ON AUTOMORPHISMS OF C*-ALGEBRAS
ASSOCIATED WITH SUBSHIFTS

KENGO MATSUMOTO

Communicated by Norberto Salinas

ABSTRACT. We prove that, for a given one-sided subshift $X_\Lambda$, any non-trivial automorphism of the subshift always yields an outer automorphism of the C*-algebra $O_\Lambda$ associated with the subshift. In particular, any non-trivial automorphism of the one-sided topological Markov shift $X_\Lambda$ for a $\{0, 1\}$-matrix $A$ yields an outer automorphism of the Cuntz-Krieger algebra $O_A$. We also determine the form of the automorphisms of the C*-algebra $O_\Lambda$ arising from automorphisms of the subshift $X_\Lambda$.

KEYWORDS: C*-algebras, automorphisms, subshifts, Cuntz-Krieger algebras.

MSC (2000): Primary 46L40; Secondary 58F03.

1. INTRODUCTION

In [22], the author has introduced and studied a class of C*-algebras associated with subshifts in the theory of symbolic dynamics. Each of the C*-algebras associated with subshifts has canonical generators of partial isometries with mutually orthogonal ranges. It also has universal properties subject to some operator relations among the generators ([22], [25]) so that it becomes purely infinite and simple in many cases including Cuntz-Krieger algebras. For a subshift $(\Lambda, \sigma)$, we denote by $X_\Lambda$ the set of all right-infinite sequences that appear in $\Lambda$. The dynamical system $(X_\Lambda, \sigma)$, simply written as $X_\Lambda$, is called the one-sided subshift for $\Lambda$. The C*-algebra $O_\Lambda$ associated with subshift $\Lambda$ is essentially constructed by the dynamics $(X_\Lambda, \sigma)$. Many dynamical property for $(X_\Lambda, \sigma)$ reflects on algebraic structure on the C*-algebra $O_\Lambda$ as in [22], [24].

We will in this paper study relationships between automorphisms of the dynamics $X_\Lambda$ and automorphisms of the algebra $O_\Lambda$. A homeomorphism $h$ of $X_\Lambda$ satisfying $h = \sigma \circ h \circ \sigma^{-1}$ is called an automorphism of $X_\Lambda$. We denote by $\text{Aut}(X_\Lambda)$ the set of all automorphisms of $X_\Lambda$. There have been many studies on
automorphisms of subshifts especially of topological Markov shifts (cf. [3], [19], . . .). Their studies are closely related to classification of subshifts (cf. [19], [29]).

Let \((\Lambda, \sigma)\) be a subshift over a finite set \(\Sigma = \{1, 2, \ldots, n\}\) with shift transformation \(\sigma\). Then the \(C^*\)-algebra \(\mathcal{O}_\Lambda\) associated with the subshift is generated by \(n\) canonical partial isometries \(S_1, S_2, \ldots, S_n\). One typical example of automorphisms of \(\mathcal{O}_\Lambda\) is defined by a mapping for \(t \in \mathbb{R}: S_j \rightarrow e^{\sqrt{-1}t}S_j, j = 1, 2, \ldots, n\). These automorphisms are called the gauge automorphisms. The fixed point algebra of the \(C^*\)-algebra \(\mathcal{O}_\Lambda\) under the gauge automorphisms is an AF-algebra which is written as \(\mathcal{F}_\Lambda\) ([22]). We denote by \(\mathcal{D}_\Lambda\) the \(C^*\)-algebra of all diagonal elements of \(\mathcal{F}_\Lambda\), which is commutative. The commutative \(C^*\)-algebra \(C(\sigma(\mathcal{A})), \sigma\), denoted by \(\mathcal{D}_\Lambda\), of all continuous functions on \(X\) is naturally embedded into the algebra \(\mathcal{D}_\Lambda\). Hence each automorphism \(h\) of \(X\) yields an automorphism \(h^*\) of the subalgebra \(\mathcal{D}_\Lambda\) of \(\mathcal{O}_\Lambda\). The induced endomorphism of \(\mathcal{D}_\Lambda\) from the shift \(\sigma\) of \(X\) is uniquely extended to an endomorphism \(\varphi\) of \(\mathcal{D}_\Lambda\) that is defined by \(\varphi(X) = \sum_{j=1}^{n} S_jX S_j^*\) for \(X \in \mathcal{D}_\Lambda\). They satisfy the relation \(h^* \circ \varphi = \varphi \circ h^*\) on \(\mathcal{D}_\Lambda\). We first see the following:

**Proposition 1.1.** (Proposition 4.2) For an automorphism \(h\) of \(X\), there exists an automorphism \(\alpha_h\) of \(\mathcal{O}_\Lambda\) such that \(\alpha_h(x) = h^*(x), x \in \mathcal{D}_\Lambda\) and the correspondence \(h \in \text{Aut}(X) \rightarrow \alpha_h \in \text{Aut}(\mathcal{O}_\Lambda)\) gives rise to a homomorphism.

Let \(\text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\) be the set of all automorphisms of \(\mathcal{O}_\Lambda\) whose restrictions to the algebra \(\mathcal{D}_\Lambda\) give rise to automorphisms of \(X\). Namely that is the group of automorphisms of \(\mathcal{O}_\Lambda\) coming from \(\text{Aut}(X)\). The extension of \(h \in \text{Aut}(X)\) to an automorphism of \(\mathcal{O}_\Lambda\) is not necessarily unique. By proving the result:

\[
\mathcal{D}_\Lambda \cap \mathcal{O}_\Lambda = \mathcal{D}_\Lambda
\]
as in Proposition 3.3, we see that any automorphism of \(X\) may be uniquely extended to an automorphism of \(\mathcal{O}_\Lambda\) modulo unitaries in \(\mathcal{D}_\Lambda\). That is, for an automorphism \(h\) of \(X\), if two automorphisms \(\alpha^h, \beta^h\) of \(\mathcal{O}_\Lambda\) coincide with \(h^*\) on \(X\), then \(\alpha^h = \beta^h \circ \lambda(U)\) for some unitary \(U \in \mathcal{D}_\Lambda\) where \(\lambda(U) \in \text{Aut}(\mathcal{O}_\Lambda)\) is defined to be \(\lambda(U)(S_i) = US_i\) (Corollary 4.10). We denote by \(\mathcal{U}(\mathcal{D}_\Lambda)\) the group of all unitaries in \(\mathcal{D}_\Lambda\). Let \(Z^2_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) \equiv \mathcal{U}(\mathcal{D}_\Lambda)\) be the set of all unitary one-cocycles for \(\varphi\) of \(\mathcal{U}(\mathcal{D}_\Lambda)\) that are defined to be \(\mathcal{U}(\mathcal{D}_\Lambda)\)-valued functions \(U\) from \(\mathbb{N}\) such that \(U(k + l) = U(k)\varphi_k(U(l))\), \(k, l \in \mathbb{N}\). We will in fact prove

**Theorem 1.2.** (Theorem 4.9) There exists a natural short exact sequence:

\[
0 \rightarrow Z^2_\sigma(\mathcal{U}(\mathcal{D}_\Lambda)) \rightarrow \text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) \rightarrow \text{Aut}(X) \rightarrow 0
\]

that splits.

We will next study outerness for the automorphisms of \(\mathcal{O}_\Lambda\) coming from automorphisms of \(X\). We introduce a condition for an automorphism of \(X\) called condition (I). The condition is considered as a relative version to the condition (I) for the original dynamics \((X, \sigma)\). We will show that if a non-trivial automorphism of \(X\) satisfies the condition (I), its extension to an automorphism of \(\mathcal{O}_\Lambda\) is outer (Theorem 5.2). We will also prove that any extension as an automorphism of \(\mathcal{O}_\Lambda\) of a non-trivial automorphism of \(X\) is always outer if \(X\) satisfies a certain aperiodicity condition called (D) (Theorem 5.12). In particular, any extension
of a non-trivial automorphism of a topological Markov shift $X_A$ for an aperiodic matrix $A$ to an automorphism of the Cuntz-Krieger algebra $O_A$ is outer. We see that the automorphism $\lambda(u)$ of $O_A$ for a unitary $u$ in $A$ is inner if and only if $u$ gives rise to a coboundary for $\varphi_A$ in $U(D_A)$. Let $B^1(U(D_A))$ be the subgroup of all coboundaries in $Z^1(U(D_A))$. Set $H^1(U(D_A)) = Z^1(U(D_A))/B^1(U(D_A))$ the one-cohomology group for $\varphi_A$ of $U(D_A)$.

**Theorem 1.3.** (Theorem 5.16) There exists a natural short exact sequence:

$$0 \rightarrow H^1(U(D_A)) \rightarrow \text{Out}_\sigma(O_A, D_A) \rightarrow \text{Aut}(X_A) \rightarrow 0$$

that splits, where $\text{Out}_\sigma(O_A, D_A)$ means the group of all outer automorphisms of $O_A$ in $\text{Aut}_\sigma(O_A, D_A)$.

We will, in the final section, present certain examples of automorphisms of the $C^*$-algebra coming from some subshifts. We will further see that if a subshift $X_A$ has a fixed point, then the non-trivial gauge automorphisms are outer (Corollary 6.5).

Slightly similar exact sequences to the above two exact sequences have appeared in a discussion of classification of von Neumann algebras arising from non-singular ergodic transformation (cf. [20]). The classification has exactly corresponds to orbit equivalences of such ergodic transformations (cf. [5], [10], [20]). $C^*$-algebraic analogies have also been discussed in [4], [15], [27], etc. If a subshift $\Lambda$ is a topological Markov shift and, in particular, a full shift, the associated $C^*$-algebra $O_\Lambda$ becomes a Cuntz-Krieger algebra and a Cuntz algebra respectively. Hence our study, in this paper, includes studies of automorphisms of these algebras from a view point of symbolic dynamical systems. Studies of automorphisms of Cuntz-Krieger algebras and Cuntz algebras are seen in many papers as in [1], [8], [12], [13], [18], [26], [28], ... The author has recently received a preprint [18] by Katayama-Takehana in which outerness of automorphisms of Cuntz-Krieger algebras are discussed by using a technique of Hilbert $C^*$-bimodules (cf. [17]).

2. **BASIC NOTATION AND THE $C^*$-ALGEBRA $O_\Lambda$**

Let $\Sigma$ be a finite set $\{1, 2, \ldots, n\}$ for $n > 1$. Let $\Sigma^Z$, $\Sigma^N$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation $\sigma$ on $\Sigma^Z$, $\Sigma^N$ given by $(\sigma(x))_i = x_{i+1}, i \in \mathbb{Z}, \mathbb{N}$ for $x = (x :)$ is called the (full) shift. Let $\Lambda$ be a shift invariant closed subset of $\Sigma^Z$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|\Lambda)$ is called a subshift. We denote $\sigma|\Lambda$ by $\sigma$ for simplicity. This class of the subshifts includes the class of the topological Markov shifts (cf. [19], [21]).

A finite sequence $\mu = (\mu_1, \ldots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length $k$ of $\mu$. A block $\mu = (\mu_1, \ldots, \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^Z$ if $x_m = \mu_1, \ldots, x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$. For $x = (x_i) \in \Sigma^Z$ or $\Sigma^N$ and $i \leq j$, we write

$$x_{[i,j]} = (x_i, x_{i+1}, \ldots, x_j), \quad x_{[i,\infty)} = (x_i, x_{i+1}, \ldots) \in \Sigma^N.$$
For a subshift \((\Lambda, \sigma)\), let \(\Lambda^k\) be the set of all words with length \(k\) in \(\Sigma^\mathbb{Z}\) occurring in some \(x \in \Lambda\). Put \(\Lambda_l = \bigcup_{k=0}^l \Lambda^k\) for \(l \in \mathbb{N}\) and \(\Lambda^* = \bigcup_{k=0}^\infty \Lambda^k\) where \(\Lambda^0\) denotes the empty word \(\emptyset\). Let \(X_\Lambda\) be the set of all right-infinite sequences that appear in \(\Lambda\). The dynamical system \((X_\Lambda, \sigma)\) is called the one-sided subshift for \(\Lambda\).

Put \(\Lambda^l(x) = \{\mu \in \Lambda^l \mid \mu x \in X_\Lambda\}\) for \(x \in X_\Lambda, l \in \mathbb{N}\). We define equivalence relations in the space \(X_\Lambda\). For \(l \in \mathbb{N}\), two points \(x, y \in X_\Lambda\) are said to be \(l\)-past equivalent if \(\Lambda^l(x) = \Lambda^l(y)\). We write this equivalence as \(x \sim_l y\) (cf. [24]).

**Definition.** ([24]) (i) A subshift \((X_\Lambda, \sigma)\) satisfies condition (I) if for any \(l \in \mathbb{N}\) and \(x \in X_\Lambda\), there exists \(y \in X_\Lambda\) such that \(y \neq x\) and \(y \sim_l x\).

(ii) A subshift \((X_\Lambda, \sigma)\) is irreducible in past equivalence if for any \(l \in \mathbb{N}\), \(y \in X_\Lambda\) and a sequence \((x^k)_{k \in \mathbb{N}}\) of \(X_\Lambda\) with \(x^k \sim_k x^{k+1}\) for \(k \in \mathbb{N}\), there exist a number \(N\) and a word \(\mu \in \Lambda^N\) such that \(y \sim_{l-N} \mu x^{l+N}\).

(iii) A subshift \((X_\Lambda, \sigma)\) is aperiodic in past equivalence if for any \(l \in \mathbb{N}\), there exists a number \(N\) such that for any pair \(x, y \in X_\Lambda\), there exists a word \(\mu \in \Lambda^N\) such that \(y \sim_l \mu x^N\).

If a subshift \((X_\Lambda, \sigma)\) is aperiodic in past equivalence or irreducible in past equivalence with an aperiodic point, then it satisfies condition (I) ([24]). If a subshift \((X_\Lambda, \sigma)\) is a topological Markov shift \((X_\Lambda, \sigma)\) determined by a square matrix \(A\) with entries in \(\{0, 1\}\), the above aperiodicity, irreducibility and condition (I) as a subshift coincide with the aperiodicity, irreducibility and condition (I) let it stand as it is in [9] for the matrix \(A\) respectively.

Now we will review the construction of the \(C^*\)-algebras associated with subshifts along [22]. We henceforth fix an arbitrary subshift \((\Lambda, \sigma)\). Let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of \(n\)-dimensional Hilbert space \(\mathbb{C}^n\).

We put

\[
F_\Lambda^0 = \mathbb{C}e_0 \quad (e_0: \text{vacuum vector});
\]

\[
F_\Lambda^k = \text{the Hilbert space spanned by the vectors } e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}, \mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k;
\]

\[
F_\Lambda = \bigoplus_{k=0}^\infty F_\Lambda^k \quad (\text{Hilbert space direct sum}).
\]

We denote by \(T_\nu, (\nu \in \Lambda^*)\) the creation operator on \(F_\Lambda\) of \(e_\nu, \nu \in \Lambda^*(\nu \neq \emptyset)\) defined by

\[
T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu & (\nu \mu \in \Lambda^*), \\ 0 & \text{else}, \end{cases}
\]

which is a partial isometry. We put \(T_0 = 1\) for \(\nu = \emptyset\). Let \(P_0\) be the rank one projection onto the vacuum vector \(e_0\). It immediately follows that \(\sum_{i=1}^n T_i T_i^* + P_0 = 1\). We then easily see that for \(\mu, \nu \in \Lambda^*\), the operator \(T_\mu P_0 T_\nu^*\) is the rank one partial isometry from the vector \(e_\nu\) to \(e_\mu\). Hence, the \(C^*\)-algebra generated by elements of the form \(T_\mu P_0 T_\nu^*\), \(\mu, \nu \in \Lambda^*\) is nothing but the \(C^*\)-algebra \(\mathcal{K}(F_\Lambda)\) of all compact operators on \(F_\Lambda\). Let \(T_\Lambda\) be the \(C^*\)-algebra on \(F_\Lambda\) generated by the elements \(T_\nu, \nu \in \Lambda^*\).
DEFINITION. ([22]) The $C^*$-algebra $O_\Lambda$ associated with subshift $(\Lambda, \sigma)$ is defined as the quotient $C^*$-algebra $T_\Lambda/K(F_\Lambda)$ of $T_\Lambda$ by $K(F_\Lambda)$.

We denote by $S_i, S_n$ the quotient images of the operators $T_i, i \in \Sigma, T_\mu, \mu \in \Lambda^*$ respectively. Hence $O_\Lambda$ is generated by partial isometries $S_1, \ldots, S_n$ with relation $\sum_{i=1}^n S_i S_i^* = 1$.

If $(\Lambda, \sigma)$ is a topological Markov shift, the $C^*$-algebra $O_\Lambda$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [9], [11], [13]).

We will present notation and basic facts for studying the $C^*$-algebra $O_\Lambda$.

Put $a_\mu = S_i^* S_\mu, \mu \in \Lambda^*$. Since $T_\nu T_\mu^*$ commutes with $T_\mu T_\nu^*, \mu, \nu \in \Lambda^*$, the following identities hold

\[
(*) \quad a_\mu S_\nu = S_\nu a_\mu, \quad \mu, \nu \in \Lambda^*.
\]

We notice that for $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$,

\[
S_\mu S_\nu \neq 0 \quad \text{if and only if} \quad \mu = \nu.
\]

We will use the following notation. Let $k, l$ be natural numbers with $k \leq l$.

- $A_l = \text{The } C^*$-subalgebra of $O_\Lambda$ generated by $a_\mu, \mu \in A_l$.
- $A_\Lambda = \text{The } C^*$-subalgebra of $O_\Lambda$ generated by $a_\mu, \mu \in \Lambda^*$.
- $D_\Lambda = \text{The } C^*$-subalgebra of $O_\Lambda$ generated by $S_\mu S_\mu^*, \mu \in \Lambda^*$.
- $F_\Lambda = \text{The } C^*$-subalgebra of $O_\Lambda$ generated by $S_\mu S_\mu^*, \mu \in \Lambda^*$.
- $F_\infty = \text{The } C^*$-subalgebra of $O_\Lambda$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^*, \mu = |\nu|, a \in A_\Lambda$.

The projections $\{T_\mu T_\nu^*; \mu \in \Lambda^*\}$ are mutually commutative so that the $C^*$-algebras $A_l, l \in \mathbb{N}$ are commutative. Thus we easily see the following lemma (cf. [22], Section 3).

**Lemma 2.1.** (i) $A_l$ is finite dimensional and commutative.
(ii) $A_l$ is naturally embedded into $A_{l+1}$ so that $A_\Lambda = \lim_{l \to \infty} A_l$ is a commutative AF-algebra.
(iii) Each element of $F_\Lambda$ is a finite linear combination of elements of the form $S_\mu S_\nu^*, \mu, \nu \in \Lambda^*, a \in A_l$. Hence $F_\Lambda$ is finite dimensional.
(iv) There are two embeddings in $\{F_\Lambda\}_{k \leq l}$:
   (a) $i_l : F_\Lambda \subset F_{k+1}^l$ through the embedding $A_l \subset A_{l+1}$ and,
   (b) $\eta_k : F_k \subset F_{k+1}^l$ through the identity

\[
S_\mu a S_\nu^* = \sum_{j=1}^n S_{\mu j} S^*_j a S_{\nu j}, \quad \mu, \nu \in \Lambda^k, a \in A_l.
\]

(v) Both $F_\infty = \lim_{l \to \infty} F_\Lambda$ and $F_\infty = \lim_{k \to \infty} F_k^\infty$ are AF-algebras.
In the preceding Hilbert space $F_\Lambda$, the transformation $e_\mu \to z^k e_\mu$, $\mu \in \Lambda^k$, $z \in \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ on each base $e_\mu$ yields a unitary representation which leaves $K(F_\Lambda)$ invariant. Thus it gives rise to an action $\alpha$ of $\mathbb{T}$ on the $C^*$-algebra $\mathcal{O}_\Lambda$. It is called the gauge action and satisfies $\alpha_z(S_i) = z S_i$, $i = 1, 2, \ldots, n$.

Each element $X$ of the $*$-subalgebra of $\mathcal{O}_\Lambda$ algebraically generated by $S_\mu, S_\nu^*$, $\mu, \nu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|\nu| \geq 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu| \geq 1} S_\mu X_\mu$$

for some $X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\Lambda$

because of the relation $(\ast)$. The map $E(X) = \int_{z \in \mathbb{T}} \alpha_z(X)dz$, $X \in \mathcal{O}_\Lambda$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_\Lambda^\alpha$ under $\alpha$. We then have (cf. [22], Proposition 3.11)

**Lemma 2.2.** $\mathcal{F}_\Lambda = \mathcal{O}_\Lambda^\alpha$.

Note that the $C^*$-algebra $\mathcal{D}_\Lambda$ is isomorphic to the commutative $C^*$-algebra $C(X_\Lambda)$ of all complex valued continuous functions on the one-sided subshift $X_\Lambda$ for $\Lambda$. Put

$$\varphi_\Lambda(X) = \sum_{j=1}^n S_j X S_j^*, \quad X \in \mathcal{D}_\Lambda \ (\text{or} \ X \in \mathcal{O}_\Lambda)$$

which corresponds to the shift $\sigma$ of $X_\Lambda$.

Consider the following condition called $(I_\Lambda)$ for the $C^*$-algebra $\mathcal{O}_\Lambda$ (cf. [22]).

$(I_\Lambda)$: For any $l, k \in \mathbb{N}$ with $l \geq k$, there exists a projection $q_k^l$ in $\mathcal{D}_\Lambda$ such that

(i) $q_k^l a q_k^l \neq 0$ for any nonzero $a \in \mathcal{A}_\Lambda$;

(ii) $q_k^l q_k^m q_k^l = 0$, $1 \leq m \leq k$.

As in [24], the subshift $(X_\Lambda, \sigma)$ satisfies condition $(I_\Lambda)$ if and only if the $C^*$-algebra $\mathcal{O}_\Lambda$ satisfies condition $(I_\Lambda)$. Hence we may describe structure theorems for the $C^*$-algebra $\mathcal{O}_\Lambda$ proved in [22].

**Lemma 2.3.** ([22], Theorems 4.9 and 5.2) Let $\mathfrak{A}$ be a unital $C^*$-algebra. Suppose that there is a unital $*$-homomorphism $\pi$ from $\mathcal{A}_\Lambda$ to $\mathfrak{A}$ and there are $n$ partial isometries $s_1, \ldots, s_n \in \mathfrak{A}$ satisfying the following relations

$$\sum_{j=1}^n s_j s_j^* = 1, \quad s_\mu^* s_\nu = s_\nu s_\mu^*, \quad \mu, \nu \in \Lambda^*,$n$$

$$s_\mu^* s_\mu = \pi(s_\mu^* s_\mu), \quad \mu \in \Lambda^*$$

where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}$, $\mu = (\mu_1, \ldots, \mu_k)$. Then there exists a unital $*$-homomorphism $\bar{\pi}$ from $\mathcal{O}_\Lambda$ to $\mathfrak{A}$ such that $\bar{\pi}(S_i) = s_i$, $i = 1, \ldots, n$ and its restriction to $\mathcal{A}_\Lambda$ coincides with $\pi$. In addition, if the subshift $X_\Lambda$ satisfies condition $(I_\Lambda)$, this extended homomorphism $\bar{\pi}$ becomes injective whenever $\pi$ is injective.

**Lemma 2.4.** ([22], Theorem 6.3 and Theorem 7.5 and [24], Theorem 5.8) If a subshift $X_\Lambda$ is irreducible in past equivalence and has an aperiodic point, then $\mathcal{O}_\Lambda$ is simple. In addition, if a subshift $X_\Lambda$ is aperiodic in past equivalence, the $C^*$-algebra $\mathcal{O}_\Lambda$ is simple and purely infinite.

We notice the following lemma.
**Lemma 3.1.**

\[ \mathcal{D}_\Lambda' \cap \mathcal{O}_\Lambda \subset \mathcal{F}_\Lambda. \]

**Proof.** Assume that \( X \in \mathcal{O}_\Lambda \) commutes with each element of \( \mathcal{D}_\Lambda \). For a nonempty word \( \mu \in \Lambda^* \), put \( X_\mu = E(S_\mu^* X) \), \( X_{-\mu} = E(X S_\mu) \). We will show that \( X_\mu = X_{-\mu} = 0 \). For \( f \in \mathcal{D}_\Lambda \), we see \( E(S_\mu^* X f) = E(S_\mu^* f S_\mu^* X) \) so that

\[ X_\mu f = S_\mu^* f S_\mu X_\mu. \]

We in particular have

\[ X_\mu = X_\mu S_\mu S_\mu^* \quad X_\mu S_\mu f S_\mu^* = f X_\mu. \]

Let \( i \) be the length of \( \mu \). It then follows that

\[ X_\mu \varphi_\Lambda(f) = X_\mu S_\mu S_\mu^* \sum_{\nu \in \Lambda^i} S_\nu f S_\nu^* = X_\mu S_\mu f S_\mu^*. \]

Thus we obtain

\[ X_\mu \varphi_\Lambda(f) = f X_\mu, \quad f \in \mathcal{D}_\Lambda. \]

Now suppose that \( X_\mu \neq 0 \). For any \( \varepsilon > 0 \), take \( X_\mu(m) \in \mathcal{F}_{m_k} \) such that

\[ \| X_\mu - X_\mu(m) \| \varepsilon \quad \text{for some} \quad m_l \geq m_k \geq i \quad \text{and assume that} \quad \| X_\mu \| = \| X_\mu(m) \| = 1. \]

We then have

\[ \| f X_\mu(m) - X_\mu(m) \varphi_\Lambda(f) \| \leq 2 \| f \| \varepsilon. \]

Since \( \mathcal{O}_\Lambda \) satisfies condition (I\( \Lambda \)), for \( m_l \geq m_k \), there exists a projection \( q_{m_k}^{m_l} \in \mathcal{D}_\Lambda \) satisfying the condition (i), (ii) in condition (I\( \Lambda \)). Put \( Q(m) = \varphi_\Lambda q_{m_k}^{m_l} \in \mathcal{D}_\Lambda \). It is easy to see that \( Q(m) \) commutes with \( X_\mu(m) \). Hence we get

\[ \| X_\mu(m) Q(m) - X_\mu(m) \varphi_\Lambda(Q(m)) \| \leq 2 \varepsilon. \]

As \( Q(m) \) is orthogonal to \( \varphi_\Lambda(Q(m)) \) because of condition (I\( \Lambda \)), the correspondence \( Y \in \mathcal{F}_{m_l} \rightarrow Q(m) Y Q(m) \in Q(m) \mathcal{F}_{m_k} Q(m) \) yields an isomorphism and hence isometric by [22], Corollary 5.4. Hence we have \( \| X_\mu(m) Q(m) \| = \| X_\mu(m) \| = 1 \) so that

\[ \| X_\mu(m) Q(m) - X_\mu(m) \varphi_\Lambda(Q(m)) \| = \max\{ \| X_\mu(m) Q(m) \|, \| X_\mu(m) \varphi_\Lambda(Q(m)) \| \} \]

\[ \geq \| X_\mu(m) Q(m) \| = 1. \]

This is a contradiction for a sufficiently small \( \varepsilon \). Thus we conclude \( X_\mu = 0 \). We similarly have \( X_{-\mu} = 0 \). This means that \( X = E(X) \in \mathcal{F}_\Lambda. \)  

**3. THE COMMUTANT OF \( \mathcal{D}_\Lambda \) IN \( \mathcal{O}_\Lambda \)**

We henceforth fix an arbitrary subshift \( (X_\Lambda, \sigma) \) which satisfies condition (I). We denote by \( \mathcal{D}_\Lambda \) the \( C^* \)-subalgebra of \( \mathcal{F}_\Lambda \) consisting of all diagonal elements of \( \mathcal{F}_\Lambda \) as in the previous section. In this section, we will show that the commutant of the commutative \( C^* \)-algebra \( \mathcal{D}_\Lambda \) in \( \mathcal{O}_\Lambda \) is exactly the algebra \( \mathcal{D}_\Lambda \).

**Lemma 2.5.** ([22], Proposition 5.8 and [24], Lemma 4.5, cf. [9], 2.17 Proposition) Suppose that both subshifts \( (X_\Lambda_1, \sigma) \) and \( (X_\Lambda_2, \sigma) \) satisfy condition (I). If they are topologically conjugate, then there exists an isomorphism \( \Phi \) from \( \mathcal{O}_{\Lambda_1} \) onto \( \mathcal{O}_{\Lambda_2} \) such that \( \Phi \circ \alpha^i_1 = \alpha^i_2 \circ \Phi, \ z \in \mathbb{T} \) where \( \alpha^i \) is the gauge action on \( \mathcal{O}_{\Lambda_i} \), \( i = 1, 2 \) respectively.
**Lemma 3.2.**
\[ \mathcal{D}_\Lambda' \cap \mathcal{F}_\Lambda = \mathcal{D}_\Lambda. \]

**Proof.** It suffices to show the inclusion relation \[ \mathcal{D}_\Lambda' \cap \mathcal{F}_\Lambda \subset \mathcal{D}_\Lambda. \]

Set the algebras:

- \[ \mathcal{D}_k = \text{The } C^*\text{-subalgebra of } \mathcal{D}_\Lambda \text{ generated by } S_\mu a S_\mu^*, \mu \in \Lambda^k, a \in A_k. \]
- \[ \mathcal{D}_\infty^k = \text{The } C^*\text{-subalgebra of } \mathcal{D}_\Lambda \text{ generated by } S_\mu a S_\mu^*, \mu \in \Lambda^k, a \in A. \]
- \[ \mathcal{D}_\Lambda = \text{The } C^*\text{-subalgebra of } \mathcal{D}_\Lambda \text{ generated by } S_\mu S_\mu^*, \mu \in \Lambda^k. \]

Put \[ P_\mu = S_\mu S_\mu^* \text{ for } \mu \in \Lambda^k. \]

The map \[ \mathcal{E}_k \text{ defined by } \mathcal{E}_k(X) = \sum_{\mu \in \Lambda^k} P_\mu XP_\mu \text{ for } X \in \mathcal{F}_k \], yields an expectation from \[ \mathcal{F}_k \] to \[ \mathcal{D}_k \]. Since the restriction of \( \mathcal{E}_{k+1} \) to \[ \mathcal{F}_k \] coincides with \( \mathcal{E}_k \), the sequence of the expectations \( \{ \mathcal{E}_k \}_{k \in \mathbb{N}} \) gives rise to an expectation \( \varepsilon_\Lambda \) from \( \mathcal{F}_\Lambda \) onto \( \mathcal{D}_\Lambda \) such that \( \varepsilon_\Lambda | \mathcal{F}_k = \mathcal{E}_k \). Similarly the sequence of the expectations \( \{ \varepsilon_k \}_{k \in \mathbb{N}} \) gives rise to an expectation \( \varepsilon_\Lambda \) from \( \mathcal{F}_\Lambda \) onto \( \mathcal{D}_\Lambda \) such that \( \varepsilon_\Lambda | \mathcal{F}_\Lambda = \mathcal{E}_\Lambda \). Now let \( X \) be an element of \( \mathcal{F}_\Lambda \) which commutes with \( \mathcal{D}_\Lambda \). Since we have \( \varepsilon_\Lambda(X) = X \) for all \( k \in \mathbb{N} \), we see \( \varepsilon(X) = X \) so that \( X \) belongs to \( \mathcal{D}_\Lambda \).

Therefore we obtain

**Proposition 3.3.**
\[ \mathcal{D}_\Lambda' \cap \mathcal{O}_\Lambda = \mathcal{D}_\Lambda. \]

We also see

**Proposition 3.4.** (i) \( \mathcal{D}_\Lambda \) is a maximal abelian \( * \)-subalgebra of \( \mathcal{O}_\Lambda \).

(ii) There exists a faithful conditional expectation \( \varepsilon_\Lambda \) from \( \mathcal{O}_\Lambda \) onto \( \mathcal{D}_\Lambda \).

**4. Automorphisms of \( \mathcal{O}_\Lambda \) Coming From \( X_\Lambda \)**

Put

\[ U_\mu = \{ (x_1, x_2, \ldots) \in X_\Lambda \mid x_1 = \mu_1, x_2 = \mu_2, \ldots, x_k = \mu_k \} \]

the cylinder set for \( \mu = \mu_1 \cdots \mu_k \in \Lambda^k \). We denote by \( \chi_{U_\mu} \) the characteristic function of \( U_\mu \) on \( X_\Lambda \). The correspondence \( S_\mu S_\mu^* \rightarrow \chi_{U_\mu} \) yields an isomorphism from \( \mathcal{D}_\Lambda \) onto \( C(X_\Lambda) \).

**Lemma 4.1.** Let \( H_\Lambda \) be the Hilbert space with complete orthonormal basis \( \{ e_x \mid x \in X_\Lambda \} \). Let \( T_1, \ldots, T_n \) be the operators on \( H_\Lambda \) defined by

\[ T_j e_x = \begin{cases} e_{jx} & \text{if } jx \in X_\Lambda; \\ 0 & \text{otherwise.} \end{cases} \]

Then \( T_1, \ldots, T_n \) are partial isometries such that the correspondence \( S_j \rightarrow T_j \) yields a faithful nondegenerate representation of \( \mathcal{O}_\Lambda \) onto the \( C^* \)-algebra generated by \( T_1, \ldots, T_n \).

**Proof.** The assertion is easily shown from Lemma 2.3.

\[ \square \]
Now suppose that $O_\Lambda$ is represented on the Hilbert space $H_\Lambda$. For words $\mu \in \Lambda^k$, $\nu \in \Lambda^l$ with $k < l$, the projection $S_\mu a_\nu S_\mu^*$ exactly corresponds to the orthogonal projection onto the subspace spanned by the vectors: $e_x$ for $x \in U_\mu \cap \sigma^{-k}(\sigma(U_\nu))$. In particular, for a word $\nu = \tilde{\nu} \mu \in \Lambda^*$ with $|\tilde{\nu}| = m$, the projection $S_\mu a_\nu S_\mu^*$ is represented by the orthogonal projection onto the subspace spanned by the vectors: $e_x$ for $x \in \sigma^m(U_\nu)$.

For an automorphism $h$ of $X_\Lambda$, we denote by $h^*$ the induced automorphism of the algebra $\mathcal{D}_\Lambda$ defined by $h^*(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_\Lambda = C(X_\Lambda)$. By Lemma 2.5, we know that the automorphism $h^*$ of $\mathcal{D}_\Lambda$ may be extended to an automorphism of the $C^*$-algebra $O_\Lambda$. In the following proposition, we will give another proof of this fact and show that an extension can be taken in a homomorphic way.

**Proposition 4.2.** For an automorphism $h$ of $X_\Lambda$, there exists an automorphism $\alpha_h$ of $O_\Lambda$ such that $\alpha_h(x) = h^*(x)$, $x \in \mathcal{D}_\Lambda$ and the correspondence $h \in \text{Aut}(X_\Lambda) \rightarrow \alpha_h \in \text{Aut}(O_\Lambda)$ gives rise to a homomorphism.

**Proof.** We assume that $O_\Lambda$ is represented on the Hilbert space $H_\Lambda$. For an automorphism $h$ of $X_\Lambda$, put a unitary $V_h$ on $H_\Lambda$:

$$V_h e_x = e_{h(x)}, \quad x \in X_\Lambda.$$

We will show that $\text{Ad}(V_h)(O_\Lambda) = O_\Lambda$. Put $S'_i = \text{Ad}(V_h)(S_i)$, $i = 1, \ldots, n$ so that we see for $x \in X_\Lambda$

$$S'_i e_x = \begin{cases} e_{h(\text{ih}^{-1}(x))} \quad &\text{if } \text{ih}^{-1}(x) \in X_\Lambda; \\ 0 \quad &\text{otherwise.} \end{cases}$$

Set $Y_i = \{x \in X_\Lambda \mid \text{ih}^{-1}(x) \in X_\Lambda\}$ so that $Y_i = h(\sigma(U_i))$. As $h$ is a sliding block code (cf. [21]), $h(U_j)$ is a finite disjoint union of cylinder sets ([16]). Hence $Y_i$ is of the form: $Y_i = \bigcup_{m=1}^p \sigma(U_{\nu_s(m)})$ for some $\nu_s(m) \in \Lambda^*$. Let $P_i$ be the orthogonal projection on $H_\Lambda$ onto the subspace corresponding to the set $Y_i$. Since the projection for the subset $\sigma(U_{\nu_s(m)})$ is written as $S_{\nu_s(m)} a_{\nu_s(m)} S_{\nu_s(m)}^*$ where $\nu_s(m) = \tilde{\nu}_s(m) \mu_s(m)$ with $|\tilde{\nu}_s(m)| = 1$, the projection $P_i$ belongs to the algebra $\mathcal{D}_\Lambda$. For $y \in Y_i$, we denote by $h(\text{ih}^{-1}(y))_1$ the first coordinate of $h(\text{ih}^{-1}(y))$. Set

$$Y_i(j) = \{y \in Y_i \mid h(\text{ih}^{-1}(y))_1 = j\} \quad \text{for } j = 1, \ldots, n.$$

We see that

$$h^{-1}(Y_i(j)) = \{x \in X_\Lambda \mid ix \in h^{-1}(U(j))\} \cap h^{-1}(Y_i).$$

The set $Y_i(j)$ is the intersection between $Y_i$ and a finite union of cylinder sets. Hence the orthogonal projection corresponding to the set $Y_i(j)$ belongs to $\mathcal{D}_\Lambda$, that we denote by $P_i(j)$. For an element $x \in X_\Lambda$, $x$ belongs to $Y_i(j)$ if and only if $e_{h(\text{ih}^{-1}(x))} = e_{jx}$ as vectors in $H_\Lambda$. Hence we have $S_i^* P_i(j) = S_i P_i(j)$. Since we have $P_i = \sum_{j=1}^n P_i(j)$ and $P_i = S_i^* S_i'$, it follows that $S_i' = \sum_{j=1}^n S_i P_i(j)$ so that $\text{Ad}(V_h)(S_i)$ belongs to the algebra $O_\Lambda$. We then write $\alpha_h = \text{Ad}(V_h)$. It defines an automorphism of $O_\Lambda$. This correspondence $h \in \text{Aut}(X_\Lambda) \rightarrow \alpha_h \in \text{Aut}(O_\Lambda)$ gives rise to a homomorphism.
We set 
\[
\text{Aut}(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) = \{ \alpha \in \text{Aut}(\mathcal{O}_\Lambda) \mid \alpha(\mathcal{D}_\Lambda) = \mathcal{D}_\Lambda \} ,
\]
\[
\text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) = \{ \alpha \in \text{Aut}(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) \mid \alpha \circ \sigma^* = \sigma^* \circ \alpha \text{ on } \mathcal{D}_\Lambda \}
\]
where \(\sigma^*\) denotes the endomorphism \(\varphi_\Lambda \left( = \sum_{j=1}^n S_j \cdot S_j^* \right)\) of \(\mathcal{D}_\Lambda\) induced by the shift \(\sigma\).

As an extension on \(\mathcal{O}_\Lambda\) of an automorphism \(h\) of \(X_\Lambda\) commutes with shift on \(\mathcal{D}_\Lambda\), we will study the group \(\text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\). We first see a difference between \(\text{Aut}(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\) and \(\text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\) as follows:

**Lemma 4.3.** An automorphism \(\alpha \in \text{Aut}(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\) belongs to \(\text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\) if and only if \((S_\mu^*)^\sigma S_\nu\) belongs to \(\mathcal{D}_\Lambda\) for all words \(\mu, \nu \in \Lambda^*\) with \(|\mu| = |\nu|\).

**Proof.** We see that \(\alpha\) commutes with \(\varphi_\Lambda\) if and only if the following equalities hold:

\[
\alpha \left( \sum_{\mu \in \Lambda^k} S_\mu S_\gamma S_\gamma^* S_\mu^* \right) = \sum_{\nu \in \Lambda^k} S_\nu \alpha(S_\gamma S_\gamma^*) S_\nu^*
\]
for any word \(\gamma \in \Lambda^*.\) The above equality is equivalent to the equality:

\[
\alpha(S_\mu^* S_\mu S_\gamma S_\gamma^*) S_\nu = \alpha(S_\mu)^* S_\nu \alpha(S_\gamma S_\gamma^*) S_\nu^*
\]
that is equivalent to the condition that \(\alpha(S_\mu)^* S_\nu\) commutes with \(\alpha(S_\gamma S_\gamma^*)\). This means that \(\alpha(S_\mu)^* S_\nu\) belongs to the algebra \(\mathcal{D}_\Lambda\) by Proposition 3.3. \(\blacksquare\)

Thus we see

**Proposition 4.4.** For an automorphism \(\alpha \in \text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)\), we have

(i) \(\alpha \circ \alpha_t = \alpha_t \circ \alpha\) for all \(t \in \mathbb{R}\), where \(\alpha_t\) is the gauge automorphism of \(\mathcal{O}_\Lambda\).

(ii) \(\alpha(\mathcal{D}_\Lambda) = \mathcal{D}_\Lambda\).

(iii) \(\alpha \circ \lambda_\Lambda = \lambda_\Lambda \circ \alpha\) on \(\mathcal{D}_\Lambda\) where \(\lambda_\Lambda\) is defined by \(\lambda_\Lambda(X) = \sum_{j=1}^n S_j X S_j\) for \(X \in \mathcal{O}_\Lambda\).

**Proof.** (i) For \(j, k = 1, \ldots, n\), put \(f_{j,k} = \alpha(S_j)^* S_k\) that belongs to \(\mathcal{D}_\Lambda\) by the previous lemma. Since \(\alpha(S_j) = \sum_{k=1}^n S_k f_{j,k}^*\), it follows that

\[
\alpha_t(\alpha(S_j)) = \sum_{k=1}^n \alpha_t(S_k) f_{j,k}^* = \alpha_t(S_j) \alpha(\alpha_t(S_j)).
\]

(ii) For \(\mu, \nu \in \Lambda^k\) and \(\gamma \in \Lambda^*,\) we put \(f_{\mu,\nu} = \alpha(S_\mu)^* S_\nu, g_\gamma = \alpha(S_\gamma S_\gamma^*) \in \mathcal{D}_\Lambda\). As the algebra \(\mathcal{D}_\Lambda\) is invariant under \(\alpha\), it commutes with \(\alpha(\mathcal{D}_\Lambda)\). Hence we have for \(\nu \neq \xi\)

\[
f_{\mu,\nu} g_\gamma f_{\mu,\xi} = S_\nu S_\nu S_\nu^* \alpha(S_\mu S_\gamma S_\gamma^*) S_\xi S_\xi^* = 0.
\]

It follows that

\[
\alpha(S_\mu a_\mu S_\mu^*) = \sum_{\nu, \xi \in \Lambda^k} S_\nu f_{\mu,\nu} g_\gamma f_{\mu,\xi} S_\xi^* = \sum_{\nu \in \Lambda^k} S_\nu f_{\mu,\nu} g_\gamma S_\nu^*.
\]
This shows that $\alpha(\mathcal{D}_A) = \mathcal{D}_A$.

(iii) For $\mu, \gamma \in \Lambda^*$ with $\mu = \mu_1\mu'$, $\mu_1 = 1, \ldots, n$, it follows that

$$\alpha \circ \lambda_\Lambda(S_\mu a, S^*_\mu) = \alpha(S_{\mu'} a, S^*_{\mu'})\alpha(S_{\mu_1}^* S_{\mu_1}).$$

On the other hand, we have

$$\lambda_\Lambda \circ \alpha(S_\mu a, S^*_\mu) = \sum_{j=1}^n S_j^* \alpha(S_{\mu_1}) \alpha(S_{\mu'} a, S^*_{\mu'}) \alpha(S_{\mu_1}^*) S_j$$

$$= \sum_{j=1}^n \alpha(S_{\mu'} S_{\mu_1}^*) \alpha(S_{\mu_1}^* S_{\mu_1}) S_j^{*}\alpha(S_{\mu_1})$$

because both the elements $S_j^* \alpha(S_{\mu_1}), \alpha(S_{\mu_1}^*) S_j$ belong to $\mathcal{D}_A$ by the previous lemma. Hence we have the assertion.

**Lemma 4.5.** If $\alpha \in \text{Aut}_\sigma(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda)$ is the identity on $\mathcal{D}_\Lambda$, it is also the identity on $\mathcal{D}_A$. Hence an extension of an automorphism of $X_\Lambda$ to an automorphism of $\mathcal{D}_A$ is unique.

**Proof.** Suppose that $\alpha$ is the identity on $\mathcal{D}_A$. As $\alpha$ commutes with $\lambda_\Lambda$, we see for $\mu \in \Lambda^i$,

$$\alpha(S_{\mu} S_{S_{\mu}}) = \alpha \circ \lambda_\Lambda(S_{\mu} S_{S_{\mu}}) = \lambda_\Lambda \circ \alpha(S_{\mu} S_{S_{\mu}}) = S_{\mu} S_{\mu}.$$ 

For $\nu \in \Lambda^k$ with $k \leq l$, it follows that by Lemma 4.3

$$\alpha(S_{\mu} a_\mu S^*_\mu) = \sum_{\xi \in \Lambda^k} S_{\xi} S^*_{\xi} \alpha(S_{\mu}) a_\mu \alpha(S^*_\mu)$$

$$= \sum_{\xi \in \Lambda^k} S_{\xi} a_\mu \alpha(S^*_\mu) \alpha(S_{\mu}) \alpha(S^*_\mu) = S_{\nu} a_\mu S^*_\mu.$$

Hence we obtain that $\alpha$ is the identity on $\mathcal{D}_A$.

**Lemma 4.6.** For an automorphism $\alpha$ of $\mathcal{O}_\Lambda$, its restriction to $\mathcal{D}_\Lambda$ is the identity if and only if there exists a unitary $U_\alpha \in \mathcal{O}_\Lambda$ such that

$$\alpha(S_i) = U_\alpha S_i, \quad i = 1, 2, \ldots, n$$

and $U_\alpha \in \mathcal{D}_\Lambda$.

**Proof.** Suppose that the restriction of an automorphism $\alpha$ of $\mathcal{O}_\Lambda$ to the subalgebra $\mathcal{D}_\Lambda$ is the identity. Set $U_\alpha = \sum_{i=1}^n \alpha(S_i) S_i^*$. Since the extension of an automorphism of $X_\Lambda$ to an automorphism of the algebra $\mathcal{D}_\Lambda$ is unique, we see

$$\alpha(S_i S_i) = S_i S_i, \quad i = 1, 2, \ldots, n.$$ 

It follows that $U_\alpha S_i = \alpha(S_i)$ and

$$U_\alpha U_\alpha^* = \sum_{i=1}^n U_\alpha S_i S_i^* U_\alpha = \sum_{i=1}^n \alpha(S_i S_i^*) = 1.$$

We also have

$$U_\alpha^* U_\alpha = \sum_{i,j=1}^n S_i \alpha(S_i^*) \alpha(S_j) S_j^* = \sum_{i=1}^n S_i S_i^* = 1.$$
For a word \( \mu = (\mu_1, \ldots, \mu_l) \in \Lambda^* \), put \( \mu' = (\mu_2, \ldots, \mu_l) \). It then follows that

\[
U_\alpha S_\mu S_\mu^* U_\alpha^* = \sum_{j,k=1}^n \alpha(S_j) S_j^* S_\mu S_\mu^* S_k \alpha(S_k^*)
\]

\[
= \alpha(S_{\mu_1}) S_{\mu_1} S_\mu S_\mu^* \alpha(S_{\mu_1}^*)
\]

\[
= \alpha(S_{\mu_1}) S_{\mu_1} S_\mu^* \alpha(S_{\mu_1}^*) = \alpha(S_{\mu_1} S_\mu S_\mu^*) = S_\mu S_\mu^*.
\]

Hence \( U_\alpha \) commutes with every element of \( \mathcal{D}_\Lambda \) so that it belongs to \( \mathcal{D}_\Lambda \) by Proposition 3.3. The converse implication is easy.

For an automorphism \( \alpha \) of \( \mathcal{O}_\Lambda \), put

\[
U_\alpha(k) = \sum_{\mu \in \Lambda^k} \alpha(S_\mu) S_\mu^* \quad \text{for} \quad k = 1, 2, \ldots.
\]

**Corollary 4.7.** For an automorphism \( \alpha \) of \( \mathcal{O}_\Lambda \), its restriction to \( \mathcal{D}_\Lambda \) is the identity if and only if \( U_\alpha(k) \) is a unitary in \( \mathcal{D}_\Lambda \) for each \( k = 1, 2, \ldots \). In this case, we have

\[
U_\alpha(k+l) = U_\alpha(k) \varphi^k_\Lambda(U_\alpha(l)) \quad \text{for} \quad k, l = 1, 2, \ldots
\]

and

\[
\alpha(S_\mu) = U_\alpha(k) S_\mu \quad \text{for} \quad \mu \in \Lambda^k, \quad k = 1, 2, \ldots.
\]

**Proof.** Suppose that \( \alpha \) is the identity on \( \mathcal{D}_\Lambda \). As in the proof of the previous lemma, we see that \( U_\alpha(k) \) commutes with every element of the algebra \( \mathcal{D}_\Lambda \) so that it belongs to \( \mathcal{D}_\Lambda \) by Proposition 3.3. The converse implication is direct. The identities (4.1) and (4.2) are straightforward.

Let \( \mathcal{U}(\mathcal{D}_\Lambda) \) be the set of all unitaries in \( \mathcal{D}_\Lambda \). A unitary one-cocycle for \( \varphi_\Lambda \) is defined as a \( \mathcal{U}(\mathcal{D}_\Lambda) \)-valued function \( U \) from \( \mathbb{N} \) satisfying

\[
U(k+l) = U(k) \varphi^k_\Lambda(U(l)) \quad \text{for} \quad k, l = 1, 2, \ldots.
\]

We denote by \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) the set of all unitary one-cocycles for \( \varphi_\Lambda \) in \( \mathcal{U}(\mathcal{D}_\Lambda) \). It is an abelian group in a natural way. For \( U \in Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \), put

\[
\lambda(U)(S_\mu) = U(k) S_\mu \quad \text{for} \quad \mu \in \Lambda^k, \quad k = 1, 2, \ldots.
\]

By Lemma 2.3, we see that \( \lambda(U) \) yields an automorphism of the \( C^* \)-algebra \( \mathcal{O}_\Lambda \) that acts identically on \( \mathcal{D}_\Lambda \). Hence \( \lambda \) gives rise to a map from \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) to \( \text{Aut}_\varphi(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) \). We notice that \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) is regarded as the unitary group \( \mathcal{U}(\mathcal{D}_\Lambda) \) by corresponding to the value at 1. We sometimes identify them.

**Lemma 4.8.** The map \( \lambda : U \in Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \rightarrow \lambda(U) \in \text{Aut}_\varphi(\mathcal{O}_\Lambda, \mathcal{D}_\Lambda) \) gives rise to an injective homomorphism.

**Proof.** Since \( \lambda(U)(v) = v \) for \( v \in \mathcal{U}(\mathcal{D}_\Lambda) \), \( \lambda \) gives rise to a homomorphism. Suppose that \( \lambda(U) = id \) on \( \mathcal{O}_\Lambda \). It follows that

\[
U(1) = \sum_{j=1}^n \lambda(U)(S_j) S_j^* = \sum_{j=1}^n S_j S_j^* = 1.
\]

Hence \( U \) is the unit of \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \).
Thus we have the following theorem.

**Theorem 4.9.** Suppose that $(X_\Lambda, \sigma)$ satisfies condition (I). There exists a natural short exact sequence:

$$0 \to Z^1_{\sigma}(\mathcal{U}(D_\Lambda)) \to \text{Aut}_\sigma(O_\Lambda, \mathcal{D}_\Lambda) \to \text{Aut}(X_\Lambda) \to 0$$

that splits. Hence we have a semidirect product:

$$\text{Aut}_\sigma(O_\Lambda, \mathcal{D}_\Lambda) = \text{Aut}(X_\Lambda) \cdot \mathcal{U}(D_\Lambda).$$

Namely we have

**Corollary 4.10.** Any automorphism of $X_\Lambda$ is uniquely extended to an automorphism of $O_\Lambda$ modulo unitaries in $D_\Lambda$. That is, for an automorphism $h$ of $X_\Lambda$, if two automorphisms $\alpha^h, \beta^h$ of $O_\Lambda$ coincide with $h^*$ on $X_\Lambda$, then $\alpha^h = \beta^h \circ \lambda(u)$ for some unitary $u$ in $D_\Lambda$ where $\lambda(u) \in \text{Aut}(O_\Lambda)$ is defined to be $\lambda(u)(S_i) = uS_i$.

Now we refer a connection to the K-theory for $O_\Lambda$ and $\mathcal{F}_\Lambda$.

**Corollary 4.11.** Any automorphism $h$ of $X_\Lambda$ induces an automorphism $h_*$ of the K-groups $K_*(O_\Lambda)$ and $K_0(\mathcal{F}_\Lambda)$ such that the maps $h \in \text{Aut}(X_\Lambda) \to h_* \in \text{Aut}(K_*(O_\Lambda))$ and $h \in \text{Aut}(X_\Lambda) \to h_* \in \text{Aut}(K_0(\mathcal{F}_\Lambda))$ give rise to homomorphisms respectively. In particular, $h_* \in \text{Aut}(K_0(\mathcal{F}_\Lambda))$ commutes with the induced automorphism $\lambda_{h_*}$ of $K_0(\mathcal{F}_\Lambda)$.

**Proof.** For $U \in \mathcal{U}(D_\Lambda)$, as $\lambda(U) = \text{id}$ on $D_\Lambda$ and hence on $A_\Lambda$, the induced homomorphism $\lambda(U)_*$ on $K_* (O_\Lambda)$ is trivial because of [23]. Hence the assertion is clear by Theorem 4.9 with Lemma 4.5.

---

5. **Outer Automorphisms**

If a subshift $\Lambda$ is the full $n$-shift $\Lambda_n$, the $C^*$-algebra $O_{\Lambda_n}$ is nothing but the Cuntz algebra $O_n$ of order $n$. Outerness of some types of automorphisms of $O_n$ have been discussed in several papers (cf. [1], [2], [8], [12], [13], [26], [28], etc.)

In this section, we will discuss on outerness of automorphisms of $O_\Lambda$ coming from automorphisms of $X_\Lambda$. Let $\text{Inn}_\sigma(O_\Lambda)$ be the set of all inner automorphisms of $O_\Lambda$. We set

$$\text{Inn}_\sigma(O_\Lambda, \mathcal{D}_\Lambda) = \text{Inn}(O_\Lambda) \cap \text{Aut}_\sigma(O_\Lambda, \mathcal{D}_\Lambda)$$

$$= \{\text{Ad}(v) \in \text{Aut}_\sigma(O_\Lambda, \mathcal{D}_\Lambda) \mid v \in O_\Lambda, \text{ unitary}\}$$

and

$$\text{Out}_\sigma(O_\Lambda, \mathcal{D}_\Lambda) = \text{Aut}_\sigma(O_\Lambda, \mathcal{D}_\Lambda)/\text{Inn}_\sigma(O_\Lambda, \mathcal{D}_\Lambda).$$

**Lemma 5.1.** For an automorphism $\alpha \in \text{Aut}(O_\Lambda)$, if there exists a unitary $v \in O_\Lambda$ such that $\alpha = \text{Ad}(v)$, then we have $U_{\alpha}(k) = v_{\Lambda}^*(v)^k$ for $k \in \mathbb{N}$.

**Proof.** For a unitary $v \in O_\Lambda$ with $\alpha = \text{Ad}(v)$, it follows that for $\mu \in \Lambda^k$,

$$U_{\alpha}(k)S_\mu S^*_\mu = vS_\mu v^* S^*_\mu.$$

Hence we get the assertion.
Now we introduce the notion of condition (I) for an automorphism of $X_{\Lambda}$.

**Definition.** An automorphism $h \in \text{Aut}(X_{\Lambda})$ satisfies condition (I) if it satisfies the following condition: For any $l, k \in \mathbb{N}$ with $l \geq k$, there exists a projection $q_{l}^{k}$ in $\mathfrak{D}_{\Lambda}$ such that:

(i) $h^{*}(q_{l}^{k})a \neq 0$ for any nonzero $a \in \mathcal{A}_{l}$;

(ii) $h^{*}(q_{l}^{k})\varphi^{*}_{\Lambda}(q_{l}^{k}) = 0$, $1 \leq m \leq k$.

Hence we see that a subshift $(X_{\Lambda}, \sigma)$ satisfies condition (I) if and only if the trivial automorphism $id \in \text{Aut}(X_{\Lambda})$ satisfies condition (I) in the above sense.

We will first verify the following theorem.

**Theorem 5.2.** If a non-trivial automorphism $h \in \text{Aut}(X_{\Lambda})$ satisfies condition (I), then any extension of $h$ to an automorphism of $\mathfrak{O}_{\Lambda}$ is always outer.

We fix an automorphism $h \in \text{Aut}(X_{\Lambda})$ satisfying condition (I) and its arbitrary extension $\alpha \in \text{Aut}_{\tau}(\mathfrak{O}_{\Lambda}, \mathfrak{D}_{\Lambda})$ to $\mathfrak{O}_{\Lambda}$. Suppose that $\alpha$ is inner in $\mathfrak{O}_{\Lambda}$ that is implemented by a unitary $v \in \mathfrak{O}_{\Lambda}$.

In order to prove the above theorem, we provide some lemmas.

**Lemma 5.3.** For $k, l = 1, 2, \ldots, n$, we put $X = S_{k}^{*}vS_{l}$. Then we have $Xf = v^{*}fX$ for all $f \in \mathfrak{D}_{\Lambda}$.

*Proof.* By Lemma 4.3, $\alpha^{-1}(S_{k}^{*})S_{l}$ commutes with $\mathfrak{D}_{\Lambda}$. This implies that $v^{*}Xf = fXv^{*}$ for all $f \in \mathfrak{D}_{\Lambda}$. \hfill \Box

**Lemma 5.4.** We have $X \in \mathcal{F}_{\Lambda}$ and hence $v \in \mathcal{F}_{\Lambda}$.

*Proof.* Although the proof given here is parallel to the proof of Lemma 3.1, we give it for the sake of completeness. Put $X_{\mu} = E(S_{\mu}^{*}X)$, $X_{\mu}E = E(XS_{\mu})$ $\mu \in \Lambda^{*}$. We will show that $X_{\mu} = X_{\mu}E = 0$ for any non-empty word $\mu$. For $f \in \mathfrak{D}_{\Lambda}$, as $Xf = h^{*}(f)X$ by the above lemma, it follows that

$$X_{\mu}f = E(S_{\mu}^{*}Xf) = S_{\mu}^{*}h^{*}(f)S_{\mu}X_{\mu}.$$ 

Put $i = |\mu|$ so that we see

$$X_{\mu}\varphi^{*}_{\Lambda}(f) = S_{\mu}^{*}\varphi^{*}_{\Lambda}(h^{*}(f))S_{\mu}X_{\mu} = S_{\mu}^{*}S_{\mu}h^{*}(f)S_{\mu}S_{\mu}X_{\mu} = h^{*}(f)X_{\mu}.$$ 

Now suppose that $X_{\mu} \neq 0$. For $\varepsilon > 0$, take $X_{\mu}(m) \in \mathcal{F}_{\mu}^{l_{m}}$ with $l_{m} \geq k_{m} \geq i$ such that $\|X_{\mu} - X_{\mu}(m)\| < \varepsilon$. We may assume $\|X_{\mu}\| = \|X_{\mu}(m)\| = 1$. It then follows that

$$\|h^{*}(f)X_{\mu}(m) - X_{\mu}(m)\| \leq 2\|f\|.$$ 

As $h$ satisfies condition (I), there exists a projection $q_{m}$ in $\mathfrak{D}_{\Lambda}$ such that

(i) $h^{*}(q_{m})a \neq 0$ for any nonzero $a \in \mathcal{A}_{l_{m}}$;

(ii) $h^{*}(q_{m})\varphi^{*}_{\Lambda}(q_{m}) = 0$, $1 \leq j \leq k_{m}$.

Put $Q_{m} = \varphi^{*}_{\Lambda}(q_{m})$. Both of the projections $h^{*}(Q_{m}), \varphi^{*}_{\Lambda}(Q_{m})$ belong to $\varphi^{*}_{\Lambda}(\mathfrak{D}_{\Lambda})$ so that $h^{*}(Q_{m}), \varphi^{*}_{\Lambda}(Q_{m})$ commute with $\mathcal{F}_{\mu}^{l_{m}}$. Since we see

$$h^{*}(Q_{m})\varphi^{*}_{\Lambda}(Q_{m}) = 0,$$

it follows that

$$\|h^{*}(Q_{m})X_{\mu}(m) - X_{\mu}(m)\| \leq \text{Max}\{\|h^{*}(Q_{m})X_{\mu}(m)\|, \|X_{\mu}(m)\|\} \geq \|h^{*}(Q_{m})X_{\mu}(m)\|.$$
By using a similar manner to the proof of [22], Corollary 5.4, we see the mapping
\[ X \in \mathcal{F}_{k_m}^m \to h^*(Q_m)X h^*(Q_m) \in h^*(Q_m)\mathcal{F}_{k_m}^m h^*(Q_m) \]
is an isomorphism. Hence we have \[ ||h^*(Q_m)X_\mu(m)|| = ||X_\mu(m)|| = 1. \] This is a contradiction for sufficiently small \( \varepsilon \). Thus we conclude that \( X_\mu = 0 \) and similarly \( X_{-\mu} = 0 \) so that \( X \in \mathcal{F}_\Lambda \). We also see that \( v \in \mathcal{F}_\Lambda \) because of the identity
\[ v = \sum_{k,j=1}^n S_k S_\ell vS_i S_j. \]

**Lemma 5.5.** For any \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that for any word \( \mu \in \Lambda^k \) we have
\[ ||vS_\mu^*S_\mu - b_\mu|| < \varepsilon \quad \text{for some } b_\mu \in \mathcal{D}_\Lambda. \]

**Proof.** By the above lemma, for \( \varepsilon > 0 \), take \( v_m \in \mathcal{F}_{k_m}^m \) with \( l_m > k_m \) such that \[ ||v - v_m|| < \varepsilon. \] Put \( k = k_m \). For any \( \mu, \nu \in \Lambda^k \), we have \( S_\mu^*v_m S_\nu \in \mathcal{A}_\Lambda \). As \( \alpha(S_\mu^*S_\nu) \in \mathcal{P}_\Lambda \), we see \( vS_\mu^*vS_\nu S_\mu v_m S_\nu \) belongs to \( \mathcal{D}_\Lambda \). Put \( b_\mu = vS_\mu^*vS_\nu S_\mu \) that belongs to \( \mathcal{D}_\Lambda \). Hence we get \[ ||vS_\mu^*S_\mu - b_\mu|| < \varepsilon. \]

**Proof of Theorem 5.2.** Keep the above notation. It suffices to show that the unitary \( v \) belongs to the algebra \( \mathcal{D}_\Lambda \). For any \( \varepsilon > 0 \), take \( v \in \mathcal{P}_\Lambda \) such that for a word \( \mu \in \Lambda^k \) there exists an element \( b_\mu \in \mathcal{D}_\Lambda \) as above. For any \( a \in \mathcal{D}_\Lambda \), we have
\[ ||(av - va)S_\mu^*S_\mu|| \leq ||a(vS_\mu^*S_\mu - b_\mu)|| + ||(b_\mu - vS_\mu^*S_\mu)a|| \leq 2\varepsilon. \]
Let \( f_1^k, \ldots, f_{n(k)}^k \) be the set of all nonzero minimal projections in the commutative
\( \mathcal{C}^* \)-algebra generated by projections \( a_\mu \) for \( \mu \in \Lambda^k \). As \( \sum_{i=1}^{n(k)} f_i^k = 1 \), we have
\[ f_i^k (av - va)^* (av - va) f_j^k = 0 \quad \text{for } i \neq j \]
so that we see
\[ ||av - va|| = \left| \left| \sum_{i=1}^{n(k)} (av - va) f_i^k \right| \right| = \max_{1 \leq i \leq n(k)} ||(av - va) f_i^k||^2. \]
Since \( f_i^k \) is majorized by a projection of the form \( S_\mu^*S_\mu \) for some \( \mu \in \Lambda^k \). We obtain that
\[ ||av - va|| \leq 2\varepsilon. \]
Now \( a \in \mathcal{D}_\Lambda \) is independent of \( \varepsilon \) and hence \( v \in \mathcal{D}_\Lambda \cap \mathcal{F}_\Lambda \). This implies \( v \in \mathcal{D}_\Lambda \) and the homeomorphism \( h \) is trivial. 

We next introduce some condition, called \( (D) \), for subshifts that guarantee condition \( (I) \) for all non-trivial automorphism of \( X_\Lambda \). A subshift \( (X_\Lambda, \sigma) \) satisfies \textit{condition} \( (D) \) if for any \( l \in \mathbb{N} \), there exists \( N_l \in \mathbb{N} \) such that for any \( x \in X_\Lambda \), there exists \( y \in X_\Lambda \) such that \( y \neq x \), \( y \sim \ell x \) and \( \sigma^{N_l}(x) = \sigma^{N_l}(y) \).

This condition is clearly a stronger condition than condition \( (I) \) for subshifts. But the following proposition shows that it is not a so strong condition.
Proposition 5.6. Suppose that $X_\Lambda$ is not a single point. If $X_\Lambda$ is aperiodic in past equivalence, then it satisfies condition (D).

To prove the above proposition, we need the following lemma.

Lemma 5.7. Suppose that $X_\Lambda$ is not a single point. If $X_\Lambda$ is aperiodic in past equivalence, there exists $K \in \mathbb{N}$ such that for any $z \in X_\Lambda$ there are words $\mu, \nu \in \Lambda^K$ satisfying $\mu \neq \nu$ and $\mu z, \nu z \in X_\Lambda$.

Proof. If $\Lambda$ is a full shift, the assertion is clear. Suppose that $\Lambda$ is not a full shift. Take $l \in \mathbb{N}$ and $a, b \in X_\Lambda$ such that $a$ is not $l$-past equivalent to $b$. Since $X_\Lambda$ is aperiodic in past equivalence, we find $K \in \mathbb{N}$ such that for any $z \in X_\Lambda$, there are words $\mu_a, \mu_b \in \Lambda^K$ satisfying $\mu_a z \sim_l a$, $\mu_b z \sim_l b$ so that we see $\mu_a \neq \mu_b$.

Proof of Proposition 5.6. For any $l \in \mathbb{N}$, as $X_\Lambda$ is aperiodic in past equivalence, take $N \in \mathbb{N}$ as in the property of aperiodicity in past equivalence and $K \in \mathbb{N}$ as in the above lemma. Set $N_1 = N + K$. For any $x \in X_\Lambda$, put $\gamma = x|_{[1, N]}$, $\xi = x|_{[N + 1, N + K]}$ and $x' = x|_{(N + K + 1, \infty)}$. By the above lemma, there exist distinct words $\mu, \nu \in \Lambda^K$ with $\mu x', \nu x' \in X_\Lambda$. We may assume that $\mu \neq \xi$ (otherwise $\nu \neq \xi$). Put $y' = \mu x' \in X_\Lambda$. Since $X_\Lambda$ is aperiodic in past equivalence, we may find $\eta \in \Lambda^N$ with $x \sim_\eta \eta y'$. Set $y = \eta y' \in X_\Lambda$. Thus we see that

$x \neq y$, $x \sim_{l} y$ and $\sigma^{N_1}(x) = \sigma^{N_1}(y)$.

We will show that every non-trivial automorphism on $X_\Lambda$ satisfies condition (I) under the condition (D) for the subshift.

The following lemma is direct.

Lemma 5.8. A subshift $X_\Lambda$ satisfies condition (D) if and only if it satisfies the following condition:

For any pair $l, m \in \mathbb{N}$, there exists $N_{l,m} \in \mathbb{N}$ such that for any $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that

(i) $x|_{[1, m]} = y|_{[1, m]}$ and $x|_{[m + N_{l,m} + 1, \infty)} = y|_{[m + N_{l,m} + 1, \infty)}$;

(ii) $x|_{[m + 1, m + N_{l,m}]} \neq y|_{[m + 1, m + N_{l,m}]}$;

(iii) $x \sim_{l} y$.

For $l \in \mathbb{N}$, let $F_{1}^{l}, \ldots, F_{m(l)}^{l}$ be the set of all $l$-past equivalence classes in $X_\Lambda$.

Hence we have a decomposition of $X_\Lambda$: $\bigcup_{i=1}^{m(l)} F_{i}^{l} = X_\Lambda$.

Lemma 5.9. Suppose that $X_\Lambda$ satisfies condition (D). Then for an automorphism $h \in \text{Aut}(X_\Lambda)$ and a natural number $l \in \mathbb{N}$ and $i = 1, \ldots, m(l)$, there exists $y \in F_{i}^{l}$ such that

$\sigma^{m}(y) \neq h(y)$ for $1 \leq m \leq l$.

Proof. Fix $l$ and $i = 1, \ldots, m(l)$. Take $x \in F_{i}^{l}$ and suppose that $\sigma(x) = h(x)$. By condition (D), there exists $N_i \in \mathbb{N}$ satisfying the property of (D). Put $\mu = x|_{[1, N_i]}$ and take $\mu' \in \Lambda^{N_i}$ such that $\mu \neq \mu'$ and $\mu' \sigma^{N_i}(x) \neq \sigma^{N_i}(x)$. As $\mu \neq \mu'$ and $\sigma^{N_i}(x') = \sigma^{N_i}(x)$, we obtain that $\sigma(x') \neq h(x')$. We in fact see that if
On automorphisms of $C^*$-algebras associated with subshifts

\[ \sigma(x') = h(x'), \quad \sigma^N(x') = h^N(x'). \] Hence $h^N(x') = h^N(x)$ because $h(x) = \sigma(x)$. This is a contradiction for $x \neq x'$. Therefore we find an element $x' \in F_l$ such that $\sigma(x') \neq h(x')$. Put $x(1) = x'$.

We will next see that there exists $x(2) \in F_l$ such that
\[ \sigma(x(2)) \neq h(x(2)), \quad \sigma^2(x(2)) \neq h(x(2)). \]

If $\sigma^2(x(1)) \neq h(x(1))$, we may take $x(2)$ as $x(1)$. Suppose that $\sigma^2(x(1)) = h(x(1))$. As $h$ and $\sigma$ are uniformly continuous on $X_\Lambda$, there exists $m_1 \in \mathbb{N}$ such that for $y \in X_\Lambda$, if $x(1)[1,m_1] = y[1,m_1]$, then $\sigma(y) \neq h(y)$. By Lemma 5.8, there exists $N_{i,m_1} \in \mathbb{N}$ and $y \in X_\Lambda$ such that
\begin{enumerate}[(i)]  
  \item $x(1)[1,m_1] = y[1,m_1]$ and $x(1)[m_1+N_{i,m_1}+1,\infty) = y[m_1+N_{i,m_1}+1,\infty]$;
  \item $x(1)[m_1+1,1+1,1+N_{i,m_1}] \neq y[m_1+1,1+1,1+N_{i,m_1}]$;
  \item $x(1) \sim_i y$.
\end{enumerate}

Hence we see $y \in F_l$ and $\sigma(y) \neq h(y)$. If $\sigma^2(y) = h(y)$, we have, by the above condition (i) and the condition $\sigma^2(x(1)) = h(x(1))$, $h_{m_1+N_{i,m_1}}(x(1)) = h_{m_1+N_{i,m_1}}(y)$ a contradiction to $x(1) \neq y$. Therefore we obtain $\sigma^2(y) \neq h(y)$. Thus by putting $x(2) = y$, we have
\[ x(2) \in F_l, \quad \sigma(x(2)) \neq h(x(2)) \quad \text{and} \quad \sigma^2(x(2)) \neq h(x(2)). \]

By continuing similar arguments to the above, we may take, for any $n \in \mathbb{N}$, an element $x(n) \in F_l$ such that $\sigma^k(x(n)) \neq h(x(n))$ for all $1 \leq k \leq n$.

\begin{lemma}
Suppose that $X_\Lambda$ satisfies condition (D). Then for an automorphism $h \in \text{Aut}(X_\Lambda)$ and natural numbers $l, k \in \mathbb{N}$ with $l \geq k$, there exists $y_l^1 \in F_l$ for each $i = 1, 2, \ldots, m(l)$ such that
\[ \sigma^m(y_l^i) \neq h(y_l^i) \quad \text{for all } 1 \leq m \leq k \text{ and } i, j = 1, 2, \ldots, m(l). \]
\end{lemma}

\begin{proof}
For $i = 1$, by the previous lemma, we may find $y_l^1 \in F_l$ such that $\sigma^m(y_l^1) \neq h(y_l^1)$ for all $1 \leq m \leq k$. Similarly find $x_2 \in F_2$ such that
\[ \sigma^n(x_2) \neq h(x_2) \quad \text{for } 1 \leq n \leq k. \]

By uniformly continuity for $h, \sigma$, there exists $K_{2,1} \in \mathbb{N}$ such that if $y \in F_2$ satisfies $x_2[1,K_{2,1}] = y[1,K_{2,1}]$, then $\sigma^n(y) \neq h(y)$ for $1 \leq n \leq K_{2,1}$. Now put $X_{K_{2,1}}$ satisfying condition (D) so that there exists $z_2^1 \in F_2$ satisfying $z_2^1[1,K_{2,1}] = x_2[1,K_{2,1}]$ and $z_2^1 \neq x_2^1$ for some $N > K_{2,1}$. If $\sigma(z_2^1) = h(z_2^1)$, we see $\sigma(z_2^1) \neq h(y_l^1)$. Hence we may find $z_2^i \in F_2$ such that
\[ \sigma^n(z_2^i) \neq h(z_2^i) \quad \text{for } 1 \leq n \leq k \quad \text{and} \quad \sigma(z_2^i) \neq h(y_l^i). \]

By using (5.2) instead of (5.1), a similar argument to the above one shows that there exists an element $w_2^1 \in F_2$ such that
\[ \sigma^m(w_2^1) \neq h(w_2^1) \quad \text{for } 1 \leq m \leq k \quad \text{and} \quad \sigma(w_2^1) \neq h(y_l^1), \quad \sigma^2(w_2^1) \neq h(y_l^1). \]

By repeating these procedure, we may find $u_2^1 \in F_2$ such that
\[ \sigma^m(u_2^1) \neq h(u_2^1), \quad \sigma^m(u_2^1) \neq h(y_l^1) \quad \text{for } 1 \leq n \leq k. \]
We next choose an element \( v'_2 \in F^l_2 \) from (5.3) such that
\[
\sigma^n(v'_2) \neq h(v'_2), \quad \sigma^n(v'_2) \neq h(y'_1) \quad \text{for } 1 \leq n \leq k \quad \text{and} \quad \sigma(y'_1) \neq h(v'_2)
\]
by using a similar argument to the preceding one. By repeating these procedure several times, we finally take an element \( y'_2 \in F^l_2 \) such that
\[
\sigma^n(y'_1) \neq h(y'_2) \quad \text{for all } 1 \leq n \leq k, \, i, j = 1, 2.
\]
Consequently we may find elements \( y'_i \in F^l_i \) for \( i = 1, \ldots, m(l) \) that satisfy the required condition by similar procedures. \( \blacksquare \)

We thus have

**Proposition 5.11.** Suppose that \( X_\Lambda \) satisfies condition (D). Then any automorphism \( h \in \text{Aut}(X_\Lambda) \) satisfies condition (I).

**Proof.** For any \( l, k \in \mathbb{N} \) with \( l \geq k \), we will first find a projection \( p_k \) in \( \mathfrak{D}_\Lambda \) satisfying the following conditions:

(i) \( p_k a \neq 0 \) for any nonzero \( a \in \mathcal{A}_l \);

(ii) \( p_k \varphi^n h^*(h^{-1} p_k) = 0 \), \( 1 \leq m \leq k \).

For any \( l, k \in \mathbb{N} \) with \( l \geq k \), take \( y'_i \in F^l_i \) as in the previous lemma. Put \( Y = \{ y'_i \mid i = 1, \ldots, m(l) \} \subset X_\Lambda \). As we see \( \sigma^{-m}(h(Y)) \cap Y = \emptyset \) for \( 1 \leq m \leq k \), there exists a clopen set \( V \), that includes \( Y \), such that \( \sigma^{-m}(h(V)) \cap V = \emptyset \) for \( 1 \leq m \leq k \). Let \( p_k \) be the characteristic function of \( V \) on \( X_\Lambda \). The projection \( p_k \) satisfies the above conditions (i), (ii). We then put \( q_k = h^{*-1}(p_k) \) that satisfies the required conditions for condition (I). \( \blacksquare \)

We reach the following theorem

**Theorem 5.12.** Suppose that \( X_\Lambda \) satisfies the condition (D). Then any extension of a non-trivial automorphism of the subshift \( X_\Lambda \) to an automorphism of the C*-algebra \( \mathfrak{O}_\Lambda \) is outer.

Let \( X_\Lambda \) be the one-sided topological Markov shift determined by an \( n \times n \) square matrix \( A \) with entries in \( \{0, 1\} \). If \( A \) is an aperiodic matrix, the subshift \( X_\Lambda \) is aperiodic in past equivalence and hence satisfies condition (D). Thus we have

**Corollary 5.13.** For an aperiodic matrix \( A \) with entries in \( \{0, 1\} \), any extension of a non-trivial automorphism of the topological Markov shift \( X_\Lambda \) to an automorphism of the Cuntz-Krieger algebra \( \mathfrak{O}_\Lambda \) is outer.

A coboundary \( U \) is defined as a \( \mathcal{U}(\mathcal{D}_\Lambda) \)-valued function \( U \) from \( \mathbb{N} \) such that there exists \( v \in \mathcal{U}(\mathcal{D}_\Lambda) \) such that
\[
U(k) = v \varphi^k(v^*) \quad \text{for } k = 1, 2, \ldots.
\]
We denote by \( B^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) the set of all coboundaries in \( \mathcal{U}(\mathcal{D}_\Lambda) \). It is a subgroup of \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \). If we identify \( Z^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) with \( \mathcal{U}(\mathcal{D}_\Lambda) \), we can regard \( B^1(\mathcal{U}(\mathcal{D}_\Lambda)) \) as the set of all unitaries \( U \) in \( \mathcal{U}(\mathcal{D}_\Lambda) \) that is of the form
\[
U = v \varphi^k(v^*) \quad \text{for some unitary } v \in \mathcal{U}(\mathcal{D}_\Lambda).
\]
We recall that for a unitary \( U \in \mathcal{U}(\mathcal{D}_\Lambda) \), an automorphism \( \lambda(U) \) of \( \mathfrak{O}_\Lambda \) is defined as \( \lambda(U)(S_i) = US_i, \, i = 1, \ldots, n \) that gives rise to an element of \( \text{Aut}_\sigma(\mathfrak{O}_\Lambda, \mathfrak{D}_\Lambda) \).
Lemma 5.14. For a unitary \( U \in \mathcal{U}(D_\Lambda) \), the automorphism \( \lambda(U) \) is of the form \( \lambda(U) = \text{Ad}(v) \) for some unitary \( v \in \mathcal{O}_\Lambda \) if and only if \( v \in \mathcal{U}(D_\Lambda) \) and
\[
U = v \varphi_\Lambda(v^*).
\]

Proof. Suppose that \( \lambda(U) = \text{Ad}(v) \) for some unitary \( v \in \mathcal{O}_\Lambda \). Since \( \lambda(U) \) is the identity on \( D_\Lambda \), \( v \) commutes with every element of \( D_\Lambda \) so that \( v \) belongs to the algebra \( D_\Lambda \) by Proposition 3.3. The condition \( \lambda(U)(S_i) = \text{Ad}(v)(S_i), \quad i = 1, \ldots, n \) is equivalent to the condition \( US_i v S_i^* = v S_i S_i^* \). That is also equivalent to the condition \( \sum_{i=1}^n S_i v S_i^* = U^* v \) that is nothing but \( U = v \varphi_\Lambda(v^*) \).

We thus have

Proposition 5.15. For a unitary \( U \in \mathcal{U}(D_\Lambda) \), the automorphism \( \lambda(U) \) belongs to \( \text{Inn}_\sigma(\mathcal{O}_\Lambda, D_\Lambda) \) if and only if \( U \) belongs to \( B_1^\sigma(\mathcal{U}(D_\Lambda)) \).

Now we set
\[
H_1^\sigma(\mathcal{U}(D_\Lambda)) = Z_1^\sigma(\mathcal{U}(D_\Lambda))/B_1^\sigma(\mathcal{U}(D_\Lambda))
\]
the one-cohomology group. Therefore we conclude

Theorem 5.16. Suppose that a subshift \((X_\Lambda, \sigma)\) satisfies condition (D). There exists a natural short exact sequence:
\[
0 \rightarrow H_1^\sigma(\mathcal{U}(D_\Lambda)) \rightarrow \text{Out}_\sigma(\mathcal{O}_\Lambda, D_\Lambda) \rightarrow \text{Aut}(X_\Lambda) \rightarrow 0
\]
that splits. Hence we have a semidirect product:
\[
\text{Out}_\sigma(\mathcal{O}_\Lambda, D_\Lambda) = \text{Aut}(X_\Lambda) \cdot \mathcal{U}(D_\Lambda)/B_1^\sigma(\mathcal{U}(D_\Lambda)).
\]

Proof. The above exact sequence is induced by the exact sequence in Theorem 4.9 and Proposition 5.15.

6. Examples

In this section, we will present some examples of automorphisms of \( \mathcal{O}_\Lambda \) coming from automorphisms of certain subshifts \( X_\Lambda \). In [3], Boyle–Franks–Kitchens have studied automorphisms of one-sided topological Markov shifts. We will use some of their results in [3].

Example 6.1. The full 2-shift \( \Lambda_2 \).

It is known that the automorphism group \( \text{Aut}(X_2) \) of the one-sided full 2-shift \( X_2 \) is the group \( \mathbb{Z}/2\mathbb{Z} \) (cf. [16], [3]). The non-trivial element is the flip-flop \( s_{12} \) that interchanges the symbols 1 and 2. Let \( \alpha_{12} \) be the automorphism of the Cuntz algebra \( \mathcal{O}_2 \) defined by
\[
\alpha_{12}(S_1) = S_2, \quad \alpha_{12}(S_2) = S_1.
\]
It is an extension of \( s_{12} \) and hence outer by Corollary 5.13. The outerness of the automorphism was first proved by Archbold in [2]. The discussion has been generalized in [12], [26] and [18].
**Example 6.2.** The topological Markov shift determined by the matrix

\[ A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \]

It was proved in [3] that the automorphism group \( \text{Aut}(X_A) \) of the one-sided topological Markov shift \( X_A \) is isomorphic to the group \( \mathfrak{S}_3 \) of all permutations of order 3. By the calculation formula for the \( K_0 \)-group \( K_0(\mathcal{O}_A) \) of the Cuntz-Krieger algebra \( \mathcal{O}_A \) in [7], we know that the group \( K_0(\mathcal{O}_A) \) is isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) through the correspondences:

\[ [1] = (0, 0), \quad [S_1 S_1^*] = (1, 0), \quad [S_2 S_2^*] = (0, 1), \quad [S_3 S_3^*] = (1, 1). \]

Let \( s(ijk) \in \mathfrak{S}_3 \) be the permutation given by \( \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \). Put

\[ \alpha(ijk)(S_i) = S_i, \quad \alpha(ijk)(S_j) = S_j, \quad \alpha(ijk)(S_k) = S_k. \]

Then \( \alpha(ijk) \) gives rise to an automorphism of \( \mathcal{O}_A \) that is an extension of an automorphism of \( X_A \) induced by the permutation \( s(ijk) \) of the symbols. It is an outer automorphism of \( \mathcal{O}_A \) by Corollary 5.13 or by [18]. Such automorphisms of \( \mathcal{O}_A \) yields automorphisms of \( K_0(\mathcal{O}_A) \) so that we see a natural isomorphism between \( \text{Aut}(X_A) \) and \( \text{Aut}(K_0(\mathcal{O}_A)) \) (cf. Corollary 4.11).

**Example 6.3** The full 3-shift \( \Lambda_3 \).

Boyle-Franks-Kitchens in [3] showed that, for \( n > 2 \), the automorphism group \( \text{Aut}(X_n) \) of the one-sided full \( n \)-shift \( X_n \) is infinite. We now treat automorphisms of the full 3-shift \( X_3 \). For \( k = 1, 2, \ldots \), let \( \tau_k \) be an automorphism of \( X_3 \) defined by exchanging words:

\[ \tau_k(3 \underbrace{2 \cdots 2}_{k \text{ times}}) = 1 \underbrace{2 \cdots 2}_{k \text{ times}}, \quad \tau_k(1 \underbrace{2 \cdots 2}_{k \text{ times}}) = 3 \underbrace{2 \cdots 2}_{k \text{ times}} \]

and \( \tau_k \) identically acts on other words in \( X_3 \). Put

\[ \alpha_{\tau_k}(S_2) = S_2, \]

\[ \alpha_{\tau_k}(S_3) = S_1 P_{2^k} + S_3 (1 - P_{2^k}), \]

\[ \alpha_{\tau_k}(S_1) = S_3 P_{2^k} + S_1 (1 - P_{2^k}), \]

where \( P_{2^k} = S_2 \underbrace{\cdots S_2}_{k \text{ times}} S_2^* \underbrace{\cdots S_2^*}_{k \text{ times}} \). It is easy to see that \( \alpha_{\tau_k} \) yields an automorphism of the Cuntz algebra \( \mathcal{O}_3 \) that is an extension of \( \tau_k \). The automorphisms are outer by Corollary 5.13 or by [26], Theorem 1.

We finally remark on outerness of the automorphisms \( \lambda(u) \) of \( \mathcal{O}_\Lambda \) coming from unitaries \( u \) of \( \mathcal{U}(\mathcal{D}_\Lambda) \). Suppose that a subshift \( X_\Lambda \) satisfies condition (I). We denote by \( \text{Per}_n^\sigma(X_\Lambda) \) the set of all \( n \) periodic points of \( X_\Lambda \) under the shift \( \sigma \). The following proposition is directly seen from Lemma 5.14.
**Proposition 6.4.** For a unitary \( u \) in \( D_\Lambda \) if there exists a point \( x \) in \( \text{Per}^n_{\sigma}(X_\Lambda) \) for some \( n \in \mathbb{N} \), such that \( u(x) \neq u^*(\sigma^{n-1}(x))u^*(\sigma^{n-2}(x)) \cdots u^*(\sigma(x)) \), the automorphism \( \lambda(u) \) is outer in \( O_\Lambda \). In particular, if \( u \) is a complex number \( z \) with modulus one such that \( z^n \neq 1 \) and \( \text{Per}^n_{\sigma}(X_\Lambda) \) is not empty, then the automorphism \( \lambda(z) \) defined by \( \lambda(z)(S_i) = zS_i \) is outer.

**Corollary 6.5.** If there exists a fixed point in \( X_\Lambda \) for \( \sigma \), the gauge action \( \alpha \) of \( O_\Lambda \) is an outer action of the one dimensional torus group \( \mathbb{T} \).

Acknowledgements. The author would like to thank Yasuo Watatani for his several discussions and suggestions and Yoshikazu Katayama for his useful suggestions and discussions on unitary cocycles and coboundaries.

REFERENCES

5. A. Connes, A classification of injective factors, Cases II\(_1\), II\(_\infty\), III\(_\lambda\), \( \lambda \neq 1 \), *Ann. of Math. (2)* **104**(1976), 73–115.

KENGO MATSUMOTO
Department of Mathematics
Joetsu University of Education
Joetsu 943–8512
JAPAN

E-mail: matsu@juen.ac.jp

Current address:

Department of Mathematical Sciences
Yokohama City University
22–2 Seto, Kanazawa-ku
Yokohama, 236–0027
JAPAN

E-mail: matsu@math.yokohama-cu.ac.jp

Received May 25, 1998; revised May 14, 1999.