) = Lg(\lambda) \quad \text{for all } \lambda \in \mathbb{D}^N. \tag{3.7}$$

Denote by $S_N^0(\mathcal{N}^-, \mathcal{N}^+)$ and $B_N^0(\mathcal{N}^-, \mathcal{N}^+)$ the subclasses of $S_N(\mathcal{N}^-, \mathcal{N}^+)$ and $B_N(\mathcal{N}^-, \mathcal{N}^+)$ respectively, consisting of all operator-valued functions vanishing at $z = 0$. By (1.30) and Theorem 2.3, the transfer functions of multiparametric dissipative scattering LSDSs belong to $B_N^0(\mathcal{N}^-, \mathcal{N}^+)$, but we have not a full description of the transfer functions class for dissipative systems. For conservative systems we have obtained such description.

THEOREM 3.2. *The holomorphic function $\theta : \mathbb{D}^N \to [\mathcal{N}^-, \mathcal{N}^+]$ is the transfer function of some multiparametric conservative scattering* LSDS *α (i.e., $\theta = \theta_\alpha$) if and only if $\theta \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$.*

Proof. Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ be a multiparametric conservative scattering LSDS. Take some $\varphi_1^-, \varphi_2^- \in \mathcal{N}^-$ and define the input multisequences for α

$$\varphi_k^-(t) = \begin{cases} \varphi_k^- & \text{for } t = 0, \\ 0 & \text{for } t \in \widetilde{\mathbb{Z}}_+^N \setminus \{0\}; \end{cases} \qquad k = 1, 2.$$

Then, by (1.27), (1.32) and (1.33), the formulas

$$\widehat{\varphi}_k^-(z) \equiv \varphi_k^-, \quad \widehat{x}_k(z) = (I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B}\varphi_k^-, \quad \widehat{\varphi}_k^+(z) = \theta_\alpha(z)\varphi_k^-, \quad k = 1, 2,$$

define holomorphic functions on $\mathbb{D}^N$ (note that by Theorem 2.3, θ_α is a holomorphic operator-valued function on $\mathbb{D}^N$). For arbitrary $\lambda, z \in \mathbb{D}^N$ by (1.31)–(1.33) we get

$$\begin{aligned}
&\langle (I_{\mathcal{N}^-} - \theta_\alpha(\lambda)^*\theta_\alpha(z))\varphi_1^-, \varphi_2^-\rangle_{\mathcal{N}^-} \\
&\quad = \langle \varphi_1^-, \varphi_2^-\rangle_{\mathcal{N}^-} - \langle \widehat{\varphi}_1^+(z), \widehat{\varphi}_2^+(\lambda)\rangle_{\mathcal{N}^+} \\
&\quad = \left\langle \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^-} - \left\langle \begin{pmatrix} \widehat{x}_1(z) \\ \widehat{\varphi}_1^+(z) \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \widehat{\varphi}_2^+(\lambda) \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^+} \\
&\quad = \left\langle \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^-} - \left\langle z\mathbf{G} \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \lambda\mathbf{G} \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^+} \\
&\quad = \left\langle (I_{\mathcal{X}\oplus\mathcal{N}^-} - (\lambda\mathbf{G})^* z\mathbf{G}) \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^-}.
\end{aligned}$$

According to Proposition 2.4 (see (2.5) and (2.6)), the last one is equal to

$$\begin{aligned}
&\left\langle \left(\sum_{k=1}^N G_k^* G_k - \sum_{k=1}^N \overline{\lambda}_k z_k G_k^* G_k \right) \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^-} \\
&= \left\langle \sum_{k=1}^N (1 - \overline{\lambda}_k z_k) G_k^* G_k \begin{pmatrix} \widehat{x}_1(z) \\ \varphi_1^- \end{pmatrix}, \begin{pmatrix} \widehat{x}_2(\lambda) \\ \varphi_2^- \end{pmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{N}^-} \\
&= \left\langle \sum_{k=1}^N (1 - \overline{\lambda}_k z_k) \begin{pmatrix} (I_{\mathcal{X}} - \lambda\mathbf{A})^{-1}\lambda\mathbf{B} \\ I_{\mathcal{N}^-} \end{pmatrix}^* G_k^* G_k \begin{pmatrix} (I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} \\ I_{\mathcal{N}^-} \end{pmatrix} \varphi_1^-, \varphi_2^- \right\rangle_{\mathcal{N}^-}.
\end{aligned}$$

Since φ_1^- and φ_2^- are arbitrary elements of $\mathcal{N}^-$, we obtain (3.4) for $\theta = \theta_\alpha$ where $\mathcal{M}_k := \mathcal{X} \oplus \mathcal{N}^+$, $F_k(z) := G_k \begin{pmatrix} (I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} \\ I_{\mathcal{N}^-} \end{pmatrix}$, thus $\theta_\alpha \in S_N(\mathcal{N}^-, \mathcal{N}^+)$. Moreover, according to (1.30), $\theta(0) = 0$ and hence $\theta_\alpha \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$.

Conversely, suppose $\theta \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$. Then for any θ the relation (3.4) holds with some separable Hilbert spaces $\mathcal{M}_k$ and some holomorphic functions $F_k : \mathbb{D}^N \to [\mathcal{N}^-, \mathcal{M}_k]$ $(k = 1, \dots, N)$. Rewrite (3.4) in the following form: $\forall \lambda, z \in \mathbb{D}^N$

$$\sum_{k=1}^N (\lambda_k F_k(\lambda))^* z_k F_k(z) + I_{\mathcal{N}^-} = \sum_{k=1}^N F_k(\lambda)^* F_k(z) + \theta(\lambda)^* \theta(z). \tag{3.8}$$

Set $\mathcal{N} := \mathcal{N}^-$, $\mathcal{M} := \bigoplus_{k=1}^N \mathcal{M}_k$, $\mathcal{K} := \mathcal{M} \oplus \mathcal{N}^-$, $\mathcal{L} := \mathcal{M} \oplus \mathcal{N}^+$ and define the functions $g : \mathbb{D}^N \to [\mathcal{N}, \mathcal{K}]$ and $f : \mathbb{D}^N \to [\mathcal{N}, \mathcal{L}]$ by

$$g(\lambda) := \begin{pmatrix} \lambda_1 F_1(\lambda) \\ \vdots \\ \lambda_N F_N(\lambda) \\ I_{\mathcal{N}^-} \end{pmatrix}, \quad f(\lambda) := \begin{pmatrix} F_1(\lambda) \\ \vdots \\ F_N(\lambda) \\ \theta(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{D}^N.$$

Then, for these functions, (3.8) means (3.6). If we define the spaces $\mathcal{G}$ and $\mathcal{F}$ by (3.5), then we arrive at the conditions of Lemma 3.1, and therefore there is a unique unitary operator $L : \mathcal{G} \to \mathcal{F}$ satisfying (3.7), i.e.,

$$L \begin{pmatrix} \lambda_1 F_1(\lambda) \\ \vdots \\ \lambda_N F_N(\lambda) \\ I_{\mathcal{N}^-} \end{pmatrix} = \begin{pmatrix} F_1(\lambda) \\ \vdots \\ F_N(\lambda) \\ \theta(\lambda) \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{D}^N. \tag{3.9}$$

Denote by $P_k := P_{\mathcal{M}_k}$ the orthogonal projection in $\mathcal{M}$ onto $\mathcal{M}_k$, and set $\mathbf{P} := (P_1, \ldots, P_N)$,

$$F(\lambda) := \begin{pmatrix} F_1(\lambda) \\ \vdots \\ F_N(\lambda) \end{pmatrix} \in [\mathcal{N}^-, \mathcal{M}], \quad \lambda \in \mathbb{D}^N.$$

Then (3.9) implies

$$L \begin{pmatrix} \lambda \mathbf{P} F(\lambda) \\ I_{\mathcal{N}^-} \end{pmatrix} = \begin{pmatrix} F(\lambda) \\ \theta(\lambda) \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{D}^N. \tag{3.10}$$

If we substitute $\lambda = 0$ into this equality, we get

$$L \begin{pmatrix} 0 \\ \varphi^- \end{pmatrix} = \begin{pmatrix} F(0)\varphi^- \\ 0 \end{pmatrix}, \quad \forall \varphi^- \in \mathcal{N}^-. \tag{3.11}$$

From (3.10) and (3.11) we conclude that

$$\begin{pmatrix} \lambda \mathbf{P} F(\lambda)\varphi^- \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{P} F(\lambda) \\ I_{\mathcal{N}^-} \end{pmatrix} \varphi^- - \begin{pmatrix} 0 \\ I_{\mathcal{N}^-} \end{pmatrix} \varphi^- \in \mathcal{G},$$

for any $\varphi^- \in \mathcal{N}^-$ and for any $\lambda \in \mathbb{D}^N$, and

$$L \begin{pmatrix} \lambda \mathbf{P} F(\lambda)\varphi^- \\ 0 \end{pmatrix} = \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda) \end{pmatrix} \varphi^-. \tag{3.12}$$

Since for any $\varphi_1^-, \varphi_2^- \in \mathcal{N}^-$ and $\lambda \in \mathbb{D}^N$ the vectors $\begin{pmatrix} 0 \\ \varphi_1^- \end{pmatrix}$ and $\begin{pmatrix} \lambda \mathbf{P} F(\lambda)\varphi_2^- \\ 0 \end{pmatrix}$ are orthogonal in $\mathcal{G}$ and L is a unitary operator, by (3.11) and (3.12), $\begin{pmatrix} F(0)\varphi_1^- \\ 0 \end{pmatrix} \perp \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda) \end{pmatrix} \varphi_2^-$ in $\mathcal{F}$, and hence in $\mathcal{M} \oplus \mathcal{N}^+$. Therefore $F(0)\varphi_1^- \perp (F(\lambda) - F(0))\varphi_2^-$ in $\mathcal{M}$. Thus

$$\bigvee_{\lambda \in \mathbb{D}^N} F(\lambda)\mathcal{N}^- = \left(\bigvee_{\lambda \in \mathbb{D}^N} (F(\lambda) - F(0))\mathcal{N}^- \right) \oplus F(0)\mathcal{N}^-. \tag{3.13}$$

The second summand is a closed lineal in $\mathcal{M}$ because $F(0)$ is an isometry (this follows from (3.4) for $\lambda = z = 0$). Set $U := L\Big|\bigvee_{\lambda\in\mathbb{D}^N} \lambda\mathbf{P}F(\lambda)\mathcal{N}^-$. Then, by (3.12), U is a unitary operator from $\bigvee_{\lambda\in\mathbb{D}^N} \lambda\mathbf{P}F(\lambda)\mathcal{N}^-$ onto $\bigvee_{\lambda\in\mathbb{D}^N} \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda)\end{pmatrix}\mathcal{N}^-$, and

$$U(\lambda\mathbf{P})F(\lambda) = \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda)\end{pmatrix} \quad \text{for all } \lambda \in \mathbb{D}^N. \tag{3.14}$$

We have the inclusions

$$\bigvee_{\lambda\in\mathbb{D}^N} \lambda\mathbf{P}F(\lambda)\mathcal{N}^- \subset \mathcal{M}$$

and

$$\bigvee_{\lambda\in\mathbb{D}^N} \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda)\end{pmatrix}\mathcal{N}^- \subset (\mathcal{M}\ominus F(0)\mathcal{N}^-)\oplus\mathcal{N}^+.$$

Enlarging if necessary $\mathcal{M}$ (for instance, by addition of an infinite countable number to its dimension) we can achieve

$$\begin{aligned}\dim&\Big(\mathcal{M}\ominus \bigvee_{\lambda\in\mathbb{D}^N} \lambda\mathbf{P}F(\lambda)\mathcal{N}^-\Big)\\ &= \dim\Big(((\mathcal{M}\ominus F(0)\mathcal{N}^-)\oplus\mathcal{N}^+)\ominus \bigvee_{\lambda\in\mathbb{D}^N} \begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda)\end{pmatrix}\mathcal{N}^-\Big)\end{aligned}$$

and extend U to a unitary operator $\widetilde{U} : \mathcal{M} \to (\mathcal{M}\ominus F(0)\mathcal{N}^-)\oplus\mathcal{N}^+$. Set $\mathcal{X} := \mathcal{M}\ominus F(0)\mathcal{N}^-$. Then $\widetilde{U}$ maps $\mathcal{X}\oplus F(0)\mathcal{N}^-$ isometrically onto $\mathcal{X}\oplus\mathcal{N}^+$, and $\widetilde{U}\begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & F(0)\end{pmatrix}$ maps $\mathcal{X}\oplus\mathcal{N}^-$ isometrically onto $\mathcal{X}\oplus\mathcal{N}^+$. Set

$$G_k := \widetilde{U}P_k\begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & F(0)\end{pmatrix}, \quad k = 1,\ldots,N.$$

As operators from $\mathcal{X}\oplus\mathcal{N}^-$ into $\mathcal{X}\oplus\mathcal{N}^+$, they have a block form

$$G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k\end{pmatrix}, \quad k = 1,\ldots,N,$$

and define the conservative scattering LSDS $\alpha = (N;\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D};\mathcal{X},\mathcal{N}^-,\mathcal{N}^+)$. Indeed, for any $\zeta\in\mathbb{T}^N$, $\zeta\mathbf{G} = \widetilde{U}(\zeta\mathbf{P})\begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & F(0)\end{pmatrix}$ is a unitary operator from $\mathcal{X}\oplus\mathcal{N}^-$ onto $\mathcal{X}\oplus\mathcal{N}^+$ because $\begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & F(0)\end{pmatrix} \in [\mathcal{X}\oplus\mathcal{N}^-,\mathcal{M}]$, $\zeta\mathbf{P}\in[\mathcal{M},\mathcal{M}]$ and

$\widetilde{U} \in [\mathcal{M}, \mathcal{X} \oplus \mathcal{N}^+]$ are unitary operators. By virtue of (3.13) and (3.14) for an arbitrary $\lambda \in \mathbb{D}^N$,

$$\begin{aligned}\begin{pmatrix} F(\lambda) - F(0) \\ \theta(\lambda) \end{pmatrix} &= U(\lambda\mathbf{P})F(\lambda) = \widetilde{U}(\lambda\mathbf{P})F(\lambda) \\ &= \widetilde{U}(\lambda\mathbf{P}) \begin{pmatrix} F(\lambda) - F(0) \\ F(0) \end{pmatrix} \\ &= \widetilde{U}(\lambda\mathbf{P}) \begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & F(0) \end{pmatrix} \begin{pmatrix} F(\lambda) - F(0) \\ I_{\mathcal{N}^-} \end{pmatrix} \\ &= \lambda\mathbf{G} \begin{pmatrix} F(\lambda) - F(0) \\ I_{\mathcal{N}^-} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{A}(F(\lambda)) - F(0)) + \lambda\mathbf{B} \\ \lambda\mathbf{C}(F(\lambda)) - F(0)) + \lambda\mathbf{D} \end{pmatrix}.\end{aligned}$$

This implies $F(\lambda) - F(0) = (I_{\mathcal{X}} - \lambda\mathbf{A})^{-1}\lambda\mathbf{B}$ and $\theta(\lambda) = \lambda\mathbf{C}(I_{\mathcal{X}} - \lambda\mathbf{A})^{-1}\lambda\mathbf{B} + \lambda\mathbf{D}$, i.e., $\theta = \theta_\alpha$, which completes the proof. ∎

Consider for $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ the following subspace

$$\mathcal{X}_1 := \bigvee_{p,k,j} p(\mathbf{A}, \mathbf{A}^*)(B_k\mathcal{N}^- + C_j^*\mathcal{N}^+) \tag{3.15}$$

of $\mathcal{X}$, where p runs the set of all monomials in $2N$ non-commuting variables, k and j run the set $\{1, \dots, N\}$ (cf. (0.6)). We shall call the *LSDS α closely connected* if $\mathcal{X}_1 = \mathcal{X}$.

Theorem 3.3. *Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ be an arbitrary multiparametric* LSDS*. Then:*

(i) *we have the decomposition $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$, where $\mathcal{X}_1$ is defined by* (3.15)*; with respect to this decomposition*

$$\begin{aligned} A_k &= \begin{pmatrix} A_k^{(0)} & 0 \\ 0 & A_k^{(1)} \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 \\ B_k^{(1)} \end{pmatrix}, \\ C_k &= (\, 0 \quad C_k^{(1)}\,), \qquad D_k = D_k^{(1)}, \end{aligned} \qquad k = 1, \dots, N \tag{3.16}$$

and $\alpha_1 := (N; \mathbf{A}^{(1)}, \mathbf{B}^{(1)}, \mathbf{C}^{(1)}, \mathbf{D}^{(1)}; \mathcal{X}_1, \mathcal{N}^-, \mathcal{N}^+)$ is a closely connected multiparametric LSDS*;*

(ii) *if α is a dissipative (respective conservative) scattering* LSDS *then α_1 is also a dissipative (respective conservative) scattering* LSDS*;*

(iii) $\theta_{\alpha_1} = \theta_\alpha$.

Proof. It is clear from (3.15) that $\mathcal{X}_1$ is the minimal subspace in $\mathcal{X}$ containing $B_k\mathcal{N}^-$ and $C_k^*\mathcal{N}^+$, and reducing A_k for all $k \in \{1, \dots, N\}$. This implies (3.16), and (i) follows. If α is a dissipative system, then for any $\zeta \in \mathbb{T}^N$

$$\zeta\mathbf{G}^{(1)} = \begin{pmatrix} \zeta\mathbf{A}^{(1)} & \zeta\mathbf{B}^{(1)} \\ \zeta\mathbf{C}^{(1)} & \zeta\mathbf{D}^{(1)} \end{pmatrix} = P_{\mathcal{X}_1 \oplus \mathcal{N}^+}(\zeta\mathbf{G})|\mathcal{X}_1 \oplus \mathcal{N}^-$$

is a contractive operator, thus α_1 is also dissipative. If α is a conservative system, then for any $\zeta \in \mathbb{T}^N$, $\zeta\mathbf{G} \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]$ is a unitary operator, and we get from (3.16):

$$\zeta\mathbf{G} = \begin{pmatrix} \zeta\mathbf{A}^{(0)} & 0 \\ 0 & \zeta\mathbf{G}^{(1)} \end{pmatrix} \in [\mathcal{X}_0 \oplus (\mathcal{X}_1 \oplus \mathcal{N}^-), \mathcal{X}_0 \oplus (\mathcal{X}_1 \oplus \mathcal{N}^+)]. \tag{3.17}$$

This implies that $\zeta\mathbf{G}^{(1)}$ is a unitary operator for any $\zeta \in \mathbb{T}^N$, thus α_1 is a conservative system, and the statement (ii) follows. Further, for $z \in \mathbb{D}^N$, $(z\mathbf{C}^{(1)})(z\mathbf{B}^{(1)}) = (z\mathbf{C})P_{\mathcal{X}_1}(z\mathbf{B}) = (z\mathbf{C})(z\mathbf{B})$ because $z\mathbf{B}\mathcal{N}^- \subset \mathcal{X}_1$. For $z \in \mathbb{D}^N$ and for $n \in \mathbb{Z}_+$, $z\mathbf{C}^{(1)}(z\mathbf{A}^{(1)})^n z\mathbf{B}^{(1)} = z\mathbf{C}P_{\mathcal{X}_1}(z\mathbf{A}P_{\mathcal{X}_1})^n z\mathbf{B} = z\mathbf{C}(z\mathbf{A})^n z\mathbf{B}$ because $z\mathbf{B}\mathcal{N}^- \subset \mathcal{X}_1$ and $z\mathbf{A}\mathcal{X}_1 \subset \mathcal{X}_1$. Therefore $\theta_{\alpha_1}(z) = z\mathbf{D}^{(1)} + z\mathbf{C}^{(1)}(I_{\mathcal{X}} - z\mathbf{A}^{(1)})^{-1} z\mathbf{B}^{(1)} = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B} = \theta_\alpha(z)$. The proof is complete. ∎

REMARK 3.4. If α is conservative, then due to (3.17) the restriction of the linear pencil $L_{\mathbf{A}}(\zeta) = \zeta\mathbf{A} \in [\mathcal{X}, \mathcal{X}]$ $(\zeta \in \mathbb{T}^N)$ to $\mathcal{X}_0 \subset \mathcal{X}$, that is the pencil $L_{\mathbf{A}^{(0)}}(\zeta) = \zeta\mathbf{A}^{(0)} \in [\mathcal{X}_0, \mathcal{X}_0]$ $(\zeta \in \mathbb{T}^N)$, consists of unitary operators.

We shall call the linear pencil $L_{\mathbf{A}}(\zeta)$ $(\zeta \in \mathbb{T}^N)$ of contractive operators in a Hilbert space $\mathcal{X}$ *completely non-unitary* if there is no any proper subspace $\mathcal{X}_0$ in $\mathcal{X}$ reducing $L_{\mathbf{A}}(\zeta)$ for each $\zeta \in \mathbb{T}^N$ (or, equivalently, reducing all A_k for $k = 1, \dots, N$) and such that the pencil $L_{\mathbf{A}^{(0)}}(\zeta) = L_{\mathbf{A}}(\zeta)|\mathcal{X}_0$ $(\zeta \in \mathbb{T}^N)$ is unitary (i.e. consisting of unitary operators).

THEOREM 3.5. *The conservative scattering* LSDS $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ *is closely connected if and only if the pencil* $L_{\mathbf{A}}(\zeta)$ $(\zeta \in \mathbb{T}^N)$ *is completely non-unitary.*

Proof. Suppose that α is closely connected and there is a proper subspace $\mathcal{X}_0 \subset \mathcal{X}$ reducing $L_{\mathbf{A}}(\zeta)$ for all $\zeta \in \mathbb{T}^N$ so that $L_{\mathbf{A}^{(0)}}(\zeta) = L_{\mathbf{A}}(\zeta)|\mathcal{X}_0$ $(\zeta \in \mathbb{T}^N)$ is a unitary pencil of operators. Then $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_0^\perp$ and, with respect to this decomposition, (3.16) and (3.17) take place. But in this case $\mathcal{X}_0^\perp = \mathcal{X}_1 \neq \mathcal{X}$ where $\mathcal{X}_1$ is defined by (3.15). So we get the contradiction with the supposition on the close connectedness of α. Thus $L_{\mathbf{A}}(\zeta)$ $(\zeta \in \mathbb{T}^N)$ is a completely non-unitary pencil of operators.

Conversely, let $L_{\mathbf{A}}(\zeta)$ $(\zeta \in \mathbb{T}^N)$ be a completely non-unitary pencil of operators. Then, by Remark 3.4, in the decomposition $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$ we have that $\mathcal{X}_0 = \{0\}$ is necessary, i.e., $\mathcal{X}_1 = \mathcal{X}$ and α is closely connected. ∎

Theorem 3.2 together with Theorem 3.3 imply the following multidimensional analogue of Theorem 1.2.

THEOREM 3.6. *An arbitrary operator-valued function* $\theta \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$ *has a closely connected conservative realization, i.e., there is a multiparametric closely connected conservative scattering* LSDS α *such that* $\theta = \theta_\alpha$.

REMARK 3.7. Theorem 3.6 is only a partial generalization of Theorem 1.2 to the case $N > 1$, since Theorem 1.2 claims also uniqueness of a closely connected conservative realization for $\theta \in S^0(\mathcal{N}^-, \mathcal{N}^+)$ up to unitary similarity; that means, in particular, that the state spaces of two such realizations are isomorphic.

The following example shows that for $N > 1$ there are closely connected conservative realizations of a $\theta \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$ with non-isomorphic state spaces.

EXAMPLE 3.8. Define $\theta(z) = z_1 z_2$ for $z \in \mathbb{D}^2$. Evidently, $\theta(z)$ is a scalar contractive holomorphic function on $\mathbb{D}^2$ and $\theta(0) = 0$, i.e., $\theta \in B_2^0(\mathbb{C}, \mathbb{C}) = S_2^0(\mathbb{C}, \mathbb{C})$. Define $\alpha := (2; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathbb{C}, \mathbb{C}, \mathbb{C})$ where $A_1 = A_2 = 0$, $B_1 = 0$, $B_2 = 1$, $C_1 = 1$, $C_2 = 0$, $D_1 = D_2 = 0$, i.e., for $z = (z_1, z_2) \in \mathbb{C}^2$

$$z\mathbf{G} = \begin{pmatrix} z\mathbf{A} & z\mathbf{B} \\ z\mathbf{C} & z\mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 & z_2 \\ z_1 & 0 \end{pmatrix}.$$

It is clear that if $z \in \mathbb{T}^2$ then $z\mathbf{G}$ is a unitary operator in $\mathbb{C}^2$, i.e., α is a conservative system. Since $B_2\mathcal{N}^- = \mathbb{C}$, from the relations $\mathbb{C} = \mathcal{X} \supset \mathcal{X}_1 \supset B_2\mathcal{N}^- = \mathbb{C}$ we obtain the close connectedness of α. Finally, it is clear that $\theta_\alpha(z) = \theta(z) = z_1 z_2$.

Now define $\alpha' = (2; \mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'; \mathbb{C}^3, \mathbb{C}, \mathbb{C})$ where

$$A_1' = \begin{pmatrix} 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}, \quad B_1' = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix},$$

$$C_1' = (0 \quad 1/\sqrt{2} \quad 0), \qquad D_1' = 0,$$

$$A_2' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 \end{pmatrix}, \quad B_2' = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix},$$

$$C_2' = (1/\sqrt{2} \quad 0 \quad 0), \qquad D_2' = 0,$$

i.e., for $z = (z_1,\ z_2) \in \mathbb{C}^2$

$$z\mathbf{G}' = \begin{pmatrix} z\mathbf{A}' & z\mathbf{B}' \\ z\mathbf{C}' & z\mathbf{D}' \end{pmatrix} = \left(\begin{array}{ccc|c} 0 & 0 & -z_1/\sqrt{2} & z_1/\sqrt{2} \\ 0 & 0 & z_2/\sqrt{2} & z_2/\sqrt{2} \\ -z_2/\sqrt{2} & z_1/\sqrt{2} & 0 & 0 \\ \hline z_2/\sqrt{2} & z_1/\sqrt{2} & 0 & 0 \end{array}\right).$$

Evidently, if $z \in \mathbb{T}^2$, then $z\mathbf{G}'$ is a unitary operator in $\mathbb{C}^4$, i.e., α' is a conservative system. Since $B_1'\mathcal{N}^- = \mathbb{C} \oplus \{0\} \oplus \{0\} \subset \mathbb{C}^3$, $B_2'\mathcal{N}^- = \{0\} \oplus \mathbb{C} \oplus \{0\} \subset \mathbb{C}^3$, $A_2'B_1'\mathcal{N}^- = \{0\} \oplus \{0\} \oplus \mathbb{C} \subset \mathbb{C}^3$, from the relations $\mathbb{C}^3 = \mathcal{X} \supset \mathcal{X}_1 \supset B_1'\mathcal{N}^- + B_2'\mathcal{N}^- + A_2'B_1'\mathcal{N}^- = \mathbb{C}^3$ we obtain the close connectedness of α'. Since for $z \in \mathbb{C}^2$ we have $(z\mathbf{C}')(z\mathbf{B}') = z_1z_2$, and for any $z \in \mathbb{C}^2$ and $n \in \mathbb{Z}_+ \setminus \{0\}$ we have $(z\mathbf{A}')^n z\mathbf{B}' = 0$, we obtain $\theta_{\alpha'}(z) = z\mathbf{D}' + z\mathbf{C}'(I_{\mathbb{C}^3} - z\mathbf{A}')^{-1}z\mathbf{B}' = (z\mathbf{C}')(z\mathbf{B}') = z_1z_2 = \theta(z)$. Thus both α and α' are closely connected conservative realizations of $\theta(z)$; however, $1 = \dim \mathcal{X} \neq \dim \mathcal{X}' = 3$.

REMARK 3.9. The lack of the uniqueness of a closely connected conservative realization for a given function from the generalized Schur class is not a specific character of our approach: a similar phenomenon appears in [8].

REMARK 3.10. The closely connected conservative realization $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ of $\theta \in S_N^0(\mathcal{N}^-, \mathcal{N}^+)$ is a minimal conservative realization in the following sense: if $\mathcal{X}'$ is some proper subspace in $\mathcal{X}$, $A_k' = P_{\mathcal{X}'}A_k|\mathcal{X}'$, $B_k' = P_{\mathcal{X}'}B_k$, $C_k' = C_k|\mathcal{X}'$, $D_k' = D_k$ for $k = 1, \dots, N$ then the system $\alpha' = (N; \mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'; \mathcal{X}', \mathcal{N}^-, \mathcal{N}^+)$ can not be conservative, not to mention a conservative realization of θ. Indeed, in the opposite case the following representation takes place:

$$\zeta\mathbf{G} = \begin{pmatrix} \zeta\mathbf{A}^{(0)} & 0 \\ 0 & \zeta\mathbf{G}' \end{pmatrix}, \quad \zeta \in \mathbb{T}^N,$$

where $\zeta\mathbf{A}^{(0)} \in [\mathcal{X} \ominus \mathcal{X}', \mathcal{X} \ominus \mathcal{X}']$ and $\zeta\mathbf{G}' \in [\mathcal{X}' \oplus \mathcal{N}^-, \mathcal{X}' \oplus \mathcal{N}^+]$ $(\zeta \in \mathbb{T}^N)$ are linear pencils of unitary operators, and we get $\mathcal{X}_1 \subset \mathcal{X}' \neq \mathcal{X}$ (see (3.15)) that contradicts the close connectedness of α.

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