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ENDOMORPHISMS OF $\mathcal{B}(\mathcal{H})$, EXTENSIONS OF PURE STATES, AND A CLASS OF REPRESENTATIONS OF $\mathcal{O}_n$

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ABSTRACT. We construct the pure states of $\mathcal{O}_n$ that extend a given pure state of the fixed point algebra $\mathcal{F}_n$ of the gauge action, and we show that the gauge group acts transitively on these extensions. We apply this to construct and classify the ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$ whose tail algebra has a minimal projection. We discuss examples arising from product states of $\mathcal{F}_n$ and from the trace on the Choi subalgebra of $\mathcal{O}_n$.

KEYWORDS: *Cuntz algebras, ergodic endomorphisms, shifts, pure states.*

MSC (2000): 46L30, 46A22.

INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. An *endomorphism* of $\mathcal{B}(\mathcal{H})$ is a homomorphism of $\mathcal{B}(\mathcal{H})$ into itself which preserves adjoints. The main goal of this paper is to find complete conjugacy invariants for a certain class of endomorphisms of $\mathcal{B}(\mathcal{H})$. Two endomorphisms are *conjugate* if there is an intertwining isomorphism of the underlying algebras. Every isomorphism of $\mathcal{B}(\mathcal{H})$ is unitarily implemented, so conjugacy for endomorphisms of $\mathcal{B}(\mathcal{H})$ is *spatial* equivalence, the strongest reasonable equivalence relation in any classification scheme.

If an endomorphism α fixes only the scalar operators it is called *ergodic*, and if its *tail algebra* $\bigcap\limits_k \alpha^k(\mathcal{B}(\mathcal{H}))$ consists only of scalars it is called a *shift*. The class of endomorphisms we shall consider includes every ergodic endomorphism whose tail algebra has a minimal projection; in particular, it includes all shifts.

We shall only consider endomorphisms which preserve the identity operator I on $\mathcal{H}$. At the other extreme are those endomorphisms α which are completely nonunital in the sense that $\alpha^k(I)$ decreases strongly to zero; we refer the reader to

Section 2 of [10] for the classification of such endomorphisms. Since any endomorphism can be decomposed into unital and completely nonunital components which determine its conjugacy class, our focus on unital endomorphisms is justified.

As is customary, for $2 \leqslant n \leqslant \infty$ we denote by $\mathcal{O}_n$ the C^*-algebra defined by Cuntz in [6]. Let $\{ v_a : 1 \leqslant a \leqslant n \}$ denote the distinguished generating isometries in $\mathcal{O}_n$, so that $\sum_{a=1}^{n} v_a v_a^* \leqslant 1$, with equality if n is finite; $\mathcal{O}_n$ is the universal C^*-algebra generated by such collections of isometries. There is a correspondence between endomorphisms of $\mathcal{B}(\mathcal{H})$ and representations of Cuntz algebras, which stems from the observation by Arveson ([3]) that every endomorphism α of $\mathcal{B}(\mathcal{H})$ can be implemented by a collection $S_1, \dots, S_n$ of isometries on $\mathcal{H}$ via $\alpha(A) = \sum S_a A S_a^*$; if α is unital then such a collection gives rise to a representation of $\mathcal{O}_n$ via $v_a \mapsto S_a$. Conversely, any representation $\pi : \mathcal{O}_n \to \mathcal{B}(\mathcal{H})$ gives rise to an endomorphism $\operatorname{Ad}\pi$ of $\mathcal{B}(\mathcal{H})$ via

$$(\dagger) \qquad \operatorname{Ad}\pi(A) := \sum_{a=1}^{n} \pi(v_a) A \pi(v_a)^*, \quad A \in \mathcal{B}(\mathcal{H});$$

for infinite n the above sum converges in the strong operator topology for every A. If $n < \infty$ then $\operatorname{Ad}\pi$ is unital, but for infinite n this need not be the case. A representation π of $\mathcal{O}_\infty$ for which $\operatorname{Ad}\pi$ is unital (i.e., for which $\sum_{a=1}^{\infty} \pi(v_a)\pi(v_a)^*$ converges strongly to I) is called *essential*.

There is an obvious way to construct endomorphisms from states of $\mathcal{O}_n$: use the GNS representation for the state to implement an endomorphism via (†). This correspondence allows us to study endomorphisms by looking at states of $\mathcal{O}_n$; e.g. ergodic endomorphisms arise from pure states, and conjugacy of endomorphisms corresponds to quasi-free equivalence of states ([10]).

A commonly used method of analyzing $\mathcal{O}_n$ is to exploit the *gauge action* γ of the circle $\mathbb{T}$ on $\mathcal{O}_n$ determined by $\gamma_\lambda(v_a) = \lambda v_a$. We will denote by $\mathcal{F}_n$ the fixed-point algebra of this action. When n is finite, $\mathcal{F}_n$ is canonically isomorphic to the UHF algebra $M_n \otimes M_n \otimes M_n \otimes \cdots$, and hence carries a canonical unital shift, given at the C^*-algebra level by a formula analogous to (†). This shift does not exist on $\mathcal{F}_\infty$ because the strong sum does not make sense at this level, but one can always shift a state ρ of $\mathcal{F}_n$ (for finite or infinite n) by defining

$$\alpha^*\rho(x) := \sum_{a=1}^{n} \rho(v_a x v_a^*), \quad x \in \mathcal{F}_n.$$

A state $\widetilde{\rho}$ of $\mathcal{O}_\infty$ that extends ρ is essential (i.e., its GNS representation is essential) if and only if each of the shifted positive linear functionals $\alpha^{*k}\rho$ is a state (Remark 2.10 of [10]); in this case we say that ρ is *essential*. We will only consider essential states of $\mathcal{F}_n$, with the understanding that the adjective is superfluous when n is finite.

The state space of $\mathcal{F}_n$ is more tractable than that of $\mathcal{O}_n$, and has often been used to study representations of Cuntz algebras ([7], [1], [16]) and unital endomorphisms of $\mathcal{B}(\mathcal{H})$ ([10], [11], [4], [5]). Having a specific procedure for extending states of $\mathcal{F}_n$ to $\mathcal{O}_n$ is extremely useful, especially if it allows one to apply Powers' criteria for states of UHF algebras ([14]) to decide when an extension is pure and when different extensions are unitarily equivalent.

Perhaps the most obvious way to extend a state is by composition with the canonical conditional expectation $\Phi : \mathcal{O}_n \to \mathcal{F}_n$ obtained by averaging over the gauge action. This gives the unique gauge-invariant state of $\mathcal{O}_n$ which extends the given state of $\mathcal{F}_n$. Gauge-invariant extensions of states of $\mathcal{F}_n$ have been considered before: e.g., by Evans ([7]), by Araki, Carey and Evans for product states and $n < \infty$ ([1]), and later by Laca for factor states and any n ([11]). Extensions of diagonal states (i.e., states of the diagonal subalgebra $\mathcal{D}$ of $\mathcal{F}_n$) have been considered by Spielberg ([16], [17]), by Archbold, Lazar, Tsui and Wright ([2]), and by Stacey ([19]) in the context of extending the trace on the Choi subalgebra of $\mathcal{O}_2$. In earlier work, Lazar, Tsui and Wright ([12]) dealt with pure state extensions of pure diagonal states, and identified the unique pure state extension of a nonperiodic (irrational) point in the spectrum of $\mathcal{D}$ ([6]).

A different procedure for extending pure states of $\mathcal{F}_n$ was given in Theorem 4.3 of [10], where it played a key rôle in the classification of shifts of $\mathcal{B}(\mathcal{H})$ up to conjugacy. Roughly, the technique used there consisted of lifting the GNS representation of a state from $\mathcal{F}_n$ to $\mathcal{O}_n$ without changing the Hilbert space; see also Section 5 of [4]. These two techniques for extending a pure state ρ of $\mathcal{F}_n$ are in a sense opposite: the first one always works, but only gives an extension which is pure if ρ is *aperiodic* in the sense that its translates by powers of the canonical shift α^* are mutually disjoint; the second only works if ρ is *quasi-invariant* in the sense that it is quasi-equivalent to $\alpha^*\rho$, but then gives extensions which are pure.

Here we show how to extend any pure state ρ of $\mathcal{F}_n$ which is *periodic* in the sense that ρ is quasi-equivalent to $\alpha^{*p}\rho$ for some positive integer p. Our procedure interpolates between the two techniques described above, and explains them as extreme cases of the same construction.

The paper is organized as follows. We begin with a preliminary section on periodicity of states of $\mathcal{F}_n$. In Section 2 we construct and analyze a class of representations of $\mathcal{O}_n$. Roughly speaking, this class is indexed by pairs (ρ, θ) consisting of a periodic pure state ρ of $\mathcal{F}_n$ and a representation θ of $C(\mathbb{T})$. The quasi-orbit of ρ and the unitary equivalence class of θ determine the unitary equivalence class of the representation up to a gauge automorphism. This ambiguity can be removed with the addition of a third parameter, called a *linking vector*, which is related to the periodicity of ρ and is determined up to a scalar multiple of modulus one.

In Section 3 we study the state extensions of a periodic pure state ρ of $\mathcal{F}_n$. Propositions 3.2 and 3.3 form the technical core of the paper, and show that the representations constructed in Section 2 include the GNS representation associated with any state which extends ρ. Our main result, Theorem 3.5, parameterizes the extensions of ρ to states of $\mathcal{O}_n$ by the Borel probability measures on the circle. In this parameterization the equivalence class of the measure is a complete invariant for unitary equivalence of state extensions. We also compare states which extend different pure states of $\mathcal{F}_n$. The invariant we use for this is the set of quasi-equivalence classes of the shifted states, called the *quasi-orbit* of ρ; see Definition 1.2. In Corollary 3.6 we answer to the affirmative a conjecture made in the final remark of [8], to the effect that a periodic pure state of $\mathcal{F}_n$ has precisely a circle of pure extensions on which the gauge group acts transitively and p-to-1, p being the period of the state. Aperiodic pure states, in contrast, have unique state extensions which are necessarily pure and fixed by the gauge action (Theorem 4.3 of [11]).

In Section 4 we use our representations to construct endomorphisms of $\mathcal{B}(\mathcal{H})$ via (†). Our main classification result is Theorem 4.1, where we obtain complete conjugacy invariants for these endomorphisms based on the parameters ρ and θ. In Corollary 4.2 we apply this theorem to classify the endomorphisms which arise from extending periodic pure states of $\mathcal{F}_n$ to $\mathcal{O}_n$, as described above. The second main result of the section is Theorem 4.3, where we characterize the endomorphisms that arise from state extensions in terms of their tail and fixed-point algebras.

Although the pure extensions of a pure state ρ are mutually disjoint, the (ergodic) endomorphisms they produce are all conjugate. In Corollary 4.4 we classify these endomorphisms using the action of quasi-free automorphisms on the quasi-orbit of ρ, and in Corollary 4.5 we characterize them as those ergodic endomorphisms whose tail algebra has a minimal projection.

In Section 5 we examine several examples arising from pure product states of $\mathcal{F}_n$. In Example A we show how our Theorem 3.5 generalizes Fowler's result on pure product states (Theorem 3.1 of [8]). In Example B we consider product states which are constructed from periodic sequences of unit vectors in n-dimensional Hilbert space. We show that the ergodic endomorphisms which correspond to such periodic sequences are completely classified up to conjugacy by a geometric invariant used in their construction. This generalizes earlier conjugacy results for shifts from [15], [18], [10], [4] and for the ergodic endomorphisms constructed in [11]. Finally, in Example C we apply our techniques to the problem of extending the trace on the Choi algebra to $\mathcal{O}_2$.

1. PRELIMINARIES

A *multi-index* is a k-tuple $s = (s_1, \ldots, s_k)$, where $1 \leqslant s_i \leqslant n$ for each i, and k is any nonnegative integer. We write $|s| := k$ and set $v_s := v_{s_1} \cdots v_{s_k}$, with the convention that $v_s := 1$ if $|s| = 0$. Then $\mathcal{O}_n$ is the closed linear span of monomials of the form $v_s v_t^*$, where s and t are arbitrary multi-indices, and $\mathcal{F}_n$ is the closed linear span of such monomials for which $|s| = |t|$. The canonical conditional expectation $\Phi : \mathcal{O}_n \to \mathcal{F}_n$ is given by

$$\Phi(v_s v_t^*) = \begin{cases} v_s v_t^* & \text{if } |s| = |t| \\ 0 & \text{otherwise.} \end{cases}$$

There are two ways to shift an essential state ρ of $\mathcal{F}_n$: "backwards" by α^*, as defined in the introduction, and "forwards" by β^*, as defined by

$$\beta^*\rho(x) = \rho(v_1^* x v_1), \quad x \in \mathcal{F}_n.$$

The arbitrary choice of v_1 is irrelevant up to unitary equivalence. The shift β^* is a quasi-inverse of α^* in the sense that $\alpha^*\beta^*\rho = \rho \overset{q}{\sim} \beta^*\alpha^*\rho$ for any essential state ρ of $\mathcal{F}_n$ (Lemma 3.1 of [11]). (We use $\overset{q}{\sim}$ and $\overset{u}{\sim}$ to denote quasi-equivalence and unitary equivalence, respectively.)

EXAMPLE 1.1. It is helpful to see how the shifts α^* and β^* act on product states. Suppose n is finite and $\mathcal{E}$ is the n-dimensional Hilbert space spanned by the v_i's, so that $\mathcal{K}(\mathcal{E})$ is isomorphic to the algebra M_n of $n \times n$ matrices. Then $\mathcal{F}_n$ is isomorphic to the UHF algebra $M_n \otimes M_n \otimes M_n \otimes \cdots$ via $v_s v_t^* \mapsto e_{s_1 t_1} \otimes e_{s_2 t_2} \otimes \cdots \otimes e_{s_k t_k} \otimes 1 \otimes 1 \otimes \cdots$, where s and t are multi-indices of the same length k, and $\{e_{ij}\}$ is the obvious system of matrix units in M_n ([6]).

Suppose ω_i is a state of M_n for each i, and let

$$\omega = \omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \cdots$$

be the corresponding product state of $\mathcal{F}_n$. Let ω_{v_1} be the pure state of M_n determined by $\omega_{v_1}(e_{11}) = 1$. Then

$$\alpha^* \omega = \omega_2 \otimes \omega_3 \otimes \omega_4 \otimes \cdots$$

and

$$\beta^* \omega = \omega_{v_1} \otimes \omega_1 \otimes \omega_2 \otimes \cdots .$$

Similar considerations apply to product states of $\mathcal{F}_\infty$ (Section 3 of [11]).

DEFINITION 1.2. The *quasi-orbit* of an essential state ρ of $\mathcal{F}_n$ is the set of quasi-equivalence classes of the states $\alpha^{*k}\rho$ and $\beta^{*k}\rho$ for $k \geqslant 0$.

Let us describe the quasi-orbit of an essential factor state ρ. The states $\alpha^{*k}\rho$ and $\beta^{*l}\rho$ for $k, l \geqslant 0$ are factor states (of the same type as ρ) (Corollary 3.5 of [11]), so any given pair of these states is either disjoint or quasi-equivalent. In the latter case, since both α^* and β^* respect quasi-equivalence of factor states (Corollary 3.6 of [11]) we can apply an appropriate power of one of the shifts to the quasi-equivalent pair to obtain $\rho \overset{q}{\sim} \alpha^{*p}\rho$ for some p.

DEFINITION 1.3. Suppose ρ is an essential factor state of $\mathcal{F}_n$. The *period* of ρ is the smallest positive integer p for which ρ is quasi-equivalent to $\alpha^{*p}\rho$. If no such p exists, we say that ρ is *aperiodic*, or that it has period $p = \infty$.

The quasi-orbit of an essential factor state ρ with finite period p is thus

$$\{[\rho], [\alpha^*\rho], \ldots, [\alpha^{*(p-1)}\rho]\},$$

or alternatively

$$\{[\rho], [\beta^*\rho], \ldots, [\beta^{*(p-1)}\rho]\},$$

where the brackets denote quasi-equivalence classes. In particular, the period of an essential factor state is the cardinality of its quasi-orbit.

REMARK 1.4. Although it would be more accurate to refer to a state which is quasi-equivalent to its p^{th} translate as *quasi-periodic*, we will adhere to the prevailing practice and use the term "periodic" in an asymptotic sense. Examples of strictly periodic states (i.e., states which are *equal* to their translate by some power of α^*) will appear in Section 5.

Quasi-equivalence of essential factor states of $\mathcal{F}_n$ is an asymptotic property (by Theorem 2.7 of [14] for $n < \infty$ and Proposition 3.6 of [10] for $n = \infty$), so two essential factor states ρ and ω have the same quasi-orbit if and only if they are *shift-equivalent* in the sense that there exists k such that

$$\|\alpha^{*(k+j)}\rho - \alpha^{*j}\omega\| \to 0 \qquad \text{as } j \to \infty.$$

When ρ and ω are pure this condition simplifies significantly. Since β^* preserves purity (Lemma 4.2 of [10]), ρ and ω have the same quasi-orbit if and only if

$$\rho \overset{u}{\sim} \beta^{*k}\omega \qquad \text{or} \qquad \omega \overset{u}{\sim} \beta^{*k}\rho \quad \text{ for some } k \geqslant 0.$$

It should be noted that $\alpha^*\rho$ need not be pure even if ρ is. Some examples of this have been given in [5].

We close this section by highlighting some relations between the shifts of a pure essential state ρ with finite period p:

$$\beta^{*k}\rho \overset{u}{\sim} \beta^{*l}\rho \iff \alpha^{*k}\rho \overset{q}{\sim} \alpha^{*l}\rho \iff p \text{ divides } k-l, \tag{1.1}$$

and

$$\alpha^{*k}\rho \overset{q}{\sim} \beta^{*l}\rho \iff p \text{ divides } k+l. \tag{1.2}$$

2. A CLASS OF REPRESENTATIONS OF $\mathcal{O}_n$.

Suppose $\widetilde{\rho}$ is a state of $\mathcal{O}_n$, ρ is the restriction of $\widetilde{\rho}$ to $\mathcal{F}_n$, and $\widetilde{\sigma} : \mathcal{O}_n \to \mathcal{B}(\widetilde{\mathcal{H}})$ is the GNS representation for $\widetilde{\rho}$ with canonical cyclic vector ξ. From the unit vectors $\widetilde{\sigma}(v_1^k)\xi$, with $k = 0, 1, 2, \ldots$, we see that the states $\beta^{*k}\rho$ are vector states in the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$:

$$\beta^{*k}\rho(x) = \langle \widetilde{\sigma}(x)\widetilde{\sigma}(v_1^k)\xi, \widetilde{\sigma}(v_1^k)\xi\rangle, \quad x \in \mathcal{F}_n.$$

As an immediate result, the GNS representations of these shifted states appear as subrepresentations of the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$. Because of this simple fact, whenever we are extending states or representations from $\mathcal{F}_n$ to $\mathcal{O}_n$, we are forced to consider the shifted states. It is therefore convenient to establish the following notation, to be used throughout this paper.

NOTATION 2.1. Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n = \infty$ assume that ρ is essential. For $i = 0, 1, \ldots, p-1$, denote by $\pi_i^\rho : \mathcal{F}_n \to \mathcal{B}(\mathcal{H}_i^\rho)$ the GNS representation for $\beta^{*i}\rho$ with canonical cyclic vector ξ_i^ρ. When there is no chance of confusion we will drop the superscript ρ.

For notational convenience we define $\mathcal{H}_p := \mathcal{H}_0$ and $\pi_p := \pi_0$. Our convention for ξ_p will be somewhat different:

DEFINITION 2.2. A *linking vector* for ρ is a vector $\xi_p \in \mathcal{H}_0$ such that

$$\beta^{*p}\rho(x) = \langle \pi_0(x)\xi_p, \xi_p\rangle, \quad x \in \mathcal{F}_n.$$

Since $\beta^{*p}\rho \overset{u}{\sim} \rho$, there is always a linking vector, and it is determined up to a scalar multiple of modulus one because ρ is pure.

In the following proposition we use a pure essential state ρ and a linking vector ξ_p to construct a representation $\widetilde{\pi}[\rho, \xi_p]$ of $\mathcal{O}_n$; we will see later (Remark 2.6) that this representation is irreducible.

PROPOSITION 2.3. *Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n=\infty$ assume that ρ is essential. Let $\xi_p \in \mathcal{H}_0$ be a linking vector for ρ.*

(i) *If $1 \leqslant a \leqslant n$ and $0 \leqslant i \leqslant p-1$, then there is an isometry $S_{a,i} : \mathcal{H}_i \to \mathcal{H}_{i+1}$ determined by*

$$S_{a,i}\pi_i(x)\xi_i = \pi_{i+1}(v_a x v_1^*)\xi_{i+1}, \quad x \in \mathcal{F}_n. \tag{2.1}$$

(ii) *Let S_a be the isometry $\bigoplus_{i=0}^{p-1} S_{a,i}$ on $\bigoplus_{i=0}^{p-1} \mathcal{H}_i$. There is a representation $\widetilde{\pi}[\rho,\xi_p]$ of $\mathcal{O}_n$, essential if $n=\infty$, such that*

$$\widetilde{\pi}[\rho,\xi_p](v_a) = S_a, \quad 1 \leqslant a \leqslant n.$$

(iii) *If $k = i + mp$ with $0 \leqslant i \leqslant p-1$ and $m \geqslant 0$, then $S_1^k\xi_0$ is a unit vector in $\mathcal{H}_i$ which implements $\beta^{*k}\rho$ as a vector state in π_i. For $0 \leqslant k \leqslant p$ we have $S_1^k\xi_0 = \xi_k$, and for $k \geqslant p+1$ we define $\xi_k := S_1^k\xi_0$.*

Proof. If $x \in \mathcal{F}_n$, then

$$\begin{aligned}\|\pi_{i+1}(v_a x v_1^*)\xi_{i+1}\|^2 &= \langle \pi_{i+1}(v_1 x^* v_a^* v_a x v_1^*)\xi_{i+1}, \xi_{i+1}\rangle = \beta^{*(i+1)}\rho(v_1 x^* x v_1^*) \\ &= \beta^{*i}\rho(x^*x) = \langle \pi_i(x^*x)\xi_i, \xi_i\rangle = \|\pi_i(x)\xi_i\|^2.\end{aligned}$$

Since vectors of the form $\pi_i(x)\xi_i$ are dense in $\mathcal{H}_i$, this gives (i).

We will next show that $S_{a,i}S_{a,i}^* = \pi_{i+1}(v_a v_a^*)$, for which we first need to find a formula for $S_{a,i}^*$. If $x, y \in \mathcal{F}_n$, then

$$\begin{aligned}&\langle S_{a,i}^*\pi_{i+1}(x)\xi_{i+1}, \pi_i(y)\xi_i\rangle = \langle \pi_{i+1}(x)\xi_{i+1}, S_{a,i}\pi_i(y)\xi_i\rangle \\ &\quad = \langle \pi_{i+1}(x)\xi_{i+1}, \pi_{i+1}(v_a y v_1^*)\xi_{i+1}\rangle = \langle \pi_{i+1}(v_1 y^* v_a^* x)\xi_{i+1}, \xi_{i+1}\rangle \\ &\quad = \beta^{*(i+1)}\rho(v_1 y^* v_a^* x) = \beta^{*i}\rho(y^* v_a^* x v_1) = \langle \pi_i(y^* v_a^* x v_1)\xi_i, \xi_i\rangle \\ &\quad = \langle \pi_i(v_a^* x v_1)\xi_i, \pi_i(y)\xi_i\rangle,\end{aligned}$$

so

$$S_{a,i}^*\pi_{i+1}(x)\xi_{i+1} = \pi_i(v_a^* x v_1)\xi_i, \quad x \in \mathcal{F}_n. \tag{2.2}$$

Using the definition of $S_{a,i}$ we have

$$\begin{aligned}S_{a,i}S_{a,i}^*\pi_{i+1}(x)\xi_{i+1} &= S_{a,i}\pi_i(v_a^* x v_1)\xi_i = \pi_{i+1}(v_a v_a^* x v_1 v_1^*)\xi_{i+1} \\ &= \pi_{i+1}(v_a v_a^*)\pi_{i+1}(x)\pi_{i+1}(v_1 v_1^*)\xi_{i+1},\end{aligned}$$

so to show that $S_{a,i}S_{a,i}^* = \pi_{i+1}(v_a v_a^*)$ we must verify that

$$\pi_{i+1}(v_1 v_1^*)\xi_{i+1} = \xi_{i+1}, \quad 0 \leqslant i \leqslant p-1. \tag{2.3}$$

Since $\pi_{i+1}(v_1 v_1^*)$ is a projection, this follows from the calculation

$$\|\pi_{i+1}(v_1 v_1^*)\xi_{i+1}\|^2 = \langle \pi_{i+1}(v_1 v_1^*)\xi_{i+1}, \xi_{i+1}\rangle = \beta^{*(i+1)}\rho(v_1 v_1^*) = \beta^{*i}\rho(1) = 1.$$

It is now easy to see that the range projections $S_a S_a^*$ sum to the identity operator, from which the existence of the representation $\widetilde{\pi}[\rho,\xi_p]$ follows immediately: since each π_i is essential,

$$\sum_{a=1}^{n} S_a S_a^* = \sum_{a=1}^{n} \bigoplus_{i=0}^{p-1} S_{a,i}S_{a,i}^* = \sum_{a=1}^{n} \bigoplus_{i=0}^{p-1} \pi_{i+1}(v_a v_a^*) = \bigoplus_{i=0}^{p-1} I_{i+1} = I.$$

This completes the proof of (ii).

To see that $S_1^k\xi_0$ implements $\beta^{*k}\rho$ as a vector state in π_i, first observe that

$$S_1^*\pi_{j+1}(x)S_1 = \pi_j(v_1^* x v_1), \quad x \in \mathcal{F}_n,\ 0 \leqslant j \leqslant p-1; \tag{2.4}$$

this is an easy consequence of (2.1) and (2.2). Thus

$$\begin{aligned}\langle \pi_i(x)S_1^k\xi_0, S_1^k\xi_0\rangle &= \langle S_1^{*k}\pi_i(x)S_1^k\xi_0, \xi_0\rangle = \langle \pi_0(v_1^{*k}xv_1^k)\xi_0, \xi_0\rangle \\ &= \rho(v_1^{*k}xv_1^k) = \beta^{*k}\rho(x).\end{aligned}$$

By (2.3) we have $S_1\xi_k = \pi_{k+1}(v_1v_1^*)\xi_{k+1} = \xi_{k+1}$ for $0 \leqslant k \leqslant p-1$, so $\xi_k = S_1^k\xi_0$ for $0 \leqslant k \leqslant p$. ∎

We are now ready to construct the representations of $\mathcal{O}_n$ that will be used to classify the state extensions of ρ to $\mathcal{O}_n$. Suppose $U_0, \ldots, U_{p-1}$ are unitary operators on a Hilbert space $\mathcal{K}$. It is immediate from Proposition 2.3 that the range projections of the isometries $\bigoplus_{i=0}^{p-1} S_{a,i} \otimes U_i$ for $1 \leqslant a \leqslant n$ sum to the identity operator on $\bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}$. Consequently there is a unique representation of $\mathcal{O}_n$, essential if $n = \infty$, which maps v_a to $\bigoplus_{i=0}^{p-1} S_{a,i} \otimes U_i$. We will restrict our attention to p-tuples $(U_0, \ldots, U_{p-1})$ in which every component but the last one is equal to the identity; up to unitary equivalence of the resulting representation there is no loss of generality in this, as we will see in Proposition 3.2.

NOTATION 2.4. Consider a triple (ρ, ξ_p, θ) in which

(i) ρ is a pure state of $\mathcal{F}_n$ with finite period p, essential if $n = \infty$;

(ii) $\xi_p \in \mathcal{H}_0$ is a linking vector for ρ; and

(iii) θ is a representation of $C(\mathbb{T})$ on a Hilbert space $\mathcal{K}_\theta$.

Let U_θ be the p-tuple $(I, I, \ldots, \theta(\mathbf{z}))$ of unitaries on $\mathcal{K}_\theta$, where $\mathbf{z}$ is the identity function on $\mathbb{T}$. We will denote by $\widetilde{\pi}[\rho, \xi_p, \theta]$ the representation of $\mathcal{O}_n$ on $\widetilde{\mathcal{K}}_\theta := \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_\theta$ which is determined by

$$\widetilde{\pi}[\rho, \xi_p, \theta](v_a) = \bigoplus_{i=0}^{p-1} S_{a,i} \otimes U_{\theta,i}, \quad 1 \leqslant a \leqslant n. \tag{2.5}$$

PROPOSITION 2.5. *Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n = \infty$ assume that ρ is essential. Suppose $\xi_p \in \mathcal{H}_0$ is a linking vector for ρ and θ is a representation of $C(\mathbb{T})$ on a Hilbert space $\mathcal{K}_\theta$.*

(i) *The restriction of $\widetilde{\pi}[\rho, \xi_p, \theta]$ to $\mathcal{F}_n$ is $\bigoplus_{i=0}^{p-1} \pi_i \otimes I_\theta$.*

(ii) *If ψ is another representation of $C(\mathbb{T})$, then $\widetilde{\pi}[\rho, \xi_p, \theta]$ and $\widetilde{\pi}[\rho, \xi_p, \psi]$ are unitarily equivalent* (*resp. disjoint*) *if and only if θ and ψ are unitarily equivalent* (*resp. disjoint*).

(iii) *$\widetilde{\pi}[\rho, \xi_p, \theta]$ is irreducible if and only if θ is irreducible* (*i.e.*, $\dim \mathcal{K}_\theta = 1$).

(iv) *If $\eta \in \mathcal{K}_\theta$, then $\xi_0 \otimes \eta$ is cyclic for $\widetilde{\pi}[\rho, \xi_p, \theta]$ if and only if η is cyclic for θ.*

(v) *For each* $\lambda \in \mathbb{T}$ *let* τ_λ *be translation by* λ *on* $C(\mathbb{T})$*; that is,* $\tau_\lambda f(z) = f(\lambda^{-1}z)$ *for* $f \in C(\mathbb{T})$ *and* $z \in \mathbb{T}$*. If* $\mu^p = \lambda \in \mathbb{T}$*, then*

$$\widetilde{\pi}[\rho, \lambda\xi_p, \theta] = \widetilde{\pi}[\rho, \xi_p, \theta \circ \tau_{\overline{\lambda}}] \overset{u}{\sim} \widetilde{\pi}[\rho, \xi_p, \theta] \circ \gamma_\mu. \tag{2.6}$$

Proof. (i) For the moment write $\widetilde{\pi}$ for $\widetilde{\pi}[\rho, \xi_p, \theta]$ and σ for $\bigoplus_{i=0}^{p-1} \pi_i \otimes I$. Since both σ and the restriction of $\widetilde{\pi}$ to $\mathcal{F}_n$ are unital representations and $\mathcal{F}_n = \overline{\operatorname{span}}\{v_s v_t^* : |s| = |t|\}$, it suffices, by induction, to show that $\widetilde{\pi}(y) = \sigma(y)$ implies that $\widetilde{\pi}(v_j y v_k^*) = \sigma(v_j y v_k^*)$ whenever $y \in \mathcal{F}_n$ and $1 \leqslant j, k \leqslant n$. If $x \in \mathcal{F}_n$, $0 \leqslant i \leqslant p-1$, and $\eta \in \mathcal{K}_\theta$, then

$$\begin{aligned}
&\widetilde{\pi}(v_j y v_k^*)(\pi_{i+1}(x)\xi_{i+1} \otimes \eta) = \widetilde{\pi}(v_j)\sigma(y)\widetilde{\pi}(v_k)^*(\pi_{i+1}(x)\xi_{i+1} \otimes \eta)\\
&\quad = \widetilde{\pi}(v_j)\sigma(y)(\pi_i(v_k^* x v_1)\xi_i \otimes U_{\theta,i}^*\eta) = \widetilde{\pi}(v_j)(\pi_i(y v_k^* x v_1)\xi_i \otimes U_{\theta,i}^*\eta)\\
&\quad = \pi_{i+1}(v_j y v_k^* x v_1 v_1^*)\xi_{i+1} \otimes U_{\theta,i}U_{\theta,i}^*\eta = \sigma(v_j y v_k^*)(\pi_{i+1}(x)\xi_{i+1} \otimes \eta) \quad \text{(by (2.3))}.
\end{aligned}$$

(ii) Let $\mathcal{I}$ be the intertwining space

$$\mathcal{I} := \{\, T \in \mathcal{B}(\widetilde{\mathcal{K}}_\theta, \widetilde{\mathcal{K}}_\psi) : T\widetilde{\pi}[\rho, \xi_p, \theta](z) = \widetilde{\pi}[\rho, \xi_p, \psi](z)T \;\forall\, z \in \mathcal{O}_n \,\},$$

and let

$$\mathcal{I}_0 := \Big\{\, T = \bigoplus_{i=0}^{p-1} I_i \otimes T_0 \in \mathcal{B}(\widetilde{\mathcal{K}}_\theta, \widetilde{\mathcal{K}}_\psi) : T_0\theta(f) = \psi(f)T_0 \;\forall f \in C(\mathbb{T}) \,\Big\}.$$

We claim that $\mathcal{I} = \mathcal{I}_0$, from which (ii) follows immediately.

As a first step we describe the space $\mathcal{J} \supseteq \mathcal{I}$ defined by

$$\mathcal{J} := \{\, T \in \mathcal{B}(\widetilde{\mathcal{K}}_\theta, \widetilde{\mathcal{K}}_\psi) : T\widetilde{\pi}[\rho, \xi_p, \theta](x) = \widetilde{\pi}[\rho, \xi_p, \psi](x)T \;\forall\, x \in \mathcal{F}_n \,\}.$$

By (i), $\mathcal{J}$ is the set of operators which intertwine $\bigoplus_{i=0}^{p-1} \pi_i \otimes I_\theta$ and $\bigoplus_{i=0}^{p-1} \pi_i \otimes I_\psi$. Since $\pi_0, \ldots, \pi_{p-1}$ are irreducible and mutually disjoint, we have

$$\mathcal{J} = \Big\{\, T = \bigoplus_{i=0}^{p-1} I_i \otimes T_i : T_i \in \mathcal{B}(\mathcal{K}_\theta, \mathcal{K}_\psi) \,\Big\}.$$

Suppose that $T = \bigoplus_{i=0}^{p-1} I_i \otimes T_i \in \mathcal{J}$. For notational convenience let $T_p := T_0$. If $x \in \mathcal{F}_n$, $0 \leqslant i \leqslant p-1$ and $\eta \in \mathcal{K}_\theta$, then

$$\begin{aligned}
T\widetilde{\pi}[\rho, \xi_p, \theta](v_a)(\pi_i(x)\xi_i \otimes \eta) &= T(\pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes U_{\theta,i}\eta)\\
&= \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes T_{i+1}U_{\theta,i}\eta,
\end{aligned} \tag{2.7}$$

whereas

$$\begin{aligned}
\widetilde{\pi}[\rho, \xi_p, \psi](v_a)T(\pi_i(x)\xi_i \otimes \eta) &= \widetilde{\pi}[\rho, \xi_p, \psi](v_a)(\pi_i(x)\xi_i \otimes T_i\eta)\\
&= \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes U_{\psi,i}T_i\eta.
\end{aligned} \tag{2.8}$$

Now suppose that $T \in \mathcal{I}$. By (2.7) and (2.8) we have

$$T_{i+1}U_{\theta,i} = U_{\psi,i}T_i, \quad 0 \leqslant i \leqslant p-1. \tag{2.9}$$

Setting $i = 0, 1, \ldots, p-2$ gives $T_0 = T_1 = \cdots = T_{p-1}$, and setting $i = p-1$ gives $T_0\theta(\mathbf{z}) = \psi(\mathbf{z})T_0$. Since $\mathbf{z}$ generates $C(\mathbb{T})$ this implies that $T \in \mathcal{I}_0$, and thus $\mathcal{I} \subseteq \mathcal{I}_0$.

Conversely, suppose T_0 intertwines θ and ψ, so that $T := \bigoplus_{i=0}^{p-1} I_i \otimes T_0 \in \mathcal{I}_0$. By setting $T_i := T_0$ for $1 \leqslant i \leqslant p$, we see that (2.9) holds. By (2.7) and (2.8) it follows that $T\widetilde{\pi}[\rho, \xi_p, \theta](v_a) = \widetilde{\pi}[\rho, \xi_p, \psi](v_a)T$ for each a, so that $T \in \mathcal{I}$. Thus $\mathcal{I} = \mathcal{I}_0$ as claimed, completing the proof of (ii).

(iii) Setting $\psi = \theta$ in the proof of (ii) gives

$$\widetilde{\pi}[\rho, \xi_p, \theta](\mathcal{O}_n)' = \Big\{ T = \bigoplus_{i=0}^{p-1} I_i \otimes T_0 : T_0 \in \theta(C(\mathbb{T}))' \Big\}, \tag{2.10}$$

from which (iii) is immediate.

(iv) Let $\mathcal{M} \subseteq \widetilde{\mathcal{K}}_\theta$ be the cyclic subspace for $\widetilde{\pi}[\rho, \xi_p, \theta]$ generated by $\xi_0 \otimes \eta$. The orthogonal projection P of $\widetilde{\mathcal{K}}_\theta$ onto $\mathcal{M}$ commutes with $\widetilde{\pi}[\rho, \xi_p, \theta](\mathcal{O}_n)$, so by (2.10) there is a projection $P_0 \in \theta(C(\mathbb{T}))'$ such that $P = \bigoplus_{i=0}^{p-1} I_i \otimes P_0$. On the other hand, $\mathcal{M}$ is the closed linear span of vectors of the form $\widetilde{\pi}[\rho, \xi_p, \theta](v_s v_t^*)(\xi_0 \otimes \eta)$, where s and t are multi-indices. Given such a vector, express $|s| - |t| = j + mp$ for $j \in \{0, \ldots, p-1\}$ and $m \in \mathbb{Z}$. By (2.5),

$$\widetilde{\pi}[\rho, \xi_p, \theta](v_s v_t^*)(\xi_0 \otimes \eta) \in \mathcal{H}_j \otimes \theta(\mathbf{z}^m)\eta,$$

from which it follows that the range of P_0 is the closure of $\theta(C(\mathbb{T}))\eta$. Assertion (iv) now follows easily.

(v) If $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_\theta$, then

$$\begin{aligned}
\widetilde{\pi}[\rho, \lambda\xi_p, \theta](v_a)(\pi_i(x)\xi_i \otimes \eta) &= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes \eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)(\lambda\xi_p) \otimes \theta(\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes \eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)\xi_p \otimes \theta(\lambda\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes \eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)\xi_p \otimes \theta \circ \tau_{\overline{\lambda}}(\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= \widetilde{\pi}[\rho, \xi_p, \theta \circ \tau_{\overline{\lambda}}](v_a)(\pi_i(x)\xi_i \otimes \eta),
\end{aligned}$$

giving the first half of (2.6). Let T be the unitary operator $\bigoplus_{i=0}^{p-1} I_i \otimes \mu^i I_\theta$ on $\widetilde{\mathcal{K}}_\theta$. Then

$$\begin{aligned}
\widetilde{\pi}[\rho, \xi_p, \theta] \circ \gamma_\mu(v_a)T(\pi_i(x)\xi_i \otimes \eta) &= \mu\widetilde{\pi}[\rho, \xi_p, \theta](v_a)(\pi_i(x)\xi_i \otimes \mu^i\eta) \\
&= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes \mu^{i+1}\eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)\xi_p \otimes \lambda\theta(\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes \mu^{i+1}\eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)\xi_p \otimes \theta \circ \tau_{\overline{\lambda}}(\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= T\widetilde{\pi}[\rho, \xi_p, \theta \circ \tau_{\overline{\lambda}}](v_a)(\pi_i(x)\xi_i \otimes \eta),
\end{aligned}$$

from which the second half of (2.6) follows. ∎

Remark 2.6. The representation $\widetilde{\pi}[\rho, \xi_p]$ of Proposition 2.3 is irreducible because it is unitarily equivalent to $\widetilde{\pi}[\rho, \xi_p, \varepsilon_1]$, where ε_1 is evaluation at $1 \in \mathbb{T}$, and $\widetilde{\pi}[\rho, \xi_p, \varepsilon_1]$ is irreducible by Proposition 2.5 (iii).

Proposition 2.7. *Suppose ρ and ω are periodic pure states of $\mathcal{F}_n$, essential if $n = \infty$. If ρ and ω have the same quasi-orbit (and hence the same period p), then there are linking vectors ξ_p^ρ and ξ_p^ω such that $\widetilde{\pi}[\rho, \xi_p^\rho, \theta]$ and $\widetilde{\pi}[\omega, \xi_p^\omega, \theta]$ are unitarily equivalent for every representation θ of $C(\mathbb{T})$.*

Proof. Define a relation $\approx$ on the pure essential states of $\mathcal{F}_n$ with finite period p as follows: $\rho \approx \omega$ if there are linking vectors ξ_p^ρ and ξ_p^ω such that $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \theta]$ for every representation θ of $C(\mathbb{T})$. Then $\approx$ is an equivalence relation. To show transitivity recall that the linking vector for a pure essential state is unique up to a scalar of modulus one and observe that if $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \theta]$ and $\mu \in \mathbb{T}$, then by (2.6)

$$\widetilde{\pi}[\rho, \mu^p \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_\mu \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \theta] \circ \gamma_\mu \overset{u}{\sim} \widetilde{\pi}[\omega, \mu^p \xi_p^\omega, \theta].$$

The proof of the proposition is based on the following two claims.

Claim 1: *If $\rho \overset{u}{\sim} \omega$, then $\rho \approx \omega$.*
Claim 2: *$\rho \approx \beta^{*k}\rho$ for every positive integer k.*

Given the claims, the proof is easy: if ρ and ω have the same quasi-orbit, then $\rho \overset{u}{\sim} \beta^{*k}\omega$ for some positive integer k, and by the two claims $\rho \approx \beta^{*k}\omega \approx \omega$.

Proof of Claim 1. If $\rho \overset{u}{\sim} \omega$, then there is a vector $\zeta \in \mathcal{H}_0^\rho$ such that $\omega(x) = \langle \pi_0^\rho(x)\zeta, \zeta\rangle$ for $x \in \mathcal{F}_n$. Let ξ_p^ρ be a linking vector for ρ, and let S_1 be the isometry on $\bigoplus_{i=0}^{p-1} \mathcal{H}_i^\rho$ defined in Proposition 2.3. If $0 \leqslant i \leqslant p$ and $x \in \mathcal{F}_n$, then by (2.4)

$$\beta^{*i}\omega(x) = \omega(v_1^{*i} x v_1^i) = \langle \pi_0^\rho(v_1^{*i} x v_1^i)\zeta, \zeta\rangle = \langle S_1^{*i}\pi_i^\rho(x) S_1^i \zeta, \zeta\rangle, = \langle \pi_i^\rho(x) S_1^i\zeta, S_1^i\zeta\rangle,$$

so $S_1^i\zeta$ implements $\beta^{*i}\omega$ as a vector state in π_i^ρ. Hence for each $i \in \{0, \dots, p-1\}$ there is a unique unitary operator $V_i : \mathcal{H}_i^\omega \to \mathcal{H}_i^\rho$ which intertwines π_i^ω and π_i^ρ and maps ξ_i^ω to $S_1^i\zeta$. Define $\xi_p^\omega := V_0^* S_1^p \zeta$; then ξ_p^ω is a linking vector for ω. Let θ be a representation of $C(\mathbb{T})$, and let $V : \bigoplus_{i=0}^{p-1} \mathcal{H}_i^\omega \otimes \mathcal{K}_\theta \to \bigoplus_{i=0}^{p-1} \mathcal{H}_i^\rho \otimes \mathcal{K}_\theta$ be the unitary operator $\bigoplus_{i=0}^{p-1} V_i \otimes I_\theta$. Then V intertwines $\widetilde{\pi}[\omega, \xi_p^\omega, \theta]$ and $\widetilde{\pi}[\rho, \xi_p^\rho, \theta]$. To see this, suppose $1 \leqslant a \leqslant n$, $0 \leqslant i \leqslant p-1$, $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_\theta$. Using Proposition 2.5 (i) and the convention $V_p := V_0$,

$$\begin{aligned}
V\widetilde{\pi}[\omega, \xi_p^\omega, \theta](v_a)(\pi_i^\omega(x)\xi_i^\omega \otimes \eta) &= V_{i+1}\pi_{i+1}^\omega(v_a x v_1^*)\xi_{i+1}^\omega \otimes U_{\theta,i}\eta \\
&= \pi_{i+1}^\rho(v_a x v_1^*) S_1^{i+1}\zeta \otimes U_{\theta,i}\eta = \widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_a x v_1^*)(S_1^{i+1}\zeta \otimes U_{\theta,i}\eta) \\
&= \widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_a x)(S_1^i\zeta \otimes \eta) = \widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_a)(\pi_i^\rho(x) S_1^i\zeta \otimes \eta) \\
&= \widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_a) V(\pi_i^\omega(x)\xi_i^\omega \otimes \eta).
\end{aligned}$$

This completes the proof of Claim 1.

Proof of Claim 2. It suffices to prove the claim for $k = 1$. Let $\omega := \beta^*\rho$. Then $\pi_i^\omega = \pi_{i+1}^\rho$, $\mathcal{H}_i^\omega = \mathcal{H}_{i+1}^\rho$ and $\xi_i^\omega = \xi_{i+1}^\rho$ for $0 \leqslant i \leqslant p-2$.

Fix a linking vector ξ_p^ρ for ρ. By Proposition 2.3 (iii), $\xi_p^\omega := \xi_{p+1}^\rho$ is a linking vector for ω. Let θ be a representation of $C(\mathbb{T})$. We claim that $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \theta]$. To construct the intertwining unitary, let $W_0 : \mathcal{H}_0^\rho \to \mathcal{H}_{p-1}^\omega$ be the unique unitary operator which intertwines π_0^ρ and π_{p-1}^ω and maps ξ_p^ρ to ξ_{p-1}^ω. Let $W : \bigoplus_{i=0}^{p-1} \mathcal{H}_i^\rho \otimes \mathcal{K}_\theta \to \bigoplus_{i=0}^{p-1} \mathcal{H}_i^\omega \otimes \mathcal{K}_\theta$ be the unitary operator which is the identity from $\mathcal{H}_{i+1}^\rho \otimes \mathcal{K}_\theta$ to $\mathcal{H}_i^\omega \otimes \mathcal{K}_\theta$ for $0 \leqslant i \leqslant p-2$, and $W_0 \otimes \theta(\mathbf{z})^*$ from $\mathcal{H}_0^\rho \otimes \mathcal{K}_\theta$ to $\mathcal{H}_{p-1}^\omega \otimes \mathcal{K}_\theta$. If $1 \leqslant i \leqslant p$, $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_\theta$, then

$$W(\pi_i^\rho(x)\xi_i^\rho \otimes \eta) = \begin{cases} \pi_{i-1}^\omega(x)\xi_{i-1}^\omega \otimes \eta & \text{if } 1 \leqslant i \leqslant p-1 \\ \pi_{p-1}^\omega(x)\xi_{p-1}^\omega \otimes \theta(\mathbf{z})^*\eta & \text{if } i = p, \end{cases}$$

so for each $a \in 1, \dots, n$ we have

$$\begin{aligned} W\widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_a)(\pi_i^\rho(x)\xi_i^\rho \otimes \eta) &= \begin{cases} W(\pi_{i+1}^\rho(v_a x v_1^*)\xi_{i+1}^\rho \otimes \eta) & \text{if } 1 \leqslant i \leqslant p-2 \\ W(\pi_0^\rho(v_a x v_1^*)\xi_p^\rho \otimes \theta(\mathbf{z})\eta) & \text{if } i = p-1 \\ W(\pi_1^\rho(v_a x v_1^*)\xi_{p+1}^\rho \otimes \eta) & \text{if } i = p \end{cases} \\ &= \pi_i^\omega(v_a x v_1^*)\xi_i^\omega \otimes \eta \\ &= \begin{cases} \widetilde{\pi}[\omega, \xi_p^\omega, \theta](v_a)(\pi_{i-1}^\omega(x)\xi_{i-1}^\omega \otimes \eta) & \text{if } 1 \leqslant i \leqslant p-1 \\ \widetilde{\pi}[\omega, \xi_p^\omega, \theta](v_a)(\pi_{p-1}^\omega(x)\xi_{p-1}^\omega \otimes \theta(\mathbf{z})^*\eta) & \text{if } i = p \end{cases} \\ &= \widetilde{\pi}[\omega, \xi_p^\omega, \theta](v_a)W(\pi_i^\rho(x)\xi_i^\rho \otimes \eta). \end{aligned}$$

Thus W intertwines $\widetilde{\pi}[\rho, \xi_p^\rho, \theta]$ and $\widetilde{\pi}[\omega, \xi_p^\omega, \theta]$, completing the proof of Claim 2. ∎

CORROLARY 2.8. *Suppose ρ and ω are pure states of $\mathcal{F}_n$ with finite periods p and q, respectively; if $n = \infty$ assume also that ρ and ω are essential. Suppose θ and ψ are representations of $C(\mathbb{T})$. Then $\widetilde{\pi}[\rho, \xi_p^\rho, \theta]$ and $\widetilde{\pi}[\omega, \xi_q^\omega, \psi]$ are unitarily equivalent (for some choice of linking vectors ξ_p^ρ and ξ_q^ω) if and only if*

(i) *ρ and ω have the same quasi-orbit, and*

(ii) *θ is unitarily equivalent to $\psi \circ \tau_\lambda$ for some $\lambda \in \mathbb{T}$.*

Proof. Suppose $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_q^\omega, \psi]$. By Proposition 2.5(i), the unitary operator which implements this equivalence also intertwines the representations $\bigoplus_{i=0}^{p-1} \pi_i^\rho \otimes I_\theta$ and $\bigoplus_{j=0}^{q-1} \pi_j^\omega \otimes I_\psi$ of $\mathcal{F}_n$. From this it is evident that $\mathcal{K}_\theta \cong \mathcal{K}_\psi$, $p = q$, and $\pi_0^\omega \overset{u}{\sim} \pi_k^\rho$ for some $k \in \{0, 1, \dots, p-1\}$. Thus $\omega \overset{u}{\sim} \beta^{*k}\rho$, so ρ and ω have the same quasi-orbit.

By Proposition 2.7 and the essential uniqueness of linking vectors, there are scalars $a, b \in \mathbb{T}$ such that $\widetilde{\pi}[\rho, a\xi_p^\rho, \varphi] \overset{u}{\sim} \widetilde{\pi}[\omega, b\xi_p^\omega, \varphi]$ for every represention φ of $C(\mathbb{T})$. By (2.6) we then have $\widetilde{\pi}[\rho, \xi_p^\rho, \varphi \circ \tau_{\overline{a}}] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \varphi \circ \tau_{\overline{b}}]$ for each φ, and taking $\varphi = \psi \circ \tau_b$ gives $\widetilde{\pi}[\rho, \xi_p^\rho, \psi \circ \tau_{b\overline{a}}] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \psi]$. Thus $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\rho, \xi_p^\rho, \psi \circ \tau_{b\overline{a}}]$, so by Proposition 2.5(ii) we have $\theta \overset{u}{\sim} \psi \circ \tau_{b\overline{a}}$.

Conversely, suppose ρ and ω have the same quasi-orbit and $\theta \overset{u}{\sim} \psi \circ \tau_\lambda$. By Proposition 2.7, there are linking vectors ξ_p^ρ and ξ_p^ω such that

$$\widetilde{\pi}[\rho, \xi_p^\rho, \psi \circ \tau_\lambda] \overset{u}{\sim} \widetilde{\pi}[\omega, \xi_p^\omega, \psi \circ \tau_\lambda].$$

The first of these representations is unitarily equivalent to $\widetilde{\pi}[\rho, \xi_p^\rho, \theta]$ (Proposition 2.5 (ii)), and the second to $\widetilde{\pi}[\omega, \overline{\lambda}\xi_p^\omega, \psi]$ (Proposition 2.5 (v)). Thus $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \overset{u}{\sim} \widetilde{\pi}[\omega, \overline{\lambda}\xi_p^\omega, \psi]$. ∎

3. EXTENSIONS OF PERIODIC PURE STATES OF $\mathcal{F}_n$ TO $\mathcal{O}_n$.

In this section we use the representations constructed in Section 2 to parameterize and classify the state extensions of periodic pure essential states of $\mathcal{F}_n$. Our main result, Theorem 3.5, is preceded by a general technical lemma and two technical propositions.

LEMMA 3.1. *Suppose π is a representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$ and $\{\pi_i : i \in I\}$ is a collection of subrepresentations of π, each of which is quasi-equivalent to a given representation φ. Let $\mathcal{H}_i$ be the representation space of π_i. If $\bigcup_{i\in I} \mathcal{H}_i$ has dense linear span in $\mathcal{H}$, then no subrepresentation of π is disjoint from φ. If in addition φ is factorial, then π is factorial and quasi-equivalent to φ.*

Proof. Suppose ψ is a subrepresentation of π, and let ξ be a nonzero vector in the representation space of ψ. Since $\bigcup_{i\in I} \mathcal{H}_i$ has dense linear span in $\mathcal{H}$, there is an $i \in I$ such that $\xi \notin \mathcal{H}_i^\perp$. Express $\xi = \xi_0 + \xi_1 \in \mathcal{H}_i \oplus \mathcal{H}_i^\perp$, and let ω_ξ, ω_{ξ_0} and ω_{ξ_1} be the corresponding vector functionals. Then $\omega_\xi = \omega_{\xi_0} + \omega_{\xi_1}$, so $\omega_{\xi_0} \leqslant \omega_\xi$. Let π_ξ and π_{ξ_0} denote the GNS representations associated with ω_ξ and ω_{ξ_0}, respectively. Then π_{ξ_0} is unitarily equivalent to a subrepresentation of π_ξ, which in turn is unitarily equivalent to a subrepresentation of ψ. But π_{ξ_0} is also unitarily equivalent to a subrepresentation of π_i, which is quasi-equivalent to φ. Thus ψ and φ are not disjoint. If φ is factorial, this means that $\pi \overset{q}{\sim} \varphi$. ∎

PROPOSITION 3.2. *Let $\widetilde{\sigma}$ be a representation of $\mathcal{O}_n$ on a separable Hilbert space $\widetilde{\mathcal{K}}$; if $n = \infty$ assume that $\widetilde{\sigma}$ is essential. Suppose there exists a pure state ρ of $\mathcal{F}_n$ with finite period p such that the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$ decomposes as a direct sum $\bigoplus_{i=0}^{p-1} \sigma_i$, where σ_i is quasi-equivalent to the* GNS *representation π_i for $\beta^{*i}\rho$. Then there is a linking vector ξ_p for ρ and a representation θ of $C(\mathbb{T})$ such that $\widetilde{\sigma}$ is unitarily equivalent to $\widetilde{\pi}[\rho, \xi_p, \theta]$. Consequently, the multiplicity of π_i in σ_i is independent of $i = 0, 1, \ldots, p-1$.*

Proof. Let $\sigma = \bigoplus_{i=0}^{p-1} \sigma_i$ be the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$. Each σ_i is unitarily equivalent to the representation $\pi_i \otimes I_i$ of $\mathcal{F}_n$ on $\mathcal{H}_i \otimes \mathcal{K}_i$ for some separable Hilbert space $\mathcal{K}_i$, so modulo a unitary equivalence we may assume that $\widetilde{\mathcal{K}} = \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_i$

and $\sigma = \bigoplus_{i=0}^{p-1} \pi_i \otimes I_i$. Let ξ_p be a linking vector for ρ, and adopt the notation convention $\mathcal{K}_p := \mathcal{K}_0$. Of course $\pi_p := \pi_0$ and $\mathcal{H}_p := \mathcal{H}_0$, as usual.

Fix $i \in \{0, 1, \ldots, p-1\}$ and $\eta \in \mathcal{K}_i$. We claim that there is a (necessarily unique) vector $U_i\eta \in \mathcal{K}_{i+1}$ such that

$$\widetilde{\sigma}(v_1)(\xi_i \otimes \eta) = \xi_{i+1} \otimes U_i\eta. \tag{3.1}$$

To begin with, note that for any $x \in \mathcal{F}_n$,

$$\begin{aligned}\langle \sigma(x)\widetilde{\sigma}(v_1)(\xi_i \otimes \eta), \widetilde{\sigma}(v_1)(\xi_i \otimes \eta)\rangle &= \langle \sigma(v_1^* x v_1)(\xi_i \otimes \eta), \xi_i \otimes \eta\rangle \\ &= \|\eta\|^2 \langle \pi_i(v_1^* x v_1)\xi_i, \xi_i\rangle = \|\eta\|^2 \beta^{*i}\rho(v_1^* x v_1) = \|\eta\|^2 \beta^{*(i+1)}\rho(x).\end{aligned}$$

On the other hand, we can express $\widetilde{\sigma}(v_1)(\xi_i \otimes \eta) = \sum_{j=1}^{p} \sum_k \zeta_{j,k} \otimes \delta_{j,k}$, where $\zeta_{j,k} \in \mathcal{H}_j$ and for each j the set $\{\delta_{j,k}\}$ is an orthonormal basis for $\mathcal{K}_j$. Then

$$\begin{aligned}\langle \sigma(x)\widetilde{\sigma}(v_1)(\xi_i \otimes \eta), \widetilde{\sigma}(v_1)(\xi_i \otimes \eta)\rangle &= \sum_{j=1}^{p} \Big\langle \sum_k \pi_j(x)\zeta_{j,k} \otimes \delta_{j,k}, \sum_l \zeta_{j,l} \otimes \delta_{j,l} \Big\rangle \\ &= \sum_{j=1}^{p} \sum_k \langle \pi_j(x)\zeta_{j,k}, \zeta_{j,k}\rangle.\end{aligned}$$

Since $\beta^{*(i+1)}\rho$ is pure, each of the positive linear functionals $\omega_{\zeta_{j,k}}(x) := \langle \pi_j(x)\zeta_{j,k}, \zeta_{j,k}\rangle$ is a multiple of $\beta^{*(i+1)}\rho$. However, $\omega_{\zeta_{j,k}}$ is also unitarily equivalent to a multiple of $\beta^{*j}\rho$, because π_j is irreducible. Since the states $\beta^{*j}\rho$ for $j = 1, 2, \ldots, p$ are mutually disjoint, we thus have $\zeta_{j,k} = 0$ unless $j = i+1$. Moreover, $\omega_{\zeta_{i+1,k}}$ is a scalar multiple of $\beta^{*(i+1)}\rho$ if and only if $\zeta_{i+1,k}$ is a scalar multiple of ξ_{i+1}, so after simplifying and rearranging, the sum $\sum_{j=1}^{p} \sum_k \zeta_{j,k} \otimes \delta_{j,k}$ turns out to be an elementary tensor; specifically, it belongs to the subspace $\xi_{i+1} \otimes \mathcal{K}_{i+1}$. Thus we can define $U_i : \mathcal{K}_i \to \mathcal{K}_{i+1}$ by (3.1), as claimed.

We next claim that U_i is a unitary operator. It is evident that U_i is linear, and since

$$\langle U_i\eta, U_i\zeta\rangle = \langle \widetilde{\sigma}(v_1)(\xi_i \otimes \eta), \widetilde{\sigma}(v_1)(\xi_i \otimes \zeta)\rangle = \langle \xi_i \otimes \eta, \xi_i \otimes \zeta\rangle = \langle \eta, \zeta\rangle, \quad \eta, \zeta \in \mathcal{K}_i,$$

U_i is an isometry. To see that U_i is surjective, suppose $\zeta \in \mathcal{K}_{i+1}$. By (2.3) we have

$$\xi_{i+1} \otimes \zeta = \pi_{i+1}(v_1 v_1^*)\xi_{i+1} \otimes \zeta = \sigma(v_1 v_1^*)(\xi_{i+1} \otimes \zeta) = \widetilde{\sigma}(v_1)\widetilde{\sigma}(v_1^*)(\xi_{i+1} \otimes \zeta).$$

Now $\widetilde{\sigma}(v_1^*)(\xi_{i+1} \otimes \zeta)$ can be approximated by a finite sum of vectors of the form $\pi_j(x)\xi_j \otimes \eta$, where $0 \leqslant j \leqslant p-1$, $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_j$, and for each such vector

$$\begin{aligned}\widetilde{\sigma}(v_1)(\pi_j(x)\xi_j \otimes \eta) &= \widetilde{\sigma}(v_1)\sigma(x)(\xi_j \otimes \eta) = \sigma(v_1 x v_1^*)\widetilde{\sigma}(v_1)(\xi_j \otimes \eta) \\ &= \sigma(v_1 x v_1^*)(\xi_{j+1} \otimes U_j\eta) = \pi_{j+1}(v_1 x v_1^*)\xi_{j+1} \otimes U_j\eta \in \mathcal{H}_{j+1} \otimes \operatorname{ran} U_j.\end{aligned} \tag{3.2}$$

Thus

$$\xi_{i+1} \otimes \zeta = \widetilde{\sigma}(v_1)\widetilde{\sigma}(v_1^*)(\xi_{i+1} \otimes \zeta) \in \bigoplus_{j=0}^{p-1} \mathcal{H}_{j+1} \otimes \operatorname{ran} U_j,$$

which shows that $\zeta \in \operatorname{ran} U_i$. Thus U_i is surjective.

Define unitary operators $T_0, \dots, T_{p-1}$ inductively by $T_0 := U_{p-1}$ and $T_{i+1} = U_i T_i$ for $0 \leqslant i \leqslant p-2$. Then T_{p-1} is a unitary operator on $\mathcal{K}_{p-1}$, so $\theta(\mathbf{z}) = T_{p-1}$ determines a representation θ of $C(\mathbb{T})$ on $\mathcal{K}_{p-1}$. We claim that $\widetilde{\sigma}$ is unitarily equivalent to $\widetilde{\pi}[\rho, \xi_p, \theta]$. For this, let $T : \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_{p-1} \to \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_i$ be the unitary $\bigoplus_{i=0}^{p-1} I_i \otimes T_i$. If $1 \leqslant a \leqslant n$, $0 \leqslant i \leqslant p-1$, $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_{p-1}$, then by (3.2)

$$\begin{aligned}
\widetilde{\sigma}(v_a)T(\pi_i(x)\xi_i \otimes \eta) &= \sigma(v_a v_1^*)\widetilde{\sigma}(v_1)(\pi_i(x)\xi_i \otimes T_i\eta) \\
&= \sigma(v_a v_1^*)(\pi_{i+1}(v_1 x v_1^*)\xi_{i+1} \otimes U_i T_i \eta) \\
&= \begin{cases} \pi_{i+1}(v_a x v_1^*)\xi_{i+1} \otimes T_{i+1}\eta & \text{if } 0 \leqslant i \leqslant p-2 \\ \pi_0(v_a x v_1^*)\xi_p \otimes T_0\theta(\mathbf{z})\eta & \text{if } i = p-1 \end{cases} \\
&= T\widetilde{\pi}[\rho, \xi_p, \theta](v_a)(\pi_i(x)\xi_i \otimes \eta),
\end{aligned}$$

so T intertwines $\widetilde{\pi}[\rho, \xi_p, \theta]$ and $\widetilde{\sigma}$.

The multiplicity of π_i in σ_i is the dimension of $\mathcal{K}_i$. Since each $U_i : \mathcal{K}_i \to \mathcal{K}_{i+1}$ is unitary, this multiplicity is constant in i. ∎

Proposition 3.3. *Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n = \infty$ assume that ρ is essential. Let $\widetilde{\sigma}$ be the* GNS *representation of $\mathcal{O}_n$ corresponding to a state $\widetilde{\rho}$ extending ρ. Then the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$ decomposes as a direct sum $\bigoplus_{i=0}^{p-1} \sigma_i$, with σ_i quasi-equivalent to the* GNS *representation π_i of $\beta^{*i}\rho$. Furthermore, the decomposition is central and the multiplicity of π_i in σ_i is independent of $i = 0, 1, \dots, p-1$.*

Proof. Let $\widetilde{\mathcal{H}}$ be the Hilbert space on which $\widetilde{\sigma}$ represents $\mathcal{O}_n$, and let $\xi \in \widetilde{\mathcal{H}}$ be the canonical cyclic vector which implements $\widetilde{\rho}$ as a vector state. Let σ denote the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$. For each $k \in \mathbb{Z}$ let

$$\mathcal{G}_k = \{z \in \mathcal{O}_n : \gamma_\lambda(z) = \lambda^k z, \lambda \in \mathbb{T}\}.$$

Notice that $\mathcal{G}_0 = \mathcal{F}_n$ and, in general, that $\mathcal{G}_k$ is the k^{th} spectral subspace of $\mathcal{O}_n$ under the action of the gauge group $\{\gamma_\lambda : \lambda \in \mathbb{T}\}$. Define

$$\mathcal{M}_k := \overline{\{\widetilde{\sigma}(z)\xi : z \in \mathcal{G}_k\}}.$$

Then $\mathcal{M}_k$ is invariant under $\sigma(\mathcal{F}_n)$ and $\widetilde{\mathcal{H}} = \overline{\operatorname{span}} \bigcup_{k \in \mathbb{Z}} \mathcal{M}_k$. Let φ_k denote the subrepresentation of σ obtained by restricting each of the operators $\sigma(x)$ to $\mathcal{M}_k$. We claim that

$$\varphi_k \overset{q}{\sim} \pi_i, \tag{3.3}$$

where i is the unique element of $\{0, 1, \dots, p-1\}$ such that $k - i \in p\mathbb{Z}$. The proof follows Lemma 3.5 of [11].

Suppose $k \geqslant 0$. Since $\mathcal{G}_k = \mathcal{F}_n v_1^k$, the vector $\widetilde{\sigma}(v_1^k)\xi \in \mathcal{M}_k$ is cyclic for φ_k. Moreover, for $x \in \mathcal{F}_n$ we have

$$\langle \varphi_k(x)\widetilde{\sigma}(v_1^k)\xi, \widetilde{\sigma}(v_1^k)\xi \rangle = \langle \sigma(v_1^{*k} x v_1^k)\xi, \xi \rangle = \beta^{*k}\rho(x).$$

By the uniqueness of the GNS representation and (1.1) it follows that $\varphi_k \overset{u}{\sim} \pi_k \overset{u}{\sim} \pi_i$.

Suppose now that $k < 0$. Using s to denote a multi-index, for $x \in \mathcal{F}_n$ we have

$$\alpha^{*|k|}\rho(x) = \sum_{|s|=|k|} \rho(v_s x v_s^*) = \sum_{|s|=|k|} \langle \sigma(v_s x v_s^*)\xi, \xi\rangle = \sum_{|s|=|k|} \langle \varphi_k(x)\widetilde{\sigma}(v_s^*)\xi, \widetilde{\sigma}(v_s^*)\xi\rangle.$$

We claim that $\{\widetilde{\sigma}(v_s^*)\xi : |s| = |k|\}$ is generating for φ_k. Since $\{v_r v_t^* : |r| - |t| = k\}$ has dense linear span in $\mathcal{G}_k$, it suffices to show that

$$\widetilde{\sigma}(v_r v_t^*)\xi \in \overline{\operatorname{span}}\,\{\sigma(x)\widetilde{\sigma}(v_s^*)\xi : x \in \mathcal{F}_n,\ |s| = |k|\}$$

for each such r, t. But this is easy: simply write $t = st'$, where $|s| = |k|$, so that $\widetilde{\sigma}(v_r v_t^*)\xi = \sigma(v_r v_{t'}^*)\widetilde{\sigma}(v_s^*)\xi$.

Since $\{\widetilde{\sigma}(v_s^*)\xi : |s| = |k|\}$ is generating for φ_k, Lemma 3.2 of [11] gives that φ_k is quasi-equivalent to the GNS representation for $\alpha^{*|k|}\rho$. By (1.2), this implies that $\varphi_k \overset{q}{\sim} \pi_i$, finishing the proof of (3.3).

For $i = 0, 1, \ldots, p-1$, let $\mathcal{S}_i = \overline{\operatorname{span}} \bigcup_{b\in\mathbb{Z}} \mathcal{M}_{i+bp}$. Each $\mathcal{S}_i$ is invariant under $\sigma(\mathcal{F}_n)$, and by Lemma 3.1 the corresponding subrepresentation σ_i of σ is quasi-equivalent to π_i. The proof will be complete once we show that $\mathcal{S}_i \perp \mathcal{S}_j$ if $i \neq j$, and hence that $\sigma = \bigoplus_{i=0}^{p-1} \sigma_i$. For this, suppose $w \in \mathcal{G}_k$ and $z \in \mathcal{G}_l$, where $k - l \notin p\mathbb{Z}$; we will show that $\widetilde{\sigma}(w)\xi \perp \widetilde{\sigma}(z)\xi$. Without loss of generality assume that $k \geqslant l$. Let $\zeta = \widetilde{\sigma}(z^*w)\xi \in \mathcal{M}_{k-l}$, and write $\zeta = \zeta_0 \oplus \zeta_1 \in \mathcal{M}_0 \oplus \mathcal{M}_0^\perp$. If $\zeta \neq 0$, then the vector functional ω_ζ is unitarily equivalent to (a nonzero multiple of) $\beta^{*(k-l)}\rho$. Since $\beta^{*(k-l)}\rho$ is pure and $\omega_\zeta = \omega_{\zeta_0} + \omega_{\zeta_1}$, either $\omega_{\zeta_0} \overset{u}{\sim} \beta^{*(k-l)}\rho$ or $\zeta_0 = 0$. Since $\omega_{\zeta_0} \overset{u}{\sim} \rho$ and p does not divide $k - l$, by (1.1) we thus have $\zeta_0 = 0$; that is, $\zeta \perp \mathcal{M}_0$. In particular,

$$\langle \widetilde{\sigma}(w)\xi, \widetilde{\sigma}(z)\xi\rangle = \langle \widetilde{\sigma}(z^*w)\xi, \xi\rangle = \langle \zeta, \xi\rangle = 0.$$

The decomposition $\sigma = \bigoplus_{i=0}^{p-1} \sigma_i$ is central because the σ_i are mutually disjoint; the multiplicity of σ_i is constant in i by Proposition 3.2. ∎

NOTATION 3.4. Let $P(\mathbb{T})$ be the space of Borel probability measures on the circle $\mathbb{T}$. For each $\mu \in P(\mathbb{T})$, let M_μ be the representation of $C(\mathbb{T})$ on $L^2(\mathbb{T}, \mu)$ by multiplication operators. Let $\mathbb{1}$ be the function of constant value 1 on $\mathbb{T}$.

THEOREM 3.5. *Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n = \infty$ assume that ρ is essential. For $i \in \{0, 1, \ldots, p-1\}$ let $\pi_i : \mathcal{F}_n \to \mathcal{B}(\mathcal{H}_i)$ be the* GNS *representation for $\beta^{*i}\rho$ with canonical cyclic vector ξ_i. Let ξ_p be a linking vector for ρ (as in Definition 2.2), and for $k = i + mp \geqslant p + 1$ let ξ_k be the corresponding vector in $\mathcal{H}_i$ which implements $\beta^{*k}\rho$ as a vector state in π_i (as in* Proposition 2.3 (iii)).

(i) *For each $\mu \in P(\mathbb{T})$ there is a unique state $\widetilde{\rho}\,[\mu, \xi_p]$ of $\mathcal{O}_n$ such that*

$$\widetilde{\rho}\,[\mu, \xi_p](v_s v_t^*) = \begin{cases} \langle \pi_0(v_s v_t^* v_1^{*k})\xi_k, \xi_0\rangle \int_{\mathbb{T}} z^{k/p}\, \mathrm{d}\mu(z) & \text{if } p \text{ divides } k \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

where s and t are multi-indices with $k := |s| - |t| \geqslant 0$. The state $\widetilde{\rho}[\mu, \xi_p]$ extends ρ.

(ii) *If $\widetilde{\rho}$ is a state of $\mathcal{O}_n$ which extends ρ, then $\widetilde{\rho} = \widetilde{\rho}[\mu, \xi_p]$ for some $\mu \in P(\mathbb{T})$.*

(iii) *With the linking vector ξ_p fixed, the map $\mu \mapsto \widetilde{\rho}[\mu, \xi_p]$ is an affine isomorphism of $P(\mathbb{T})$ onto the states of $\mathcal{O}_n$ which extend ρ, and $\widetilde{\rho}[\mu, \xi_p]$ is pure if and only if μ is a unit point mass.*

(iv) *$\widetilde{\rho}[\mu, \xi_p]$ and $\widetilde{\rho}[\nu, \xi_p]$ are unitarily equivalent (resp. disjoint) if and only if the measures μ and ν are equivalent (resp. disjoint).*

(v) *Suppose ω is another pure essential state of $\mathcal{F}_n$.*

(a) *If ρ and ω have the same quasi-orbit (and hence the same period p), then there are linking vectors ξ_p^ρ and ξ_p^ω such that $\widetilde{\rho}[\mu, \xi_p^\rho]$ and $\widetilde{\omega}[\mu, \xi_p^\omega]$ are unitarily equivalent for every $\mu \in P(\mathbb{T})$.*

(b) *If $\widetilde{\rho}[\mu, \xi_p^\rho]$ and $\widetilde{\omega}[\nu, \xi_q^\omega]$ are unitarily equivalent, then ρ and ω have the same quasi-orbit and μ is equivalent to a translation of ν.*

Proof. (i) Suppose s and t are multi-indices such that $k := |s| - |t| \geqslant 0$. Since elements of the form $v_s v_t^*$ and their adjoints have dense linear span in $\mathcal{O}_n$, there is at most one state $\widetilde{\rho}[\mu, \xi_p]$ which satisfies (3.4). For existence, express $k = i + mp$ with $0 \leqslant i \leqslant p-1$, and let $\widetilde{\pi}$ be the representation $\widetilde{\pi}[\rho, \xi_p, M_\mu]$ of $\mathcal{O}_n$ on $\bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes L^2(\mathbb{T}, \mu)$ defined in (2.5). Then

$$
\begin{aligned}
&\langle \widetilde{\pi}(v_s v_t^*)(\xi_0 \otimes \mathbb{1}), \xi_0 \otimes \mathbb{1} \rangle \\
&\quad = \langle \widetilde{\pi}(v_s v_t^* v_1^{*k}) \widetilde{\pi}(v_1^k)(\xi_0 \otimes \mathbb{1}), \xi_0 \otimes \mathbb{1} \rangle \\
&\quad = \langle \pi_i(v_s v_t^* v_1^{*k}) \xi_k \otimes \mathbf{z}^m, \xi_0 \otimes \mathbb{1} \rangle = \langle \pi_i(v_s v_t^* v_1^{*k}) \xi_k, \xi_0 \rangle \langle \mathbf{z}^m, \mathbb{1} \rangle \\
&\quad = \begin{cases} \langle \pi_0(v_s v_t^* v_1^{*k}) \xi_k, \xi_0 \rangle \displaystyle\int_{\mathbb{T}} z^{k/p} \, \mathrm{d}\mu(z) & \text{if } p \text{ divides } k \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

so $\widetilde{\rho}[\mu, \xi_p]$ is the vector state in $\widetilde{\pi}[\rho, \xi_p, M_\mu]$ implemented by $\xi_0 \otimes \mathbb{1}$. Setting $|s| - |t| = 0$ shows that $\widetilde{\rho}[\mu, \xi_p]$ extends ρ.

(ii) Suppose $\widetilde{\rho}$ is a state of $\mathcal{O}_n$ which extends ρ. By Propositions 3.3 and 3.2, there is a linking vector ζ for ρ and a representation ψ of $C(\mathbb{T})$ such that the GNS representation for $\widetilde{\rho}$ is unitarily equivalent to $\widetilde{\pi}[\rho, \zeta, \psi]$. Let $\lambda \in \mathbb{T}$ be such that $\zeta = \lambda \xi_p$, and let $\theta := \psi \circ \tau_{\overline{\lambda}}$; by (2.6), $\widetilde{\pi}[\rho, \zeta, \psi] \overset{u}{\sim} \widetilde{\pi}[\rho, \xi_p, \theta]$. Hence there is a unit vector $\xi \in \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_\theta$ which is cyclic for $\widetilde{\pi}[\rho, \xi_p, \theta]$ and which implements $\widetilde{\rho}$ as a vector state. By the argument used in Proposition 3.2 to derive (3.1), there is a vector $\eta \in \mathcal{K}_\theta$ such that $\xi = \xi_0 \otimes \eta$. Thus

$$
\widetilde{\rho}(x) = \langle \widetilde{\pi}[\rho, \xi_p, \theta](x)(\xi_0 \otimes \eta), \xi_0 \otimes \eta \rangle, \quad x \in \mathcal{O}_n. \tag{3.5}
$$

Since $\xi_0 \otimes \eta$ is cyclic for $\widetilde{\pi}[\rho, \xi_p, \theta]$, by Proposition 2.5 (iv) the vector η is cyclic for θ. It follows that if we define $\mu \in P(\mathbb{T})$ by

$$
\int_{\mathbb{T}} f \, \mathrm{d}\mu = \langle \theta(f)\eta, \eta \rangle, \quad f \in C(\mathbb{T}),
$$

then $T_0 f = \theta(f)\eta$ for $f \in C(\mathbb{T})$ determines a unitary operator $T_0 : L^2(\mathbb{T}, \mu) \to \mathcal{K}_\theta$. Let $T : \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes L^2(\mathbb{T}, \mu) \to \bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes \mathcal{K}_\theta$ be the unitary operator $\bigoplus_{i=0}^{p-1} I \otimes T_0$. Routine calculations show that T intertwines $\widetilde{\pi}[\rho, \xi_p, M_\mu]$ and $\widetilde{\pi}[\rho, \xi_p, \theta]$ and maps $\xi_0 \otimes \mathbb{1}$ to $\xi_0 \otimes \eta$. It now follows immediately from (3.5) that $\xi_0 \otimes \mathbb{1}$ implements $\widetilde{\rho}$ as a vector state in $\widetilde{\pi}[\rho, \xi_p, M_\mu]$, so by the proof of (i) we have $\widetilde{\rho} = \widetilde{\rho}[\mu, \xi_p]$.

(iii) From (3.4) and part (ii) it is clear that $\mu \mapsto \widetilde{\rho}[\mu, \xi_p]$ is affine and surjective. To see that it is injective, suppose $\mu, \nu \in P(\mathbb{T})$ and $\mu \neq \nu$. Then there is a positive integer m such that $\int z^m \, \mathrm{d}\mu(z) \neq \int z^m \, \mathrm{d}\nu(z)$. Let $k := mp$. Since π_0 is irreducible and $\mathcal{F}_n = \overline{\operatorname{span}}\{ v_a v_b^* : |a| = |b| \}$, there are multi-indices a and b of equal length such that $\langle \pi_0(v_a v_b^*)\xi_k, \xi_0\rangle \neq 0$. Using $v_j^* v_i = \delta_{ij}\mathbb{1}$, the element $v_a v_b^* v_1^k$ can be written in the form $v_s v_t^*$. Since $\xi_k = \pi_0(v_1^k v_1^{*k})\xi_k$ we then have

$$\langle \pi_0(v_s v_t^* v_1^{*k})\xi_k, \xi_0\rangle = \langle \pi_0(v_a v_b^*)\pi_0(v_1^k v_1^{*k})\xi_k, \xi_0\rangle = \langle \pi_0(v_a v_b^*)\xi_k, \xi_0\rangle \neq 0.$$

By (3.4) it follows that $\widetilde{\rho}[\mu, \xi_p](v_s v_t^*) \neq \widetilde{\rho}[\nu, \xi_p](v_s v_t^*)$, completing the proof of injectivity. Since $\mu \mapsto \widetilde{\rho}[\mu, \xi_p]$ is an affine isomorphism it preserves extreme points; hence point masses correspond to pure states.

(iv) Since $\mathbb{1}$ is cyclic for M_μ, by Proposition 2.5 (iv) the vector $\xi_0 \otimes \mathbb{1}$ is cyclic for $\widetilde{\pi}[\rho, \xi_p, M_\mu]$. Thus $\widetilde{\pi}[\rho, \xi_p, M_\mu]$ is unitarily equivalent to the GNS representation for $\widetilde{\rho}[\mu, \xi_p]$. Since M_μ and M_ν are unitarily equivalent (resp. disjoint) if and only if μ and ν are equivalent (resp. disjoint) measures, (iv) follows immediately from Proposition 2.5 (ii).

Finally, since μ is equivalent to a translate of ν if and only if $M_\mu \overset{u}{\sim} M_\nu \circ \tau_\lambda$ for some $\lambda \in \mathbb{T}$, (v) is an immediate consequence of Proposition 2.7 and Corollary 2.8. ∎

COROLLARY 3.6. *Suppose ρ is a pure state of $\mathcal{F}_n$ with finite period p; if $n = \infty$ assume that ρ is essential. The gauge group acts p-to-1 and transitively on the pure extensions of ρ to $\mathcal{O}_n$, and distinct pure extensions are disjoint.*

Proof. Fix a linking vector ξ_p for ρ. By Theorem 3.5 (iii), the pure extensions of ρ are $\{ \widetilde{\rho}[\mu_c, \xi_p] : c \in \mathbb{T} \}$, where μ_c denotes the unit point mass at c. Since no two different point masses are equivalent, it follows from Theorem 3.5 (iv) that no two different pure extensions are unitarily equivalent; that is, distinct pure extensions are disjoint.

If s and t are multi-indices with $k := |s| - |t| \geqslant 0$, then by (3.4)

$$\widetilde{\rho}[\mu_c, \xi_p](v_s v_t^*) = \begin{cases} c^{k/p}\langle \pi_0(v_s v_t^* v_1^{*k})\xi_k, \xi_0\rangle & \text{if } p \text{ divides } k \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $\lambda \in \mathbb{T}$, then

$$\widetilde{\rho}[\mu_c, \xi_p] \circ \gamma_\lambda(v_s v_t^*) = \begin{cases} (\lambda^p c)^{k/p}\langle \pi_0(v_s v_t^* v_1^{*k})\xi_k, \xi_0\rangle & \text{if } p \text{ divides } k \\ 0 & \text{otherwise,} \end{cases}$$

so $\widetilde{\rho}[\mu_c, \xi_p] \circ \gamma_\lambda = \widetilde{\rho}[\mu_{\lambda^p c}, \xi_p]$. Thus the gauge action is transitive on the pure extensions of ρ, and for any pure extension $\widetilde{\rho}$ we have $\widetilde{\rho} \circ \gamma_\lambda = \widetilde{\rho}$ if and only if $\lambda^p = 1$. ∎

4. ENDOMORPHISMS OF $\mathcal{B}(\mathcal{H})$.

We are now ready to construct and classify endomorphisms of $\mathcal{B}(\mathcal{H})$ using the representations from Section 2. We will use $\overset{c}{\sim}$ to denote conjugacy of endomorphisms.

Recall that a representation $\varphi : \mathcal{O}_n \to \mathcal{B}(\mathcal{H})$ gives rise to an endomorphism of $\mathcal{B}(\mathcal{H})$ via

$$\operatorname{Ad}\varphi(A) = \sum_{k=1}^{n} \varphi(v_k) A \varphi(v_k)^*, \quad A \in \mathcal{B}(\mathcal{H}).$$

Recall also that the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{O}_n)$ extends to an action of the unitary group $\mathcal{U}(\mathcal{E})$ by *quasi-free* automorphisms, determined by $\gamma_W(v_a) = Wv_a$. Modifying φ by a quasi-free automorphism does not change $\operatorname{Ad}\varphi$, and modifying it by a unitary equivalence only changes $\operatorname{Ad}\varphi$ to a conjugate endomorphism. This is indeed all the collapsing there is in the map $\varphi \mapsto \operatorname{Ad}\varphi$: the endomorphisms $\operatorname{Ad}\varphi_1$ and $\operatorname{Ad}\varphi_2$ are conjugate if and only if $\varphi_2 \overset{u}{\sim} \varphi_1 \circ \gamma_W$ for some $W \in \mathcal{U}(\mathcal{E})$ (Proposition 2.4 of [10]).

Suppose ρ is a periodic pure essential state of $\mathcal{F}_n$ and θ is a representation of $C(\mathbb{T})$. For each choice of linking vector ξ_p for ρ we can form the representation $\widetilde{\pi}[\rho, \xi_p, \theta]$ as in (2.5). By Proposition 2.5(v), two representations of the form $\widetilde{\pi}[\rho, *, \theta]$ differ by at most a gauge automorphism and a unitary equivalence, so the conjugacy class of $\operatorname{Ad}\widetilde{\pi}[\rho, \xi_p, \theta]$ does not depend on the choice of ξ_p. We will denote this conjugacy class, or a representative thereof, by

$$\alpha_{\rho,\theta} := \operatorname{Ad}\widetilde{\pi}[\rho, \xi_p, \theta].$$

Since we only look at endomorphisms modulo conjugacy, this slight abuse of notation will not cause problems.

Two endomorphisms coming from different states of $\mathcal{F}_n$ and representations of $C(\mathbb{T})$ can be conjugate, and the following theorem determines exactly when this happens.

THEOREM 4.1. *Suppose ρ and ω are periodic pure states of $\mathcal{F}_n$, essential if $n = \infty$, and let θ and ψ be representations of $C(\mathbb{T})$. Then $\alpha_{\rho,\theta}$ and $\alpha_{\omega,\psi}$ are conjugate if and only if*

(i) *there is a unitary operator W on $\mathcal{E}$ such that $\rho \circ \gamma_W$ and ω have the same quasi-orbit, and*

(ii) *θ is unitarily equivalent to $\psi \circ \tau_\lambda$ for some $\lambda \in \mathbb{T}$.*

Proof. Let ξ_p^ρ and ξ_q^ω be linking vectors for ρ and ω, respectively. The endomorphisms $\alpha_{\rho,\theta}$ and $\alpha_{\omega,\psi}$ are conjugate if and only if there is a unitary operator W on $\mathcal{E}$ such that $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W$ and $\widetilde{\pi}[\omega, \xi_q^\omega, \psi]$ are unitarily equivalent. The proof will be by direct application of Corollary 2.8 once $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W$ has been changed to an appropriate form. It suffices to prove the following:

CLAIM. *For every unitary W on $\mathcal{E}$ there is a linking vector $\xi_p^{\rho\circ\gamma_W}$ for $\rho \circ \gamma_W$ such that*

$$\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W \overset{u}{\sim} \widetilde{\pi}[\rho \circ \gamma_W, \xi_p^{\rho\circ\gamma_W}, \theta].$$

Proof of Claim. Fix a unitary operator W on $\mathcal{E}$, let ξ_p^ρ be a linking vector for ρ, and let S_1 be the isometry on $\bigoplus_{i=0}^{p-1} \mathcal{H}_i^\rho$ defined in Proposition 2.3. If $0 \leqslant i \leqslant p$ and $x \in \mathcal{F}_n$, then by (2.4) and Proposition 2.3 (iii)

$$\begin{aligned}(\beta^{*i}(\rho \circ \gamma_W))(x) &= \rho \circ \gamma_W(v_1^{*i} x v_1^i) = \langle \pi_0^\rho \circ \gamma_W(v_1^{*i} x v_1^i)\xi_0^\rho, \xi_0^\rho\rangle \\ &= \langle S_1^{*i}\pi_i^\rho(v_1^i\gamma_W(v_1^{*i}xv_1^i)v_1^{*i})S_1^i\xi_0^\rho, \xi_0^\rho\rangle = \langle \pi_i^\rho(v_1^i\gamma_W(v_1^{*i}xv_1^i)v_1^{*i})\xi_i^\rho, \xi_i^\rho\rangle \\ &= \langle \pi_i^\rho \circ \gamma_W(x)\pi_i^\rho(\gamma_W(v_1^i)v_1^{*i})\xi_i^\rho, \pi_i^\rho(\gamma_W(v_1^i)v_1^{*i})\xi_i^\rho\rangle,\end{aligned}$$

so $\pi_i^\rho(\gamma_W(v_1^i)v_1^{*i})\xi_i^\rho$ implements $\beta^{*i}(\rho \circ \gamma_W)$ as a vector state in $\pi_i^\rho \circ \gamma_W$. Hence for each $i \in \{0, \dots, p-1\}$ there is a unique unitary operator $V_i : \mathcal{H}_i^{\rho\circ\gamma_W} \to \mathcal{H}_i^\rho$ which intertwines $\pi_i^{\rho\circ\gamma_W}$ and $\pi_i^\rho \circ \gamma_W$ and satisfies

$$V_i \xi_i^{\rho\circ\gamma_W} = \pi_i^\rho(\gamma_W(v_1^i)v_1^{*i})\xi_i^\rho.$$

Define $\xi_p^{\rho\circ\gamma_W} := V_0^*\pi_0^\rho(\gamma_W(v_1^p)v_1^{*p})\xi_p^\rho$; then $\xi_p^{\rho\circ\gamma_W}$ is a linking vector for $\rho \circ \gamma_W$. Let θ be a representation of $C(\mathbb{T})$, and let $V : \bigoplus_{i=0}^{p-1} \mathcal{H}_i^{\rho\circ\gamma_W} \otimes \mathcal{K}_\theta \to \bigoplus_{i=0}^{p-1} \mathcal{H}_i^\rho \otimes \mathcal{K}_\theta$ be the unitary operator $\bigoplus_{i=0}^{p-1} V_i \otimes I_\theta$. Then V intertwines $\widetilde{\pi}[\rho \circ \gamma_W, \xi_p^{\rho\circ\gamma_W}, \theta]$ and $\widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W$. To see this, suppose $1 \leqslant a \leqslant n$, $0 \leqslant i \leqslant p-1$, $x \in \mathcal{F}_n$ and $\eta \in \mathcal{K}_\theta$. Using Proposition 2.5 (i) and the convention $V_p := V_0$,

$$\begin{aligned}V\widetilde{\pi}[\rho \circ \gamma_W, &\xi_p^{\rho\circ\gamma_W}, \theta](v_a)(\pi_i^{\rho\circ\gamma_W}(x)\xi_i^{\rho\circ\gamma_W} \otimes \eta) = V_{i+1}\pi_{i+1}^{\rho\circ\gamma_W}(v_a x v_1^*)\xi_{i+1}^{\rho\circ\gamma_W} \otimes U_{\theta,i}\eta \\ &= \pi_{i+1}^\rho \circ \gamma_W(v_a x v_1^*)\pi_{i+1}^\rho(\gamma_W(v_1^{i+1})v_1^{*(i+1)})\xi_{i+1}^\rho \otimes U_{\theta,i}\eta \\ &= \pi_{i+1}^\rho(\gamma_W(v_a x v_1^i)v_1^{*(i+1)})\xi_{i+1}^\rho \otimes U_{\theta,i}\eta \\ &= \widetilde{\pi}[\rho, \xi_p^\rho, \theta](\gamma_W(v_a x v_1^i)v_1^{*(i+1)})\widetilde{\pi}[\rho, \xi_p^\rho, \theta](v_1)(\xi_i^\rho \otimes \eta) \\ &= \widetilde{\pi}[\rho, \xi_p^\rho, \theta](\gamma_W(v_a x v_1^i)v_1^{*i})(\xi_i^\rho \otimes \eta) \\ &= \widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W(v_a)(\pi_i^\rho \circ \gamma_W(x)\pi_i^\rho(\gamma_W(v_1^i)v_1^{*i})\xi_i^\rho \otimes \eta) \\ &= \widetilde{\pi}[\rho, \xi_p^\rho, \theta] \circ \gamma_W(v_a)V(\pi_i^{\rho\circ\gamma_W}(x)\xi_i^{\rho\circ\gamma_W} \otimes \eta).\end{aligned}$$

This completes the proof of the claim, and hence the theorem. ∎

When θ is the representation M_μ by multiplication operators on $L^2(\mathbb{T}, \mu)$ we write simply $\alpha_{\rho,\mu}$ in place of α_{ρ,M_μ}. As an immediate corollary to Theorem 3.5 we can now parameterize and classify the endomorphisms constructed using the strategy of [10], wherein one starts with a pure essential state ρ of $\mathcal{F}_n$, extends ρ to a state $\widetilde{\rho}$ of $\mathcal{O}_n$, and then uses the GNS representation for $\widetilde{\rho}$ to implement an endomorphism of $\mathcal{B}(\mathcal{H})$.

COROLLARY 4.2. *Suppose ρ is a periodic pure state of $\mathcal{F}_n$, essential if $n = \infty$, and $\widetilde{\sigma}$ is the* GNS *representation for a state $\widetilde{\rho}$ of $\mathcal{O}_n$ which extends ρ. Then there is a Borel probability measure μ on the circle $\mathbb{T}$ such that* $\mathrm{Ad}\,\widetilde{\sigma}$ *is conjugate to $\alpha_{\rho,\mu}$.*

Let ω be another periodic pure essential state of $\mathcal{F}_n$, and let $\nu \in P(\mathbb{T})$. Then $\alpha_{\rho,\mu}$ and $\alpha_{\omega,\nu}$ are conjugate if and only if

(i) *there is a unitary operator* W *on* $\mathcal{E}$ *such that* $\rho \circ \gamma_W$ *and* ω *have the same quasi-orbit, and*

(ii) μ *is equivalent to a translate of* ν.

Proof. Fix a linking vector ξ_p for ρ. By Theorem 3.5 (ii), $\widetilde{\rho} = \widetilde{\rho}[\mu, \xi_p]$ for some $\mu \in P(\mathbb{T})$, so that $\widetilde{\sigma} \overset{u}{\sim} \widetilde{\pi}[\rho, \xi_p, M_\mu]$. Thus $\mathrm{Ad}\,\widetilde{\sigma} \overset{c}{\sim} \alpha_{\rho,\mu}$.

Since the measure μ is equivalent to a translate of ν if and only if $M_\mu \overset{u}{\sim} M_\nu \circ \tau_\lambda$ for some $\lambda \in \mathbb{T}$, the second part follows directly from Theorem 4.1. ∎

There are two von Neumann algebras naturally associated with an endomorphism α of $\mathcal{B}(\mathcal{H})$: its *tail algebra*

$$\mathrm{Tail}\,(\alpha) := \bigcap_{k=1}^{\infty} \alpha^k(\mathcal{B}(\mathcal{H})),$$

and its *fixed-point algebra*

$$\mathrm{FPA}\,(\alpha) := \{\, A \in \mathcal{B}(\mathcal{H}) : \alpha(A) = A \,\}.$$

Assuming as we are that α is unital, one can always realize α as $\mathrm{Ad}\,\varphi$ for some (essential) representation φ of $\mathcal{O}_n$. By Proposition 3.1 of [10], $\mathrm{FPA}\,(\alpha)$ is the commutant of $\varphi(\mathcal{O}_n)$ and $\mathrm{Tail}\,(\alpha)$ is the commutant of $\varphi(\mathcal{F}_n)$. If φ is the GNS representation of some state $\widetilde{\rho}$ of $\mathcal{O}_n$, then the canonical cyclic vector ξ for φ is separating for $\mathrm{FPA}\,(\alpha)$. In the following theorem we show that when the restriction of $\widetilde{\rho}$ to $\mathcal{F}_n$ is pure, much more is true: $\mathrm{FPA}\,(\alpha)$ is abelian, and the projection onto the closure of $\mathrm{Tail}\,(\alpha)'\xi$ is minimal in $\mathrm{Tail}\,(\alpha)$. Moreover, the latter condition characterizes these endomorphisms.

THEOREM 4.3. *Suppose* α *is a unital endomorphism of* $\mathcal{B}(\mathcal{H})$ *with Powers index* n $(2 \leqslant n \leqslant \infty)$. *Then* (i) *and* (ii) *below are equivalent:*

(i) α *is conjugate to* $\mathrm{Ad}\,\widetilde{\sigma}$ *for* $\widetilde{\sigma}$ *the* GNS *representation of a state extending a pure essential state* ρ *of* $\mathcal{F}_n$.

(ii) $\mathrm{Tail}\,(\alpha)$ *has a minimal projection whose range contains a separating vector for* $\mathrm{FPA}\,(\alpha)$.

If α *satisfies* (i) *and* (ii), *then the center of* $\mathrm{Tail}\,(\alpha)$ *is finite-dimensional if and only if* ρ *has finite period, in which case* $\dim Z(\mathrm{Tail}\,(\alpha))$ *is the period of* ρ. *Moreover,*

(a) *If* ρ *has finite period* p, *then there is a Borel probability measure* μ *on the circle* $\mathbb{T}$ *such that* $\mathrm{Tail}\,(\alpha)$ *is spatially isomorphic to*

$$\Big\{ \bigoplus_{i=0}^{p-1} I_i \otimes T_i : T_i \in \mathcal{B}(L^2(\mathbb{T},\mu)) \Big\} \subset \mathcal{B}\Big(\bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes L^2(\mathbb{T},\mu) \Big) \tag{4.1}$$

and $\mathrm{FPA}\,(\alpha)$ *is spatially isomorphic to the abelian algebra*

$$\Big\{ \bigoplus_{i=0}^{p-1} I_i \otimes T_0 : T_0 \in M_\mu(L^\infty(\mathbb{T},\mu)) \Big\} \subset \mathcal{B}\Big(\bigoplus_{i=0}^{p-1} \mathcal{H}_i \otimes L^2(\mathbb{T},\mu) \Big), \tag{4.2}$$

where as usual $\mathcal{H}_i$ *denotes the GNS Hilbert space for* $\beta^{*i}\rho$.

(b) *If* ρ *is aperiodic, then* $\mathrm{Tail}\,(\alpha)$ *is isomorphic to* $\ell^\infty(\mathbb{Z})$.

Proof. (ii) ⇒ (i) Suppose $\mathrm{Tail}\,(\alpha)$ has a minimal projection P whose range contains a vector ξ which is separating for $\mathrm{FPA}\,(\alpha)$. Let φ be a representation of $\mathcal{O}_n$ such that $\alpha = \mathrm{Ad}\,\varphi$, and let $\widetilde{\rho}$ be the vector state of $\mathcal{O}_n$ in φ implemented by ξ. Since ξ is separating for $\mathrm{FPA}\,(\alpha) = \varphi(\mathcal{O}_n)'$ it is cyclic for φ, so $\alpha \overset{c}{\sim} \mathrm{Ad}\,\widetilde{\sigma}$ with $\widetilde{\sigma}$ the GNS representation for $\widetilde{\rho}$. The restriction of $\widetilde{\rho}$ to $\mathcal{F}_n$ is pure because P is minimal in $\mathrm{Tail}\,(\alpha) = \varphi(\mathcal{F}_n)'$.

(i) ⇒ (ii) If (i) holds and ρ has finite period p, then by Corollary 4.2 there is a Borel probability measure μ on $\mathbb{T}$ such that $\alpha \overset{c}{\sim} \alpha_{\rho,\mu}$. Let ξ_p be a linking vector for ρ. The tail and fixed-point algebras of α are then spatially equivalent to those of $\mathrm{Ad}\,\widetilde{\pi}[\rho, \xi_p, M_\mu]$, which by the proof of Proposition 2.5(ii) are given by (4.1) and (4.2), respectively; the second of these requires the extra observation that $M_\mu(C(\mathbb{T}))' = M_\mu(L^\infty(\mathbb{T}, \mu))$. Let P_0 be the rank-one projection onto the constant function $\mathbb{1} \in L^2(\mathbb{T}, \mu)$, let $P_i = 0$ for $i = 1, \dots, p-1$, and let $P = \bigoplus_{i=0}^{p-1} I_i \otimes P_i$; then P is a minimal projection in the tail algebra. Since $\mathbb{1}$ is cyclic for $M_\mu(C(\mathbb{T}))$ it is separating for $M_\mu(C(\mathbb{T}))'$, so any nonzero vector in the range of P is separating for the fixed-point algebra.

If ρ is aperiodic, then $\widetilde{\rho}$ must be the gauge-invariant state $\rho \circ \Phi$. Let σ be the restriction of $\widetilde{\sigma}$ to $\mathcal{F}_n$. By Propositions 2.2 and 3.4 of [11], σ decomposes as a direct sum $\bigoplus_{i=-\infty}^{\infty} \sigma_i$, where σ_i is irreducible and quasi-equivalent to the GNS representation of $\beta^{*i}\rho$ (resp. $\alpha^{*|i|}\rho$) if $i \geqslant 0$ (resp. $i < 0$). Since these irreducible summands are mutually disjoint,

$$\mathrm{Tail}\,(\mathrm{Ad}\,\widetilde{\sigma}) = \sigma(\mathcal{F}_n)' \cong \ell^\infty(\mathbb{Z}).$$

Let P be any minimal projection in this algebra. Any nonzero vector in the range of P is separating for $\mathrm{FPA}\,(\alpha)$ since α is ergodic. ∎

ERGODIC ENDOMORPHISMS. By Proposition 3.1 of [10], $\mathrm{Ad}\,\varphi$ is ergodic if and only if φ is irreducible. Thus the pure extensions of a pure essential state ρ yield ergodic endomorphisms via their GNS representations. Since these pure extensions are in the same gauge orbit, the corresponding endomorphisms are all conjugate:

COROLLARY 4.4. *Suppose ρ is a pure state of $\mathcal{F}_n$, essential if $n = \infty$.*

(i) *Let $\widetilde{\sigma}$ be the* GNS *representation for a pure extension of ρ. Then the ergodic endomorphism* $\mathrm{Ad}\,\widetilde{\sigma}$ *depends only on ρ up to conjugacy, so we denote it by $\alpha_\rho := \mathrm{Ad}\,\widetilde{\sigma}$.*

(ii) *If ω is another pure essential state of $\mathcal{F}_n$, then α_ρ and α_ω are conjugate if and only if there is a unitary operator W on $\mathcal{E}$ such that $\rho \circ \gamma_W$ and ω have the same quasi-orbit.*

Proof. First suppose ρ is periodic. Let ξ_p be a linking vector for ρ. If $\widetilde{\rho}$ is a pure state of $\mathcal{O}_n$ which extends ρ, then by Theorem 3.5(iii) there is a unit point mass μ on $\mathbb{T}$ such that $\widetilde{\rho} = \widetilde{\rho}\,[\mu, \xi_p]$. The GNS representation $\widetilde{\sigma}$ for $\widetilde{\rho}$ is thus unitarily equivalent to $\widetilde{\pi}[\rho, \xi_p, M_\mu]$, so that $\mathrm{Ad}\,\widetilde{\sigma}$ is conjugate to $\alpha_{\rho,\mu}$. Since condition (ii) of Corollary 4.2 is automatic for point masses, all such endomorphisms $\mathrm{Ad}\,\widetilde{\sigma}$ are conjugate. The second assertion also follows from Corollary 4.2.

Suppose now that ρ is aperiodic. By Theorem 4.3 of [11], the gauge-invariant extension $\rho\circ\Phi$ is the only state of $\mathcal{O}_n$ which extends ρ, and $\rho\circ\Phi$ is pure, so (i) is trivial. Part (ii) follows from remarks following the proof of Theorem 4.3 of [11]. ∎

Finally we give an intrinsic characterization of the class of ergodic endomorphisms arising from pure states of $\mathcal{F}_n$ in terms of the tail algebra.

COROLLARY 4.5. (i) *Suppose ρ is a pure state of $\mathcal{F}_n$, essential if $n=\infty$. Let α_ρ be the ergodic endomorphism associated with ρ as in Corollary* 4.4. *Then* $\mathrm{Tail}\,(\alpha_\rho)$ *is isomorphic to $\mathbb{C}^p$ if ρ has finite period p, and $\ell^\infty(\mathbb{Z})$ if ρ is aperiodic.*

(ii) *Suppose α is an ergodic endomorphism of $\mathcal{B}(\mathcal{H})$ whose tail algebra has a minimal projection. Then there is a pure essential state ρ of $\mathcal{F}_n$ such that α is conjugate to α_ρ. In particular, if α is a shift then it is conjugate to α_ρ for some pure essential quasi-invariant state ρ.*

Proof. (i) By definition, α_ρ satisfies condition (i) of Theorem 4.3. The result is thus immediate from this theorem for aperiodic ρ. If ρ has finite period p then $\mathrm{Tail}\,(\alpha_\rho)$ is given by (4.1) for some point mass μ, and is hence isomorphic to $\mathbb{C}^p$.

(ii) Let P be a minimal projection in $\mathrm{Tail}\,(\alpha)$. Every nonzero vector in the range of P is separating for $\mathrm{FPA}\,(\alpha)=\mathbb{C}I$, so by (ii)$\Rightarrow$(i) of Theorem 4.3 and Corollary 4.4(i), $\alpha=\alpha_\rho$ for some pure essential state ρ of $\mathcal{F}_n$. If α is a shift then $\mathrm{Tail}\,(\alpha)$ consists of scalar operators so the identity is a minimal projection. Thus $\alpha=\alpha_\rho$ for some pure essential state ρ, and by Theorem 4.5 of [10], ρ must be quasi-invariant. ∎

We finish the section by pointing out that, as a consequence of the corollary, there is an interesting restriction on the possible tail algebras of ergodic endomorphisms:

SCHOLIUM 4.6. *If the tail algebra of an ergodic endomorphism has a minimal projection, then it is isomorphic to either $\mathbb{C}^p$ or $\ell^\infty(\mathbb{Z})$, depending on the period p of the state arising from a vector in the range of the minimal projection.*

5. EXAMPLES

Our main source of examples are the pure *product states* $\omega=\bigotimes_{i=1}^{\infty}\omega_i$ of $\mathcal{F}_n$, where each ω_i is a pure state of $\mathcal{K}(\mathcal{E})$; cf. Example 1.1. For each unit vector v in $\mathcal{E}$ let ω_v be the pure state of $\mathcal{K}(\mathcal{E})$ given by $\omega_v(T)=\langle Tv,v\rangle$; strictly speaking, ω_v depends only on the one-dimensional subspace $[v]:=\mathbb{C}v$ and not on v itself. If $f=(f_1,f_2,\ldots)$ is a sequence of unit vectors we let $\omega_f:=\bigotimes_i\omega_{f_i}$ be the corresponding pure product state of $\mathcal{F}_n$. Thus

$$\omega_f(v_{s_1}\cdots v_{s_k}v_{t_k}^*\cdots v_{t_1}^*)=\langle v_{s_1},f_1\rangle\cdots\langle v_{s_k},f_k\rangle\langle f_k,v_{t_k}\rangle\cdots\langle f_1,v_{t_1}\rangle.$$

A. PERIODIC PURE ESSENTIAL PRODUCT STATES. Suppose ω_f has finite period p; this is equivalent to p being the smallest positive integer for which the series

$\sum(1-|\langle f_i, f_{i+p}\rangle|)$ converges (Section 4 of [10]). The GNS triple for ω_f is unitarily equivalent to $(\pi_0', \mathcal{H}_0', \xi_0')$, where $\mathcal{H}_0'$ is the infinite tensor product $\mathcal{E}^{\otimes\infty}$ with canonical unit vector $\xi_0' := f_1 \otimes f_2 \otimes \cdots$, and

$$\begin{aligned}\pi_0'(v_{s_1}\cdots v_{s_k}v_{t_k}^*\cdots v_{t_1}^*)&(h_1\otimes h_2\otimes\cdots)\\ &= \langle h_1, v_{t_1}\rangle\cdots\langle h_k, v_{t_k}\rangle v_{s_1}\otimes\cdots\otimes v_{s_k}\otimes h_{k+1}\otimes h_{k+2}\otimes\cdots.\end{aligned}$$

(See [13] and [9] for the definitions and basic properties of infinite tensor products.) The state $\beta^{*i}\omega_f$ corresponds to the sequence $(\underbrace{v_1,\ldots,v_1}_{i}, f_1, f_2, \ldots)$, so we can similarly define $(\pi_i', \mathcal{H}_i', \xi_i')$. Replacing $(\pi_i, \mathcal{H}_i, \xi_i)$ with $(\pi_i', \mathcal{H}_i', \xi_i')$ for $0 \leqslant i \leqslant p-1$ in Theorem 3.5,

$$\xi_p' := \underbrace{v_1\otimes\cdots\otimes v_1}_{p}\otimes f_1\otimes f_2\otimes\cdots\in\mathcal{H}_0'$$

is a linking vector for ω_f. For this choice of ξ_p', the vectors ξ_k for $k = i+mp \geqslant p+1$ are similarly given by

$$\xi_k' := \underbrace{v_1\otimes\cdots\otimes v_1}_{k}\otimes f_1\otimes f_2\otimes\cdots\in\mathcal{H}_i'.$$

It is routine to check that (3.4) yields the same formula for extensions of ω_f as that given in Theorem 3.1 of [8].

B. GENERALIZED CUNTZ STATES. Next we use periodic sequences to construct and classify examples along the lines of those from Section 4 of [10] and Corollary 5.5 of [11]. When f is a constant sequence, the pure extensions of ω_f to $\mathcal{O}_n$ are the Cuntz states ([6]), and lead to shifts which admit a pure normal invariant state ([15]). We consider here the more general case where f has period p, so that it is determined by the p-tuple $(f_1, \ldots, f_p)$. For such a sequence, ω_f is periodic in a stronger sense than that of Definition 1.3: indeed $\alpha^{*p}\omega_f = \omega_f$.

Although the pure extensions of ω_f are mutually disjoint, by Corollary 4.4(i) they induce the same endomorphism of $\mathcal{B}(\mathcal{H})$ up to conjugacy. In order to compare the endomorphisms coming from two different sequences we use Corollary 4.4(ii). The criterion is particularly easy to apply in this strictly periodic situation because the quasi-orbit of ω_f is obtained by simply taking the cyclic permutations of the p-tuple $(f_1, \ldots, f_p)$.

COROLLARY 5.1. *Suppose f and g are periodic sequences of unit vectors in n-dimensional Hilbert space $\mathcal{E}$. Let α_f (resp. α_g) be the ergodic endomorphism associated to a pure extension of ω_f (resp. ω_g). Then α_f is conjugate to α_g if and only if there is a unitary W on $\mathcal{E}$ such that the p-tuple of 1-dimensional subspaces $([Wf_1], \ldots, [Wf_p])$ is a cyclic permutation of $([g_1], \ldots, [g_p])$.*

Proof. Corollary 4.4 (ii) reduces the question of conjugacy to finding a quasi-free automorphism γ_W of $\mathcal{F}_n$ that superimposes the quasi-orbit of ω_f to that of ω_g. By Corollary 5.3 of [11] the quasi-orbits of $\omega_f \circ \gamma_W$ and ω_g coincide if and only if the series $\sum_j(1-|\langle Wf_j, g_{j+k}\rangle|)$ converges for some k. Since the sequences f and g are periodic, this series converges if and only if all its terms vanish; i.e. if and only if $[Wf_j] = [g_{j+k}]$ for some fixed k and every j. ∎

REMARK 5.2. The orbit of the p-tuple $([f_1], \dots, [f_p])$ of one-dimensional subspaces of $\mathcal{E}$ under the joint action of cyclic permutations and of the unitary group $\mathcal{U}(\mathcal{E})$ (acting diagonally on p-tuples) is thus a *complete conjugacy invariant* for the class of ergodic endomorphisms arising from pure essential product states of $\mathcal{F}_n$ which are strictly periodic under α^*.

This invariant also classifies the larger class of ergodic endomorphisms associated with pure essential product states which are *eventually* strictly periodic, in the sense that there exists $p \geqslant 1$ such that for large enough k one has $\alpha^{*(k+p)}\omega_f = \alpha^{*k}\omega_f$.

C. PURE EXTENSIONS OF DIAGONAL STATES. Assume n is finite. The diagonal $\mathcal{D}$ in $\mathcal{F}_n$ is the abelian subalgebra generated by the projections $v_s v_s^*$, where s is any multi-index. The spectrum $\widehat{\mathcal{D}}$ of $\mathcal{D}$ is canonically isomorphic to the totally disconnected compact space $\{1, 2, \dots, n\}^{\mathbb{N}}$. A *rational point* in $\widehat{\mathcal{D}}$ is one which corresponds to a sequence which is eventually periodic, and *irrational points* correspond to aperiodic sequences ([6]).

When the sequence $f = (f_i)$ consists of basis vectors (that is, each $f_i \in \{v_k : 1 \leqslant k \leqslant n\}$), the state ω_f of $\mathcal{F}_n$ is a *diagonal* pure state; i.e. it corresponds to a point in $\widehat{\mathcal{D}}$. It was observed by Cuntz that if the sequence is aperiodic then the state ω_f has a unique pure extension. Using our Corollary 3.6 we can say what happens at the rational points.

COROLLARY 5.3. *Suppose f is a sequence of basis vectors with periodic tail, so that $\omega_f|\mathcal{D}$ is a rational point in the spectrum of $\mathcal{D}$. Then the pure extensions of ω_f to $\mathcal{O}_n$ are mutually disjoint and indexed by the circle $\mathbb{T}$ (via the composition of $\mathrm{e}^{2\pi \mathrm{i}t} \mapsto \mathrm{e}^{2\pi \mathrm{i}t/p}$ and the gauge action).*

Setting $n = 2$ gives uncountably many inequivalent pure extensions of the trace on the Choi subalgebra of $\mathcal{O}_2$ arising from each rational point in $\widehat{\mathcal{D}}$.

Proof. The first assertion follows from Corollary 3.6; the second one is immediate because if a state of $\mathcal{O}_2$ restricts to a diagonal state on $\mathcal{F}_n$, then it extends the trace on the Choi algebra ([2] and [19]). ∎

REFERENCES

1. H. ARAKI, A.L. CAREY, D.E. EVANS, On O_{n+1}, *J. Operator Theory* **12**(1984), 247–264.
2. R.J. ARCHBOLD, A. LAZAR, S.-K. TSUI, S. WRIGHT, Factor state extensions of type III, *J. Funct. Anal.* **95**(1991), 219–230.
3. W.B. ARVESON, *Continuous Analogues of Fock Space*, *Mem. Amer. Math. Soc.* **409**(1989), 1–66.
4. O. BRATTELI, P.E.T. JØRGENSEN, G.L. PRICE, Endomorphisms of $\mathcal{B}(\mathcal{H})$, in *Quantization, Nonlinear Partial Differential Equations, and Operator Algebra* (Cambridge, MA, 1994), Proc. Sympos. Pure Math., vol. 59, Amer. Math. Soc., Providence, RI 1996, pp. 93–138.
5. O. BRATTELI, P.E.T. JØRGENSEN, Endomorphisms of $\mathcal{B}(\mathcal{H})$. II: Finitely correlated states on $\mathcal{O}_n$, *J. Funct. Anal.* **145**(1997), 323–373.

6. J. CUNTZ, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.
7. D.E. EVANS, On O_n, *Publ. Res. Inst. Math. Sci.* **16**(1980), 915–927.
8. N. FOWLER, States of Toeplitz-Cuntz algebras, *J. Operator Theory* **42**(1999), 121–144.
9. A. GUICHARDET, Produits tensoriels infinis et représentations des relations d'anti-commutation, *Ann. Sci. École Norm. Sup.* **83**(1966), 1–52.
10. M. LACA, Endomorphisms of $B(H)$ and Cuntz algebras, *J. Operator Theory* **30** (1993), 85–108.
11. M. LACA, Gauge invariant states of $\mathcal{O}_\infty$, *J. Operator Theory* **30**(1993), 381–396.
12. A. LAZAR, S.-K. TSUI, S. WRIGHT, Pure state extensions and restrictions in $\mathcal{O}_2$, *J. Operator Theory* **23**(1990), 171–178.
13. J. VON NEUMANN, On infinite direct products, *Comp. Math.* **6**(1938), 1–77.
14. R. POWERS, Representations of uniformly hyperfinite algebras and their associated von Neumann rings, *Ann. Math.* **86**(1967), 138–171.
15. R. POWERS, An index theory for semigroups of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ and type II_1 factors, *Canad. J. Math.* **40**(1988), 86–114.
16. J. SPIELBERG, Diagonal states on $\mathcal{O}_2$, *Pacific J. Math.* **144**(1990), 361–382.
17. J. SPIELBERG, Type III factor states on $\mathcal{O}_2$ which extend the trace on Choi's algebra, *J. Operator Theory* **25**(1991), 215–236.
18. P.J. STACEY, Product shifts on $\mathcal{B}(\mathcal{H})$, *Proc. Amer. Math. Soc.* **113**(1991), 955–963.
19. P.J. STACEY, A family of type III extensions of the trace on the Choi algebra, *Proc. Amer. Math. Soc.* **114**(1992), 683–686.

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