THE CLOSURE OF THE UNITARY ORBIT OF THE SET OF STRONGLY IRREDUCIBLE OPERATORS IN NON-WELL ORDERED NEST ALGEBRA

YOU QING JI, CHUN LAN JIANG and ZONG YAO WANG

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Abstract. A bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \) is strongly irreducible if \( T \) does not commute with any non-trivial idempotent. A nest \( \mathcal{N} \) is a chain of subspaces of \( \mathcal{H} \) containing \{0\} and \( \mathcal{H} \), which is closed under intersection and closed span. The nest algebra \( \text{alg}\mathcal{N} \) associated with \( \mathcal{N} \) is the set of all operators which leave each subspace in \( \mathcal{N} \) invariant. This paper proves that the norm closure of the unitary orbit of the strongly irreducible operators in a nest algebra is the set of operators whose spectrum is connected if and only if \( \mathcal{N} \) or \( \mathcal{N}^\perp \) are not well-ordered.

Keywords: Strongly irreducible operator, nest, nest algebra, unitary orbit, spectrum.

MSC (2000): 47A, 47B, 47C.

1. INTRODUCTION

Let \( \mathcal{H} \) be a complex, separable, infinite dimensional Hilbert space. \( \mathcal{L}(\mathcal{H}) \) denotes the algebra of all bounded linear operators acting on \( \mathcal{H} \). An operator \( T \) on \( \mathcal{H} \) is called strongly irreducible, or briefly, \( T \in (SI) \), if \( T \) does not commute with any nontrivial idempotent. A nest \( \mathcal{N} \) is a chain of subspaces of \( \mathcal{H} \) containing \{0\} and \( \mathcal{H} \), which is closed under intersection and closed span. It is well known that for a nest \( \mathcal{N} \) there is a spectral measure \( E(t) \) on \([0, 1]\), such that \( \mathcal{N} = \{E([0, t])\mathcal{H}; t \in [0, 1]\} \) and the compact subset \( \text{supp}E \) of \([0, 1]\) is order-isomorphic to and topologically homeomorphic to \( \mathcal{N} \) when \( \mathcal{N} \) is given the order topology and \( \text{supp}E \) has the order and the related topology induced on it by the usual topology of the real line. In what follows we will denote \( M_{[c,d]} = E([c,d])\mathcal{H} \) when \([c,d] \subset [0, 1]\) and \( M_t = M_{[0,t]} \). For each \( M \in \mathcal{N} \), let \( M_- = \bigcup\{M' \in \mathcal{N}: M' \subseteq M\} \). If \( M_- \neq M \), \( M \odot M' \) is called an atom of \( \mathcal{N} \) and the cardinal number \( \dim M \odot M_- \) is called the dimension of the atom. A nest is called continuous if it has no atoms. The nest algebra \( \text{alg}\mathcal{N} \)
associated with $N$ is the family of operators defined by $\text{alg} N = \{ T \in \mathcal{L}(\mathcal{H}) : TM \subseteq M \text{ for all } M \in N \}$.

D.A. Herrero proved the following theorem ([7]):

**Theorem H.** (i) If $N$ is well ordered with finite dimensional atoms, then $U(\text{alg} N)^{-} = (\text{QT})$.

(ii) If $N \perp$ is well ordered with finite dimensional atoms, then $U(\text{alg} N)^{-} = (\text{QT})^*$.

(iii) If neither (i) nor (ii) holds, then $U(\text{alg} N)^{-} = \mathcal{L}(\mathcal{H})$ when $d = \infty$, $U(\text{alg} N)^{-} = \mathcal{L}(\mathcal{H})_d$ when $d < \infty$,

where $U(\text{alg} N)^{-}$ is the norm closure of the unitary orbit $U(\text{alg} N)$ of $\text{alg} N$, $(\text{QT})$ is the set of quasitriangular operators on $\mathcal{H}$, $(\text{QT})^* := \{ T \in \mathcal{L}(\mathcal{H}) : T^* \in (\text{QT}) \}$, $d = \sum_{A \in \Lambda} \dim A$, $\Lambda$ denotes the set of atoms of $N$;

$$
\mathcal{L}(\mathcal{H})_d = \left\{ T \in \mathcal{L}(\mathcal{H}) : \sum_{\lambda \in \sigma_{\circ}(T) \wedge \sigma_{\circ}(T)^{\ast}} \dim \mathcal{H}(\lambda, T) \leq d \right\},
$$

$\sigma_{\circ}(T)$ is the set of normal eigenvalues of $T$, $\sigma_{\circ}(T)^{\ast}$ is the polynormally convex hull of the essential spectrum $\sigma_{\circ}(T)$ of $T$ and $\mathcal{H}(\lambda, T)$ is the Riesz spectral subspace of $T$ associated with $\lambda$.

In [12], the authors of this paper proved that each nest algebra contains strongly irreducible operators, i.e., $\text{alg} N \cap (\text{SI}) \neq \emptyset$. Furthermore, the authors proved that $U(\text{alg} N \cap (\text{SI}))^{-} = (\text{QT})_{\circ}$ if $N$ is a well ordered nest, where $(\text{QT})_{\circ} := \{ T \in (\text{QT}) : \sigma(T) \text{ and the Weyl spectrum, } \sigma_{W}(T) \text{ of } T \text{ are connected} \}$ (see [13]) and $U(\text{alg} N \cap (\text{SI}))^{-} = \{ T \in \mathcal{L}(\mathcal{H}) : \sigma(T) \text{ is connected} \}$ if $N$ is a continuous nest [14]. The following is the main result of this paper.

**Theorem 1.1.** Let $N$ be a maximal nest. Then $U(\text{alg} N \cap (\text{SI}))^{-} = \{ T \in \mathcal{L}(\mathcal{H}) : \sigma(T) \text{ is connected} \}$ if and only if $N$ and $N \perp$ are not well-ordered.

2. Preparation

**Lemma 2.1.** ([11], Lemma 2) Let $A, B \in \mathcal{L}(\mathcal{H})$. Assume that

$$
\mathcal{H} = \bigvee \{ \ker(\lambda - B)^{k} : \lambda \in \Gamma, k \geq 1 \}
$$

for a certain subset $\Gamma$ of the point spectrum $\sigma_{p}(B)$ of $B$, and $\sigma_{p}(A) \cap \Gamma = \emptyset$; then $\tau_{AB}$ is injective.

**Lemma 2.2.** Let $\sigma$ be the closure of a connected Cauchy domain and $\Omega$ is an open disc in $\sigma$. Then there exists an operator $A \in \mathcal{L}(\mathcal{H}) \cap (\text{SI})$ such that:

(i) $\sigma(A) = \sigma_{\text{re}}(A) = \sigma$;

(ii) $\sigma_{p}(A) = \Omega$, $\text{mul} (A - \lambda) = 1(\lambda \in \Omega)$, and $\sigma_{p}(A^{\ast}) = \emptyset$;

(iii) If $\{ \lambda_{k} \}_{k=1}^{\infty} \subseteq \Omega$, pairwise distinct and $\lim_{k \to \infty} \lambda_{k} = \lambda_{0} \in \Omega$, then $\bigvee \{ \ker(A - \lambda_{k}) : k \geq 1 \} = \mathcal{H}$;
(iv) \( \| (A - \lambda)^{-1}\| \leq 2/\dist(\lambda, \sigma) \) for \( \lambda \notin \sigma \).

**Proof.** Without loss of generality we may assume that \( \Omega \) is the unit disc. Let \( S \) be the backward lateral shift, i.e., \( S^* = T^*_2 \in \mathcal{L}(\mathcal{H}_1) \), where \( \mathcal{H}_1 \) is the Hardy space \( H^2 \). Let \( M \) be a diagonal operator on \( \mathcal{H}_1 \) with \( \sigma(M) = \sigma_{re}(M) = \sigma \).

Set \( T = S^* \oplus M \). By a result of J. Agler, E. Franks and D.A. Herrero ([1]), for each \( \varepsilon > 0 \), there is a compact operator \( K, \| K \| < \varepsilon \), such that \( A - T + K \) is quasisimilar to \( T^*_2 \in \mathcal{B}_1(\Omega) \). By a result of C.L. Jiang ([15]), \( A \in (\text{SI}) \). Choose \( \varepsilon \) small enough, then \( A \) satisfies (i)–(iv).

**THEOREM 2.3.** ([9], Theorem 3.53) Let \( A, B \in \mathcal{L}(\mathcal{H}) \), then the following are equivalent for \( \tau_{AB} \):

(i) \( \tau_{AB} \) is surjective;
(ii) \( \sigma_t(A) \cap \sigma_t(B) = \emptyset \);
(iii) \( \text{ran} \tau_{AB} \) contains the set of finite rank operators;
(iv) \( \tau_{AB}|J \) is surjective for every norm ideal \( J \);

where \( \tau_{AB} \in \mathcal{L}(\mathcal{L}(\mathcal{H})) \) is given by \( \tau_{AB}(X) = AX - XB \) for \( X \in \mathcal{L}(\mathcal{H}) \).

**LEMMA 2.4.** Let \( \sigma \) be the closure of a connected Cauchy domain and \( \Omega \) be a connected open subset of \( \sigma \). Then there exists an operator \( W \in \mathcal{L}(\mathcal{H}) \cap (\text{SI}) \) satisfying:

(i) \( \sigma(W) = \sigma_{re}(W) = \sigma \);
(ii) \( \sigma_p(W) \subseteq \Omega \), \( \sigma_p(W^*) = \emptyset \);
(iii) There exists \( \{\lambda_k\}_{k=1}^\infty \subseteq \Omega \) such that \( \lim_{k \to \infty} \lambda_k = \lambda_0 \in \Omega \), \( \text{mul}(W - \lambda_k) = \infty \) \((k \geq 1)\) and \( \sqrt{\ker(W - \lambda_k)} : k \geq 1 = \mathcal{H} \).

**Proof.** Choose a sequence \( \{D_n\}_{n=0}^\infty \) of open discs in \( \Omega \) satisfying \( D_n \setminus \overline{D}_m \neq \emptyset \) \((n \neq m, n \neq \emptyset) \) and \( D_0 \subseteq \bigcap_{n=1}^\infty D_n \).

Without loss of generality we may assume that \( D_0 \) is the unit disc and \( D_1 = \alpha_1 + rD_0 \). Let \( S^* = T^*_2 \in \mathcal{L}(\mathcal{H}_1) \), where \( \mathcal{H}_1 = H^2 \). Set \( A_1 = \alpha_1 + rS^* \). Let \( \mathcal{H} = \bigoplus_{n=1}^\infty \mathcal{H}_n \), where \( \mathcal{H}_n = \mathcal{H}_1 \) \((n \geq 2) \). For each \( n \geq 2 \), by Lemma 2.2, we can construct \( A_n \in \mathcal{L}(\mathcal{H}_n) \cap (\text{SI}) \) satisfying:

(a) \( \sigma(A_n) = \sigma_{re}(A_n) = \sigma \), \( \sigma_p(A_n) = D_n \), \( \sigma_p(A_n^*) = \emptyset \) and \( \text{mul}(A_n - \lambda) = 1 \) for \( \lambda \in D_n \);
(b) If \( \{\mu_k\}_{k=1}^\infty \subseteq D_n \), pairwise distinct and \( \lim_{k \to \infty} \mu_k = \mu_0 \in D_n \), then \( \sqrt{\ker(A_n - \mu_k)} : k \geq 1 \) = \( \mathcal{H}_n \);
(c) \( \|(A_n - \lambda)^{-1}\| \leq \frac{2}{\dist(\lambda, \sigma)} \) for \( \lambda \notin \sigma \).

It follows from \( D_n \setminus \overline{D}_m \neq \emptyset \), (b) and Lemma 2.1 that \( \ker A_{n}A_{n} = \{0\} \) \((n \neq m) \). Since \( \sigma_t(A_1) \cap \sigma_t(A_n) \neq \emptyset \), by Theorem 2.3, we can find a compact operator \( W \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_1) \), \( \|W\| < 2^{-n} \), such that \( W_n \notin \text{ran} \tau_{A_1A_n} \) \((n \geq 2) \).

Define

\[
W = \begin{bmatrix}
A_1 & W_2 & W_3 & \cdots \\
A_2 & 0 & & \\
& A_3 & & \\
& & \ddots & \\
0 & & & \\
\end{bmatrix} \in \mathcal{L}(\mathcal{H}).
\]
Let \( P \in \mathcal{A}'(W) \) be an idempotent and consider the representation
\[
P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
\]

Since \( PW = WP \), then \( A_l P_{21} = P_{21} A_l \). Moreover, \( \ker \tau_{A_l A_1} = \{0\} \) implies that \( P_{21} = 0 \). Similarly, \( P_{1k} = 0 \) (\( k > 1 \)). Thus \( P_{2l} A_l = A_l P_{2l} \) and \( P_{2l}^2 = P_{2l} \) (\( l = 1, 2, \ldots \)). Since \( A_1 \in (\text{SI}) \), \( P_{2l} = 0 \) or 1 (\( l = 1, 2, \ldots \)). Assume that \( P_{21} = 0 \) (otherwise, consider 1 – \( P \)). If \( P_{22} = 1 \), \( W_2 \in \text{ran} \tau_{A_1 A_2} \), a contradiction. Thus \( P_{22} = 0 \) and therefore \( P_{12} = 0 \). By the same argument, \( P_{2l} = 0 \) (\( l = 3, 4, \ldots \)) and \( P = 0 \), i.e., \( W \in (\text{SI})(\mathcal{H}) \). Let \( \{\lambda_k\}^{\infty}_{k=1} \subset D_0 \) be an arbitrary sequence such that \( \lim_{k \to \infty} \lambda_k = \lambda_0 \in D_0 \), pairwise distinct, then \( \bigvee \left\{ \ker \left( \bigoplus_{n=1}^{\infty} A_n - \lambda_k \right) : k \geq 1 \right\} = \bigoplus_{n=2}^{\infty} \mathcal{H}_n \) and \( \bigvee \{ \ker (A_1 - \lambda_k) : k \geq 1 \} = \mathcal{H}_1 \). Note that \( \{\lambda_k\}^{\infty}_{k=1} \subset \rho_r(A_1) \), thus \( \bigvee \{ \ker (W - \lambda_n) : n \geq 1 \} = \mathcal{H} \) and \( \ker (W - \lambda_n) = \infty \) (\( n = 0, 1, 2, \ldots \)). Since \( \sigma_p(A_k) \subset D_k \) and \( \sigma_p(A_k^*) = \emptyset \) (\( k = 1, 2, \ldots \)), computation indicates that \( \sigma_p(W) \subset \Omega \) and \( \sigma_p(W^*) = \emptyset \). Observe that \( W = \bigoplus A_n + K \), where \( K \) is a compact operator and \( \| (A_n - \lambda)^{-1} \| < \frac{2}{\text{dist} (\lambda, \sigma)} \) for \( \lambda \not\in \sigma \) and \( n \geq 1 \), we have
\[
\sigma \left( \bigoplus_{n=1}^{\infty} A_n \right) = \sigma_r \left( \bigoplus_{n=1}^{\infty} A_n \right) = \sigma. \quad \text{Since } \sigma(W) \text{ is connected and } \sigma_p(W^*) = \emptyset, \quad \sigma(W) = \sigma_r(W) = \sigma.
\]

**Example 2.5.** ([10]) Define \( \gamma_1 = 1, \gamma_2 = \frac{1}{2}, \gamma_3 = (\gamma_1 \gamma_2)^2, \ldots, \gamma_n = (\gamma_1 \cdots \gamma_{n-1})^n, \ldots \), and let \( \{\alpha_n\} \) be the sequence
\[
\gamma_1, \gamma_2, \ldots, \gamma_9, \gamma_1, \gamma_2, \ldots, \gamma_9, \gamma_1, \gamma_2, \ldots, \gamma_900, \gamma_1, \gamma_2, \ldots, \gamma_9000, \gamma_1, \ldots.
\]
Let \( V \) be the unilateral weighted shift defined by \( V e_n = \alpha_n e_{n+1} \) (\( n \geq 1 \)) with respect to an ONB \( \{e_n\}^{\infty}_{n=1} \) of the Hilbert space \( \mathcal{H} \). Then \( V \) is a quasinilpotent unicyclic operator and \( V^k \) is not compact for all \( k = 1, 2, \ldots, \).

**Theorem 2.6.** ([8]) Let \( R \in \mathcal{L}(\mathcal{H}) \) satisfy:

(i) \( \sigma(R) \) and \( \sigma_{W}(R) \) are connected and contain a connected open set \( \Omega \);

(ii) \( \text{ind} (\lambda - R) \geq 0 \) for all \( \lambda \in \rho_{r}(R) \) (i.e., \( R \) is a quasitriangular operator);

(iii) \( \rho_{r}(R) \supset \Omega \) and \( \text{ind} (\lambda - R) = n \) for all \( \lambda \in \Omega \).

Then for \( \varepsilon > 0 \), there exists a compact operator \( K_{\varepsilon} \), \( \| K_{\varepsilon} \| < \varepsilon \), such that \( R - K_{\varepsilon} \in \mathcal{B}_{n}(\Omega) \) (see the next definition).

**Definition 2.7.** Let \( \Omega \) be a bounded connected open set in \( \mathbb{C} \), \( n \) is a positive integer or \( \infty \). The set \( \mathcal{B}_{n}(\Omega) \) of Cowen-Douglas operators of index \( n \) is the set of operators \( B \) in \( \mathcal{L}(\mathcal{H}) \) satisfying:

(i) \( \sigma(B) \supset \Omega \);

(ii) \( \text{ran} (\lambda - B) = \mathcal{H} \) for all \( \lambda \in \Omega \);

(iii) \( \text{nul} (\lambda - B) = n \) for all \( \lambda \in \Omega \);

(iv) \( \bigvee \{ \ker (\lambda_0 - B^k) : k \geq 1 \} = \mathcal{H} \) for each \( \lambda_0 \in \Omega \).

Note that (iv) can be replaced by (iv)' or (iv)'' ([3]):

(iv)' \( \bigvee \{ \ker (\lambda_0 - B^k) : k \geq 1 \} = \mathcal{H} \) for each \( \lambda_0 \in \Omega \).
The closure of the unitary orbit 29

(iv)

\[ \lim_{n \to \infty} \{ \ker(\lambda_n - B) : n \geq 1 \} = \mathcal{H} \] for all sequences \( \{\lambda_n\}_{n=0}^{\infty} \subset \Omega \) such that \( \lim_{n \to \infty} \lambda_n = \lambda_0 \).

Consider \( B_1, B_2 \in B_1(\Omega), (0 \in \Omega) \). By Lemma 2.2 of [17], \( B_1 \) and \( B_2 \) admit the following matrix representations

\[
B_1 = \begin{bmatrix}
0 & b_{21}^1 & b_{31}^1 & 0 & b_{41}^1 & 0 & \cdots
0 & b_{21}^2 & b_{31}^2 & 0 & b_{41}^2 & 0 & \cdots
0 & b_{21}^3 & b_{31}^3 & 0 & b_{41}^3 & 0 & \cdots
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & b_{21}^1 & 0 & 0 & b_{41}^1 & 0 & \cdots
0 & b_{21}^2 & 0 & 0 & b_{41}^2 & 0 & \cdots
0 & b_{21}^3 & 0 & 0 & b_{41}^3 & 0 & \cdots
\end{bmatrix}
\]

where \( \{e_n\}_{n=1}^{\infty} \) and \( \{f_n\}_{n=1}^{\infty} \) are ONB’s of \( \mathcal{H} \), and \( |b_{nn+1}^l| > r > 0 \) (\( i = 1, 2; n = 1, 2, \ldots \)) for some \( r \).

Define \( r(B_1, B_2) = \lim_{n \to \infty} \left[ \prod_{k=1}^{n} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} \right]^{\frac{1}{n}} \).

**Proposition 2.8.** (i) If \( r(B_1, B_2) > 1 \), then \( \ker \tau_{B_1 B_2} = \{0\} \).

(ii) If \( r(B_1, B_2) = 1 \), then given \( \varepsilon > 0 \) \( (\varepsilon < r) \), there exists a compact operator \( K \) satisfying:

(a) \( \|K\| < \varepsilon \);

(b) \( \ker \tau_{B_1 B_2 + K} = \ker \tau_{B_1 + K} = \{0\} \);

(c) \( B_2 + K \in B_1(\Omega) \) and \( r(B_1, B_2 + K) = 1 \).

**Proof.** (ii) Denote \( \delta_i = 1 - \varepsilon/2^i \) (\( i = 1, 2, \ldots \)). Since

\[
\lim_{n \to \infty} \left[ \prod_{k=1}^{n} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} \right]^{\frac{1}{n}} = d_1 > 1,
\]

there exists \( n_1 \) such that

\[
\prod_{k=1}^{n_1} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} > 2.
\]

Set \( \beta_k = 1 - d_1 \) (\( 1 \leq k \leq n_1 \)). Since

\[
\lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} \right) \left( \prod_{k=n_1+1}^{n} \frac{|b_{kk+1}^l d_2|}{\|b_{kk+1}^l\|} \right) = d_2 < 1,
\]

we can find \( n_2 > n_1 \) such that

\[
\prod_{k=1}^{n_1} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} \cdot \prod_{k=n_1+1}^{n_2} \frac{|b_{kk+1}^l d_2|}{\|b_{kk+1}^l\|} < \frac{1}{2}.
\]

Set \( \beta_k = 1 - 1/d_2 \) (\( n_1 + 1 \leq k \leq n_2 \)). Inductively, we can define

\[
\beta_k = \begin{cases} 
1 - \frac{d_2 - 1}{d_2}, & 1 \leq k \leq n_2,
1 - \frac{1}{d_2}, & n_2 + 1 \leq k \leq n_{2l},
\end{cases}
\]

such that

\[
(2.1) \quad \prod_{k=1}^{n_{2l-1}} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} > 2^l, \quad \prod_{k=1}^{n_{2l}} \frac{|b_{kk+1}^l|}{\|b_{kk+1}^l\|} < 2^{-l}, \quad l = 1, 2, \ldots
\]
and \( \lim_{k \to \infty} \beta_k = 0 \) and \( \sup_k |\beta_k| < \frac{\epsilon}{2} \).

Define \( K' e_k = -b_{k+1}^2 \beta_k e_k - 1 \) \((k = 2, 3, \ldots)\) and \( K' e_1 = 0 \). Then \( K' \) is compact and \( ||K'|| < \epsilon/2 \). It is easily seen that \( B'_2 + K' \) is \( B_1(\Omega) \) by (2.1), \( X \) for some \( X \in L(\mathcal{H}) \), we can prove that

\[
X = \begin{bmatrix}
x_{11} & x_{12} & \ldots \\
x_{21} & x_{22} & \ldots \\
0 & \ddots & \ddots
\end{bmatrix}
\]

with respect to \( \{e_n\} \) and

\[
x_{nn} = \prod_{k=1}^{n-1} \frac{b_{kk+1}^2 (1 - \beta_k)}{b_{kk+1}^2} x_{11}, \quad n = 1, 2, \ldots.
\]

By (2.1), \( x_{nn} = 0 \) \((n = 1, 2, \ldots)\). Similarly, a computation indicates that

\[
x_{nn+1} = \frac{b_{n+1}^2 (1 - \beta_1)}{b_{12}^2 (1 - \beta_1)} \prod_{k=1}^{n} \frac{b_{kk+1}^2 (1 - \beta_k)}{b_{kk+1}^2} x_{12}, \quad k = 2, 3, \ldots.
\]

By (2.1), \( x_{n+1} = 0 \) \((n = 1, 2, \ldots)\). Generally, we can prove that \( x_{ij} = 0 \) \((i < j)\) and therefore, \( \ker \tau_{B'_1 B'_2 B'_3} = \{0\} \). By the same argument, \( \ker \tau_{B'_1 B'_2 B'_3} = \{0\} \).

From the definition of \( \{\beta_k\} \), it is easy to see that \( r(B'_1, B'_2 + K') = 1 \). Since \( B_1 \approx B'_1 \) and \( B_2 \approx B'_2 \), we can find a compact operator \( K \) satisfies all requirements of (ii).

(i) If \( r(B_1, B_2) > 1 \), then there is a subsequence \( \{n_i\}_{i=1}^{\infty} \) of natural numbers such that \( n_1 < n_2 < \cdots \) and

\[
\prod_{k=1}^{n_i} \frac{b_{kk+1}^2 (1 - \beta_k)}{b_{kk+1}^2} > k, \quad k = 1, 2, \ldots.
\]

By the same argument of (ii), \( \ker \tau_{B_1 B_2} = \{0\} \).

Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{C} \) with \( \Omega^c = \Omega \). Let \( N(\Omega) \) be the “multiplication by \( \lambda \)” operator acting on \( L^2(\Omega, dm) \). The subspace \( A^2(\Omega) \) spanned by the rational functions with poles outside \( \Omega \) is invariant under \( N(\Omega) \). By \( N_+(\Omega) \) and \( N_-(\Omega) \) we shall denote the restriction of \( N(\Omega) \) to \( A^2(\Omega) \) and its compression to \( L^2(\Omega, dm) \) \( \oplus A^2(\Omega) \), respectively, i.e.,

\[
N(\Omega) = \begin{bmatrix}
N_+(\Omega) & G \\
0 & N_-(\Omega)
\end{bmatrix} A^2(\Omega) \oplus \begin{bmatrix}
L^2(\Omega, dm) \oplus A^2(\Omega)
\end{bmatrix},
\]

where \( N_+(\Omega) \) is called Bergmann operator.

**Lemma 2.9.** Consider a connected compact subset \( \sigma \) of \( \mathbb{C} \) and pairwise disjoint connected open subsets \( \{\Omega_k\}_{k=1}^{l} \) \((0 \leq k \leq l, 0 \leq l \leq \infty)\) of \( \sigma \) and given a sequence \( \{n_k\}_{k=1}^{l} \) of numbers such that \( \{n_k\}_{k=0}^{l} \subset \mathbb{N} \cup \{\infty\}, n_0 = \infty, 1 \leq n_k \leq \infty \) \((k \geq 1)\). Then there exists an operator \( A \) in \( \mathcal{B}_{\infty} (\Omega_0) \cap (\text{SI}) \) satisfying:

(i) \( \sigma(A) = \sigma, \sigma_{\text{loc}}(A) = \sigma \setminus \bigcup_{k=0}^{l} \Omega_k; \)

(ii) \( \text{ind}(A - \lambda) = \text{mul}(A - \lambda) = n_k \) for all \( \lambda \in \Omega_k \) \((k = 0, 1, \ldots, l)\).
Proof. Denote $\Phi_k = (\Omega_k)\circ$, let $N_+ (\Phi_k^*)$ be the Bergmann operator on $A^2 (\Phi_k^*)$ and denote $A_0 = N_+ (\Phi_0^*)$ and $A_k = N_+ (\Phi_k^*)^{(n_k)} (k = 1, 2, \ldots, l)$. Thus $\sigma (A_0) = \Omega_0$, $A_0 \in \mathcal{B}_1 (\Phi_0) \cap (\text{SI})$, $\sigma (A_k) = \Omega_k$ and $A_k \in \mathcal{B}_{n_k} (\Phi_k) (k = 1, 2, \ldots, l)$.

Let $\{\lambda_k\}_{k=1}^{\infty}$ be a dense subset of $\sigma \setminus \bigcup_{k=0}^{l} \Omega_k$. Set $T_k = \lambda_k + V^*$, where $V$ is given in Example 2.5, and define

$$G = A_0 \oplus \left( \bigoplus_{k=1}^{l} A_k \right) \oplus \left( \bigoplus_{k=1}^{\infty} T_k \right).$$

Then $G$ satisfies:

(a) $\sigma (G) = \sigma_{\mathcal{W}} (G) = \sigma$, $\sigma_{\text{re}} (G) = \sigma \setminus \bigcup_{k=0}^{l} \Omega_k$;

(b) $\text{ind} (G - \lambda) = \text{null} (G - \lambda) = 1$ for $\lambda \in \Theta_0$;

(c) $\text{ind} (G - \lambda) = \text{null} (G - \lambda) = n_k$ for $\lambda \in \Omega_k (k = 1, 2, \ldots, l)$.

By Theorem 2.6, for each $\varepsilon > 0$, there exists a compact operator $K$ with $\|K\| < \varepsilon$ such that $G + K \in \mathcal{B}_1 (\Theta_0)$. It is completely apparent that $G + K$ satisfies (a), (b) and (c).

Without loss of generality, we may assume that $0 \in \Omega_0$.

Note that $\mathcal{B}_1 (\Phi_0) \subset \mathcal{B}_1 (\Theta_0)$. By Proposition 2.8 and Theorem 2.3, there exists a compact operator $K_1$ with $\|K_1\| < \varepsilon$ such that if $r (G + K, A_0) \geq 1$,

$$(G + K) \oplus A_0^{(\infty)} + K_1 = \left[ \begin{array}{ccc} G + K & D_1 & D_2 & \cdots \\
 & B_1 & B_2 & \\
 & 0 & \ddots & \end{array} \right],$$

where $B_i \in \mathcal{B}_1 (\Theta_0)$, $D_i \notin \text{ran} \tau_{G + K, B_i}$, $\text{ker} \tau_{B_i, G + K} = \{0\} (i \geq 1)$ and $\text{ker} \tau_{B_i, B_j} = \{0\} (i \neq j)$. If $r (G + K, A_0) < 1$,

$$(G + K) \oplus A_0^{(\infty)} + K_1 = \left[ \begin{array}{ccc} B_1 & D_1 & \\
 & B_2 & \\
 & 0 & G + K \end{array} \right],$$

where $B_i \in \mathcal{B}_1 (\Theta_0)$, $D_i \in \text{ran} \tau_{B_i, G + K}$, $\text{ker} \tau_{G + K, B_i} = \{0\} (i \geq 1)$ and $\text{ker} \tau_{B_i, B_j} = \{0\} (i \neq j)$. By the same argument of Lemma 2.4, $A := (G + K) \oplus A_0^{(\infty)} + K_1 \in \mathcal{B}_{\infty} (\Theta_0) \cap (\text{SI})$. Thus $A$ satisfies the requirements of the lemma.

The spectral picture $\Lambda (T)$ of the operator $T$ is the compact set $\sigma_{\text{re}} (T)$, plus the data corresponding to the indices of $\lambda - T$ for $\lambda$ in the bounded components of $\rho_{\sigma-F} (T)$.

Lemma 2.10. Let $T \in \mathcal{L} (\mathcal{H})$ with connected spectrum $\sigma (T)$ and let $\sigma_{\text{re}} (T)$ be the closure of an analytic Cauchy domain. Then there exists an operator $A \in (\text{SI})$ satisfying:

(i) $\Lambda (A) = \Lambda (T)$;

(ii) $\min \text{ind} (A - \lambda) = \begin{cases} 0, & \text{ind} (T - \lambda) \neq 0, \\ 1, & \lambda \in \rho_{\sigma-F} (T) \cap \sigma (T); \end{cases}$
(iii) \( A \) admits a representation \( A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \mathcal{K}_1 \mathcal{K}_2 \) and there is a subset \( \{ \lambda_k : k = 0, \pm 1, \pm 2, \ldots \} \) of complex numbers such that \( \text{null} (A_1 - \lambda_k) = \infty \) \((k \geq 0)\), \( \text{null} (A_2 - \lambda_k) = \infty \) \((k < 0)\), \( \bigvee \{ \ker (A_1 - \lambda_k) : k \geq 0 \} = \mathcal{K}_1 \) and \( \bigvee \{ \ker (A_2 - \lambda_k) : k < 0 \} = \mathcal{K}_2 \), where \( \mathcal{K}_1, \mathcal{K}_2 \) are infinite dimensional Hilbert spaces;

(iv) There is an open disc \( \mathcal{G} \subset \sigma_{\text{loc}}(A) \) such that \( G \cap \sigma_p(A_1) = G^* \cap \sigma_p(A_2^*) = \emptyset \).

Proof. Choose an open disc \( G_1 \) such that \( \overline{G}_1 \subset \sigma_{\text{loc}}(T)^\circ \). Denote \( \sigma = \sigma(T) \setminus G_1 \), then \( \sigma \) is connected and \( \sigma \cap \sigma_{\text{loc}}(T) \) is still the closure of an analytic Cauchy domain. Let \( \{ \sigma_k \}_{k=0}^1 \) and \( \{ \sigma_k \}_{k=0}^1 \) be the components of \( \sigma \setminus \rho_p^*(T) \) and, respectively, \( \sigma \setminus \rho_p^*(T) \). For each \( k \) \((-l_2 \leq k \leq l_1)\) choose an open disc \( \Omega_k \) such that \( \overline{\Omega_k} \subset [\sigma_k \cap \sigma_{\text{loc}}(T)]^\circ \) \( (\text{if for more than one } k, (\sigma_k \cap \sigma_{\text{loc}}(T)) \cap (\sigma_j \cap \sigma_{\text{loc}}(T)) \neq \emptyset, \) let \( \Omega_{-j} \) equal one of the \( \Omega_k \)'s.) By Lemma 2.9 there is a \( B_k \) \((-l_2 \leq k \leq l_1)\) such that:

(i) if \( k \geq 0 \), \( B_k \in B_\infty(\Omega_k) \cap (\mathcal{SI}(l) \Omega_k), \) \( \sigma(B_k) = \sigma_k, \sigma_{\text{loc}}(B_k) = \sigma_k \cap [\sigma_{\text{loc}}(T) \setminus \Omega_k], \text{ind} (B_k - \lambda) = \text{null} (B_k - \lambda) = \text{ind} (T - \lambda) \) for \( \lambda \in \sigma_k \setminus \rho_p^*(T), \text{ind} (B_k - \lambda) = \text{null} (B_k - \lambda) = 1 \) for \( \lambda \in \sigma_k \setminus \rho_p^*(T); \)

(ii) if \( k < 0 \), \( B_k \in B_\infty(\Omega_k^\circ) \cap (\mathcal{SI}(l) \Omega_k), \) \( \sigma(B_k) = \sigma_k, \sigma_{\text{loc}}(B_k) = \sigma_k \cap [\sigma_{\text{loc}}(T) \setminus \Omega_k], \text{ind} (B_k - \lambda) = -\text{null} (B_k - \lambda)^* = \text{ind} (T - \lambda) \) for \( \lambda \in \sigma_k \cap \rho_p^*(T), \text{ind} (B_k - \lambda) = -\text{null} (B_k - 1)^* = -1 \) for \( \lambda \in \sigma_k \cap \rho_p^*(T). \)

Choose open discs \( G_1 \) and \( G_2 \) such that \( \overline{G}_1 \cup \overline{G}_2 \subset G_1 \) and \( \overline{G}_1 \cap \overline{G}_2 = \emptyset \). By Lemma 2.4, we can construct an operator \( W \in (\mathcal{SI}(l)) \) satisfying:

(i) \( \sigma(W) = \sigma_{\text{loc}}(W) = \overline{G}_1; \)

(ii) \( \sigma_g(W) \subset G_2, \sigma_g(W^*) = \emptyset; \)

(iii) There exists a sequence \( \{ \mu_k \}_{k=0}^\infty \subset G_2 \) of distinct numbers such that \( \lim_{k \to \infty} \mu_k = \mu_0, \text{null} (W - \mu_k) = \infty \) \((k \geq 1)\) and \( \bigvee \{ \ker (W - \mu_k) : k \geq 1 \} = \mathcal{K}. \)

For each \( k \) \((0 \leq k \leq l_1)\), choose \( R_k \in \mathcal{L}(\mathcal{H}_k, \mathcal{K}) \) by

\[
R_k \begin{cases} 0, \\
\notin \text{ran} \tau_{W,B_k} \text{ and } R_k \text{ is compact}, & \text{if } \sigma(B_k) \cap \sigma(W) = \emptyset, \\
\text{otherwise (Theorem 2.3).} & \text{otherwise (Theorem 2.3).}
\end{cases}
\]

Set \( R = (R_0, R_1, \ldots, R_l) \).

For each pair \((i, j)\) \((0 \leq i \leq l_1; 1 \leq j \leq l_2)\) choose \( Q_{ij} \in \mathcal{L}(\mathcal{H}_{-j}, \mathcal{H}_i) \) by

\[
Q_{ij} \begin{cases} 0, \\
\notin \text{ran} \tau_{B_i,B_{-j}} \text{, } Q_{ij} \text{ is compact,} & \text{if } \sigma_i \cap \sigma_{-j} = \emptyset, \\
\text{if } \sigma_i \cap \sigma_{-j} \neq \emptyset. & \text{if } \sigma_i \cap \sigma_{-j} \neq \emptyset.
\end{cases}
\]

Set

\[
Q = \begin{bmatrix} Q_{00} & Q_{01} & \cdots & Q_{0l_2} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{l_11} & Q_{l_12} & \cdots & Q_{l_1l_2} \end{bmatrix} \in \mathcal{L}\left( \bigoplus_{k=1}^{l_2} \mathcal{H}_{-k}, \bigoplus_{k=0}^{l_1} \mathcal{H}_k \right).
\]

Define

\[
A = \begin{bmatrix} W & R \\
0 & \bigoplus_{k=0}^{l_2} B_k \end{bmatrix} = \begin{bmatrix} A_1 & * \\
0 & A_2 \end{bmatrix} \begin{bmatrix} \mathcal{K}_1 \mathcal{K}_2 \end{bmatrix}.
\]
where \( K_1 = K \oplus \bigoplus_{k=0}^{l_1} \mathcal{H}_k \), \( K_2 = \bigoplus_{k=1}^{l_2} \mathcal{H}_{-k} \), \( A_1 = \begin{bmatrix} W & R \\ 0 & l_1 \bigoplus_{k=0}^{l_1} B_k \end{bmatrix} \) and \( A_2 = \bigoplus_{k=1}^{l_2} B_{-k} \). It follows from the properties of \( W, B_k (\sim l_2 \leq k \leq l_1) \) and Lemma 2.1 that \( \ker \tau_{B_k B_{k'}} = \ker \tau_{B_{-k} B_{-k'}} = 0 \) \((k \neq k')\), \( \ker \tau_{l_1 \bigoplus_{k=1}^{l_1} B_{-k}} = \ker \tau_{l_2 \bigoplus_{k=0}^{l_1} B_k} = \ker \tau_{l_1 \bigoplus_{k=0}^{l_1} B_k W} = [0] \). Since \( W \) and each \( B_k (\sim l_2 \leq k \leq l_1) \) are strongly irreducible, by Lemma 3.1 of [16] \( A \in (SI) \). From the construction of \( A \), we can get (i) and (ii).

Note that \( \sigma \left( \bigoplus_{k=0}^{l_1} B_k \right) \cap \mathcal{G} \subset \sigma \left( \bigoplus_{k=0}^{l_1} B_k \right) \cap G_1 \subset \sigma \cap G_1 = \emptyset \) and \( \sigma \left( \bigoplus_{k=1}^{l_2} B_{-k} \right) \cap \mathcal{G} \subset \sigma \left( \bigoplus_{k=1}^{l_2} B_{-k} \right) \cap G_1 \subset \sigma \cap G_1 = \emptyset \). Since \( \sigma_p(W) \subset G_2 \) and \( \sigma_p(W^*) = \emptyset \), \( \sigma_p(A_1) \cap \mathcal{G} = \sigma_p(A_2) \cap \mathcal{G}^* = \emptyset \). Since \( \Omega_k \cap G_1 = \emptyset (l_2 \leq k \leq l_1) \), there are \( \{\lambda_k\}_{k=1}^{\infty} \subset \sigma_p(A_1) \) and \( \{\lambda^*_k\}_{k=1}^{\infty} \subset \sigma_p(A_2) \) satisfying (iii).

**Lemma 2.11.** Let \( \sigma \) be the closure of a connected Cauchy domain and let \( \{\sigma_k\}_{k=0}^{\infty} \) and \( \{\Omega_k\}_{k=1}^{\infty} \) be two classes of subsets of \( \sigma^\circ \) satisfying:

(i) each \( \sigma_k \) is a connected Cauchy domain;

(ii) \( \sigma_k \subset \sigma_{k+1} \) and \( \sigma_{k+1} \setminus \sigma_k \) is a connected Cauchy domain \((k = 0, 1, \ldots)\);

(iii) \( \sigma = \bigcup_{k=0}^{\infty} \sigma_k \);

(iv) each \( \Omega_k \) is an open disc and \( \Omega_k \subset \sigma_{k+1} \setminus \sigma_k \) \((k = 1, 2, \ldots)\).

Then there exists an operator \( T \in (SI)_{(H)} \) satisfying:

(a) \( \sigma(T) = \sigma_{\text{ev}}(T) = \sigma_p(T) \subset \bigcup_{k=1}^{\infty} \Omega_k \) and \( \sigma_p(T^*) = \emptyset \);

(b) there is a subset \( \{\mu_n\}_{n=1}^{\infty} \) of \( \sigma_p(T) \) such that \( \text{null}(T - \mu_n) = \infty \) \((n = 1, 2, \ldots)\) and \( \bigvee \{\ker(T - \mu_n) : n \geq 1\} = \mathcal{H} \);

(c) if \( A \in \mathcal{L}(\mathcal{H}) \) such that \( \sigma(A) \cap \sigma^\circ = \emptyset \), then \( \ker \tau_{AT} = \ker \tau_{TA} = \{0\} \).

**Proof.** According to Lemma 2.4 we can construct an operator \( T_k \in (SI)_{(H_k)} \) such that \( \sigma(T_k) = \sigma_{\text{ev}}(T_k) = \sigma_k, \sigma_p(T_k) \subset \Omega_k, \sigma_p(T_k^*) = \emptyset \) and there is a sequence \( \{\lambda^k_n\}_{n=0}^{\infty} \subset \Omega_k \) satisfying \( \lim_{n \to \infty} \lambda^k_n = \lambda_0, \text{null}(T_k - \lambda^k_n) = \infty \) \((n = 1, 2, \ldots)\)

and \( \bigvee \{\ker(T_k - \lambda^k_n) : n \geq 1\} = \mathcal{H}_k (k = 1, 2, \ldots) \). Since \( \sigma_1(T_k) \cap \sigma_1(T_k) = \sigma_1 \cap \sigma_1 = \emptyset \) \((k \geq 2)\), there is a compact operator \( D_k \notin \text{ran} \tau_{T_kT_k^*} \), \( \|D_k\| < 2^{-k} (k \geq 2) \).

Set

\[
T = \begin{bmatrix}
T_1 & D_2 & D_3 & \ldots \\
T_2 & T_3 & \ldots \\
0 & & & \\
\end{bmatrix} \in \mathcal{L}(\mathcal{H}),
\]

where \( \mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k \). Since \( \{\Omega_k\}_{k=1}^{\infty} \) are pairwise disjoint, \( \ker \tau_{T_iT_j} = \{0\} \) \((i \neq j)\). By the same argument of Lemma 2.4, \( T \in (SI) \). It follows from the construction of \( T \) that \( T \) satisfies (i) and (ii). By Lemma 2.1, \( \ker \tau_{AT} = \{0\} \). If there is an
operator $X \in L(H)$ such that $TX = XA$, let $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}$; then we have $T_2X_2 = X_2A, \ldots, T_nX_n = X_nA$, $(n \geq 2)$. Since $\sigma(A) \cap \sigma^o = \emptyset$ and $\sigma(T_n) = \sigma_n \subset \sigma^o$, $\sigma(A) \cap \sigma(T_n) = \emptyset$. Thus $X_n = 0$, $(n \geq 2)$ and $T_1X_1 = X_1A$. For the same reason $X_1 = 0$ and $X = 0$, i.e., $\ker \tau_{TA} = \{0\}$. 

**Lemma 2.12.** Let $n \in \mathbb{N}$ or $n = \infty$, let $\sigma$ be a connected compact subset of $\mathbb{C}$ and $\Omega$ be a connected open subset of $\sigma^o$ such that $\sigma^o \setminus \Omega \neq \emptyset$. Then there exists an operator $A \in (SI)(\mathcal{H})$ satisfying:

(i) $\sigma(A) = \sigma$, $\sigma_{\text{ve}}(A) = \sigma \setminus \Omega$, $\sigma_p(A^*) = \emptyset$;

(ii) $\text{ind} (A - \lambda) = n$ for $\lambda \in \Omega$;

(iii) there exists a subset $\{\lambda_k\}_{k=1}^\infty$ of $\sigma$ such that $\ker (A - \lambda_k) = \infty$ $(k \geq 1)$ and $\sqrt{\ker (A - \lambda_k)} : k \geq 1 = \mathcal{H}$.

**Proof.** Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$. Choose open discs $G_1, G_2$ such that $\overline{G}_2 \subset G_1 \subset \overline{G}_1 \subset \sigma^o \setminus \Omega$. According to Lemma 2.9, we can construct an operator $A_1 \in \mathcal{B}_\infty(G_1) \cap (SI)(\mathcal{H}_1)$ satisfying $\sigma(A_1) = \sigma$, $\sigma_{\text{ve}}(A_1) = \sigma \setminus (G_1 \cup \Omega)$ and $\text{ind} (A_1 - \lambda) = n$ for $\lambda \in \Omega$. By Lemma 2.4, we can find an operator $A_2 \in (SI)(\mathcal{H}_2)$ satisfying $\sigma(A_2) = \sigma_{\text{ve}}(A_2) = \overline{G}_1$, $\sigma_p(A_2) \subset G_2$, $\sigma_p(A_2^*) = \emptyset$ and there exists a sequence $\{\mu_i\}_{i=1}^\infty \subset G_2$ such that $\ker (A_2 - \mu_i) = \infty$ $(i \geq 1)$ and $\sqrt{\ker (A_2 - \mu_i)} : i \geq 1 = \mathcal{H}_2$. By Lemma 2.1 $\ker \tau_{A_1A_2} = \{0\}$. By Theorem 2.3, there is a compact operator $K \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that $K \notin \text{ran} \tau_{A_1A_2}$.

Define $A = \begin{bmatrix} A_1 & K \\ 0 & A_2 \end{bmatrix} \mathcal{H}_1 \mathcal{H}_2$. By the same argument of Lemma 2.4, $A \in (SI)(\mathcal{H})$ and satisfies (i), (ii) and (iii). 

**Lemma 2.13.** Let $T \in L(\mathcal{H})$ with connected spectrum $\sigma(T)$ and assume that $\sigma_{\text{ve}}(T)$ is the closure of an analytic Cauchy domain, then there exists an operator $W \in (SI)(\mathcal{H})$ satisfying:

(i) $\Lambda(W) = \Lambda(T)$;

(ii) $\text{min ind} (W - \lambda) = \begin{cases} 0, & \text{if } \lambda \in \rho_{\text{ve}}^-(W), \\ 1, & \text{if } \lambda \in \sigma(W) \cap \rho_{\text{ve}}^+(W), \end{cases}$

(iii) $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix} \mathcal{H}_1 \mathcal{H}_2$, where $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$, and there is a sequence $\{\lambda_k : k = 0, \pm 1, \pm 2, \ldots\}$ of numbers such that $\sqrt{\ker (W_1 - \lambda_k)^*} : k \geq 0 = \mathcal{H}_1$ and $\sqrt{\ker (W_2 - \lambda_k)} : k < 0 = \mathcal{H}_2$;

(iv) there is an open disc $G \subset \sigma_{\text{ve}}(W)$ such that $G \cap \sigma_p(W_2) = G^* \cap \sigma_p(W_1^*) = \emptyset$.

**Proof.** Assume that

$\{\Omega_{1i}\}_{i=1}^{\infty}$ are the components of $\rho_{\text{ve}}^-(T)$,

$\{\Omega_{2j}\}_{j=1}^{\infty}$ are the components of $\rho_{\text{ve}}^+(T) \cap \sigma(T)$,

$\{\Omega_{3k}\}_{k=1}^{\infty}$ are the components of $\rho_{\text{ve}}^-(T)$.

Choose connected Cauchy domains $\Phi_{ij}$ in $\sigma(T)$ $(i = 1, 2, 3; j = 1, 2, \ldots)$ such that $\Phi_{ij} \supset \Omega_{ij}$, $\Phi_{ij} \setminus \overline{\Omega_{ij}}$ are connected Cauchy domains, $\{\Phi_{ij}\}$ are pairwise disjoint and $\sigma(T) \setminus \bigcup \Phi_{ij}$ is the closure of an analytic Cauchy domain.
Choose an open disc $\sigma_0 \subset [\sigma(T) \setminus \bigcup \Phi_{ij}]^o$. Let $\{\sigma_k\}_{k=1}^4$ be the components of $\sigma(T) \setminus [\sigma_0^o \cup (\bigcup \Phi_{ij})]$. Choose an open disc $G$ such that $\overline{G} \subset \sigma_0^o$. For each $k$ ($0 \leq k \leq l_4$), according to Lemma 2.11, we can construct an operator $E_k \in (SI)(\mathcal{H})$ satisfying:

(i) $\sigma(E_k) = \sigma_{\text{re}}(E_k) = \sigma_k$;

(ii) $\sigma_p(E_0) = \emptyset$ and there is a subset $\{\mu_n : n \geq 1\}$ of $\sigma_0 \setminus G$ such that $\text{nul} \ (E_0 - \mu_n)^* = \infty$, $\bigvee \{\ker(E_0 - \mu_n)^* : n \geq 1\} = \mathcal{H}$ and $G^* \cap \sigma_p(E_0^* \sigma) = \emptyset$;

(iii) For each $k \geq 1$, $\sigma_p(E_k^*) = \emptyset$ and there is a subset $\{\mu_{kn} : n \geq 1\}$ of $\sigma_k$ such that $\text{nul} \ (E_k - \mu_{kn}) = \infty$, $\bigvee \{\ker(E_k - \mu_{kn}) : n \geq 1\} = \mathcal{H}$;

(iv) For each $k$ and each operator $F$, if $\sigma(F) \cap \sigma_k = \emptyset$, then $\ker \tau_{E_k} = \ker \tau_{F E_k} = \{0\}$.

According to Lemma 2.12, we construct the following (SI) operators.

Step 1. Construct $A_i \in (SI)(\mathcal{H})$ ($1 \leq i \leq l_1$) such that $\sigma(A_i) = \overline{\mathcal{H}}_{1i}$, $\sigma_p(A_i) = \emptyset$, $\sigma_{\text{re}}(A_i) = \overline{\mathcal{H}}_{1i} \setminus \Omega_{1i}$, $\text{ind} (A_i - \lambda) = \text{ind} (T - \lambda)$ for $\lambda \in \Omega_{1i}$, and there is a countable subset $\Lambda_{1i}$ of $\sigma(A_i)$ such that $\text{nul} \ (A_i - \lambda)^* = \infty$ ($\lambda \in \Lambda_{1i}$) and $\bigvee \{\ker(A_i - \lambda)^* : \lambda \in \Lambda_{1i}\} = \mathcal{H}$.

Step 2. Construct $B_k \in (SI)(\mathcal{H})$ ($1 \leq k \leq l_3$) such that $\sigma(B_k) = \overline{\mathcal{H}}_{3k}$, $\sigma_p(B_k) = \emptyset$, $\sigma_{\text{re}}(B_k) = \overline{\mathcal{H}}_{3k} \setminus \Omega_{3k}$, $\text{ind} (B_k - \lambda) = \text{ind} (T - \lambda)$ for $\lambda \in \Omega_{3k}$, and there is a countable subset $\Lambda_{3k}$ of $\sigma(B_k)$ such that $\text{nul} \ (B_k - \lambda) = \infty$ ($\lambda \in \Lambda_{3k}$) and $\bigvee \{\ker(B_k - \lambda) : \lambda \in \Lambda_{3k}\} = \mathcal{H}$.

Step 3. Construct $C_j \in (SI)(\mathcal{H})$ ($1 \leq j \leq l_2$) such that $\sigma(C_j) = \overline{\mathcal{H}}_{2j}$, $\sigma_p(C_j) = \emptyset$, $\sigma_{\text{re}}(C_j) = \overline{\mathcal{H}}_{2j} \setminus \Omega_{2j}$, $\text{ind} (C_j - \lambda) = 1$ for $\lambda \in \Omega_{2j}$, and there is a countable subset $\Lambda_{2j}$ of $\sigma(C_j)$ such that $\text{nul} \ (C_j - \lambda)^* = \infty$ ($\lambda \in \Lambda_{2j}$) and $\bigvee \{\ker(C_j - \lambda)^* : \lambda \in \Lambda_{2j}\} = \mathcal{H}$.

Step 4. Construct $D_h \in (SI)(\mathcal{H})$ ($1 \leq h \leq l_2$) such that $\sigma(D_h) = \overline{\mathcal{H}}_{2h}$, $\sigma_p(D_h) = \emptyset$, $\sigma_{\text{re}}(D_h) = \overline{\mathcal{H}}_{2h} \setminus \Omega_{2h}$, $\text{ind} (D_h - \lambda) = 1$ for $\lambda \in \Omega_{2h}$, and there is a countable subset $\Lambda_{2h}$ of $\sigma(D_h)$ such that $\text{nul} \ (D_h - \lambda) = \infty$ ($\lambda \in \Lambda_{2h}$) and $\bigvee \{\ker(D_h - \lambda) : \lambda \in \Lambda_{2h}\} = \mathcal{H}$.

By the definitions, it is easily seen that

$$\ker \tau_{A_i A_j} = \ker \tau_{B_i B_j} = \ker \tau_{C_i C_j} = \ker \tau_{D_i D_j} = \ker \tau_{E_i E_j} = \{0\}, \quad i \neq j.$$
$X_2 \in \mathcal{L}(\mathcal{H}(t_2), \mathcal{H}(t_1))$, and $X_1 = \mathcal{L}(\mathcal{H}(t_2), \mathcal{H}(t_1))$ are defined similarly. $X_3 = (M_{ij})_{t_2 \times t_1} \in \mathcal{L}(\mathcal{H}(t_2), \mathcal{H}(t_1))$ is defined as follows: $M_{ij}$ is compact and $M_{ij} + K \not\in \text{ran } \tau_{D,E_i}$ for all $K \in \mathcal{K}(\mathcal{H})$ if $\sigma(D_i) \cap \sigma(E_j) = \emptyset$ (Theorem 2.3) and $M_{ij} = 0$ if $\sigma(D_i) \cap \sigma(E_j) \neq \emptyset$.

Define

$$W = \begin{bmatrix} E_0 & A & 0 & X_0 \\ A & X_1 \\ C & D & X_2 \\ 0 & B & X_3 \\ E & X_4 \end{bmatrix}$$

Assume that $P \in \mathcal{A}(W)$ is an idempotent. It follows from Lemma 2.1 and the properties of $\{E_k\}$ that $P$ admits the following representation

$$P = \begin{bmatrix} P_1 & 0 & P_{16} \\ P_2 & 0 & P_{26} \\ P_3 & 0 & P_{36} \\ P_{43} & P_4 & P_{46} \\ 0 & P_5 & P_{56} \end{bmatrix}$$

Since $E_0 \in (SI)$ and since $A, B, C, D, E$ are direct sums of $(SI)$ operators with disjoint spectrum respectively, $P_1 = 0$ or 1, $P_2 = \bigoplus_{i=1}^{l_2} \delta_{2i}$, $P_3 = \bigoplus_{i=1}^{l_1} \delta_{3i}$, $P_4 = \bigoplus_{i=1}^{l_1} \delta_{4i}$, $P_5 = \bigoplus_{i=1}^{l_1} \delta_{5i}$ and $P_6 = \bigoplus_{i=1}^{l_1} \delta_{6i}$, where $\delta_{ji} = 0$ or 1. Without loss of generality, we can assume that $P_1 = 0$. By the argument of Lemma 3.1 of [15], we can get $P_2 = P_3 = P_5 = P_6 = 0$. Since $PW = WP, P_{43}X_2 + P_4X_3 + P_{46}E = DP_{46}$. Note that $X_3$ is compact, thus $P_{43}X_2$ is compact. For each $j (1 \leq j \leq l_2)$, there must exist an integer $k$ such that $\sigma_{\text{re}}(D_j) \cap \sigma_{\text{re}}(E_k) = \emptyset$ (Theorem 2.3) and $P_{46} = (L_{ik})_{l_2 \times l_1}$, then

$$D_jL_{jk} - L_{jk}E_k = \delta_{4j}M_{jk} + K,$$

where $K$ is a compact operator. By the choice of $M_{jk}$, $\delta_{4j} = 0$. Thus $P_4 = 0$. Since $P^2 = P$, $P = 0$ and $W \in (SI)$.

Set $W_1 = \begin{bmatrix} E_0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}$, $W_2 = \begin{bmatrix} D & 0 & X_3 \\ 0 & B & X_4 \\ 0 & 0 & E \end{bmatrix}$, then $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix}$, where $\mathcal{H}_1 = \mathcal{H}(t_2 + l_2 + 1)$, $\mathcal{H}_2 = \mathcal{H}(t_1 + l_2 + 1)$. By the properties of $\{A_i\}$ and $\{C_i\}$, we have $\min \text{ ind}(W_1 - \lambda) = 0$ for $\lambda \in \rho_{\text{re}}(T) \cap \sigma(T)$ and

$$\text{ind}(W_1 - \lambda) = \begin{cases} \min \text{ ind}(T - \lambda), & \lambda \in \rho_{\text{re}}(T), \\ 1, & \lambda \in \rho_{\text{re}}(T) \cap \sigma(T). \end{cases}$$

By the properties of $E_0$, $\{A_i\}$ and $\{C_i\}$, we can find a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of numbers such that $\min \{W_1 - \lambda_k\} = \infty$ for $\lambda \geq 0$ and $\mathcal{H}_1 = \mathcal{H}(t_2 + l_2 + 1)$. By the properties of $\{A_i\}$ and $\{C_i\}$, we have $\min \text{ ind}(W_2 - \lambda) = 0$ for $\lambda \in \rho_{\text{re}}(T) \cap \sigma(T)$,

$$\text{ind}(W_2 - \lambda) = \begin{cases} \min \text{ ind}(T - \lambda), & \lambda \in \rho_{\text{re}}(T), \\ 1, & \lambda \in \rho_{\text{re}}(T) \cap \sigma(T). \end{cases}$$
and there is a sequence \( \{ \lambda_k \}_{k=-1}^{\infty} \) of numbers such that \( \text{null} (W_2 - \lambda_k) = \infty \) \( (k \leq -1) \)
and \( \sqrt{\text{ker}(W_2 - \lambda_k)} : k \leq -1 \) = \( \mathcal{H}_2 \).

It follows from \( G \cap \left( \bigcup_{k=1}^{\infty} \sigma_k \right) \cup \left( \bigcup \{ \Phi_{ij} : i = 1, 2, 3; j = 1, 2, \ldots, l_i \} \right) \) and the properties of \( E_0 \) that we have \( G \cap \sigma_p(W_2) = \emptyset \) and \( G^* \cap \sigma_p(W_1^*) = \emptyset \). Thus \( W \) satisfies (iii) and (iv) of the lemma. It is easy to see that \( W \) satisfies (i) and (ii). Thus the proof of the lemma is now complete.

3. PROOF OF THEOREM 1.1

In [13], we have proved that if \( \mathcal{N} \) is well-ordered with finite dimensional atoms, then \( \mathcal{U}(\text{alg} \mathcal{N} \cap (\text{SI})) = (\text{QT})c \). Thus we only need to show that if \( \mathcal{N} \) is maximal and \( \mathcal{N}^\perp \) are not well-ordered, then

\[
\mathcal{U}(\text{alg} \mathcal{N} \cap (\text{SI}))^\perp = \{ T \in \mathcal{L}(\mathcal{H}) : \sigma(T) \text{ connected} \}.
\]

Given an operator \( T \in \mathcal{L}(\mathcal{H}) \) with connected \( \sigma(T) \) and given \( \varepsilon > 0 \), by the theory of approximation of Hilbert space operators, there is an operator \( T_\varepsilon \in \mathcal{L}(\mathcal{H}) \) with \( \sigma(T_\varepsilon) \) connected such that \( \sigma_{\text{re}}(T_\varepsilon) \) is the closure of an analytic Cauchy domain and \( \| T - T_\varepsilon \| < \varepsilon \). Thus for the maximal nest \( \mathcal{N} \), with \( \mathcal{N} \) and \( \mathcal{N}^\perp \) not well-ordered, it suffices to show that for each operator \( T \) with connected \( \sigma(T) \) and whose \( \sigma_{\text{re}}(T) \) is the closure of an analytic Cauchy domain, we always can find an \( (\text{SI}) \) operator \( A \) in \( \text{alg} \mathcal{N} \) such that \( \| UAU^* - T \| < \varepsilon \), where \( U \) is a unitary operator, i.e., it is needed to show that

\[
\Delta := \{ T \in \mathcal{L}(\mathcal{H}) : \sigma(T) \text{ is connected and } \sigma_{\text{re}}(T) \text{ is the closure of an analytic Cauchy domain} \} \subset \mathcal{U}(\text{alg} \mathcal{N} \cap (\text{SI}))^\perp.
\]

If \( \mathcal{N} \) and \( \mathcal{N}^\perp \) are not well-ordered, there are three possibilities.

**Case A.** There are \( \{ t_n \}_{n=-\infty}^{\infty} \subset [0, 1] \) such that

\[
0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_{-n} < \cdots < t_{-2} < t_{-1} = 1,
\]

\[
\lim_{n \to -\infty} t_n = \lim_{n \to -\infty} t_{-n} \text{ and } \dim M_{(t_{n-1}, t_n)} = \infty \text{ (} n = \pm 1, \pm 2, \ldots \text{)}, \text{ where}
\]

\[
M_{(t_{n-1}, t_n)} = E((t_{n-1}, t_n]) \mathcal{H}
\]

and \( E \) is the spectral measure associated with \( \mathcal{N} \).

**Case B.** There are \( t_0, t_1, t_2, t_3 \in [0, 1] \), such that \( 0 < t_0 < t_1 < t_2 < t_3 < 1 \) and

\[
\mathcal{N}_0 := \{ M_t : 0 \leq t \leq t_0 \} \text{ is atomic,}
\]

\[
\mathcal{N}_1 := \{ M_t \ominus M_{t_0} : t \leq t_1 \} \text{ has the type } \omega + 1,
\]

\[
\mathcal{N}_2 := \{ M_t \ominus M_{t_1} : t_1 \leq t \leq t_2 \} \text{ is atomic,}
\]

\[
\mathcal{N}_3 := \{ M_t \ominus M_{t_2} : t_2 \leq t \leq t_3 \} \text{ has the type } 1 + \omega^*,
\]

\[
\mathcal{N}_4 := \{ M_t \ominus M_{t_3} : t_3 \leq t \leq 1 \} \text{ is atomic,}
\]

where \( M_t = M_{[0,t]} = E([0,t]) \mathcal{H} \).
Case C. There are \( t_0, t_1, t_2, t_3 \in [0, 1] \) such that \( 0 < t_0 < t_1 < t_2 < t_3 < 1 \) and
\[
\mathcal{N}_0 := \{ M_t : 0 \leq t \leq t_0 \} \text{ is atomic,}
\mathcal{N}_1 := \{ M_t \ominus M_{t_0} : 0 \leq t \leq t_1 \} \text{ has the type } 1 + \omega^*,
\mathcal{N}_2 := \{ M_t \ominus M_{t_1} : t_1 \leq t \leq t_2 \} \text{ is atomic,}
\mathcal{N}_3 := \{ M_t \ominus M_{t_2} : t_2 \leq t \leq t_3 \} \text{ has the type } \omega + 1,
\mathcal{N}_4 := \{ M_t \ominus M_{t_3} : t_3 \leq t \leq 1 \} \text{ is atomic.}
\]

In Case A, according to Lemma 2.10, there exists an operator \( A \in \text{(SI)} \) such that \( \Lambda(A) = \Lambda(T) \), \( \min \text{ ind } (A - \lambda) \leq \min \text{ ind } (T - \lambda) \) for \( \lambda \in \rho_{a-p}(A) \) and \( A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \mathcal{K}_1 \mathcal{K}_2 \), where
\[
A_1 = \begin{bmatrix} \lambda_1 & * \\ \lambda_2 & \lambda_3 \\ 0 & \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & * \\ \vdots & \lambda_{-2} \\ 0 & \lambda_{-1} \end{bmatrix},
\]

\( \mathcal{H}_n = \bigvee \{ \ker(A_1 - \lambda_k) : 1 \leq k \leq n \} \ominus \mathcal{H}_{n-1} \), \( \mathcal{H}_{-n} = \bigvee \{ \ker(A_2 - \lambda_k) : -n \leq k \leq -1 \} \ominus \mathcal{H}_{-n+1} (n = 1, 2, \ldots) \), \( \mathcal{H}_0 = \{0\} \), \( \dim \mathcal{H}_n = \infty (n = \pm 1, \pm 2, \ldots) \), \( \mathcal{K}_1 = \bigoplus_{n=1}^{\infty} \mathcal{H}_n \) and \( \mathcal{K}_2 = -\bigoplus_{n=-1}^{\infty} \mathcal{H}_n \), \( \{ \lambda_k : k = \pm 1, \pm 2, \ldots \} \) are given in Lemma 2.10 (iii).

By Similarity Orbit Theorem ([2]), \( T \in S(A)^{-} \), i.e., for each \( \varepsilon > 0 \), there exists an invertible operator \( X \) such that \( \| XAX^{-1} - T \| < \varepsilon \). It is easily seen that \( XAX^{-1} \) admits a same matrix representation with respect to another decomposition of the space,
\[
XAX^{-1} = \begin{bmatrix} \lambda_1 & * \\ \lambda_2 & \lambda_3 \\ & \ddots \\ & & \lambda_{-2} \\ 0 & & & \lambda_{-1} \end{bmatrix}
\]
where \( \dim M_n = \infty (n = \pm 1, \pm 2, \ldots) \).

Choose a unitary operator \( U \) so that \( U M_n = M_{(t_n, t_n]} \) \( (n = \pm 1, \pm 2, \ldots) \), then \( UXAX^{-1} U^* \in \text{alg} \mathcal{N} \cap \text{(SI)} \), i.e., \( T \in U(\text{alg} \mathcal{N} \cap \text{(SI)})^{-}. \)

If B is the case, for simplicity we only prove the conclusion of the theorem when \( t_0 = 0 \) and \( t_3 = 1 \). Denote the operator \( A \) in Case A by \( A_1 \) which satisfies (i), (ii), (iii) and (iv) of Lemma 2.10. Let \( \{ f_\alpha \}_{\alpha \in \Lambda} \) be the unit vectors of the atoms of \( \mathcal{N}_2 \), \( \bigvee \{ f_\alpha : \alpha \in \Lambda \} = M_{t_2} \ominus M_{t_1} \). Assume that \( G \) is the open disc contained in \( \sigma_{\text{pre}}(A) \) given in Lemma 2.10 (iv), then choose \( c_\alpha \in G \) \( (\alpha \in \Lambda) \) such that \( \{ c_\alpha \} \) is pairwise distinct and define \( A_3 = \sum c_\alpha f_\alpha \ominus f_\alpha. \) By the construction of \( A_1 \) in Lemma 2.10, \( G \subset \sigma_{\text{pre}}(A_1) \). Thus for each \( \alpha \) there is a unit vector \( g_\alpha \in \mathcal{K}_1 \) such
that \( g_\alpha \notin \text{ran} (A_1 - c_\alpha) \). Let \( \{d_\alpha\}_{\alpha \in \Lambda} \) be positive numbers satisfying \( \sum_{\alpha \in \Lambda} d_\alpha = 1 \).

Set \( K = \sum_{\alpha \in \Lambda} d_\alpha g_\alpha \otimes f_\alpha \) and

\[
A = \begin{bmatrix} A_1 & K & A_{12} \\ 0 & A_3 & 0 \\ 0 & 0 & A_2 \end{bmatrix},
\]

Then it is easily seen that \( \Lambda(A) = \Lambda(T) \) and \( \text{min ind } (A - \lambda) \leq \text{min ind } (T - \lambda) \) for \( \lambda \in \rho_{s-T}(T) \). By Lemma 2.10 (iii), (iv) we have \( \ker \tau_{A_3 A_1} = \ker \tau_{A_2 A_3} = \{0\} \).

Assume that \( P \) is an idempotent commuting with \( A \) and

\[
P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}
\]

then by Lemma 2.1, \( P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \). Observe that \( P' = \begin{bmatrix} P_{11} & 0 & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \)

is an idempotent commuting with \( \begin{bmatrix} A_1 & 0 & A_{12} \\ 0 & 0 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \) and \( \Lambda' = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \in (SI) \), thus \( P' = 0 \) or 1. Without loss of generality we can assume that \( P' = 0 \), or \( P = \begin{bmatrix} 0 & P_{12} & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Since \( PA = AP_1 \), \( P_{12} A_3 = A_1 P_{12} + KP_2 \). It follows from \( P_{22} A_3 = A_3 P_{22} \) and pairwise distinction of \( c_\alpha \)'s that \( P_{22} = \bigoplus_{\alpha \in \Lambda} \delta_\alpha \), where \( \delta_\alpha = 0 \) or 1. Thus for each \( \alpha \in \Lambda \)

\[
(A_1 P_{12} - P_{12} A_3) f_\alpha = A_1 P_{12} f_\alpha - c_\alpha P_{12} f_\alpha = -\delta_\alpha d_\alpha g_\alpha.
\]

Since \( g_\alpha \notin \text{ran} (A_1 - c_\alpha) \), \( \delta_\alpha = 0 \). Therefore \( P = 0 \) and \( A \in (SI) \). By Similarity Orbit Theorem ([2]), \( T \in S(A)^- \), i.e., for each \( \varepsilon > 0 \) there exists an invertible operator \( X \) such that \( \|XAX^{-1} - T\| < \varepsilon \). By Lemma 2.10 (iii), \( A_1 \) and \( A_2 \) admit upper triangular matrix representations

\[
A_1 = \begin{bmatrix} \lambda_0 & * & e_1^1 \\ \lambda_1 & * & e_1^2 \\ 0 & \cdots & \lambda_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda_{-3} & * & e_2^2 \\ \lambda_{-2} & * & e_2^1 \\ 0 & \cdots & \lambda_{-1} \end{bmatrix}
\]

with respect to some \( \text{ONB}\{e_1^i\}_{i=0}^\infty \) of \( K_1 \) and, respectively, \( \text{ONB}\{e_2^i\}_{i=0}^\infty \) of \( K_2 \).

Set

\[
\mathcal{M} = \begin{cases} \bigvee_{i=1}^\infty \{e_1^i\} (n = 0, 1, 2, \ldots); & \bigvee_{i=1}^\infty \{e_2^i\} \oplus N (N \in \mathcal{N}_2); \\
\bigvee_{i=1}^\infty \{e_1^i\} \oplus (M_{t_2} \oplus M_{t_1}) \oplus \bigvee_{j=n}^\infty \{e_2^j\} (n = 0, 1, 2, \ldots). & \end{cases}
\]

then \( \mathcal{M} \) is a maximal atomic nest, and unitarily equivalent to \( \mathcal{N} \). Thus, there exists a unitary operator \( U \) such that \( UXAX^{-1}U^* \in \text{alg} \mathcal{N} \). Therefore \( T \in \text{U(}\text{alg} \mathcal{N} \cap (SI))^\perp \).
For Case C, we only prove the conclusion of the theorem when \( t_1 = t_2 \).

According to Lemma 2.13 we get an operator \( W \in (SI) \) satisfying (i)–(iv) of Lemma 2.13. Let \( W = \begin{bmatrix} W_1 & W_{12} \\ 0 & W_2 \end{bmatrix} \mathcal{H}_1 \) \( \mathcal{H}_2 \).

Let \( N = \bigcap \{ M_n : -\infty < n < \infty \} \), \( N = \bigvee \{ M_n : -\infty < n < \infty \} \). Let \( \mathcal{N}_c = \{ M_\ell \cap M_{t_1} : t_3 \leq t \leq 1 \} \). Let \( \{ f_\alpha \}_{\alpha \in \Lambda_1} \) and \( \{ g_\beta \}_{\beta \in \Lambda_2} \) be the unit vectors of the atoms of \( \mathcal{N}_c \), and, respectively, \( \mathcal{N}_c \). Define \( B_1 = \sum \alpha \in \Lambda_1 c_\alpha f_\alpha \otimes f_\alpha \) and \( B_2 = \sum \beta \in \Lambda_2 d_\beta g_\beta \otimes g_\beta \), where \( \{ c_\alpha, \alpha \in \Lambda_1 ; d_\beta, \beta \in \Lambda_2 \} \subseteq G \subseteq \sigma_{\text{loc}}(W) \) are pairwise distinct and \( G \) is given in Lemma 2.13 (iv). By the similar way of Case B, construct operators \( E_1 \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \bigvee \{ f_\alpha : \alpha \in \Lambda_1 \}) \) and \( E_2 \in \mathcal{L}(\bigvee \{ g_\beta : \beta \in \Lambda_2 \}, \mathcal{H}_1 \oplus \mathcal{H}_2) \) such that \( E_1 f_\alpha \notin \text{ran} (W_1 - c_\alpha)^*, E_2 g_\beta \notin \text{ran} (W_2 - d_\beta) \) \( \alpha \in \Lambda_1, \beta \in \Lambda_2 \).

Set

\[
A = \begin{bmatrix} B_1 & E_1 & 0 \\ 0 & W & E_2 \\ 0 & 0 & B_2 \end{bmatrix} \bigvee \{ f_\alpha : \alpha \in \Lambda_1 \} \bigvee \{ g_\beta : \beta \in \Lambda_2 \}.
\]

By the same argument of Case B, \( A \in (SI) \) and \( T \in S(A)^- \). Thus for each \( \epsilon > 0 \), \( \| XAX^{-1} - T \| < \epsilon \) for some invertible operator \( X \). Note that by (i), (ii) and (iii) of Lemma 2.13

\[
W = \begin{bmatrix} * & \cdots \\ \lambda_{-2} & * \\ \lambda_{-1} & \cdots \\ \lambda_0 & \cdots \\ 0 & \lambda_1 & \cdots \\ \lambda_2 & \cdots \\ \cdots & \cdots \end{bmatrix}
\]

with respect to some ONB \( \{ e_n \}_{n=\infty} \) of \( \mathcal{H}_1 \oplus \mathcal{H}_2 \). Thus by the argument of Case B, there is a unitary operator \( U \) such that \( UXAX^{-1}U^* \in \text{alg} \mathcal{N} \) and therefore \( T \in \mathcal{U}(\text{alg} \mathcal{N} \cap (SI))^- \). The proof of the theorem is now complete.

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