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REGULAR C^*-VALUED WEIGHTS

JOHAN KUSTERMANS

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Abstract. We introduce the notion of a C^*-valued weight between two C^*-algebras as a generalization of an ordinary weight on a C^*-algebra and as a C^*-version of operator valued weights on von Neumann algebras. Also, some form of lower semi-continuity will be discussed together with an extension to the multiplier algebra.

A strong but useful condition for C^*-valued weights, the so-called regularity, is introduced. At the same time, we propose a construction procedure for such regular C^*-valued weights. This construction procedure will be used to define the tensor product of regular C^*-valued weights.

Keywords: *C^*-algebra, Hilbert C^*-module, weight,* KSGNS-*construction.*

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INTRODUCTION

In the papers [6] and [7], Uffe Haagerup introduced the notion of operator valued weights on von Neumann algebras as a generalization of weights which take values in another von Neumann algebra.

An application of this theory can be found in the theory of the Kac algebras (M, Δ, φ) (see [2]). There, we have a semi-finite faithful normal weight φ on the von Neumann algebra M which is left invariant in the sense that $(\iota \,\overline{\otimes}\, \varphi)(\Delta(a)) = \varphi(a)1$ for every $a \in M^+$. But one has to give a meaning to the expression $(\iota \,\overline{\otimes}\, \varphi)(\Delta(a))$. This can be done using the theory of operator valued weights. In this case, $\iota \,\overline{\otimes}\, \varphi$ is an operator valued weight from $M \,\overline{\otimes}\, M$ into M.

Now we turn to the case of quantum groups in the framework of C^*-algebras instead of the framework of von Neumann algebras. In this case, we have a C^*-algebra A, a comultiplication Δ from A into the multiplier algebra $M(A \otimes A)$ and a densely defined lower semi-continuous weight φ on A which is left invariant in the sense that $(\iota \otimes \varphi)(\Delta(a)) = \varphi(a)1$ for every $a \in \mathcal{M}_\varphi$. Also in this case we have to give a meaning to the expression $(\iota \otimes \varphi)(\Delta(a))$ for $a \in \mathcal{M}_\varphi$. This can be done by regarding $\iota \otimes \varphi$ as a C^*-valued weight from $A \otimes A$ into A (which can be extended to certain elements of $M(A \otimes A)$).

We want to perform the construction of $\iota\otimes\varphi$ in a purely C^*-algebraic setting, not in a von Neumann algebra one. For instance, we do not want to extend φ to a normal weight on a bigger von Neumann algebra and work with this extension. Partly, because there are no guarantees that φ can be extended in the case of C^*-algebraic quantum groups.

If we look at the left invariance of φ, we see that $(\iota\otimes\varphi)(\Delta(a))$ has to belong to $M(A)$ for every $a \in \mathcal{M}_\varphi$. But also in the case of quantum groups there is some interest to let C^*-valued weights take values in the set of affiliated elements. This can be seen as follows.

A C^*-algebraic quantum group will (possibly) have the analogue of a modular function for classical groups. This will be a strictly positive element δ affiliated with A such that $(\varphi \otimes \iota)(\Delta(a)) = \delta\varphi(a)$ for every $a \in \mathcal{M}_\varphi^+$. We have to give a meaning to the expression $(\varphi \otimes \iota)(\Delta(a))$ but we see already that it will be unbounded in many cases.

In the first section, we will give a possible definition for C^*-valued weights. Loosely speaking, they will be completely positive linear mappings between C^*-algebras which are unbounded. We will restrict the domain of our C^*-valued weights and let them take values in the multiplier algebra of another C^*-algebra. Later, we will discuss some extensions and even let them take values in the set of affiliated elements.

We will also introduce a KSGNS-construction of a C^*-valued weight, modelled on the KSGNS-construction for completely positive mappings (for instance, see [5]).

In the second section we introduce a special family of completely positive mappings relative to a C^*-valued weight (and even a little bit more general). These completely positive mappings allow us to introduce a notion of lower semi-continuity for C^*-valued weights in the third section. We also introduce the notion of regular C^*-valued weights in this third section.

In the fourth section, the extension of a lower semi-continuous weight to the multiplier algebra is discussed.

The fifth section serves as a first step in a construction procedure for C^*-valued weights.

We propose a construction procedure for regular C^*-valued weights in the sixth section. Along the way, we prove an important result which will be used in the next section to prove some nice results about regular C^*-valued weights. There is also a short discussion about a further extension of regular C^*-valued weights which takes values in the set of affiliated elements.

In the last section, we introduce the tensor product of two regular C^*-valued weights using the construction procedure of Section 6.

At the end of this introduction, we will fix some notation and conventions. The main technical tools for this paper come from the theory of Hilbert C^*-modules over C^*-algebras. A nice survey of this theory can be found in [5]. All our Hilbert C^*-modules are right modules and have inner products which are linear in the first variable. Consider Hilbert C^*-modules E, F over a C^*-algebra A. We will use the following notation:

(i) $L(E, F)$ is the set of linear mappings from E into F;

(ii) $\mathcal{B}(E, F)$ is the set of bounded linear mappings from E into F;

(iii) $\mathcal{L}(E,F)$ is the set of adjointable mappings from E into F.

Consider two C^*-algebras A and B and ρ a completely positive mapping from A into $M(B)$. We call ρ *strict* if and only if it is strictly continuous on bounded sets. Let $(e_k)_{k\in K}$ be an approximate unit for A. Theorem 6.5 of [5] implies that ρ is strict if and only if $(\rho(e_k))_{k\in K}$ is strictly convergent. If ρ is strict, then it has a unique extension $\overline{\rho}$ which is a completely positive linear mapping, strictly continuous on bounded sets (e.g., the appendix of [4], or use Theorem 6.5 of [5]). For every $a \in M(A)$, we define $\rho(a) = \overline{\rho}(a)$.

1. The definition of a C^*-valued weight and its KSGNS-construction

In this first section, we will introduce the definition of a C^*-valued weight between C^*-algebras. This definition is a generalization of the definition of usual weights on C^*-algebras but we will assume complete positivity instead of just positivity (which is to be expected). After that, we will introduce a KSGNS-construction for such a C^*-valued weight similar to the KSGNS-construction for completely positive mappings.

We will start of with the definition.

Definition 1.1. Consider two C^*-algebras A and B and a hereditary cone P in A^+. Put $\mathcal{N} = \{a \in A \mid a^*a \in P\}$ and $\mathcal{M} = \operatorname{span} P = \mathcal{N}^*\mathcal{N}$. Suppose that φ is a linear mapping from $\mathcal{M}$ into $M(B)$ such that

$$\sum_{i,j=1}^{n} b_j^* \varphi(a_j^* a_i) b_i \geqslant 0$$

for every $n \in \mathbb{N}$ and all $a_1,\ldots,a_n \in \mathcal{N}$, $b_1,\ldots,b_n \in B$. Then we call φ a *C^*-valued weight from A into $M(B)$*.

We will introduce the following notation.

Notation 1.2. Consider two C^*-algebras A and B and a C^*-valued weight φ from A into $M(B)$. We will use the following notation:

(i) the domain of φ will be denoted by $\mathcal{M}_\varphi$;

(ii) we define $\mathcal{N}_\varphi = \{a \in A \mid a^*a \in \mathcal{M}_\varphi^+\}$.

We have that $\mathcal{M}_\varphi$ is a sub-$*$-algebra of A, $\mathcal{M}_\varphi^+$ is a hereditary cone in A^+ and $\mathcal{M}_\varphi = \operatorname{span} \mathcal{M}_\varphi^+$. Furthermore, $\mathcal{N}_\varphi$ is a left ideal in $M(A)$ and $\mathcal{M}_\varphi = \mathcal{N}_\varphi^*\mathcal{N}_\varphi$.

Remark 1.3. Consider two C^*-algebras A, B and a C^*-valued weight from A into $M(B)$. As usual, we say that φ is densely defined if $\mathcal{M}_\varphi$ is dense in A, which is equivalent to $\mathcal{N}_\varphi$ being dense in A or to $\mathcal{M}_\varphi^+$ being dense in A^+.

As the generalization of the GNS-construction for weights, we have the KSGNS-construction for C^*-valued weights:

Definition 1.4. Consider two C^*-algebras A and B and a C^*-valued weight φ from A into $M(B)$. *A KSGNS-construction for* φ is by definition a triplet (E, Λ, π) where:

(i) E is a Hilbert C^*-module over B;

(ii) Λ is a linear mapping from $\mathcal{N}_\varphi$ into $\mathcal{L}(B, E)$ such that:

(a) the set $\{\Lambda(a)b \mid a \in \mathcal{N}_\varphi, b \in B\}$ is dense in E, and

(b) we have for every $a_1, a_2 \in \mathcal{N}_\varphi$ and $b_1, b_2 \in B$ that $\langle \Lambda(a_1)b_1, \Lambda(a_2)b_2 \rangle = b_2^* \varphi(a_2^* a_1) b_1$;

(iii) π is a $*$-homomorphism from A into $\mathcal{L}(E)$ such that $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in A$ and $a \in \mathcal{N}_\varphi$.

It is clear that a KSGNS-construction is unique up to unitary equivalence.

Result 1.5. *Consider two C^*-algebras A and B and a C^*-valued weight φ from A into $M(B)$ and let (E, Λ, π) be a* KSGNS-*construction for φ. Then*

(i) $\varphi(a_2^* a_1) = \Lambda(a_2)^* \Lambda(a_1)$ *for every* $a_1, a_2 \in \mathcal{N}_\varphi$*;*

(ii) $\|\Lambda(a)\|^2 = \|\varphi(a^* a)\|$ *for every* $a \in \mathcal{N}_\varphi$*;*

(iii) *if π is non-degenerate, then $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in M(A)$ and $a \in \mathcal{N}_\varphi$.*

Proof. The first property follows immediately from the definition of the KSGNS-construction. The second follows from the first. For the proof of the third, suppose that π is non-degenerate. Take $x \in M(A)$ and $a \in \mathcal{N}_\varphi$. We have for every $e \in A$ that ex belongs to A, thus

$$\pi(e)(\pi(x)\Lambda(a)) = \pi(ex)\Lambda(a) = \Lambda(exa) = \pi(e)\Lambda(xa).$$

From the non-degeneracy of π, we infer that $\pi(x)\Lambda(a) = \Lambda(xa)$. ∎

The proof of the existence of a KSGNS-construction is a generalization of the KSGNS-construction for completely positive maps. For a large part, we will mimic the proof of Theorem 5.6 of [5] but we will also add some features for this specific case (Lemma 1.6 and Definition 1.7).

For the most part of this section, we will fix two C^*-algebras A and B and a C^*-valued weight φ from A into $M(B)$. In the next part, we will gradually construct a KSGNS-construction for φ.

First, we define the complex vector space $F = \mathcal{N}_\varphi \odot B$. We turn F into a semi-inner product module over B such that:

(i) for every $a \in \mathcal{N}_\varphi$, $b, c \in B$ we have $(a \otimes b)c = a \otimes (bc)$;

(ii) for every $a_1, a_2 \in \mathcal{N}_\varphi$, $b_1, b_2 \in B$ we have $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = b_2^* \varphi(a_2^* a_1) b_1$.

Put $N = \{x \in F \mid \langle x, x \rangle = 0\}$; then N is a submodule of F. By the discussion in Chapter 1 of [5], we know that $\frac{F}{N}$ can be naturally turned into an inner product module over B. We define $A \otimes_\varphi B$ to be the completion of $\frac{F}{N}$. So $A \otimes_\varphi B$ is a Hilbert C^*-module over B.

For every $a \in \mathcal{N}_\varphi$ and $b \in B$, we define $a \dot{\otimes} b$ to be the equivalence class in $\frac{F}{N}$ associated with $a \otimes b$. Then we have the following properties:

(i) the mapping $\mathcal{N}_\varphi \times B \to A \otimes_\varphi B : (a, b) \mapsto a \dot{\otimes} b$ is bilinear;

(ii) for every $a \in \mathcal{N}_\varphi$ and $b, c \in B$, we have that $(a \dot{\otimes} b)c = a \dot{\otimes} (bc)$;

(iii) for every $a_1, a_2 \in \mathcal{N}_\varphi$, $b_1, b_2 \in B$, we have that $\langle a_1 \dot{\otimes} b_1, a_2 \dot{\otimes} b_2 \rangle = b_2^* \varphi(a_2^* a_1) b_1$;

(iv) the set $\langle a \dot{\otimes} b \mid a \in \mathcal{N}_\varphi, b \in B \rangle$ is dense in $A \otimes_\varphi B$.

These properties completely determine the Hilbert C^*-module $A \otimes_\varphi B$.

It is easy to see that $\|a \dot{\otimes} b\| \leqslant \|\varphi(a^*a)\|^{\frac{1}{2}} \|b\|$ for every $a \in \mathcal{N}_\varphi$ and $b \in B$. Therefore, we have for every $a \in \mathcal{N}_\varphi$ that the mapping $B \to A \otimes_\varphi B : b \mapsto a \dot{\otimes} b$ is continuous.

First, we prove the following lemma:

Lemma 1.6. *Let a be an element in $\mathcal{N}_\varphi$ and define the mapping t from B into $A \otimes_\varphi B$ such that $t(b) = a \dot{\otimes} b$ for every $b \in B$. Then t belongs to $\mathcal{L}(B, A \otimes_\varphi B)$ and $t^*(c \dot{\otimes} d) = \varphi(a^*c)d$ for every $c \in \mathcal{N}_\varphi$, $d \in B$.*

Proof. We have for every $c \in \mathcal{N}_\varphi$ and $b, d \in B$ that

$$\langle t(b), c \dot{\otimes} d \rangle = \langle a \dot{\otimes} b, c \dot{\otimes} d \rangle = d^* \varphi(c^*a) b.$$

Therefore Proposition 10.3 implies that t is an element of $\mathcal{L}(B, A \otimes_\varphi B)$. It is also clear from the above equation that $t^*(c \dot{\otimes} d) = \varphi(a^*c)d$ for every $c \in \mathcal{N}_\varphi$ and $d \in B$. ∎

This lemma justifies the following definition.

Definition 1.7. We define the linear mapping Λ_φ from $\mathcal{N}_\varphi$ into $\mathcal{L}(B, A \otimes_\varphi B)$ such that $\Lambda_\varphi(a)b = a \dot{\otimes} b$ for every $a \in \mathcal{N}_\varphi$ and $b \in B$. We also have that $\Lambda_\varphi(a)^*(c \dot{\otimes} d) = \varphi(a^*c)d$ for every $a, c \in \mathcal{N}_\varphi$ and $d \in B$.

Using this definition, it is straightforward to check the following properties:

(i) the set $\langle \Lambda_\varphi(a)b \mid a \in \mathcal{N}_\varphi, b \in B \rangle$ is dense in $A \otimes_\varphi B$;

(ii) we have for every $a_1, a_2 \in \mathcal{N}_\varphi$ and $b_1, b_2 \in B$ that $\langle \Lambda_\varphi(a_1)b_1, \Lambda_\varphi(a_2)b_2 \rangle = b_2^* \varphi(a_2^* a_1) b_1$.

Lemma 1.8. *Consider $a_1, \ldots, a_n \in \mathcal{N}_\varphi$, $b_1, \ldots, b_n \in B$ and $x \in M(A)$. Then*

$$\sum_{i,j=1}^{n} b_j^* \varphi((xa_j)^*(xa_i)) b_i \leqslant \|x\|^2 \sum_{i,j=1}^{n} b_j^* \varphi(a_j^* a_i) b_i.$$

Proof. There exists an element $y \in M(A)$ such that $\|x\|^2 \mathbb{1} - x^*x = y^*y$. Because $\mathcal{N}_\varphi$ is a left ideal in $M(A)$, we have that $ya_1, \ldots, ya_n$ belong to $\mathcal{N}_\varphi$. This implies that

$$\begin{aligned}\|x\|^2 \sum_{i,j=1}^{n} b_j^* \varphi(a_j^* a_i) b_i - \sum_{i,j=1}^{n} b_j^* \varphi((xa_j)^*(xa_i)) b_i &= \sum_{i,j=1}^{n} b_j^* \varphi(a_j^* (\|x\|^2 \mathbb{1} - x^*x) a_i) b_i \\ &= \sum_{i,j=1}^{n} b_j^* \varphi(a_j^* y^* y a_i) b_i = \sum_{i,j=1}^{n} b_j^* \varphi((ya_j)^*(ya_i)) b_i \geqslant 0. \quad ∎\end{aligned}$$

Lemma 1.9. *Consider $x \in M(A)$. Then there exists a unique $T \in \mathcal{L}(A \otimes_\varphi B)$ such that $T \Lambda_\varphi(a) = \Lambda_\varphi(xa)$ for every $a \in \mathcal{N}_\varphi$. Moreover, $T^* \Lambda_\varphi(a) = \Lambda_\varphi(x^*a)$ for every $a \in \mathcal{N}_\varphi$.*

Proof. Choose $y \in M(A)$. Take $a_1, \ldots, a_n \in \mathcal{N}_\varphi$ and $b_1, \ldots, b_n \in B$. We can restate the previous lemma in the following form:

$$\Big\langle \sum_{i=1}^{n} \Lambda_\varphi(ya_i)b_i, \sum_{i=1}^{n} \Lambda_\varphi(ya_i)b_i \Big\rangle \leqslant \|y\|^2 \Big\langle \sum_{i=1}^{n} \Lambda_\varphi(a_i)b_i, \sum_{i=1}^{n} \Lambda_\varphi(a_i)b_i \Big\rangle,$$

which implies that

$$\Big\| \sum_{i=1}^{n} \Lambda_\varphi(ya_i)b_i \Big\| \leqslant \|y\| \Big\| \sum_{i=1}^{n} \Lambda_\varphi(a_i)b_i \Big\|.$$

As usual, this last equality implies the existence of a unique continuous linear map T_y from $A \otimes_\varphi B$ into $A \otimes_\varphi B$ such that $T_y(\Lambda_\varphi(a)b) = \Lambda_\varphi(ya)b$ for every $a \in \mathcal{N}_\varphi$ and $b \in B$. We have for every $a_1, a_2 \in \mathcal{N}_\varphi$ and $b_1, b_2 \in B$ that

$$\begin{aligned} &\langle T_x(\Lambda_\varphi(a_1)b_1), \Lambda_\varphi(a_2)b_2 \rangle = \langle \Lambda_\varphi(xa_1)b_1, \Lambda_\varphi(a_2)b_2 \rangle = b_2^* \varphi(a_2^*(xa_1))b_1 \\ &\quad = b_2^* \varphi((x^*a_2)^* a_1)b_1 = \langle \Lambda_\varphi(a_1)b_1, \Lambda_\varphi(x^*a_2)b_1 \rangle = \langle \Lambda_\varphi(a_1)b_1, T_{x^*}(\Lambda_\varphi(a_2)b_2) \rangle. \end{aligned}$$

This implies that $\langle T_x(v), w \rangle = \langle v, T_{x^*}(w) \rangle$ for every $v, w \in A \otimes_\varphi B$. Therefore, we get that T_x belongs to $\mathcal{L}(A \otimes_\varphi B)$ and $T_x^* = T_{x^*}$. ∎

This lemma justifies the following definition.

DEFINITION 1.10. We define the mapping π_φ from A into $\mathcal{L}(A \otimes_\varphi B)$ such that $\pi_\varphi(x)\Lambda_\varphi(a) = \Lambda_\varphi(xa)$ for every $x \in A$, $a \in \mathcal{N}_\varphi$. Then π_φ is a $*$-homomorphism.

This discussion allows us to formulate the following proposition.

PROPOSITION 1.11. *We have that* $(A \otimes_\varphi B, \Lambda_\varphi, \pi_\varphi)$ *is a* KSGNS*-construction for* φ*. This triplet is called the canonical* KSGNS*-construction for* φ*.*

In a last part, we look at the case where φ takes values in B and investigate the connection with the mapping Λ_φ. First, we prove the following lemma.

LEMMA 1.12. *Consider a* C^**-algebra* C *and a Hilbert* C^**-module* F *over* C*. Let* t *be an element in* $\mathcal{L}(C, F)$*. Then* t^*t *belongs to* C *if and only if there exists an element* $x \in F$ *such that* $tc = xc$ *for every* $c \in C$ *which is equivalent to* t *belongs to* $\mathcal{K}(C, F)$*.*

Proof. Suppose that t^*t belongs to C. Define $S \in \mathcal{L}(C \oplus F, C \oplus F)$ with $S^* = S$ such that

$$S = \begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix}.$$

We have that

$$S^2 = \begin{pmatrix} t^*t & 0 \\ 0 & tt^* \end{pmatrix}.$$

By Proposition 1.4.5 of [8], there exists an element $U \in \mathcal{L}(C \oplus F, C \oplus F)$ such that $S = U(S^2)^{\frac{1}{4}}$. This implies that

$$\begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix} = U \begin{pmatrix} (t^*t)^{\frac{1}{4}} & 0 \\ 0 & (tt^*)^{\frac{1}{4}} \end{pmatrix}.$$

From this, we get the existence of $u \in \mathcal{L}(C,F)$ such that $t = u(t^*t)^{\frac{1}{4}}$. Because t^*t belongs to C, we have that $(t^*t)^{\frac{1}{4}}$ belongs to C. Put $x = u\big((t^*t)^{\frac{1}{4}}\big) \in F$. Then $t(c) = xc$ for every $c \in C$.

Suppose there exists an element $x \in F$ such that $t(c) = xc$ for every $c \in C$. In this case, we have for every $c \in C$ that

$$c^*(t^*t)c = \langle c, (t^*t)c\rangle = \langle tc, tc\rangle = \langle xc, xc\rangle = c^*\langle x, x\rangle c$$

which implies that $t^*t = \langle x, x\rangle \in C$. ∎

This immediately implies the following proposition.

PROPOSITION 1.13. *Consider two C^*-algebras A, B and a C^*-valued weight φ from A into $M(B)$ with* KSGNS-*construction (E, Λ, π). Let a be an element of $\mathcal{N}_\varphi$. Then $\varphi(a^*a)$ belongs to B iff there exists an element $x \in E$ such that $\Lambda(a)b = xb$ for every $b \in B$ iff $\Lambda(a)$ belongs to $\mathcal{K}(B, E)$.*

Let a be an element in $\mathcal{N}_\varphi$ such that $\varphi(a^*a) \in B$. By the previous proposition, we get the existence of a unique element $x \in E$ such that $\Lambda(a)b = xb$ for every $b \in B$. As we have seen in the previous lemma, we have in this case that $\varphi(a^*a) = \langle x, x\rangle$ which looks like the GNS-construction for weights.

2. A SPECIAL FAMILY OF COMPLETELY POSITIVE MAPPINGS

In this section, we will consider two C^*-algebras A and B, a Hilbert C^*-module E over B and a dense left ideal N in A. Furthermore, let Λ be a linear mapping from N into $\mathcal{L}(B, E)$ and π a $*$-homomorpism from A into $\mathcal{L}(E)$ such that:

(i) the set $\langle \Lambda(a)b \mid a \in N, b \in B\rangle$ is dense in E;

(ii) we have that $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in A$ and $a \in N$.

We introduce a special family of completely positive mappings relative to this objects and investigate some properties of them.

DEFINITION 2.1. We define $\mathcal{H}$ to be the set

$$\{\rho \text{ a strict completely positive linear mapping from } A \text{ into } M(B) \mid \text{ there exists an operator } T \in \mathcal{L}(E)^+ \text{ such that } b_2^*\rho(a_2^*a_1)b_1 = \langle T\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle \text{ for every } a_1, a_2 \in N, b_1, b_2 \in B\}.$$

NOTATION 2.2. Consider $\rho \in \mathcal{H}$. Then there exists a unique element $T \in \mathcal{L}(E)^+$ such that $\langle T\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho(a_2^*a_1)b_1$ for every $a_1, a_2 \in N$ and $b_1, b_2 \in B$. We define $T_\rho = T$. This implies that $\rho(a_2^*a_1) = \Lambda(a_2)^*T_\rho\Lambda(a_1)$ for every $a_1, a_2 \in N$.

RESULT 2.3. *Consider $\rho \in \mathcal{H}$. Then T_ρ belongs to $\pi(A)'$.*

Proof. Choose $x \in A$. Using the definition of T_ρ, we have for every $a_1, a_2 \in N$ and $b_1, b_2 \in B$ that

$$\begin{aligned}\langle T_\rho\pi(x)\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle &= \langle T_\rho\Lambda(xa_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho(a_2^*(xa_1))b_1\\ &= b_2^*\rho((x^*a_2)^*a_1)b_1 = \langle T_\rho\Lambda(a_1)b_1, \Lambda(x^*a_2)b_2\rangle\\ &= \langle T_\rho\Lambda(a_1)b_1, \pi(x^*)\Lambda(a_2)b_2\rangle\\ &= \langle \pi(x)T_\rho\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle.\end{aligned}$$

This implies that $T_\rho\pi(x) = \pi(x)T_\rho$. ∎

RESULT 2.4. *Consider $\rho \in \mathcal{H}$ and $a \in N$. Then $\|T_\rho^{\frac{1}{2}}\Lambda(a)\|^2 = \|\rho(a^*a)\| \leqslant \|\rho\|\,\|a\|^2$.*

This follows immediately from the fact that

$$\big(T_\rho^{\frac{1}{2}}\Lambda(a)\big)^*\big(T_\rho^{\frac{1}{2}}\Lambda(a)\big) = \Lambda(a)^*T_\rho\Lambda(a) = \rho(a^*a).$$

The following result has its analogue in the theory of weights (see Proposition 2.4 of [1]).

PROPOSITION 2.5. *Assume that π is non-degenerate. Consider $\rho \in \mathcal{H}$ and $S \in \mathcal{L}(E) \cap \pi(A)'$ such that $S^*S = T_\rho$. Then there exists a unique element $v \in \mathcal{L}(B,E)$ such that $S\Lambda(a) = \pi(a)v$ for every $a \in N$. Furthermore, the equality $\|v\|^2 = \|\rho\|$ holds. We have also that $\rho(x) = v^*\pi(x)v$ for every $x \in M(A)$.*

Proof. The non-degeneracy of π clearly implies the unicity of v. We will turn our attention to the existence. Take an approximate unit $(e_k)_{k\in K}$ for A in N (which is possible because N is a dense left ideal in A).

STEP 1. Choose $b \in B$. We have for every $k,l \in K$ that

$$\begin{aligned}\|S\Lambda(e_k)b - S\Lambda(e_l)b\|^2 &= \|\langle S\Lambda(e_k-e_l)b, S\Lambda(e_k-e_l)b\rangle\|\\ &= \|\langle S^*S\Lambda(e_k-e_l)b, \Lambda(e_k-e_l)b\rangle\|\\ &= \|\langle T_\rho\Lambda(e_k-e_l)b, \Lambda(e_k-e_l)b\rangle\|\\ &= \|b^*\rho((e_k-e_l)^2)b\|.\end{aligned}$$

Because the net $((e_k-e_l)^2)_{(k,l)\in K\times K}$ is bounded and converges strictly to 0, we have also that the net $(\rho((e_k-e_l)^2))_{(k,l)\in K\times K}$ converges strictly to 0 (remember that ρ is assumed to be strict). Therefore, the net $(b^*\rho((e_k-e_l)^2)b)_{(k,l)\in K\times K}$ converges to 0, which implies that the net $(S\Lambda(e_k)b - S\Lambda(e_l)b)_{(k,l)\in K\times K}$ converges to 0. Consequently, the net $(S\Lambda(e_k)b)_{k\in K}$ is Cauchy and hence convergent in B. From this all, we get the existence of a linear mapping v from B into E such that $(S\Lambda(e_k))_{k\in K}$ converges strongly to v. (We borrowed the idea for the preceding part from Theorem 5.6 of [5].)

STEP 2. Choose $a \in N$. It is clear that $(\pi(a)S\Lambda(e_k))_{k\in K}$ converges strongly to $\pi(a)v$. On the other hand, we have for every $k \in K$ that

$$\begin{aligned}\|\pi(a)S\Lambda(e_k) - S\Lambda(a)\|^2 &= \|S\pi(a)\Lambda(e_k) - S\Lambda(a)\|^2 = \|S\Lambda(ae_k) - S\Lambda(a)\|^2\\ &= \|\Lambda(ae_k-a)^*S^*S\Lambda(ae_k-a)\|\\ &= \|\Lambda(ae_k-a)^*T_\rho\Lambda(ae_k-a)\|\\ &= \|\rho((ae_k-a)^*(ae_k-a))\|.\end{aligned}$$

This implies that the net $(\pi(a)S\Lambda(e_k))_{k\in K}$ converges to $S\Lambda(a)$.

Combining these results we get the equality $\pi(a)v = S\Lambda(a)$.

STEP 3. We have for every $k \in K$ that

$$\|S\Lambda(e_k)\|^2 = \|\Lambda(e_k)^*S^*S\Lambda(e_k)\| = \|\Lambda(e_k)^*T_\rho\Lambda(e_k)\| = \|\rho((e_k)^2)\|.$$

Hence, $\|S\Lambda(e_k)\|^2 \leqslant \|\rho\|$ for every $k \in K$. Because $(S\Lambda(e_k))_{k\in K}$ converges strongly to v, this implies that v is bounded and $\|v\|^2 \leqslant \|\rho\|$. Using Step 2, we have for every $k \in K$ that

$$\|v\|^2 \geqslant \|\pi(e_k)v\|^2 = \|S\Lambda(e_k)\|^2 = \|\rho((e_k)^2)\|.$$

Because ρ is positive, we know that $(\|\rho((e_k)^2)\|)_{k\in K}$ converges to $\|\rho\|$. This implies that $\|v\|^2 \geqslant \|\rho\|$.

Combining these two inequalities, we get that $\|v\|^2 = \|\rho\|$.

STEP 4. We still have to prove that v is adjointable. Therefore, choose $a \in N$ and $x \in E$. We know from the second result of Step 2 that $((S\Lambda(e_k))^*\pi(a^*))_{k\in K}$ converges to $(S\Lambda(a))^*$. This implies that the net $((S\Lambda(e_k))^*\pi(a^*)x)_{k\in K}$ is convergent.

Because π is assumed to be non-degenerate, the set $\langle \pi(a^*)x \mid a \in N, x \in E\rangle$ is dense in E. By Step 3, we also know that $((S\Lambda(e_k))^*)_{k\in K}$ is bounded.

These three facts imply easily the existence of a unique linear map w from E into B such that $((S\Lambda(e_k))^*)_{k\in K}$ converges strongly to w. By definition, we have that $(S\Lambda(e_k))_{k\in K}$ converges strongly to v.

These two results imply that $\langle vb, x\rangle = \langle b, wx\rangle$ for $x \in E$, $b \in B$. Consequently, v belongs to $\mathcal{L}(B, E)$ and $v^* = w$.

STEP 5. We have for every $a_1, a_2 \in N$ that

$$\begin{aligned}\rho(a_2^*a_1) &= \Lambda(a_2)^*T_\rho\Lambda(a_1) = \Lambda(a_2)^*S^*S\Lambda(a_1) = (S\Lambda(a_2))^*(S\Lambda(a_1))\\ &= (\pi(a_2)v)^*(\pi(a_1)v) = v^*\pi(a_2^*a_1)v.\end{aligned}$$

This implies that $\rho(x) = v^*\pi(x)v$ for every $x \in N^*N$. The usual continuity and strict continuity arguments imply that $\rho(x) = v^*\pi(x)v$ for every $x \in M(A)$. ∎

NOTATION 2.6. Suppose that π is non-degenerate. Considering $\rho \in \mathcal{H}$, we define v_ρ to be the unique element in $\mathcal{L}(B, E)$ such that $T_\rho^{\frac{1}{2}}\Lambda(a) = \pi(a)v_\rho$ for every $a \in N$. Furthermore, $\|v_\rho\|^2 = \|\rho\|$. We have also that $\rho(x) = v_\rho^*\pi(x)v_\rho$ for every $x \in M(A)$.

REMARK 2.7. We have the following obvious properties:

(i) 0 belongs to $\mathcal{H}$ and $T_0 = 0$;

(ii) for every $\rho_1, \rho_2 \in \mathcal{H}$, it follows that $\rho_1 + \rho_2$ belongs to $\mathcal{H}$ and $T_{\rho_1+\rho_2} = T_{\rho_1} + T_{\rho_2}$;

(iii) for every $\rho \in \mathcal{H}$ and every $\lambda \in \mathbb{R}^+$, it follows that λT belongs to $\mathcal{H}$ and $T_{\lambda\rho} = \lambda T_\rho$.

We will need the following ordering $\leqslant$ on $\mathcal{H}$.

DEFINITION 2.8. Consider $\rho_1, \rho_2 \in \mathcal{H}$. We say that $\rho_1 \leqslant \rho_2$ if one of the following conditions fulfilled:

(i) $\rho_2 - \rho_1$ is completely positive;

(ii) we have for every $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ that

$$\sum_{i,j=1}^{n} b_j^*\rho_1(a_j^*a_i)b_i \leqslant \sum_{i,j=1}^{n} b_j^*\rho_2(a_j^*a_i)b_i;$$

(iii) $T_{\rho_1} \leqslant T_{\rho_2}$.

It is straigthforward to check the equivalence of the different conditions in the previous definition.

Remark 2.9. We also have the property: let ρ_1, ρ_2 be elements in $\mathcal{H}$ with $\rho_1 \leqslant \rho_2$. Then $\rho_2 - \rho_1$ belongs to $\mathcal{H}$ and $T_{\rho_2 - \rho_1} = T_{\rho_1} - T_{\rho_2}$.

The following sets are also generalizations of objects known in the ordinary weight theory (see, e.g., Definition 2.1.7 of [10]).

Definition 2.10. We define the following sets

$$\mathcal{F} = \{\rho \in \mathcal{H} \mid T_\rho \leqslant 1\} \quad \text{and} \quad \mathcal{G} = \{\lambda\rho \mid \lambda \in [0,1[, \rho \in \mathcal{F}\} \subseteq \mathcal{F}.$$

Remark 2.11. Let ρ be an element in $\mathcal{H}$. It is not difficult to check that ρ belongs to $\mathcal{F}$ if and only if

$$\sum_{i,j=1}^{n} b_j^* \rho(a_j^* a_i) b_i \leqslant \Big\langle \sum_{i=1}^{n} \Lambda(a_i) b_i, \sum_{i=1}^{n} \Lambda(a_i) b_i \Big\rangle$$

for every $n \in \mathbb{N}$, $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in B$.

Like in the case of weights, the reason for introducing the set $\mathcal{G}$ is to obtain a set which is directed for the ordering $\leqslant$. This is the content of the following proposition. The basic idea of its proof is the same as the proof for ordinary weights (see Proposition 2.1.8 of [10]), but some extra work has to be done in this case.

First, we prove the following lemma (which we took from Proposition 2.1.8 of [10]).

Lemma 2.12. *Consider a unital C^*-algebra C. Let T_1, T_2 be elements in C with $0 \leqslant T_1$, $T_2 \leqslant 1$, and let γ be a number in $[0,1[$. Then there exists an element $T \in C$ with $0 \leqslant T \leqslant 1$ and such that $\gamma T_1 \leqslant T$, $\gamma T_2 \leqslant T$ and $T \leqslant \frac{\gamma}{1-\gamma}(T_1 + T_2)$.*

Proof. For the moment, fix $i \in \{1,2\}$. We define

$$S_i = \frac{\gamma T_i}{1 - \gamma T_i} \in C.$$

It is then easy to check that

$$\gamma T_i = \frac{S_i}{1 + S_i} \tag{2.1}$$

and

$$S_i \leqslant \frac{\gamma}{1-\gamma} T_i. \tag{2.2}$$

Next, we define

$$T = \frac{S_1 + S_2}{1 + S_1 + S_2} \in C.$$

We infer immediately that $0 \leqslant T \leqslant 1$. By (2.2), we have that $T \leqslant S_1 + S_2 \leqslant \frac{\gamma}{1-\gamma}(T_1 + T_2)$.

We know that the function $\mathbb{R}^+ \to \mathbb{R}^+ : t \mapsto \frac{t}{1+t}$ is operator monotone (see [8]). This implies for every $i = 1, 2$ that

$$T \geqslant \frac{S_i}{S_i + 1} = \gamma T_i,$$

where we used (2.1) in the last equality. ∎

PROPOSITION 2.13. *The set $\mathcal{G}$ is directed for $\leqslant$.*

Proof. Choose $\rho_1, \rho_2 \in \mathcal{F}$ and $\lambda_1, \lambda_2 \in [0,1[$. Then there exists a number $\gamma \in [0,1[$ such that $\lambda_1, \lambda_2 < \gamma$. We also know that T_{ρ_1}, T_{ρ_2} belong to $\pi(A)' \cap \mathcal{L}(E)$ and $0 \leqslant T_{\rho_1}, T_{\rho_2} \leqslant 1$. The previous lemma implies the existence of $T \in \pi(A)' \cap \mathcal{L}(E)$ with $0 \leqslant T \leqslant 1$ and such that $\gamma T_{\rho_1} \leqslant T$, $\gamma T_{\rho_2} \leqslant T$ and $T \leqslant \frac{\gamma}{1-\gamma}(T_{\rho_1} + T_{\rho_2})$.

Put $\lambda = \max(\frac{\lambda_1}{\gamma}, \frac{\lambda_2}{\gamma}) \in [0,1[$. Then

$$\lambda T \geqslant \lambda\gamma T_{\rho_1} \geqslant \lambda_1 T_{\rho_1} = T_{\lambda_1\rho_1} \quad \text{and analogously,} \quad \lambda T \geqslant T_{\lambda_2\rho_2}. \tag{2.3}$$

In the rest of the proof, we want to construct an element ρ in $\mathcal{F}$ such that $T_\rho = T$.

In two steps, we will prove the following inequality:

Consider $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in B$. Then

$$\Big\| \sum_{i=1}^n \Lambda(b_i)^* T \Lambda(a_i) \Big\| \leqslant 4 \frac{\gamma}{1-\gamma}(\|\rho_1\| + \|\rho_2\|) \Big\| \sum_{i=1}^n b_i^* a_i \Big\|.$$

STEP 1. Choose $\omega \in B_+^*$. Define the function θ from A into $\mathbb{C}$ such that $\theta(x) = \frac{\gamma}{1-\gamma}(\omega(\rho_1(x)) + \omega(\rho_2(x)))$ for every $x \in A$. Then θ belongs to A_+^* and

$$\|\theta\| \leqslant \frac{\gamma}{1-\gamma}(\|\rho_1\| + \|\rho_2\|)\|\omega\|. \tag{2.4}$$

Next, define the mapping S from $N \times N$ into $\mathbb{C}$ such that $S(x,y) = \omega(\Lambda(y)^* \cdot T\Lambda(x))$ for every $x, y \in N$. Then S is sesquilinear. Furthermore:

(1) We have for $x \in N$ that

$$0 \leqslant \Lambda(x)^* T \Lambda(x) \leqslant \frac{\gamma}{1-\gamma} \Lambda(x)^*(T_{\rho_1} + T_{\rho_2})\Lambda(x) \leqslant \frac{\gamma}{1-\gamma}(\rho_1(x^*x) + \rho_2(x^*x)),$$

which implies that $0 \leqslant S(x,x) \leqslant \theta(x^*x)$.

(2) We have for every $x, y \in N$ and $a \in A$ that

$$\begin{aligned} \Lambda(y)^* T \Lambda(ax) &= \Lambda(y)^* T \pi(a) \Lambda(x) = \Lambda(y)^* \pi(a) T \Lambda(x) \\ &= (\pi(a^*)\Lambda(y))^* T \Lambda(x) = \Lambda(a^*y)^* T \Lambda(x), \end{aligned}$$

which implies that $S(ax, y) = S(x, a^*y)$.

This allows us to apply Lemma 9.5. Therefore, we get the existence of $\psi \in B_+^*$ with $\psi \leqslant \theta$ and such that $S(x,y) = \psi(y^*x)$ for every $x, y \in N$.

This implies that

$$\begin{aligned} \Big|\omega\Big(\sum_{i=1}^n \Lambda(b_i)^* T \Lambda(a_i)\Big)\Big| &= \Big|\sum_{i=1}^n S(a_i, b_i)\Big| = \Big|\psi\Big(\sum_{i=1}^n b_i^* a_i\Big)\Big| \leqslant \|\psi\| \Big\|\sum_{i=1}^n b_i^* a_i\Big\| \\ &\leqslant \|\theta\| \Big\|\sum_{i=1}^n b_i^* a_i\Big\| \leqslant \frac{\gamma}{1-\gamma}(\|\rho_1\| + \|\rho_2\|)\|\omega\| \Big\|\sum_{i=1}^n b_i^* a_i\Big\| \end{aligned}$$

where we used inequality (2.4) in the last inequality.

STEP 2. Choose $\omega \in B^*$ with $\|\omega\| \leqslant 1$. Then there exist $\omega_1, \ldots, \omega_4 \in B^*_+$ such that $\|\omega_1\|, \ldots, \|\omega_4\| \leqslant 1$ and $\omega = \sum\limits_{k=1}^{4} \mathrm{i}^k \omega_k$. Referring to Step 1, it is not difficult to see that

$$\Big|\omega\Big(\sum_{i=1}^{n} \Lambda(b_i)^* T \Lambda(a_i)\Big)\Big| \leqslant 4 \frac{\gamma}{1-\gamma} (\|\rho_1\| + \|\rho_2\|) \Big\| \sum_{i=1}^{n} b_i^* a_i \Big\|.$$

This implies that

$$\Big\| \sum_{i=1}^{n} \Lambda(b_i)^* T \Lambda(a_i) \Big\| \leqslant 4 \frac{\gamma}{1-\gamma} (\|\rho_1\| + \|\rho_2\|) \Big\| \sum_{i=1}^{n} b_i^* a_i \Big\|.$$

As usual, this inequality guarantees the existence of a unique continuous linear map ρ from A into $M(B)$ such that $\rho(a_2^* a_1) = \Lambda(a_2)^* T \Lambda(a_1)$ for $a_1, a_2 \in N$. It is clear that ρ is completely positive.

Choose $x \in N$. We have for every $b \in B$ that

$$\begin{aligned} b^* \rho(x^* x) b = \langle T\Lambda(x)b, \Lambda(x)b \rangle &\leqslant \frac{\gamma}{1-\gamma} (\langle T_{\rho_1} \Lambda(x)b, \Lambda(x)b \rangle + \langle T_{\rho_2} \Lambda(x)b, \Lambda(x)b \rangle) \\ &= \frac{\gamma}{1-\gamma} (b^* \rho_1(x^* x) b + b^* \rho_2(x^* x) b), \end{aligned}$$

which implies that $\rho(x^* x) \leqslant \frac{\gamma}{1-\gamma}(\rho_1(x^* x) + \rho_2(x^* x))$. Hence, $\rho \leqslant \frac{\gamma}{1-\gamma}(\rho_1 + \rho_2)$. Therefore, the strictness of ρ_1, ρ_2 implies that ρ is strict.

It is now easy to see that ρ belongs to $\mathcal{F}$ and $T_\rho = T$. Moreover, inequality (2.3) implies that $T_{\lambda\rho} = \lambda T_\rho = \lambda T \geqslant T_{\lambda_1 \rho_1}, T_{\lambda_2 \rho_2}$, which implies that $\lambda \rho \geqslant \lambda_1 \rho_1, \lambda_2 \rho_2$. ∎

The following easy lemma will be used several times.

LEMMA 2.14. *Consider $a \in M(A)^+$ and $b \in B$. Let x be an element in B such that $b^* \rho(a) b \leqslant x$ for every $\rho \in \mathcal{G}$. Then the net $(b^* \rho(a) b)_{\rho \in \mathcal{G}}$ converges to x if and only if for every $\varepsilon > 0$ there exists an element $\eta \in \mathcal{F}$ such that $\|x - b^* \eta(a) b\| \leqslant \varepsilon$.*

3. LOWER SEMI-CONTINUITY OF C^*-VALUED WEIGHTS

In this section, we will introduce a possible definition for lower semi-continuity of a C^*-valued weight. Also, some easy consequences will be derived for such lower semi-continuous C^*-valued weights. At the end of this section, we will introduce an ever stronger condition than lower semi-continuity, the so-called regularity.

NOTATION 3.1. Consider two C^*-algebras A and B and a densely defined C^*-valued weight φ from A into $M(B)$ with KSGNS-construction (E, Λ, π). The ingredients $A, B, E, \mathcal{N}_\varphi, \Lambda, \pi$ satisfy the conditions of the beginning of Section 2. We define $\mathcal{H}_\varphi, \mathcal{F}_\varphi, \mathcal{G}_\varphi$ to be the objects $\mathcal{H}, \mathcal{F}, \mathcal{G}$ defined in Definitions 2.1 and 2.10, using these specific ingredients.

It is not difficult to see that the definition of $\mathcal{H}_\varphi, \mathcal{F}_\varphi$ and $\mathcal{G}_\varphi$ is independent of the choice of the KSGNS-construction. The notations T_ρ and v_ρ however (see Notations 2.2 and 2.6), will always be relative to a specific KSGNS-construction.

We will use the following definition of lower semi-continuity in this paper. It should be noted however that this definition will exclude some interesting C^*-valued weights as being lower semi-continuous while they are lower semi-continuous in some way.

DEFINITION 3.2. Consider two C^*-algebras A and B and a densely defined C^*-valued weight φ from A into $M(B)$. We call φ *lower semi-continuous* if and only if:

(i) we have that $\mathcal{M}_\varphi^+ = \{x \in A \mid (\rho(x))_{\rho\in\mathcal{G}_\varphi}$ is strictly convergent in $M(B)\}$;

(ii) for every $x \in \mathcal{M}_\varphi^+$, the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ converges strictly to $\varphi(x)$.

REMARK 3.3. Consider two C^*-algebras A and B and a densely defined lower semi-continuous C^*-valued weight φ from A into $M(B)$. Because every element of $\mathcal{M}_\varphi$ is a linear combination of elements from $\mathcal{M}_\varphi^+$, the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ converges strictly to $\varphi(x)$ for every $x \in \mathcal{M}_\varphi$.

This definition is rather heavy. In fact, in the ordinary weight theory this definition is a major result proved by Combes (see [1]). Therefore, it might be an interesting question whether it is possible to give some kind of lower semi-continuity definition in the classical sense which is equivalent to this one. On the other hand, this definition is probably workable because most of the C^*-valued weights will be defined starting from some sort of family of completely positive mappings.

For the most part of this section, we consider two C^*-algebras A and B and a densely defined lower semi-continuous C^*-valued weight from A into $M(B)$. We also fix a KSGNS-construction (E, Λ, π) for φ.

Referring to Lemma 9.3, the following proposition follows immediately.

PROPOSITION 3.4. *Let x be an element in A^+. Then x belongs to $\mathcal{M}_\varphi^+$ if and only if for every $b \in B$ the net $(b^*\rho(x)b)_{\rho\in\mathcal{G}_\varphi}$ is convergent in B.*

As usual, we have some kind of monotone convergence properties:

RESULT 3.5. *Consider a net $(x_i)_{i\in I}$ in $\mathcal{M}_\varphi^+$ and an element x in $\mathcal{M}_\varphi^+$ such that $(x_i)_{i\in I}$ converges strictly to x and $x_i \leqslant x$ for every $i \in I$. Then $(\varphi(x_i))_{i\in I}$ converges strictly to $\varphi(x)$.*

Proof. It is clear that $0 \leqslant \varphi(x_i) \leqslant \varphi(x)$ for every $i \in I$. Choose $b \in B$. Take $\varepsilon > 0$. Then there exists $\rho \in \mathcal{G}_\varphi$ such that $\|b^*\varphi(x)b - b^*\rho(x)b\| \leqslant \frac{\varepsilon}{2}$. Because $(\rho(x_i))_{i\in I}$ converges strictly to $\rho(x)$, there exists an element $i_0 \in I$ such that $\|b^*\rho(x_i)b - b^*\rho(x)b\| \leqslant \frac{\varepsilon}{2}$ for every $i \in I$ with $i \geqslant i_0$.

Choose $j \in J$ with $j \geqslant i_0$. Then

$$\|b^*\rho(x_j)b - b^*\varphi(x)b\| \leqslant \|b^*\rho(x_j)b - b^*\rho(x)b\| + \|b^*\rho(x)b - b^*\varphi(x)b\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Because $0 \leqslant b^*\rho(x_j)b \leqslant b^*\varphi(x_j)b \leqslant b^*\varphi(x)b$, we get that

$$\|b^*\varphi(x_j)b - b^*\varphi(x)b\| \leqslant \|b^*\rho(x_j)b - b^*\varphi(x)b\| \leqslant \varepsilon.$$

Consequently, we see that $(b^*\varphi(x_i)b)_{i\in I}$ converges to $b^*\varphi(x)b$. Lemma 9.2 implies that $(\varphi(x_i))_{i\in I}$ converges strictly to $\varphi(x)$. ∎

RESULT 3.6. *Consider a net $(x_i)_{i\in I}$ in $\mathcal{M}_\varphi^+$ and an element x in A^+ such that $(x_i)_{i\in I}$ converges strictly to x and $x_i \leqslant x$ for every $i \in I$. Then x belongs to $\mathcal{M}_\varphi^+$ if and only if the net $(b^*\varphi(x_i)b)_{i\in I}$ is convergent for every $b \in B$.*

Proof. As one implication follows from the previous result, we will turn to the other one. Therefore, assume that the net $(b^*\varphi(x_i)b)_{i\in I}$ is convergent for every $b \in B$. Choose $c \in B$. By assumption, there exists an element $d \in B^+$ such that

$$(c^*\varphi(x_i)c)_{i\in I} \quad \text{converges to } d. \tag{3.1}$$

First, we will prove that

$$c^*\rho(x)c \leqslant d \quad \text{for every } \rho \in \mathcal{G}_\varphi. \tag{3.2}$$

Take $\rho \in \mathcal{G}_\varphi$. Choose $n \in \mathbb{N}$. From (3.1), we get the existence of an element $i_0 \in I$ such that $\|d - c^*\varphi(x_i)c\| \leqslant \frac{1}{n}$ for every $i \in I$ with $i \geqslant i_0$. Therefore, we have for every $i \in I$ with $i \geqslant i_0$ that $c^*\rho(x_i)c \leqslant c^*\varphi(x_i)c \leqslant d + \frac{1}{n}\mathbb{1}$. Because $(c^*\rho(x_i)c)_{i\in I}$ converges to $c^*\rho(x)c$, this implies that $c^*\rho(x)c \leqslant d + \frac{1}{n}\mathbb{1}$. If we let n tend to ∞, we get that $c^*\rho(x)c \leqslant d$. Choose $\varepsilon > 0$. By (3.1), there exists an element $j \in I$ such that $\|c^*\varphi(x_j)c - d\| \leqslant \frac{\varepsilon}{2}$. The lower semi-continuity of φ implies the existence of $\rho_0 \in \mathcal{G}_\varphi$ such that $\|c^*\varphi(x_j)c - c^*\rho(x_j)c\| \leqslant \frac{\varepsilon}{2}$ for every $\rho \in \mathcal{G}_\varphi$ with $\rho \geqslant \rho_0$. So we get that

$$\|c^*\rho(x_j)c - d\| \leqslant \varepsilon \quad \text{for every } \rho \in \mathcal{G}_\varphi \text{ with } \rho \geqslant \rho_0. \tag{3.3}$$

Take $\eta \in \mathcal{G}_\varphi$ with $\eta \geqslant \rho_0$. Using (3.2), we have that $0 \leqslant c^*\eta(x_j)c \leqslant c^*\eta(x)c \leqslant d$. Consequently, (3.3) implies that

$$\|c^*\eta(x)c - d\| \leqslant \|c^*\eta(x_j)c - d\| \leqslant \varepsilon.$$

Hence, we see that $(c^*\rho(x)c)_{\rho\in\mathcal{G}_\varphi}$ converges to d. The lower semi-continuity of φ guarantees that x belongs to $\mathcal{M}_\varphi^+$. ∎

Like for weights, these kind of convergence properties imply that the representation of a KSGNS-construction is non-degenerate (Lemma 2.1 of [1]).

RESULT 3.7. *The mapping π is non-degenerate and $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in M(A)$ and $a \in \mathcal{N}_\varphi$.*

Proof. Take an approximate unit $(e_i)_{i\in I}$ for A. Choose $a \in \mathcal{N}_\varphi$ and $b \in B$. Then $(a^*e_ia)_{i\in I}$ is a net in $\mathcal{M}_\varphi^+$ such that $a^*e_ia \leqslant a^*a$ for every $i \in I$. Therefore, Result 3.5 implies that

$$(b^*\varphi(a^*e_ia)b)_{i\in I} \quad \text{converges to } \varphi(a^*a). \tag{3.4}$$

Similarly, we get that

$$(b^*\varphi(a^*e_i^2a)b)_{i\in I} \quad \text{converges to } b^*\varphi(a^*a)b. \tag{3.5}$$

We have for every $i \in I$ that

$$\begin{aligned}\|\pi(e_i)\Lambda(a)b - \Lambda(a)b\|^2 &= \|\langle\Lambda(e_ia)b - \Lambda(a)b, \Lambda(e_ia)b - \Lambda(a)b\rangle\| \\ &= \|b^*\varphi(a^*e_i^2a)b - b^*\varphi(a^*e_ia)b - b^*\varphi(a^*e_ia)b + b^*\varphi(a^*a)b\|.\end{aligned}$$

Therefore, (3.4) and (3.5) imply that $(\pi(e_i)\Lambda(a)b)_{i\in I}$ converges to $\Lambda(a)b$. Because $(\pi(e_i))_{i\in I}$ is bounded, we can conclude from this that $(\pi(e_i))_{i\in I}$ converges strongly to 1. ∎

RESULT 3.8. *The net $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ converges strongly to* 1.

Proof. Choose $a_1, a_2 \in \mathcal{N}_\varphi$ and $b_1, b_2 \in B$. We have for every $\rho \in \mathcal{G}_\varphi$ that $\langle T_\rho\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho(a_2^*a_1)b_1$. Therefore, the lower semi-continuity of φ implies that $(\langle T_\rho\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle)_{\rho\in\mathcal{H}_\varphi}$ converges to $b_2^*\varphi(a_2^*a_1)b_1$, which is equal to $\langle\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle$. Because $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ is bounded, this implies that $(\langle T_\rho v, w\rangle)_{\rho\in\mathcal{G}_\varphi}$ converges to $\langle v, w\rangle$ for all $v, w \in E$.

Using Lemma 9.2, we conclude that $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ converges strongly to 1. ∎

PROPOSITION 3.9. *The mapping Λ is closed for the strict topology on A and the strong topology on $L(B, E)$.*

Proof. Choose a net $(x_i)_{i\in I}$ in $\mathcal{N}_\varphi$, $x \in A$ and $t \in L(B, E)$ such that $(x_i)_{i\in I}$ converges strictly to x and $(\Lambda(x_i))_{i\in I}$ converges strongly to t. Take $\rho \in \mathcal{G}_\varphi$, $c \in \mathcal{N}_\varphi$ and $b, d \in B$. We have for every $i \in I$ that $\langle T_\rho(\Lambda(x_i)b), \Lambda(c)d\rangle = d^*\rho(c^*x_i)b$, which implies that $(\langle T_\rho(\Lambda(x_i)b), \Lambda(c)d\rangle)_{i\in I}$ converges to $d^*\rho(c^*x)b$. It is also clear that $(\langle T_\rho(\Lambda(x_i)b), \Lambda(c)d\rangle)_{i\in I}$ converges to $\langle T_\rho t(b), \Lambda(c)d\rangle$. Combining these two facts, we conclude that

$$\langle T_\rho t(b), \Lambda(c)d\,\rangle = d^*\rho(c^*x)b. \tag{3.6}$$

Now we will use this last equality to prove that $x \in \mathcal{N}_\varphi$ and $\Lambda(x) = t$.

STEP 1. Take $b \in B$. For the moment, fix $\rho \in \mathcal{G}_\varphi$. We have immediately that the net $(b^*\rho(x_i^*x)b)_{i\in I}$ converges to $b^*\rho(x^*x)b$. Using (3.6), we have that $b^*\rho(x_i^*x)b = \langle T_\rho t(b), \Lambda(x_i)b\rangle$ for every $i \in I$. This implies that the net $(b^*\rho(x_i^*x)b)_{i\in I}$ converges to $\langle T_\rho t(b), t(b)\rangle$. Combining these two results, we conclude that $\langle T_\rho t(b), t(b)\rangle = b^*\rho(x^*x)b$.

This last equality implies that $(b^*\rho(x^*x)b)_{\rho\in\mathcal{G}_\varphi}$ converges to $\langle t(b), t(b)\rangle$. Because φ is assumed to be lower semi-continuous, this implies that x^*x belongs to $\mathcal{M}_\varphi^+$. So x belongs to $\mathcal{N}_\varphi$.

STEP 2. Choose $c \in \mathcal{N}_\varphi$ and $b, d \in B$.

By (3.6), we have for every $\rho \in \mathcal{G}_\varphi$ that

$$\langle T_\rho t(b), \Lambda(c)d\rangle = d^*\rho(c^*x)b = \langle T_\rho\Lambda(x)b, \Lambda(c)d\rangle.$$

Because $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ converges strongly to 1, we can conclude that $\langle t(b), \Lambda(c)d\rangle = \langle\Lambda(x)b, \Lambda(c)d\rangle$. Consequently, $t = \Lambda(x)$. ∎

At the end of this section, we introduce a stronger condition than lower semi-continuity, the so-called regularity condition. A first version of this condition was introduced for weights by J. Verding (see Definition 2.1.13 of [10]).

DEFINITION 3.10. Consider two C^*-algebras A and B and a C^*-valued weight φ from A into $M(B)$ with KSGNS-construction (E, Λ, π). We say that φ is *regular* if and only if

(i) φ is densely defined and lower semi-continuous;

(ii) there exists a net $(u_i)_{i\in I}$ in $M(A)$ satisfying the following properties:

(a) we have for every $i \in I$ that:

(1) $\mathcal{N}_\varphi u_i \subseteq \mathcal{N}_\varphi$;

(2) there exists a unique operator $S_i \in \mathcal{L}(E)$ such that $S_i\Lambda(a) = \Lambda(au_i)$ for every $a \in \mathcal{N}_\varphi$;

(3) $\|u_i\| \leqslant 1$ and $\|S_i\| \leqslant 1$;

(4) there exists a unique strict completely positive linear mapping ρ_i from A into $M(B)$ such that $b_2^*\rho_i(a_2^*a_1)b_1 = \langle S_i\Lambda(a_1)b_1, S_i\Lambda(a_2)b_2\rangle$ for every $a_1, a_2 \in \mathcal{N}_\varphi$ and $b_1, b_2 \in B$.

(b) Moreover,

(1) $(u_i)_{i\in I}$ converges strictly to 1;

(2) $(S_i)_{i\in I}$ converges strongly to 1.

Such a net $(u_i)_{i\in I}$ is called a *truncating net for* φ. If the truncating net can be chosen in such a way that every element belongs to A, then we call φ *strongly regular*.

Again, it is not too difficult to check that this definition of regularity (and strong regularity) is independent of the choice of the KSGNS-construction. This regularity condition is a very strong condition. Nevertheless, there are some interesting C^*-valued weights, which satisfy this condition.

(1) Consider two C^*-algebras A and B and a strict completely positive mapping ρ from A into $M(B)$. Then ρ is a regular C^*-valued weight from A into B with a truncating net consisting of one element, the unit in $M(A)$.

In particular, the non-degenerate $*$-homomorphisms from A into $M(B)$ are regular.

(2) Also, the previous example implies that all positive linear functionals on a C^*-algebra are regular.

(3) Consider a C^*-algebra A and a KMS-weight φ on A. These means that φ is a densely defined lower semi-continuous weight on A such that there exists a norm-continuous one-paramater group σ on A such that:

(a) φ is invariant under σ;

(b) we have that $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*)$ for every $a \in D(\sigma_{\frac{i}{2}})$.

Then φ is strongly regular and has a truncating net consisting of elements which are analytic with respect to σ.

This fact was proven by Jan Verding. The proof of Proposition 2.1.18 in [10] contains a mistake but has been corrected by himself. A modified proof of this fact can be found in [3].

It is not yet clear whether every densely defined lower semi-continuous weight is regular, but intuition tells us that this will probably not be true. It is easily true if A is commutative.

4. EXTENSION OF A LOWER SEMI-CONTINUOUS C^*-VALUED WEIGHT TO THE MULTIPLIER ALGEBRA

It is natural to look for extensions of C^*-valued weights to the multiplier algebra. In this section, we will do this for lower semi-continuous C^*-valued weights. The KSGNS-construction of this extension will also be investigated.

We start with a lemma which will also be used in a later section.

LEMMA 4.1. *Consider two C^*-algebras A and B and an increasing net $(\rho_i)_{i\in I}$ of strict completely mappings from A into $M(B)$. Let b be an element in B and define*

$$P = \{x \in M(A)^+ \mid \textit{the net } (b^*\rho_i(x)b)_{i\in I} \textit{ is convergent}\}.$$

Then P is a hereditary cone in $M(A)^+$.

Proof. It is easy to see that P is a cone in $M(A)^+$. We turn now to the hereditarity of P. Choose $x \in P$ and $y \in M(A)^+$ such that $y \leqslant x$. Take $\varepsilon > 0$. Then there exists an element $i_0 \in I$ such that we have for every $i \in I$ with $i \geqslant i_0$ that $\|b^*\rho_i(x)b - b^*\rho_{i_0}(x)b\| \leqslant \varepsilon$. We have for every $i \in I$ with $i \geqslant i_0$ that

$$0 \leqslant b^*(\rho_i(y) - \rho_{i_0}(y))b \leqslant b^*(\rho_i(x) - \rho_{i_0}(x))b,$$

which implies that

$$\|b^*\rho_i(y)b - b^*\rho_{i_0}(y)b\| \leqslant \|b^*\rho_i(x)b - b^*\rho_{i_0}(x)b\| \leqslant \varepsilon.$$

Therefore, we see that the net $(b^*\rho_i(y)b)_{i\in I}$ is Cauchy and hence convergent in B.

Hence, we get that $y \in P$. ∎

For the rest of this section, we consider two C^*-algebras A and B and a densely defined lower semi-continuous C^*-valued weight φ from A into $M(B)$. We will also fix a KSGNS-construction (E, Λ, π) for φ.

REMARK 4.2. Define

$$P = \{x \in M(A)^+ \mid \text{the net } (b^*\rho(x)b)_{\rho\in\mathcal{G}_\varphi} \text{ is convergent for every } b \in B\}.$$

We know by the previous lemma that P is a hereditary cone in $M(A)^+$ such that $\mathcal{M}_\varphi^+ \subseteq P$. Lemma 9.3 implies for every $x \in P$ that the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ is strictly convergent in $M(B)$. So we get for every $x \in \operatorname{span} P$ that the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ is strictly convergent in $M(B)$.

If we define a mapping ψ from $\operatorname{span} P$ into $M(B)$ such that the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ converges strictly to $\psi(x)$ for every $x \in \operatorname{span} P$, we get a C^*-valued weight ψ from $M(A)$ into $M(B)$.

These remarks justify the following definition.

DEFINITION 4.3. We define the C^*-valued weight $\overline{\varphi}$ from $M(A)$ into $M(B)$ such that:

(i) we have that

$$\mathcal{M}_{\overline{\varphi}}^+ = \{x \in M(A)^+ \mid \text{the net } (b^*\rho(x)b)_{\rho\in\mathcal{G}_\varphi} \text{ is convergent for every } b \in B\};$$

(ii) the net $(\rho(x))_{\rho\in\mathcal{G}_\varphi}$ converges strictly to $\overline{\varphi}(x)$ for every $x \in \mathcal{M}_{\overline{\varphi}}$.

Because φ is lower semi-continuous, it is clear that $\overline{\varphi}$ extends φ.

NOTATION 4.4. We will use the following notation:
(i) we define $\overline{\mathcal{M}}_\varphi = \mathcal{M}_{\overline{\varphi}}$ and $\overline{\mathcal{N}}_\varphi = \mathcal{N}_{\overline{\varphi}}$;
(ii) for every $x \in \overline{\mathcal{M}}_\varphi$, we put $\varphi(x) = \overline{\varphi}(x)$.
It is clear that $\mathcal{M}_\varphi^+ = \overline{\mathcal{M}}_\varphi^+ \cap A$ and $\mathcal{N}_\varphi = \overline{\mathcal{N}}_\varphi \cap A$.

The proof of the following results is the same as the proofs of Result 3.5 and Result 3.6.

RESULT 4.5. *Let $(x_i)_{i\in I}$ be a net in $\overline{\mathcal{M}}_\varphi^+$ and x an element in $\overline{\mathcal{M}}_\varphi^+$ such that $(x_i)_{i\in I}$ converges strictly to x and $x_i \leqslant x$ for every $i \in I$. Then $(\varphi(x_i))_{i\in I}$ converges strictly to $\varphi(x)$.*

RESULT 4.6. *Let $(x_i)_{i\in I}$ be a net in $\overline{\mathcal{M}}_\varphi^+$ and x an element in $M(A)^+$ such that $(x_i)_{i\in I}$ converges strictly to x and $x_i \leqslant x$ for every $i \in I$. Then x belongs to $\overline{\mathcal{M}}_\varphi^+$ if and only if the net $(b^*\varphi(x_i)b)_{i\in I}$ is convergent for every $b \in B$.*

We want to obtain a KSGNS-construction for $\overline{\varphi}$. Therefore, we want to extend the map Λ. This will be done in the next part.

LEMMA 4.7. *The mapping Λ is closable for the strict topology on $M(A)$ and the strong topology on $L(B,E)$.*

This follows immediately from the fact that Λ is closed for the strict topology on A and the strong topology on $L(B,E)$ (Proposition 3.9).

DEFINITION 4.8. We define $\overline{\Lambda}$ to be the closure of Λ with respect to the strict topology on $M(A)$ and the strong topology on $L(B,E)$. For every a in the domain of $\overline{\Lambda}$, we define $\Lambda(a) = \overline{\Lambda}(a)$.

PROPOSITION 4.9. *We have that $\overline{\Lambda}$ is a linear mapping from $\overline{\mathcal{N}}_\varphi$ into $\mathcal{L}(B,E)$ such that:*
(i) $\pi(x)\Lambda(a) = \Lambda(xa)$ *for every* $x \in M(A)$ *and* $a \in \overline{\mathcal{N}}_\varphi$*;*
(ii) $\langle \Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\varphi(a_2^*a_1)b_1$ *for every* $a_1, a_2 \in \overline{\mathcal{N}}_\varphi$ *and* $b_1, b_2 \in B$.
Consequently, the triplet $(E, \overline{\Lambda}, \overline{\pi})$ is a KSGNS*-construction for $\overline{\varphi}$.*

Proof. We split the proof in several parts.
STEP 1. Choose $x \in A$ and a in the domain of $\overline{\Lambda}$. Then there exists a net $(a_k)_{k\in K}$ in $\mathcal{N}_\varphi$ such that $(a_k)_{k\in K}$ converges strictly to a and $(\Lambda(a_k))_{k\in K}$ converges strongly to $\Lambda(a)$. It is clear that $(xa_k)_{k\in K}$ converges to xa. We also have for every $k \in K$ that xa_k belongs to $\mathcal{N}_\varphi$ and $\Lambda(xa_k) = \pi(x)\Lambda(a_k)$. So we get that the net $(\Lambda(xa_k))_{k\in K}$ converges strongly to $\pi(x)\Lambda(a)$. By the norm-strong closedness of Λ, we see that xa belongs to $\mathcal{N}_\varphi$ and $\Lambda(xa) = \pi(x)\Lambda(a)$.
STEP 2. Choose a in the domain of $\overline{\Lambda}$. Take an approximate unit $(e_k)_{k\in K}$ of A. By Step 1, we know for every $k \in K$ that $e_k a$ belongs to $\mathcal{N}_\varphi$ and $\Lambda(e_k a) = \pi(e_k)\Lambda(a)$. From this we conclude that $a^*e_k^2a$ belongs to $\mathcal{M}_\varphi^+$ and

$$b^*\varphi(a^*e_k^2a)b = \langle \Lambda(e_ka)b, \Lambda(e_ka)b\rangle = \langle \pi(e_k)\Lambda(a)b, \pi(e_k)\Lambda(a)b\rangle$$

for every $b \in B$ and $k \in K$. Hence, we see that the net $(b^*\varphi(a^*e_k^2a)b)_{k\in K}$ converges to $\langle \Lambda(a)b, \Lambda(a)b\rangle$ for every $b \in B$.

Because we also have that $a^*e_k^2a \leqslant a^*a$ for every $k \in K$ and because the net $(a^*e_k^2a)_{k\in K}$ converges strictly to a^*a, Results 4.5 and 4.6 imply that a^*a belongs to $\overline{\mathcal{M}}_\varphi^+$ and $b^*\varphi(a^*a)b = \langle \Lambda(a)b, \Lambda(a)b\rangle$ for every $b \in B$. Of course, we get also that a belongs to $\overline{\mathcal{N}}_\varphi$.

Step 3. Choose $a \in \overline{\mathcal{N}}_\varphi$. Take an approximate unit $(e_k)_{k\in K}$ for A. Then $e_k a$ belongs to $\mathcal{N}_\varphi$ for every $k \in K$. We see that $(a^*e_k a)_{k\in K}$ is a net in $\mathcal{M}_\varphi^+$ such that $a^*e_k a \leqslant a^*a$ for every $k \in K$. Because $(a^*e_k a)_{k\in K}$ converges strictly to a^*a, Result 4.5 implies that the net $(\varphi(a^*e_k a))_{k\in K}$ converges strictly to $\varphi(a^*a)$. Choose $b \in B$. Take $k, l \in K$ with $l \geqslant k$. Then $0 \leqslant e_l - e_k \leqslant 1$, which implies that $0 \leqslant (e_l - e_k)^2 \leqslant e_l - e_k$. Therefore,

$$\begin{aligned}\|\Lambda(e_l a)b - \Lambda(e_k a)b\|^2 &= \|\langle \Lambda((e_l - e_k)a)b, \Lambda((e_l - e_k)a)b\rangle\| = \|b^*\varphi(a^*(e_l - e_k)^2 a)b\| \\ &\leqslant \|b^*\varphi(a^*(e_l - e_k)a)b\| = \|b^*\varphi(a^*e_l a)b - b^*\varphi(a^*e_k a)b\|.\end{aligned}$$

This easily implies that $(\Lambda(e_k a)b)_{k\in K}$ is Cauchy and hence convergent in B.

From this all, we infer the existence of a linear operator t from B into E such that $(\Lambda(e_k a))_{k\in K}$ converges strongly to t. By the definition of $\overline{\Lambda}$, we find that a belongs to the domain of $\overline{\Lambda}$.

In the preceding part of the proof, we have proven the following facts:

(a) $\overline{\mathcal{N}}_\varphi$ equals the domain of $\overline{\Lambda}$;

(b) for every $a \in \overline{\mathcal{N}}_\varphi$ and $b \in B$, it follows that $b^*\varphi(a^*a)b = \langle \Lambda(a)b, \Lambda(a)b\rangle$;

(c) for every $x \in A$ and $a \in \overline{\mathcal{N}}_\varphi$, we have that $\Lambda(xa) = \pi(x)\Lambda(a)$.

We still have to proof some minor details. Statement (ii) of the proposition follows from (b) by polarization. Statement (i) of the proposition follows from (c) by the same method as in the proof of Result 1.5 (iii).

Finally, choose $a \in \overline{\mathcal{N}}_\varphi$. We have for every $c \in \mathcal{N}_\varphi$, $d \in B$ and $b \in B$ that $\langle \Lambda(a)(b), \Lambda(c)d\rangle = d^*\varphi(c^*a)b$. Therefore, Lemma 10.3 of Section 10 implies that $\Lambda(a)$ belongs to $\mathcal{L}(B, E)$. ∎

Result 4.10. *We have the following equalities:*

(i) *for every $a_1, a_2 \in \overline{\mathcal{N}}_\varphi$, it follows that $\varphi(a_2^* a_1) = \Lambda(a_2)^*\Lambda(a_1)$;*

(ii) *for every $a \in \overline{\mathcal{N}}_\varphi$, we have that $\|\Lambda(a)\|^2 = \|\varphi(a^*a)\|$.*

Remark 4.11. By definition, we have that $\mathcal{N}_\varphi$ is a strict-strong core for $\overline{\Lambda}$. But we have even more: consider $a \in \overline{\mathcal{N}}_\varphi$. Then there exists a net $(a_k)_{k\in K}$ in $\mathcal{N}_\varphi$ such that:

(i) we have for every $k \in K$, that $\|a_k\| \leqslant \|a\|$ and $\|\Lambda(a_k)\| \leqslant \|\Lambda(a)\|$;

(ii) the net $(a_k)_{k\in K}$ converges strictly to a and the net $(\Lambda(a_k))_{k\in K}$ converges strongly* to $\Lambda(a)$.

This follows immediately by multiplying a to the left by an approximate unit of A.

Looking at Proposition 2.5, we have the following generalization to the multiplier algebra.

PROPOSITION 4.12. *Consider $\rho \in \mathcal{F}_\varphi$. Let S be an element in $\mathcal{L}(E) \cap \pi(A)'$ such that $S^*S = T_\rho$ and let v be the unique element in $\mathcal{L}(B, E)$ such that $S\Lambda(a) = \pi(a)v$ for every $a \in \mathcal{N}_\varphi$. Then $S\Lambda(a) = \pi(a)v$ for every $a \in \overline{\mathcal{N}}_\varphi$.*

Proof. For every $e \in A$, we have that ea belongs to $\mathcal{N}_\varphi$ and $\Lambda(ea) = \pi(e)\Lambda(a)$, which implies that

$$\pi(e)S\Lambda(a) = S\pi(e)\Lambda(a) = S\Lambda(ea) = \pi(ea)v = \pi(e)\pi(a)v.$$

The non-degeneracy of π implies that $S\Lambda(a) = \pi(a)v$. ∎

COROLLARY 4.13. *Consider $\rho \in \mathcal{F}_\varphi$. Then $T_\rho^{\frac{1}{2}}\Lambda(a) = \pi(a)v_\rho$ for every $a \in \overline{\mathcal{N}}_\varphi$. Moreover, we have that $\Lambda(b)^*T_\rho\Lambda(a) = \rho(b^*a)$ for every $a, b \in \overline{\mathcal{N}}_\varphi$.*

The last statement of this corollary follows easily from the last statement of Notation 2.6.

We also want to mention the following result.

LEMMA 4.14. *Let $(a_j)_{j\in J}$ be a net in $\overline{\mathcal{N}}_\varphi$ and let $a \in \overline{\mathcal{N}}_\varphi$ such that $(a_j)_{j\in J}$ converges strictly to a and $(\Lambda(a_j))_{j\in J}$ is bounded. Then $(\Lambda(a_j)^*)_{j\in J}$ converges strongly to $\Lambda(a)^*$.*

Proof. Choose $c \in \mathcal{N}_\varphi$, $d \in B$ and $\rho \in \mathcal{G}_\varphi$. We have for every $j \in J$ that $\Lambda(a_j)^*T_\rho\Lambda(c)d = \rho(a_j^*c)d$. Therefore $(\Lambda(a_j)^*T_\rho\Lambda(c)d)_{j\in J}$ converges to $\rho(a^*c)d$, which is equal to $\Lambda(a)^*T_\rho\Lambda(c)d$. Because $(\Lambda(a_j)^*)_{j\in J}$ is also bounded, this implies that $(\Lambda(a_j)^*)_{j\in J}$ converges strongly to $\Lambda(a)^*$. ∎

Referring to Lemma 1.12, we get the following proposition.

PROPOSITION 4.15. *Let a be an element in $\overline{\mathcal{N}}_\varphi$. Then $\varphi(a^*a)$ belongs to B if and only if there exists an element $x \in E$ such that $\Lambda(a)b = xb$ for every $b \in B$, which is equivalent to $\Lambda(a)$ belongs to $\mathcal{K}(B, E)$.*

As before, the following remark applies. Let a be an element in $\overline{\mathcal{N}}_\varphi$ such that $\varphi(a^*a)$ belongs to B. Then there exists $x \in E$ such that $\Lambda(a)b = xb$ for every $b \in B$. In this case, we have that $\varphi(a^*a) = \langle x, x\rangle$.

PROPOSITION 4.16. *Let a be an element in $\overline{\mathcal{N}}_\varphi$ such that $\varphi(a^*a)$ belongs to B. Then we have for every $\rho \in \mathcal{F}_\varphi$ that $\rho(a^*a)$ belongs to B and the net $(\rho(a^*a))_{\rho\in\mathcal{G}_\varphi}$ converges to $\varphi(a^*a)$.*

Proof. Take a KSGNS-construction (E, Λ, π) for φ. By the previous proposition, we know that there exists an element $x \in E$ such that $\Lambda(a)b = xb$ for every $b \in B$. Choose $\rho \in \mathcal{F}_\varphi$. Then we have for every $b \in B$ that

$$b^*\langle T_\rho x, x\rangle b = \langle T_\rho xb, xb\rangle = \langle T_\rho\Lambda(a)b, \Lambda(a)b\rangle = b^*\rho(a^*a)b$$

which implies that $\langle T_\rho x, x\rangle = \rho(a^*a)$. Hence, we see immediately that $\rho(a^*a)$ belongs to B. We have also that $(\rho(a^*a))_{\rho\in\mathcal{G}_\varphi}$ converges to $\langle x, x\rangle$ which is equal to $\varphi(a^*a)$. ∎

COROLLARY 4.17. *Let a be an element in $\overline{\mathcal{N}}_\varphi$. Then $\varphi(a^*a)$ belongs to B if and only if for every $\rho \in \mathcal{G}_\varphi$, $\rho(a^*a)$ belongs to B and $(\rho(a^*a))_{\rho\in\mathcal{G}_\varphi}$ is convergent in B.*

5. A FIRST STEP TOWARDS A CONSTRUCTION PROCEDURE FOR C^*-VALUED WEIGHTS

We will consider the following objects in this section. Suppose that A and B are two C^*-algebras and that E is a Hilbert C^*-module over B. Let N_0 be a dense subalgebra of A and Λ_0 a linear mapping from N_0 into $L(B,E)$ such that $\langle \Lambda_0(a)b \mid a \in N_0, b \in N\rangle$ is dense in E.

Furthermore, we assume the existence of a net $(T_i)_{i\in I}$ in $\mathcal{L}(E)^+$ such that:

(1) for every $i \in I$, $\|T_i\| \leqslant 1$;

(2) $(T_i)_{i\in I}$ converges strongly to 1;

(3) for every $i \in I$ there exists a unique strict completely positive mapping ρ_i from A into $M(B)$ such that $b_2^*\rho_i(a_2^*a_1)b_1 = \langle T_i\Lambda_0(a_1)b_1, \Lambda_0(a_2)b_2\rangle$ for every $a_1, a_2 \in N_0$ and $b_1, b_2 \in B$.

It is clear from Proposition 10.3 that $\Lambda_0(a)$ belongs to $\mathcal{L}(B,E)$ for every $a \in N_0$.

LEMMA 5.1. *We have that Λ_0 is closable for the strict topology on A and the strong topology on $L(B,E)$.*

Proof. Choose a net $(a_j)_{j\in J}$ in N_0, $t \in L(B,E)$ such that $(a_j)_{j\in J}$ converges strictly to 0 and $(\Lambda_0(a_j))_{j\in J}$ converges strongly to t. Choose $b, d \in B$ and $c \in N_0$. Fix $i \in I$. We have clearly that $(\langle T_i\Lambda_0(a_j)b, \Lambda_0(c)d\rangle)_{j\in J}$ converges to $\langle T_it(b), \Lambda_0(c)d\rangle$. Because $\langle T_i\Lambda_0(a_j)b, \Lambda_0(c)d\rangle = d^*\rho_i(c^*a_j)b$ for every $j \in J$, we also have that the net $(\langle T_i\Lambda_0(a_j)b, \Lambda_0(c)d\rangle)_{j\in J}$ converges to 0. This implies that $\langle T_it(b), \Lambda_0(c)d\rangle = 0$. The fact that $(T_i)_{i\in I}$ converges strongly to 1 implies that $\langle t(b), \Lambda_0(c)d\rangle = 0$. Consequently, $t = 0$. ∎

NOTATION 5.2. We define Λ to be the closure of Λ_0 for the norm topology on A and the strong topology on $L(B,E)$. We also define N to be the domain of Λ.

LEMMA 5.3. *Consider $a_1, a_2 \in N$, $b_1, b_2 \in B$ and $i \in I$. Then*

$$\langle T_i\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho_i(a_2^*a_1)b_1.$$

Proof. STEP 1. Choose $c \in N_0$. There exists a net $(d_j)_{j\in J}$ in N_0 such that $(d_j)_{j\in J}$ converges to a_2 and $(\Lambda_0(d_j))_{j\in J}$ converges strongly to $\Lambda(a_2)$. We have clearly that the net $(\langle T_i\Lambda_0(c)b_1, \Lambda_0(d_j)b_2\rangle)_{j\in J}$ converges to $\langle T_i\Lambda_0(c)b_1, \Lambda(a_2)b_2\rangle$. We have also that $\langle T_i\Lambda_0(c)b_1, \Lambda_0(d_j)b_2\rangle = b_2^*\rho_i(d_j^*c)b_1$ for every $j \in J$. This implies that the net $(\langle T_i\Lambda_0(c)b_1, \Lambda_0(d_j)b_2\rangle)_{j\in J}$ converges to $b_2^*\rho_i(a_2^*c)b_1$. Combining these two results, we get that $\langle T_i\Lambda_0(c)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho_i(a_2^*c)b_1$.

STEP 2. There exists a net $(c_k)_{k\in K}$ in N_0 such that $(c_k)_{k\in K}$ converges to a_1 and $(\Lambda_0(c_k))_{k\in K}$ converges strongly to $\Lambda(a_1)$. We have clearly that the net $(\langle T_i\Lambda_0(c_k)b_1, \Lambda(a_2)b_2\rangle)_{k\in K}$ converges to $\langle T_i\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle$. By Step 1, we have also that $\langle T_i\Lambda_0(c_k)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho_i(a_2^*c_k)b_1$ for every $k \in K$. This implies that the net $(\langle T_i\Lambda_0(c_k)b_1, \Lambda(a_2)b_2\rangle)_{j\in J}$ converges to $b_2^*\rho_i(a_2^*a_1)b_1$. Combining these two results, we get that $\langle T_i\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = b_2^*\rho_i(a_2^*a_1)b_1$. ∎

REMARK 5.4. Using Proposition 10.3 once again, we arrive at the following conclusion: the set N is a dense subspace of A and Λ is a linear mapping from N into $\mathcal{L}(B,E)$ which is closed for the norm topology on A and the strong topology on $L(B,E)$. Furthermore, N_0 is a core for Λ in the norm-strong sense.

Consider $a_1, a_2 \in N$. Then $\rho_i(a_2^* a_1) = \Lambda(a_2)^* T_i \Lambda(a_1)$ for every $i \in I$. This implies that $(\rho_i(a_2^* a_1))_{i\in I}$ converges strictly to $\Lambda(a_2)^*\Lambda(a_1)$ for every $a_1, a_2 \in N$.

LEMMA 5.5. *Consider $x \in N_0$, $a_1, \ldots, a_n \in N_0$ and $b_1, \ldots, b_2 \in B$. Then*

$$\Big\| \sum_{k=1}^{n} \Lambda_0(x a_k) b_k \Big\| \leqslant \|x\| \, \Big\| \sum_{k=1}^{n} \Lambda_0(a_k) b_k \Big\|.$$

Proof. We have for every $i \in I$ that

$$\Big\langle T_i \Big(\sum_{k=1}^{n} \Lambda_0(x a_k) b_k \Big), \sum_{k=1}^{n} \Lambda_0(x a_k) b_k \Big\rangle = \sum_{k,l=1}^{n} \langle T_i \Lambda_0(x a_k) b_k, \Lambda_0(x a_l) b_l \rangle$$

$$= \sum_{k,l=1}^{n} b_l^* \rho_i((x a_l)^* (x a_k)) b_k \leqslant \|x\|^2 \sum_{k,l=1}^{n} b_l^* \rho_i(a_l^* a_k) b_k$$

$$= \|x\|^2 \sum_{k,l=1}^{n} \langle T_i \Lambda_0(a_k) b_k, \Lambda_0(a_l) b_l \rangle = \|x\|^2 \Big\langle T_i \Big(\sum_{k=1}^{n} \Lambda_0(a_k) b_k \Big), \sum_{k=1}^{n} \Lambda_0(a_k) b_k \Big\rangle,$$

where in inequality, we used the complete positivity of ρ_i. Because $(T_i)_{i\in I}$ converges strongly to 1, we get that

$$\Big\langle \sum_{k=1}^{n} \Lambda_0(x a_k) b_k, \sum_{k=1}^{n} \Lambda_0(x a_k) b_k \Big\rangle \leqslant \|x\|^2 \Big\langle \sum_{k=1}^{n} \Lambda_0(a_k) b_k, \sum_{k=1}^{n} \Lambda_0(a_k) b_k \Big\rangle. \quad \blacksquare$$

This lemma implies for every $x \in N_0$ the existence of a continuous linear operator L_x from E into E such that $L_x(\Lambda_0(a)b) = \Lambda_0(xa)b$ for every $a \in N_0$ and $b \in B$. It is also clear that $\|L_x\| \leqslant \|x\|$ for every $x \in N_0$.

It is not difficult to check that the mapping $N_0 \to \mathcal{B}(E) : x \mapsto L_x$ is a continuous algebra-homomorphism. This justifies the following definition.

NOTATION 5.6. We define π to be the continuous algebra-homomorphism from A into $\mathcal{B}(E)$ such that $\pi(x)\Lambda_0(a) = \Lambda_0(xa)$ for every $x, a \in N_0$.

PROPOSITION 5.7. *The set N is a left ideal in A and $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in A$ and $a \in N$.*

Proof. STEP 1. Choose $x \in N_0$ and $a \in N$. Then there exists a net $(a_j)_{j\in J}$ in N_0 such that $(a_j)_{j\in J}$ converges to a and $(\Lambda_0(a_j))_{j\in J}$ converges strongly to $\Lambda(a)$. It is clear that $(x a_j)_{j\in J}$ converges to xa. We have also for every $j \in J$ that $x a_j$ belongs to N_0 and $\Lambda_0(x a_j) = \pi(x)\Lambda(a_j)$. This implies that $(\Lambda_0(x a_j))_{j\in J}$ converges strongly to $\pi(x)\Lambda(a)$ in $\mathcal{B}(B,E)$. By definition, we find that xa belongs to N and $\Lambda(xa) = \pi(x)\Lambda(a)$.

STEP 2. Choose $x \in A$ and $a \in N$. Then there exists a sequence $(x_k)_{k=1}^{\infty}$ in N_0 such that $(x_k)_{k=1}^{\infty}$ converges to x. This implies immediately that $(x_k a)_{k=1}^{\infty}$ converges to xa. By Step 1, we know for every $k \in \mathbb{N}$ that $x_k a$ belongs to N and $\Lambda(x_k a) = \pi(x_k)\Lambda(a)$. So we get that $(\Lambda(x_k a))_{k=1}^{\infty}$ converges to $\pi(x)\Lambda(a)$. The norm-strong closedness of Λ implies that xa belongs to N and $\Lambda(xa) = \pi(x)\Lambda(a)$. $\blacksquare$

Proposition 5.8. *The mapping π is a $*$-homomorphism from A into $\mathcal{L}(E)$.*

Proof. Choose $x \in A$. Take $a, c \in N$ and $b, d \in B$. We have for every $i \in I$ that

$$\begin{aligned}\langle T_i \pi(x)\Lambda(a)b, \Lambda(c)d\rangle &= \langle T_i \Lambda(xa)b, \Lambda(c)d\rangle = d^* \rho_i(c^* xa)b = d^* \rho_i((x^* c)^* a)b \\ &= \langle T_i \Lambda(a)b, \Lambda(x^* c)d\rangle = \langle T_i \Lambda(a)b, \pi(x^*)\Lambda(c)d\rangle.\end{aligned}$$

From this, we infer that

$$\langle \pi(x)\Lambda(a)b, \Lambda(c)d\rangle = \langle \Lambda(a)b, \pi(x^*)\Lambda(c)d\rangle.$$

This implies that $\langle \pi(x)v, w\rangle = \langle v, \pi(x^*)w\rangle$ for every $v, w \in E$. So we arrive at the conclusion that $\pi(x)$ belongs to $\mathcal{L}(E)$ and $\pi(x)^* = \pi(x^*)$. ∎

Proposition 5.9. *The $*$-homomorphism π is non-degenerate.*

Proof. Choose an approximate unit $(e_j)_{j\in J}$ of A. Then we have for every $j \in J$ that $0 \leqslant \pi(e_j) \leqslant 1$. Take $i \in I$, $a, c \in N$ and $b, d \in B$. We have for every $j \in J$ that

$$\langle \pi(e_j)T_i\Lambda(a)b, \Lambda(c)d\rangle = \langle T_i\Lambda(a)b, \pi(e_j)\Lambda(c)d\rangle = \langle T_i\Lambda(a)b, \Lambda(e_j c)d\rangle = d^* \rho_i(c^* e_j a)d.$$

This implies that $(\langle \pi(e_j)T_i\Lambda(a)b, \Lambda(c)d\rangle)_{j\in J}$ converges to $d^*\rho_i(c^*a)b$, which is equal to $\langle T_i\Lambda(a)b, \Lambda(c)d\rangle$. Because $(\pi(e_j))_{j\in J}$ is bounded, this last result implies that $(\langle \pi(e_j)v, w\rangle)_{j\in J}$ converges to $\langle v, w\rangle$ for every $v, w \in E$. Using Lemma 9.2, we see that $(\pi(e_j))_{j\in J}$ converges strongly to 1. ∎

Proposition 5.10. *The set N is a left ideal in $M(A)$ and $\Lambda(xa) = \pi(x)\Lambda(a)$ for every $x \in M(A)$ and $a \in N$.*

Proof. Choose $x \in M(A)$ and $a \in N$. Take an approximate unit $(e_k)_{k\in K}$ for A. Then $(e_k xa)_{k\in K}$ converges to xa. By Proposition 5.7, we have for every $k \in K$ that $e_k xa$ belongs to N and

$$\Lambda(e_k xa) = \pi(e_k x)\Lambda(a) = \pi(e_k)\pi(x)\Lambda(a).$$

Using the previous proposition, this implies that $(\Lambda(e_k xa))_{k\in K}$ converges strongly to $\pi(x)\Lambda(a)$. The norm-strong closedness of Λ implies that xa belongs to N and $\Lambda(xa) = \pi(x)\Lambda(a)$. ∎

Proposition 5.11. *The mapping Λ is closed for the strict topology on A and the strong topology on $L(B, E)$.*

Proof. Take $a \in A$, $t \in L(B, E)$ such that there exists a net $(a_j)_{j\in J}$ in N_0 such that $(a_j)_{j\in J}$ converges strictly to a and $(\Lambda_0(a_j))_{j\in J}$ converges strongly to t.

Take $e \in N_0$. It is clear that $(ea_j)_{j\in J}$ converges to ea. We have for every $j \in J$ that ea_j belongs to N_0 and $\Lambda_0(ea_j) = \pi(e)\Lambda_0(a_j)$. This implies that $(\Lambda_0(ea_j))_{j\in J}$ converges strongly to $\pi(e)t$. By definition, we see that ea belongs to N and $\Lambda(ea) = \pi(e)t$.

We know that there exists a bounded net $(e_k)_{k\in K}$ in N_0 such that $(e_k)_{k\in K}$ converges strictly to 1. This implies immediately that $(e_k a)_{k\in K}$ converges to a. By the first part of this proof, we know for every $k \in K$ that $e_k a$ belongs to N and $\Lambda(e_k a) = \pi(e_k)\Lambda(a)$. This implies that $(\Lambda(e_k a))_{j\in J}$ converges strongly to t. The norm-strong closedness of Λ implies that a belongs to N and $\Lambda(a) = t$.

From this all, we conclude that Λ is equal to the closure of Λ_0 for the strict topology on A and the strong topology on $L(B, E)$. ∎

REMARK 5.12. We are now in a position to use the results and terminology of Section 2 using the ingredients A, B, E, N, Λ, π. So we use the notations $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and so on.

Consider $i \in I$. In the terminology of Section 2, we have that ρ_i belongs to $\mathcal{F}$ and $T_{\rho_i} = T_i$. Therefore T_i belongs to $\pi(A)'$.

Furthermore, there exists a unique element $v_i \in \mathcal{L}(B, E)$ such that $T_i^{\frac{1}{2}} \Lambda(a) = \pi(a) v_i$ for every $a \in N$. We also know that $\|v_i\|^2 = \|\rho_i\|$ and that $\rho_i(x) = v_i^* \pi(x) v_i$ for every $x \in M(A)$.

PROPOSITION 5.13. *Consider $a_1, a_2 \in N$. We have that $(\rho(a_2^* a_1))_{\rho \in \mathcal{G}}$ converges strictly to $\Lambda(a_2)^* \Lambda(a_1)$.*

Proof. Choose $a \in N$. Take $b \in B$. We have for every $\rho \in \mathcal{G}$ that $\rho(a^* a) \leqslant \Lambda(a)^* \Lambda(a)$. Hence, because $(b^* \rho_i(a^* a) b)_{i \in I}$ converges to $b^* \Lambda(a)^* \Lambda(a) b$, Lemma 2.14 implies that $(b^* \rho(a^* a) b)_{\rho \in \mathcal{G}}$ converges to $b^* \Lambda(a)^* \Lambda(a) b$.

Lemma 9.2 implies that $(\rho(a^* a))_{\rho \in \mathcal{G}}$ converges strictly to $\Lambda(a)^* \Lambda(a)$. The proposition follows by polarisation. ∎

Similar as in Result 3.8 one proves the following corollary.

COROLLARY. *The net $(T_\rho)_{\rho \in \mathcal{G}}$ converges strongly to* 1.

The last lemma is useful for the material in the next sections.

LEMMA 5.15. *Let u be an element in $M(A)$ such that $u N_0 \subseteq N_0$ and such that there exists an element $S \in \mathcal{L}(E)$ such that $S \Lambda_0(a) = \Lambda_0(au)$ for every $a \in N_0$. Then $uN \subseteq N$ and $\Lambda(au) = S \Lambda(a)$ for every $a \in N$.*

Proof. Choose $a \in N$. Then there exists a net $(a_j)_{j \in J}$ such that $(a_j)_{j \in J}$ converges to a and $(\Lambda_0(a_j))_{j \in J}$ converges strongly to $\Lambda(a)$. It is clear that $(a_j u)_{j \in J}$ converges to au. We have for every $j \in J$ that $a_j u$ belongs to N_0 and $\Lambda_0(a_j u) = S \Lambda_0(a_j)$. This implies that $(\Lambda_0(a_j u))_{j \in J}$ converges strongly to $S \Lambda(a)$. By definition, we get that au belongs to N and $\Lambda(au) = S \Lambda(a)$. ∎

6. A CONSTRUCTION PROCEDURE FOR REGULAR C^*-VALUED WEIGHTS

In this section, we will propose a construction procedure for regular C^*-valued weights out of some sort of KSGNS-construction which generalizes a similar construction procedure in Section 2.2 of [10]. We will also prove a result (Proposition 6.23) which plays a major role in the next section.

For the rest of this section, we fix the following ingredients.

Consider two C^*-algebras A and B and a Hilbert-C^*-module E over B. Let N be a dense left ideal of A and Λ a linear map from N into $L(B, E)$ such that:

(1) Λ is norm-strongly closed;

(2) the set $\langle \Lambda(a) b \mid a \in N, b \in B \rangle$ is dense in E.

Furthermore, we assume the existence of a net $(u_i)_{i \in I}$ in $M(A)$ such that we have for every $i \in I$ that:

(1) $N u_i \subseteq N$;

(2) there exists a unique element $S_i \in \mathcal{L}(E)$ such that $S_i \Lambda(a) = \Lambda(a u_i)$ for every $a \in N$;

(3) $\|u_i\| \leqslant 1$ and $\|S_i\| \leqslant 1$;

(4) there exists a unique strict completely positive linear mapping ρ_i from A into $M(B)$ such that $b_2^*\rho_i(a_2^*a_1)b_1 = \langle S_i\Lambda(a_1)b_1, S_i\Lambda(a_2)b_2\rangle$ for every $a_1, a_2 \in N$ and $b_1, b_2 \in B$;

(5) $(u_i)_{i\in I}$ converges strictly to 1;

(6) $(S_i)_{i\in I}$ converges strongly to 1.

Fix $i \in I$ and define $T_i = S_i^*S_i \in \mathcal{L}(E)$; it is clear that $0 \leqslant T_i \leqslant 1$. We also have immediately that $b_2^*\rho_i(a_2^*a_1)b_1 = \langle T_i\Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle$ for every $a_1, a_2 \in N$ and $b_1, b_2 \in B$.

Because $(S_i)_{i\in I}$ converges strongly to 1, we get that $(\langle T_i v, v\rangle)_{i\in I}$ converges to $\langle v, v\rangle$ for every $v \in E$. Therefore, Lemma 9.2 implies that $(T_i)_{i\in I}$ converges strongly to 1.

These properties imply that our ingredients A, B, E, N, Λ satisfy the conditions of the beginning of the previous section, but we do not have to close Λ anymore. So, we can use the results of the previous section. We will give a summary of them:

(1) For every $a \in N$, $\Lambda(a)$ belongs to $\mathcal{L}(B, E)$;

(2) N is a left ideal in $M(A)$;

(3) There exists a unique non-degenerate $*$-homomorphism π from A into $\mathcal{L}(E)$ such that $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in M(A)$ and $a \in N$;

(4) The mapping Λ is closed for the strict topology on A and the strong topology on $L(B, E)$;

(5) It is possible to introduce the objects $\mathcal{H}, \mathcal{F}, \mathcal{G}$ like in Section 2; in this terminology, we have for every $i \in I$ that ρ_i belongs to $\mathcal{F}$ and $T_{\rho_i} = \rho_i$;

(6) We have for every $i \in I$ that T_i belongs to $\pi(A)'$; therefore there exists a unique $v_i \in \mathcal{L}(B, E)$ such that $T_i^{\frac{1}{2}}\Lambda(a) = \pi(a)v_i$. Moreover, $\|v_i\|^2 = \|\rho_i\|$. Furthermore, $\rho_i(x) = v_i^*\pi(x)v_i$ for every $x \in M(A)$;

(7) The net $(T_\rho)_{\rho\in\mathcal{G}}$ converges strongly to 1;

(8) For every $a_1, a_2 \in N$, the net $(\rho_i(a_2^*a_1))_{i\in I}$ converges strictly to $\Lambda(a_2)^*\Lambda(a_1)$;

(9) For every $a_1, a_2 \in N$, the net $(\rho(a_2^*a_1))_{\rho\in\mathcal{G}}$ converges strictly to $\Lambda(a_2)^*\Lambda(a_1)$.

In the rest of this section, we want to construct a regular C^*-valued weight with these ingredients. At the same time, we will prove some extra properties. Essentially, we will define a C^*-valued weight φ on A such that $\mathcal{N}_\varphi = N$ and $\varphi(b^*a) = \Lambda(b)^*\Lambda(a)$ for every $a, b \in N$. There are three problems with this:

(1) Is this mapping well defined? If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements in N such that $\sum_{k=1}^n b_k^*a_k = 0$, do we have that $\sum_{k=1}^n \Lambda(b_k)^*\Lambda(a_k) = 0$?

(2) Is the positive part of N^*N a hereditary cone in A^+?

(3) Is the resulting φ lower semi-continuous?

The first question can be easily answered. If $\sum_{k=1}^n b_k^*a_k = 0$, then

$$\sum_{k=1}^n \Lambda(b_k)^*T_i\Lambda(a_k) = \rho_i\Big(\sum_{k=1}^n b_k^*a_k\Big) = 0$$

for every $i \in I$. Because $(T_i)_{i \in I}$ converges strongly to 1, we get that $\sum_{k=1}^{n} \Lambda(b_k)^* \Lambda(a_k) = 0$.

The other problems are more difficult to deal with. But this will be done in the rest of this section.

We start of with the following lemma.

Lemma 6.1. *Consider $i \in I$. Then S_i belongs to $\pi(A)'$.*

Proof. Choose $x \in A$. Take $a \in N$. By assumption, we have that au_i belongs to N and $S_i \Lambda(a) = \Lambda(au_i)$. This implies that $x(au_i)$ belongs to N and $\Lambda(x(au_i)) = \pi(x)\Lambda(au_i) = \pi(x) S_i \Lambda(a)$. We know also that xa belongs to N. Again, this implies that $(xa)u_i$ belongs to N and $\Lambda((xa)u_i) = S_i \Lambda(xa) = S_i \pi(x) \Lambda(a)$. Comparing these two results, we get that $\pi(x) S_i \Lambda(a) = S_i \pi(x) \Lambda(a)$. From this, we infer that $\pi(x) S_i = S_i \pi(x)$. ∎

This lemma allows us to introduce the following notation (see Proposition 2.5).

Notation 6.2. Consider $i \in I$. Then there exists a unique element $w_i \in \mathcal{L}(B, E)$ such that $S_i \Lambda(a) = \pi(a) w_i$ for every $a \in N$. Moreover, we have that $\|w_i\|^2 = \|\rho_i\|$ and $\rho_i(x) = w_i^* \pi(x) w_i$ for all $x \in M(A)$.

Using this result, we can prove the following one.

Result 6.3. *Consider $i \in I$. Then $Au_i \subseteq N$ and $\Lambda(au_i) = \pi(a) w_i$ for every $a \in A$.*

Proof. Choose $a \in A$. Then there exists a sequence $(a_j)_{j=1}^{\infty}$ in N such that $(a_j)_{j=1}^{\infty}$ converges to a. We get immediately that $(a_j u_i)_{j=1}^{\infty}$ converges to au_i. By the previous notation, we have for every $j \in \mathbb{N}$ that $a_j u_i$ belongs to N and $\Lambda(a_j u_i) = \pi(a_j) w_i$. This implies that $(\Lambda(a_j u_i))_{j=1}^{\infty}$ converges to $\pi(a) w_i$. Because Λ is norm-strongly closed, we get that au_i belongs to N and $\Lambda(au_i) = \pi(a) w_i$. ∎

Remark 6.4. We want to use some Hilbert space theory to prove some equalities. In this respect, we will have to make a transition from Hilbert C^*-modules to Hilbert spaces. This will be done in the following way.

(1) Consider a C^*-algebra C and a Hilbert C^*-module G over C. Let ω be an element in C^*_+.

First, we define the positive sesquilinear mapping $(\,\cdot\,,\cdot\,)$ from $G \times G$ into $\mathbb{C}$ such that $(v, w) = \omega(\langle v, w \rangle)$ for every $v, w \in G$. Then G, $(\,\cdot\,,\cdot\,)$ becomes a semi-inner product space. We define $K = \{ v \in G \mid (v, v) = 0 \}$. Then K is a subspace of G and $\frac{G}{K}$ is in a natural way an inner product space. Then $\overline{G}$ is defined as the completion of $\frac{G}{K}$, so $\overline{G}$ is a Hilbert space. The inner product on $\overline{G}$ will also be denoted by $\langle\,\cdot\,,\cdot\,\rangle$. For every $v \in G$, we define $\overline{v}$ as the equivalence class of v in $\frac{G}{N}$. Then we have the following properties:

(i) the mapping $G \to \overline{G} : v \mapsto \overline{v}$ is linear;
(ii) the set $\{ \overline{v} \mid v \in G \}$ is dense in $\overline{G}$;
(iii) we have for every $v, w \in G$ that $\langle \overline{v}, \overline{w} \rangle = \omega(\langle v, w \rangle)$.

(If $G = C$, we get nothing else but the GNS-space of ω.)

(2) Consider a C^*-algebra C and two Hilbert C^*-modules G_1, G_2 over C. Let ω be an element in C^*_+. We use ω to create two Hilbert spaces in the way described

above. Now let f be a linear mapping from G_1 into G_2 such that there exists a positive number M satisfying $\langle f(v), f(v)\rangle \leqslant M\langle v, v\rangle$ for every $v \in G_1$. (This is the case for elements in $\mathcal{L}(G_1, G_2)$.) Then it is not difficult to see that there exists a unique continuous linear map $\overline{f}$ from $\overline{G}_1$ into $\overline{G}_2$ such that $\overline{f}(\overline{v}) = \overline{f(v)}$ for every $v \in G_1$. Moreover, $\|f\| \leqslant M$.

We will now give a first application of this transition method. The following two results are inspired by Proposition 2.1.15 of [10].

LEMMA 6.5. *Let $\omega \in B^*_+$ and define the Hilbert space $\overline{E}$ with respect to this functional ω as described above. Consider $a \in A$, $b \in B$ such that $(b^*\rho_i(a^*a)b)_{i\in I}$ is eventually bounded. Then there exist a sequence $(a_n)_{n=1}^\infty$ in N and an element $v \in \overline{E}$ such that $(a_n)_{n=1}^\infty$ converges to a and $(\overline{\Lambda(a_n)b})_{n=1}^\infty$ converges to v.*

Proof. Using Result 6.3, we have for every $i \in I$ that au_i belongs to N and $\Lambda(au_i) = \pi(a)w_i$, which implies that

$$\begin{aligned}\|\overline{\Lambda(au_i)b}\|^2 &= \langle \overline{\Lambda(au_i)b}, \overline{\Lambda(au_i)b}\rangle = \omega(\langle \Lambda(au_i)b, \Lambda(au_i)b\rangle)\\ &= \omega(\langle \pi(a)w_ib, \pi(a)w_ib\rangle) = \omega(b^*\rho_i(a^*a)b),\end{aligned}$$

where we used Notation 6.2 in the last equality. Consequently, the net $(\overline{\Lambda(au_i)b})_{i\in I}$ is eventually bounded. Because we also have that the net $(au_i)_{i\in I}$ converges to a, Lemma 9.4 implies the existence of a sequence $(a_n)_{n=1}^\infty$ in the convex hull of $\{au_i \mid i \in I\} \subseteq N$ and $v \in \overline{E}$ such that $(a_n)_{n=1}^\infty$ converges to a and $(\overline{\Lambda(a_n)b})_{n=1}^\infty$ converges to v. ∎

PROPOSITION 6.6. *Consider $a \in M(A)^+$, $b \in B$ and $x \in M(B)^+$ such that there exists an element $i_0 \in I$ such that $b^*\rho_i(a)b \leqslant x$ for every $i \in I$ with $i \geqslant i_0$. Then $b^*\rho(a)b \leqslant x$ for every $\rho \in \mathcal{F}$.*

Proof. Fix $\eta \in \mathcal{F}$. Choose $e \in A^+$ with $\|e\| \leqslant 1$. Take $\omega \in B^*_+$. We have for every $i \in I$ with $i \geqslant i_0$ that $b^*\rho_i(a^{\frac12}ea^{\frac12})b \leqslant b^*\rho_i(a)b \leqslant x$. Therefore, the previous lemma implies the existence of a sequence $(c_n)_{n=1}^\infty$ in N and an element $v \in \overline{E}$ such that

$$(c_n)_{n=1}^\infty \quad \text{converges to } e^{\frac12}a^{\frac12} \qquad \text{and} \qquad (\overline{\Lambda(c_n)b})_{n=1}^\infty \quad \text{converges to } v. \tag{6.1}$$

We have for every $\rho \in \mathcal{F}$ and $n \in \mathbb{N}$ that

$$\langle \overline{T_\rho}\,\overline{\Lambda(c_n)b}, \overline{\Lambda(c_n)b}\rangle = \langle \overline{T_\rho\Lambda(c_n)b}, \overline{\Lambda(c_n)b}\rangle = \omega(\langle T_\rho\Lambda(c_n)b, \Lambda(c_n)b\rangle) = \omega(b^*\rho(c_n^*c_n)b).$$

From this equation and (6.1), we easily infer that

$$\langle \overline{T_\rho}v, v\rangle = \omega(b^*\rho(a^{\frac12}ea^{\frac12})b) \tag{6.2}$$

for every $\rho \in \mathcal{F}$.

In particular, we have for every $i \in I$ with $i \geqslant i_0$ that

$$\langle \overline{T_i}v, v\rangle = \omega(b^*\rho_i(a^{\frac12}ea^{\frac12})b) \leqslant \omega(x).$$

Because $(\overline{T_i})_{i\in I}$ converges strongly to 1, the previous inequality implies that $\|v\|^2 \leqslant \omega(x)$. Hence, using (6.2) with ρ equal to η, we get that

$$\omega(b^*\eta(a^{\frac12}ea^{\frac12})b) = \langle \overline{T_\eta}v, v\rangle \leqslant \|v\|^2 \leqslant \omega(x).$$

Because ω was chosen arbitrarily, we conclude that $b^*\eta(a^{\frac{1}{2}}ea^{\frac{1}{2}})b \leqslant x$. Next, we take an approximate unit $(e_k)_{k\in K}$ in A. By the previous part, we know that $b^*\eta(a^{\frac{1}{2}}e_k a^{\frac{1}{2}})b \leqslant x$ for every $k \in K$. The convergence of $(b^*\eta(a^{\frac{1}{2}}e_k a^{\frac{1}{2}})b)_{k\in K}$ to $b^*\eta(a)b$ implies that $b^*\eta(a)b \leqslant x$. ∎

PROPOSITION 6.7. *Consider $a \in M(A)^+$ and $b \in B$ such that $(b^*\rho_i(a)b)_{i\in I}$ converges to an element $x \in B^+$. Then the net $(b^*\rho(a)b)_{\rho\in\mathcal{G}}$ converges also to x.*

Proof. First, we prove that $b^*\rho(a^*a)b \leqslant x$ for every $\rho \in \mathcal{F}$. For this, fix $\eta \in \mathcal{F}$. Choose $\varepsilon > 0$. Then there exists an element $i_0 \in I$ such that $\|b^*\rho_i(a^*a)b - x\| \leqslant \varepsilon$ for every $i \in I$ with $i \geqslant i_0$. This implies that $b^*\rho_i(a^*a)b \leqslant x + \varepsilon\mathbb{1}$ for every $i \in I$ with $i \geqslant i_0$. Therefore, the previous proposition guarantees that $b^*\eta(a^*a)b \leqslant x + \varepsilon\mathbb{1}$. Because ε was chosen arbitrarily, this implies that $b^*\eta(a^*a)b \leqslant x$. Now, because $(b^*\rho_i(a)b)_{i\in I}$ converges to x, Lemma 2.14 implies that $(b^*\rho(a)b)_{\rho\in\mathcal{G}}$ converges to x. ∎

In the next part we will prove a major result (Proposition 6.23) of this paper which allows us to solve both problems mentioned in the beginning of this section. At the same time, this result will be very useful in the following section. We will split the proof of Proposition 6.23 in several parts.

Fix a dense right ideal D of B. We define the set

$$\mathcal{P} = \{a \in M(A)^+ \mid \text{for every } d \in D, \text{ the net } (d^*\rho(a)d)_{\rho\in\mathcal{G}} \text{ is convergent in } B\}.$$

By Lemma 4.1, we know that $\mathcal{P}$ is a hereditary cone in $M(A)^+$.

We will denote $\mathcal{N} = \{a \in M(A) \mid a^*a \in \mathcal{P}\}$ and $\mathcal{M} = \mathcal{N}^*\mathcal{N} = \operatorname{span} P$. As usual, $\mathcal{N}$ is a left ideal in $M(A)$, $\mathcal{M}$ is a sub-$*$-algebra of $M(A)$ and $\mathcal{M}^+ = \mathcal{P}$.

The following lemma follows from polarisation.

LEMMA 6.8. *Consider $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$. Then we have that the net $(b_2^*\rho(a_2^*a_1)b_1)_{\rho\in\mathcal{G}}$ is convergent in B.*

We are now going to construct a Hilbert C^*-module over B in a similar way as we did for the KSGNS-construction for C^*-valued weights.

Define the complex vector space $F = \mathcal{N} \odot D$. Moreover,

(1) we turn F into a right B-module such that $(a \otimes b)c = a \otimes (bc)$ for every $a \in \mathcal{N}$, $b \in D$ and $c \in B$;

(2) we turn F into a semi-inner product module over B such that $(b_2^*\rho(a_2^*a_1) b_1)_{\rho\in\mathcal{G}}$ converges to $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle$ for every $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$.

As before, we define $L = \{x \in F \mid \langle x, x\rangle = 0\}$. Then L is a submodule of F and $\frac{F}{L}$ turns into an inner product module over B. We define $M(A) \dot{\otimes} D$ to be the completion of $\frac{F}{L}$, so $M(A) \dot{\otimes} D$ is a Hilbert C^*-module over B.

For every $a \in \mathcal{N}$ and every $b \in D$, we define $a \dot{\otimes} b$ to be the equivalence class of $a \otimes b$ in $\frac{F}{L}$. Then we have the following properties:

(i) the function $\mathcal{N} \times D \to M(A) \dot{\otimes} D : (a, b) \mapsto a \dot{\otimes} b$ is bilinear;

(ii) the set $\langle a \dot{\otimes} b \mid a \in \mathcal{N}, b \in D\rangle$ is dense in $M(A) \dot{\otimes} D$;

(iii) for every $a \in \mathcal{N}$, $b \in D$ and $c \in B$, we have that $(a \dot{\otimes} b)c = a \dot{\otimes} (bc)$;

(iv) for every $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$, we have that the net $(b_2^*\rho(a_2^*a_1) b_1)_{\rho\in\mathcal{G}}$ converges to $\langle a_1 \dot{\otimes} b_1, a_2 \dot{\otimes} b_2\rangle$.

From the fourth property, we have immediately that

$$\sum_{i,j=1}^{n} b_j^* \rho(a_j^* a_i) b_i \leqslant \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \tag{6.3}$$

for every $\rho \in \mathcal{G}$, $a_1, \ldots, a_n \in \mathcal{N}$ and $b_1, \ldots, b_n \in B$. Of course, this inequality remains true for $\rho \in \mathcal{F}$. Also, the fourth property implies that $\langle (xa_1) \dot{\otimes} b_1, a_2 \dot{\otimes} b_2 \rangle = \langle a_1 \dot{\otimes} b_1, (x^* a_2) \dot{\otimes} b_2 \rangle$ for every $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$. We will use this fact in the following lemma.

Lemma 6.9. *Consider $x \in M(A)$, $a_1, \ldots, a_n \in \mathcal{N}$ and $b_1, \ldots, b_n \in D$. Then*

$$\Big\| \sum_{i=1}^{n} (xa_i) \dot{\otimes} b_i \Big\| \leqslant \|x\| \, \Big\| \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\|.$$

Proof. By the remark before the lemma, we get immediately that

$$\Big\langle \sum_{i=1}^{n} (za_i) \dot{\otimes} b_i, \sum_{i=1}^{n} (za_i) \dot{\otimes} b_i \Big\rangle = \Big\langle \sum_{i=1}^{n} (z^* z a_i) \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle$$

for every $z \in M(A)$. We know that there exists an element $y \in M(A)$ such that $\|x\|^2 1\!\!1 - x^* x = y^* y$. So we see that

$$\begin{aligned}
\|x\|^2 \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle &- \Big\langle \sum_{i=1}^{n} (xa_i) \dot{\otimes} b_i, \sum_{i=1}^{n} (xa_i) \dot{\otimes} b_i \Big\rangle \\
&= \|x\|^2 \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle - \Big\langle \sum_{i=1}^{n} (x^* x a_i) \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \\
&= \Big\langle \sum_{i=1}^{n} (y^* y a_i) \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle = \Big\langle \sum_{i=1}^{n} (ya_i) \dot{\otimes} b_i, \sum_{i=1}^{n} (ya_i) \dot{\otimes} b_i \Big\rangle \geqslant 0.
\end{aligned}$$

This implies that

$$\|x\|^2 \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \geqslant \Big\langle \sum_{i=1}^{n} (xa_i) \dot{\otimes} b_i, \sum_{i=1}^{n} (xa_i) \dot{\otimes} b_i \Big\rangle. \quad \blacksquare$$

Choose $x \in M(A)$. Then there exists a unique continuous linear mapping L_x from $M(A) \dot{\otimes} D$ into $M(A) \dot{\otimes} D$ such that $L_x(a \dot{\otimes} b) = (xa) \dot{\otimes} b$ for every $a \in M(A)$ and $b \in D$.

Using the remark before the previous lemma, it follows for an element $x \in M(A)$ that $\langle L_x v, w \rangle = \langle v, L_{x^*} w \rangle$ for every $v, w \in M(A) \dot{\otimes} D$. This implies that L_x belongs to $\mathcal{L}(M(A) \dot{\otimes} D)$ and $L_x^* = L_{x^*}$. Therefore, the following notation is justified.

Notation 6.10. We define the mapping θ from A into $\mathcal{L}(M(A) \dot{\otimes} D)$ such that $\theta(x)(a \dot{\otimes} b) = (xa) \dot{\otimes} b$ for $x \in A$, $a \in \mathcal{N}$ and $b \in D$. Then θ is a $*$-homomorphism.

Lemma 6.11. *The $*$-homomorphism θ is non-degenerate and $\theta(x)(a \dot{\otimes} b) = (xa) \dot{\otimes} b$ for every $a \in \mathcal{N}$, $b \in D$ and $x \in M(A)$.*

Proof. Choose an approximate unit $(e_k)_{k\in K}$ for A. Take $a_1,\dots,a_n \in \mathcal{N}$, $b_1,\dots,b_n \in D$. Choose $\varepsilon > 0$. By definition of the inner product $\langle\,\cdot\,,\cdot\,\rangle$, there exists an element $\rho \in \mathcal{G}$ such that

$$\Big\| \sum_{i,j=1}^{n} b_j^* \rho(a_j^* a_i) b_i - \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \Big\| \leqslant \frac{\varepsilon}{2}.$$

Because ρ is strictly continuous on bounded sets, we also have the existence of an element $k_0 \in K$ such that

$$\Big\| \sum_{i,j=1}^{n} b_j^* \rho(a_j^* e_k a_i) b_i - \sum_{i,j=1}^{n} b_j^* \rho(a_j^* a_i) b_i \Big\| \leqslant \frac{\varepsilon}{2}$$

for every $k \in K$ with $k \geqslant k_0$. Choose $l \in K$ with $l \geqslant k_0$. Combining the previous two results, we get that

$$\Big\| \sum_{i,j=1}^{n} b_j^* \rho(a_j^* e_l a_i) b_i - \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \Big\| \leqslant \varepsilon.$$

Inequality (6.3) implies that

$$\sum_{i,j=1}^{n} b_j^* \rho(a_j^* e_l a_i) b_i \leqslant \Big\langle \sum_{i=1}^{n} \big(e_l^{\frac12} a_i\big) \dot{\otimes} b_i, \sum_{i=1}^{n} \big(e_l^{\frac12} a_i\big) \dot{\otimes} b_i \Big\rangle.$$

Therefore, we get that

$$\begin{aligned} \sum_{i,j=1}^{n} b_j^* \rho(a_j^* e_l a_i) b_i &\leqslant \Big\langle \theta\big(e_l^{\frac12}\big)\Big(\sum_{i=1}^{n} a_i \dot{\otimes} b_i\Big), \theta\big(e_l^{\frac12}\big)\Big(\sum_{i=1}^{n} a_i \dot{\otimes} b_i\Big) \Big\rangle \\ &= \Big\langle \theta(e_l)\Big(\sum_{i=1}^{n} a_i \dot{\otimes} b_i\Big), \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \leqslant \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle. \end{aligned}$$

From this all, we infer that

$$\begin{aligned} &\Big\| \Big\langle \theta(e_l)\Big(\sum_{i=1}^{n} a_i \dot{\otimes} b_i\Big), \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle - \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \Big\| \\ &\qquad \leqslant \Big\| \sum_{i,j=1}^{n} b_j^* \rho(a_j^* e_l a_i) b_i - \Big\langle \sum_{i=1}^{n} a_i \dot{\otimes} b_i, \sum_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle \Big\| \leqslant \varepsilon. \end{aligned}$$

Hence, we see that the net $\Big(\Big\langle \theta(e_k)\Big(\sum\limits_{i=1}^{n} a_i \dot{\otimes} b_i\Big), \sum\limits_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle\Big)_{k\in K}$ converges to $\Big\langle \sum\limits_{i=1}^{n} a_i \dot{\otimes} b_i, \sum\limits_{i=1}^{n} a_i \dot{\otimes} b_i \Big\rangle$. Because $(\theta(e_k))_{k\in K}$ is bounded, this implies that $(\langle \theta(e_k)v, v\rangle)_{k\in K}$ converges to $\langle v, v\rangle$ for every $v \in M(A) \dot{\otimes} D$. Using Lemma 9.2 once more, we get that $(\theta(e_k))_{k\in K}$ converges strongly to 1. So θ is non-degenerate.

It is easy to check that the mapping $M(A) \to \mathcal{L}(M(A) \dot{\otimes} D) : x \mapsto L_x$ is a $*$-homomorphism extending θ. By uniqueness of the extension, we must have that $\theta(x) = L_x$ for every $x \in M(A)$. ∎

From the beginning of this section, we know that $(\rho(a_2^* a_1))_{\rho\in\mathcal{G}}$ converges strictly to $\Lambda(a_2)^*\Lambda(a_1)$ for every $a_1, a_2 \in N$. This immediately implies that N is a subset of $\mathcal{N}$ and $\langle \Lambda(a_1)b_1, \Lambda(a_2)b_2\rangle = \langle a_1 \dot{\otimes} b_1, a_2 \dot{\otimes} b_2\rangle$ for every $a_1, a_2 \in N$ and $b_1, b_2 \in D$. From this, we get the existence of a unique isometry U from E into $M(A) \dot{\otimes} D$ such that $U(\Lambda(a)b) = a \dot{\otimes} b$ for every $a \in N$ and $d \in D$. Then U is B-linear and $\langle Uv, Uw\rangle = \langle v, w\rangle$ for every $v, w \in E$. Eventually, we will prove that U is also surjective and this will be the main result of this section.

We want to prove the equality in Lemma 6.17 (shortly after we have proven it, it will become clear why we need it). In order to do so, we make a transition from Hilbert C^*-module theory to Hilbert space theory. For the next part, we will fix a positive linear functional ω on B and introduce Hilbert spaces with respect to ω as described in Remark 6.4. The same notation will be used.

LEMMA 6.12. *Consider $a_1, \ldots, a_m, c_1, \ldots, c_n \in \mathcal{N}$ and $b_1, \ldots, b_m, d_1, \ldots, d_n \in D$. Let ρ be an element in $\mathcal{F}$. Then*

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}\omega(d_k^*\rho(c_k^* a_j)b_j)\Big| \leqslant \Big\|\sum_{j=1}^{m}\overline{a_j \dot{\otimes} b_j}\Big\|\,\Big\|\sum_{k=1}^{n}\overline{c_k \dot{\otimes} d_k}\Big\|.$$

Proof. It is possible to define a sesquilinear mapping s from $M(A) \odot B$ into $\mathbb{C}$ such that $s(a\otimes b, c\otimes d) = \omega(d^*\rho(c^*a)b)$ for every $a, c \in M(A)$ and $b, d \in B$. It is easy to check that s is positive. Therefore, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}\omega(d_k^*\rho(c_k^* a_j)b_j)\Big|^2 &= \Big|s\Big(\sum_{j=1}^{m} a_j\otimes b_j, \sum_{k=1}^{n} c_k\otimes d_k\Big)\Big|^2\\
&\leqslant s\Big(\sum_{j=1}^{m} a_j\otimes b_j, \sum_{j=1}^{m} a_j\otimes b_j\Big) s\Big(\sum_{k=1}^{n} c_k\otimes d_k, \sum_{k=1}^{n} c_k\otimes d_k\Big)\\
&= \sum_{j,k=1}^{m}\omega(b_k^*\rho(a_k^* a_j)b_j)\sum_{j,k=1}^{n}\omega(d_k^*\rho(c_k^* c_j)d_j) \qquad (6.4)\\
&\leqslant \omega\Big(\Big\langle\sum_{j=1}^{m} a_j\dot{\otimes} b_j, \sum_{j=1}^{m} a_j\dot{\otimes} b_j\Big\rangle\Big)\omega\Big(\Big\langle\sum_{k=1}^{n} c_k\dot{\otimes} d_k, \sum_{k=1}^{n} c_k\dot{\otimes} d_k\Big\rangle\Big) \text{ by (6.3)}\\
&= \Big\|\sum_{j=1}^{m}\overline{a_j\dot{\otimes} b_j}\Big\|^2\Big\|\sum_{k=1}^{n}\overline{c_k\dot{\otimes} d_k}\Big\|^2. \quad ∎
\end{aligned}$$

Consider $\rho \in \mathcal{F}$. By the previous lemma, there exists a unique continuous positive sesquilinear form s_ρ on $\overline{M(A)\dot{\otimes} D}$ such that $s_\rho(\overline{a\dot{\otimes} b}, \overline{c\dot{\otimes} d}) = \omega(d^*\rho(c^*a)b)$ for every $a, b \in \mathcal{N}$ and $b, d \in D$. The previous lemma also implies that $\|s_\rho\| \leqslant 1$. These remarks justify the following notation.

NOTATION 6.13. Consider $\rho \in \mathcal{F}$. Then there exists a unique element D_ρ in $\mathcal{B}(\overline{M(A)\dot{\otimes} D})$ such that $\langle D_\rho\overline{a\dot{\otimes} b}, \overline{c\dot{\otimes} d}\rangle = \omega(d^*\rho(c^*a)b)$ for every $a, c \in \mathcal{N}$ and $b, d \in D$. We also have that $0 \leqslant D_\rho \leqslant 1$.

We have for every $a, c \in \mathcal{N}$ and $b, d \in D$ that the net $(\langle D_\rho \overline{a \dot{\otimes} b}, \overline{c \dot{\otimes} d}\rangle)_{\rho \in \mathcal{G}}$ converges to $\omega(\langle a \dot{\otimes} d, c \dot{\otimes} d\rangle)$ and this last expression equals $\langle \overline{a \dot{\otimes} b}, \overline{c \dot{\otimes} d}\rangle$. Because $(D_\rho)_{\rho \in \mathcal{G}}$ is bounded, this implies that $(\langle D_\rho v, v\rangle)_{\rho \in \mathcal{G}}$ converges to $\langle v, v\rangle$ for every $v \in \overline{A \dot{\otimes} B}$. Referring to Lemma 9.2, we get that $(D_\rho)_{\rho \in \mathcal{G}}$ converges strongly to 1.

LEMMA 6.14. *Consider a bounded net $(a_k)_{k \in K}$ in $\mathcal{N}$, $a \in \mathcal{N}$, $b \in D$ and $v \in \overline{M(A) \dot{\otimes} D}$ such that $(a_k)_{k \in K}$ converges strictly to a and $(\overline{a_k \dot{\otimes} b})_{k \in K}$ converges to v. Then $\overline{a \dot{\otimes} b} = v$.*

Proof. Choose $\rho \in \mathcal{F}$. We have for every $k \in K$ that

$$\begin{aligned}\big\| D_\rho^{\frac{1}{2}} \overline{a_k \dot{\otimes} b} - D_\rho^{\frac{1}{2}} \overline{a \dot{\otimes} b} \big\|^2 &= \big\| D_\rho^{\frac{1}{2}} \overline{(a_k - a) \dot{\otimes} b} \big\|^2 = \big\langle D_\rho \overline{(a_k - a) \dot{\otimes} b}, \overline{(a_k - a) \dot{\otimes} b} \big\rangle \\ &= \omega(b^* \rho((a_k - a)^*(a_k - a))b).\end{aligned}$$

This implies that $\big(D_\rho^{\frac{1}{2}} \overline{a_k \dot{\otimes} b}\big)_{k \in K}$ converges to $D_\rho^{\frac{1}{2}} \overline{a \dot{\otimes} b}$. It is also clear that $(D_\rho^{\frac{1}{2}} \overline{a_k \dot{\otimes} b})_{k \in K}$ converges to $D_\rho^{\frac{1}{2}} v$. Therefore, $D_\rho^{\frac{1}{2}} \overline{a \dot{\otimes} b} = D_\rho^{\frac{1}{2}} v$. Because $(D_\rho^{\frac{1}{2}})_{\rho \in \mathcal{G}}$ converges strongly to 1, we get that $\overline{a \dot{\otimes} b} = v$. ∎

PROPOSITION 6.15. *$\overline{U}$ is a unitary transformation from $\overline{E}$ to $\overline{M(A) \dot{\otimes} D}$.*

Proof. Because $\langle Ux, Ux\rangle = \langle x, x\rangle$ for every $x \in E$, we get that $\overline{U}$ is isometric. We turn to the surjectivity of $\overline{U}$.

STEP 1. Choose $a \in \mathcal{N} \cap A$ and $b \in D$. Inequality (6.3) implies for every $i \in I$ that $b^* \rho_i(a^* a) b \leqslant \langle a \dot{\otimes} b, a \dot{\otimes} b\rangle$. Therefore, Lemma 6.5 implies the existence of a sequence $(a_n)_{n=1}^\infty$ in N and $v \in \overline{E}$ such that $(a_n)_{n=1}^\infty$ converges to a and $(\overline{\Lambda(a_n) b})_{n \in N}$ converges to v. We have for every $n \in \mathbb{N}$ that

$$\overline{U}\, \overline{\Lambda(a_n) b} = \overline{U \Lambda(a_n) b} = \overline{a_n \dot{\otimes} b},$$

which implies that $(\overline{a_n \dot{\otimes} b})_{n=1}^\infty$ converges to $\overline{U} v$.

Using the previous lemma, we see that $\overline{a \dot{\otimes} b} = \overline{U} v$.

STEP 2. Next, we choose an element $a \in \mathcal{N}$. Take an approximate unit $(e_k)_{k \in K}$ in A. We have for every $k \in K$ that $e_k a$ belongs to $\mathcal{N} \cap A$, so $\overline{e_k a \dot{\otimes} b}$ belongs to $\operatorname{Ran} \overline{U}$ by the first part of the proof. We have for every $k \in K$ that $e_k a \dot{\otimes} b = \theta(e_k)(a \dot{\otimes} b)$. Therefore, the non-degeneracy of θ implies that $(e_k a \dot{\otimes} b)_{k \in K}$ converges to $a \dot{\otimes} b$. So we get that $(\overline{e_k a \dot{\otimes} b})_{k \in K}$ converges to $\overline{a \dot{\otimes} b}$. Because $\operatorname{Ran} \overline{U}$ is closed in $\overline{M(A) \dot{\otimes} D}$, we conclude that $\overline{a \dot{\otimes} b}$ belongs to $\operatorname{Ran} \overline{U}$.

Because $\operatorname{Ran} \overline{U}$ is closed in $\overline{M(A) \dot{\otimes} D}$ and the set $\langle \overline{a \dot{\otimes} b} \mid a \in \mathcal{N}, b \in D\rangle$ is dense in $\overline{M(A) \dot{\otimes} D}$, we get that $\operatorname{Ran} \overline{U} = \overline{M(A) \dot{\otimes} D}$. ∎

LEMMA 6.16. *Consider $a \in \mathcal{N}$, $b \in D$, $c \in M(A)$, $d \in B$ and $\rho \in \mathcal{F}$. Then $\omega(d^* \rho(c^* a) b) = \omega\big(\big\langle a \dot{\otimes} b, U T_\rho^{\frac{1}{2}} \pi(c) v_\rho d \big\rangle\big)$.*

Proof. It is enough to prove the equality for $c \in N$ and $d \in D$. We have for every $a_1, a_2 \in N$, $b_1, b_2 \in B$ that

$$\begin{aligned}\big\langle \overline{T}_\rho \overline{\Lambda(a_1)b_1}, \overline{\Lambda(a_2)b_2} \big\rangle &= \big\langle \overline{T_\rho \Lambda(a_1)b_1}, \overline{\Lambda(a_2)b_2} \big\rangle = \omega\big(\big\langle T_\rho \Lambda(a_1)b_1, \Lambda(a_2)b_2 \big\rangle\big) \\ &= \omega(b_2^* \rho(a_2^* a_1) b_1) = \big\langle D_\rho \overline{a_1 \dot{\otimes} b_1}, \overline{a_2 \dot{\otimes} b_2} \big\rangle \\ &= \big\langle D_\rho \overline{U\Lambda(a_1)b_1}, \overline{U\Lambda(a_2)b_2} \big\rangle = \big\langle D_\rho \overline{U}\, \overline{\Lambda(a_1)b_1}, \overline{U}\, \overline{\Lambda(a_2)b_2} \big\rangle \\ &= \big\langle \overline{U}^* D_\rho \overline{U}\, \overline{\Lambda(a_1)b_1}, \overline{\Lambda(a_2)b_2} \big\rangle.\end{aligned}$$

This implies that $\overline{T_\rho} = \overline{U}^* D_\rho \overline{U}$. Therefore, we have that

$$\begin{aligned}\omega(d^* \rho(c^* a) b) &= \langle D_\rho \overline{a \dot{\otimes} b}, \overline{c \dot{\otimes} d} \rangle = \langle \overline{a \dot{\otimes} b}, D_\rho \overline{c \dot{\otimes} d} \rangle = \langle \overline{a \dot{\otimes} b}, \overline{U}\, \overline{T_\rho U}^* \overline{c \dot{\otimes} d} \rangle \\ &= \langle \overline{a \dot{\otimes} b}, \overline{U}\, \overline{T_\rho} \overline{\Lambda(c) d} \rangle = \langle \overline{a \dot{\otimes} b}, \overline{U T_\rho \Lambda(c) d} \rangle = \omega(\langle a \dot{\otimes} b, U T_\rho \Lambda(c) d \rangle) \\ &= \omega(\langle a \dot{\otimes} b, U T_\rho^{\frac{1}{2}} \pi(c) v_\rho d \rangle)\end{aligned}$$

where we used Notation 2.6 in the last equality. ∎

Remembering that this lemma is true for every $\omega \in B_+^*$, we get the lemma which we wanted to prove.

LEMMA 6.17. *Consider $a \in \mathcal{N}$, $b \in D$, $c \in M(A)$, $d \in B$ and $\rho \in \mathcal{F}$. Then* $d^* \rho(c^* a) b = \langle a \dot{\otimes} b, U T_\rho^{\frac{1}{2}} \pi(c) v_\rho d \rangle$.

Before using this lemma, we need to introduce some extra notation.

LEMMA 6.18. *Consider $\rho \in \mathcal{F}$, $a_1, \ldots, a_n \in \mathcal{N}$ and $b_1, \ldots, b_n \in D$. Then*

$$\Big\langle \sum_{j=1}^n \pi(a_j) v_\rho b_j, \sum_{j=1}^n \pi(a_j) v_\rho b_j \Big\rangle \leqslant \Big\langle \sum_{j=1}^n a_j \dot{\otimes} b_j, \sum_{j=1}^n a_j \dot{\otimes} b_j \Big\rangle.$$

Proof. We have that

$$\Big\langle \sum_{j=1}^n \pi(a_j) v_\rho b_j, \sum_{j=1}^n \pi(a_j) v_\rho b_j \Big\rangle = \sum_{j,k=1}^n b_k^* \rho(a_k^* a_j) b_l \leqslant \Big\langle \sum_{j=1}^n a_j \dot{\otimes} b_j, \sum_{j=1}^n a_j \dot{\otimes} b_j \Big\rangle. \quad ∎$$

This lemma justifies the following notation.

NOTATION 6.19. Consider $\rho \in \mathcal{F}$. We define the continuous linear mapping F_ρ from $M(A) \dot{\otimes} D$ into E such that $F_\rho(a \dot{\otimes} b) = \pi(a) v_\rho b$ for every $a \in \mathcal{N}$ and $b \in D$. We have that F_ρ is B-linear and $\langle F_\rho v, F_\rho v \rangle \leqslant \langle v, v \rangle$ for every $v \in M(A) \dot{\otimes} D$.

DEFINITION 6.20. Consider $\rho \in \mathcal{F}$. We define $R_\rho = U T_\rho^{\frac{1}{2}} F_\rho$. Therefore, R_ρ is a continuous B-linear mapping from $M(A) \dot{\otimes} D$ into $M(A) \dot{\otimes} D$ such that $\|R_\rho\| \leqslant 1$.

Now we will use Lemma 6.17 to prove that R_ρ belongs to $\mathcal{L}(M(A) \dot{\otimes} D)$.

LEMMA 6.21. *Consider $\rho \in \mathcal{F}$. We have for every $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$ that $\langle R_\rho a_1 \dot\otimes b_1, a_2 \dot\otimes b_2 \rangle = b_2^* \rho(a_2^* a_1) b_1$. This implies that R_ρ belongs to $\mathcal{L}(M(A) \dot\otimes D)$ and $0 \leqslant R_\rho \leqslant 1$.*

Proof. By Lemma 6.17, we have for every $a_1, a_2 \in \mathcal{N}$ and $b_1, b_2 \in D$ that

$$\begin{aligned} b_1^* \rho(a_1^* a_2) b_2 &= \langle a_2 \dot\otimes b_2, U T_\rho^{\frac{1}{2}} \pi(a_1) v_\rho b_1 \rangle = \langle a_2 \dot\otimes b_2, U T_\rho^{\frac{1}{2}} F_\rho a_1 \dot\otimes b_1 \rangle \\ &= \langle a_2 \dot\otimes b_2, R_\rho a_1 \dot\otimes b_1 \rangle. \end{aligned}$$

Taking the adjoint of this equation, we obtains that

$$\langle R_\rho a_1 \dot\otimes b_1, a_2 \dot\otimes b_2 \rangle = b_2^* \rho(a_2^* a_1) b_1.$$

Using the complete positivity of ρ, this equality implies that $\langle R_\rho v, v \rangle \geqslant 0$ for every $v \in M(A) \dot\otimes D$. This equality implies that R_ρ belongs to $\mathcal{L}(M(A) \dot\otimes D)$ and $R_\rho \geqslant 0$. ∎

LEMMA 6.22. *We have that $(R_\rho)_{\rho \in \mathcal{G}}$ converges strongly to 1.*

Proof. Choose $a, c \in \mathcal{N}$ and $b, d \in D$. The definition of $\langle \cdot, \cdot \rangle$ and the previous lemma imply that $(\langle R_\rho a \dot\otimes b, c \dot\otimes d \rangle)_{\rho \in \mathcal{G}}$ converges to $\langle a \dot\otimes b, c \dot\otimes d \rangle$. Because $(R_\rho)_{\rho \in \mathcal{G}}$ is bounded, this implies that $(\langle R_\rho v, w \rangle)_{\rho \in \mathcal{G}}$ converges to $\langle v, w \rangle$ for every $v, w \in M(A) \dot\otimes D$. Lemma 9.2 guarantees that $(R_\rho)_{\rho \in \mathcal{G}}$ converges strongly to 1. ∎

At last, we can prove that U is a unitary transformation from E to $M(A) \dot\otimes D$.

PROPOSITION 6.23. *U is a unitary tansformation from E to $M(A) \dot\otimes D$.*

Proof. Take $v \in M(A) \dot\otimes D$. For every $\rho \in \mathcal{G}$, the definition of R_ρ guarantees that $R_\rho v$ belongs to $\operatorname{Ran} U$. The previous lemma implies that $(R_\rho v)_{\rho \in \mathcal{G}}$ converges to v. Because $\operatorname{Ran} U$ is closed in $M(A) \dot\otimes D$, we get that v belongs to $\operatorname{Ran} U$. ∎

LEMMA 6.24. (i) *Consider $a \in M(A)$. Then $U^* \theta(a) U = \pi(a)$.*
(ii) *Consider $\rho \in \mathcal{G}$. Then $U^* R_\rho U = T_\rho$.*

Proof. (i) We have for every $b \in N$ and $c \in D$ that

$$U^* \theta(a) U \Lambda(b) c = U^* \theta(a)(b \dot\otimes c) = U^*(ab \dot\otimes c) = \Lambda(ab) c = \pi(a) \Lambda(b) c.$$

This implies that $U^* \theta(a) U = \pi(a)$.

(ii) We have for every $a_1, a_2 \in N$ and $b_1, b_2 \in D$ that

$$\begin{aligned} \langle U^* R_\rho U \Lambda(a_1) b_1, \Lambda(a_2) b_2 \rangle &= \langle R_\rho U \Lambda(a_1) b_1, U \Lambda(a_2) b_2 \rangle = \langle R_\rho (a_1 \dot\otimes b_1), a_2 \dot\otimes b_2 \rangle \\ &= b_2^* \rho(a_2^* a_1) b_1 \quad \text{by Lemma 6.21} \\ &= \langle T_\rho \Lambda(a_1) b_1, \Lambda(a_2) b_2 \rangle. \end{aligned}$$

This implies that $U^* R_\rho U = T_\rho$. ∎

LEMMA 6.25. *Consider $\rho \in \mathcal{F}$ and $S \in \pi(A)' \cap \mathcal{L}(E)$ such that $S^* S = T_\rho$. Let v be the unique element in $\mathcal{L}(B, E)$ such that $S \Lambda(c) = \pi(c) v$ for every $c \in N$. Then $S U^*(a \dot\otimes b) = \pi(a) v b$ for every $a \in \mathcal{N}$ and $b \in D$.*

Proof. Choose $a \in \mathcal{N}$ and $b \in D$. Using the previous lemma and Lemma 6.21, we find for every $c \in \mathcal{N} \cap A$ that

$$\begin{aligned}\|SU^*(c \dot{\otimes} b)\|^2 &= \|\langle SU^*(c \dot{\otimes} b), SU^*(c \dot{\otimes} b)\rangle\| = \|\langle T_\rho U^*(c \dot{\otimes} b), U^*(c \dot{\otimes} b)\rangle\| \\ &= \|\langle UT_\rho U^*(c \dot{\otimes} b), c \dot{\otimes} b\rangle\| = \|\langle R_\rho(c \dot{\otimes} b), c \dot{\otimes} b\rangle\| \\ &= \|b^* \rho(c^* c) b\| \leqslant \|b\|^2 \|c\|^2 \|\rho\|.\end{aligned}$$

So, we get that the mapping $\mathcal{N} \cap A \to E : c \mapsto SU^*(c \dot{\otimes} b)$ is continuous. We also have for every $c \in N$ that $SU^*(c \dot{\otimes} b) = S\Lambda(c)b = \pi(c)vb$. Because N is dense in A, we get that $SU^*(c \dot{\otimes} b) = \pi(c)vb$ for every $c \in \mathcal{N} \cap A$.

Take an approximate unit $(e_k)_{k \in K}$ for A. We have for every $k \in K$ that $e_k a$ belongs to $\mathcal{N} \cap A$, which implies that $SU^*(e_k a \dot{\otimes} b) = \pi(e_k a)vb$ by the first part of the proof. So, we get for every $k \in K$ that

$$\begin{aligned}\pi(e_k)(SU^*(a \dot{\otimes} b)) &= S\pi(e_k)U^*(a \dot{\otimes} b) = SU^*\theta(e_k)(a \dot{\otimes} b) \\ &= SU^*(e_k a \dot{\otimes} b) = \pi(e_k a)vb = \pi(e_k)(\pi(a)vb).\end{aligned}$$

The non-degeneracy of π implies that $SU^*(a \dot{\otimes} b) = \pi(a)vb$. ∎

As a special case of the previous lemma, we get the following one.

Lemma 6.26. *Consider $\rho \in \mathcal{F}$. Then $T_\rho^{\frac{1}{2}} U^*(a \dot{\otimes} b) = \pi(a)v_\rho b$ for every $a \in \mathcal{N}$, $b \in D$.*

Now we will apply this theory for the first time.

Proposition 6.27. *Consider $a \in A$. Then a belongs to N if and only if the net $(b^* \rho(a^* a) b)_{\rho \in \mathcal{G}}$ is convergent for every $b \in B$.*

Proof. One implication is trivial. We prove the other one. In this respect, suppose that $(b^* \rho(a^* a) b)_{\rho \in \mathcal{G}}$ is convergent for every $b \in B$. We will apply the previous theory (which starts after Proposition 6.7) with $D = B$. By the definition of $\mathcal{N}$, it is clear that a belongs to $\mathcal{N}$. Choose $b \in B$. By Result 6.3 and Lemma 6.25, we have for every $i \in I$ that au_i belongs to N and

$$\Lambda(au_i)b = \pi(a)w_i b = S_i U^*(a \dot{\otimes} b).$$

(The mapping U was introduced after the proof of Lemma 6.11.) This implies that $(\Lambda(au_i)b)_{\rho \in \mathcal{G}}$ converges to $U^*(a \dot{\otimes} b)$. It is also clear that $(au_i)_{i \in i}$ converges to a. Therefore, the norm-strong closedness of Λ implies that a belongs to N (and $\Lambda(a)b = U^*(a \dot{\otimes} b)$ for every $b \in B$). ∎

The remarks at the beginning of this section guarantee that the following mapping φ is well defined:

Definition 6.28. We define the linear mapping φ from N^*N into $M(B)$ such that $\varphi\big(\sum_{j=1}^n b_j^* a_j\big) = \sum_{j=1}^n \Lambda(b_j)^* \Lambda(a_j)$ for every $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in N$.

Proposition 6.29. *We have that φ is a densely defined C^*-valued weight from A to B such that $\mathcal{N}_\varphi = N$ and $\varphi(b^* a) = \Lambda(b)^* \Lambda(a)$ for every $a, b \in N$.*

Proof. Define

$$P = \{a \in A^+ \mid \text{for every } b \in B \text{ the net } (b^* \rho(a^* a) b)_{\rho \in \mathcal{G}} \text{ is convergent in } B\}.$$

Then P is an hereditary cone in A^+ (this follows from Lemma 4.1). From Proposition 6.27, we know that $N = \{a \in A^+ \mid a^* a \in P\}$. Define $M = \operatorname{span} P = N^* N$. Then φ is a linear mapping from M into $M(B)$. We have for every $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in N$, $b_1, \ldots, b_n \in B$ that

$$\sum_{j,k=1}^{n} b_k^* \varphi(a_k^* a_j) b_j = \sum_{j,k=1}^{n} b_k^* \Lambda(a_k)^* \Lambda(a_j) b_j = \sum_{j,k=1}^{n} \langle \Lambda(a_j) b_j, \Lambda(a_k) b_k \rangle$$
$$= \Big\langle \sum_{j=1}^{n} \Lambda(a_j) b_j, \sum_{j=1}^{n} \Lambda(a_j) b_j \Big\rangle \geqslant 0.$$

So, we see that φ is a C^*-valued weight from A to B such that $\mathcal{N}_\varphi = N$. We also have immediately that φ is densely defined. ∎

COROLLARY 6.30. *The triplet (E, Λ, π) is a* KSGNS*-construction for φ. This implies that $\mathcal{H}_\varphi = \mathcal{H}$, $\mathcal{F}_\varphi = \mathcal{F}$ and $\mathcal{G}_\varphi = \mathcal{G}$.*

The last statement of this corollary follows immediately from Notation 3.1.

From the beginning of this section, we know for every $a \in N$ that $(\rho(a^* a))_{\rho \in \mathcal{G}}$ converges strictly to $\Lambda(a)^* \Lambda(a)$ which is equal to $\varphi(a^* a)$ by definition. Referring to Proposition 6.27, the definition of φ and Definition 3.2, we have also:

COROLLARY 6.31. *The C^*-valued weight φ is lower semi-continuous.*

Of course, we also have:

COROLLARY 6.32. *The C^*-valued weight φ is regular and has $(u_i)_{i \in I}$ as truncating net.*

In the next section, we will use even more, the theory introdeced in this section.

7. REGULAR C^*-VALUED WEIGHTS

In this section, we will prove some results about regular C^*-valued weights. In fact, we have already proven the most difficult steps in the previous section. We will work with two C^*-algebras A and B and a regular C^*-valued weight φ from A into $M(B)$. Take a KSGNS-construction (E, Λ, π) for φ. We will also fix a truncating net $(u_i)_{i \in I}$ for φ. First, we recapitulate some notations and properties. Choose $i \in I$.

(1) We define S_i to be the unique element in $\mathcal{L}(E) \cap \pi(A)'$ such that $S_i \Lambda(a) = \Lambda(a u_i)$ for every $a \in \mathcal{N}_\varphi$. Furthermore, we put $T_i = S_i^* S_i \in \mathcal{L}(E)^+ \cap \pi(A)'$.

(2) We have that $\|u_i\| \leqslant 1$, $\|S_i\| \leqslant 1$ and $0 \leqslant T_i \leqslant 1$.

(3) Define the strict completely mapping ρ_i from A into $M(B)$ such that $\rho_i(a_2^* a_1) = \Lambda(a_2)^* T_i \Lambda(a_1)$ for every $a_1, a_2 \in A$. Then, ρ_i belongs to $\mathcal{F}_\varphi$ and $T_{\rho_i} = T_i$.

(4) We define $v_i, w_i \in \mathcal{L}(B,E)$ such that $\pi(a)v_i = T_i^{\frac{1}{2}}\Lambda(a)$ and $\pi(a)w_i = S_i\Lambda(a)$ for every $a \in \mathcal{N}_\varphi$. Then the equalities $\|v_i\|^2 = \|w_i\|^2 = \|\rho_i\|$ hold. We have also that $\rho_i(x) = v_i^*\pi(x)v_i = w_i^*\pi(x)w_i$ for every $x \in M(A)$. This implies that $v_i^*v_i = w_i^*w_i$.

(5) We have also that $(u_i)_{i\in I}$ converges strictly to 1 and that $(S_i)_{i\in I}$, $(T_i)_{i\in I}$ converge strongly to 1.

Lemma 6.3 of the previous section implies that

RESULT 7.1. *Consider $i \in I$. We have that $Au_i \subseteq \mathcal{N}_\varphi$ and $\Lambda(au_i) = \pi(a)w_i$ for every $a \in A$.*

This result can be generalized to

RESULT 7.2. *Consider $i \in I$. We have that $M(A)u_i \subseteq \overline{\mathcal{N}}_\varphi$ and $\Lambda(au_i) = \pi(a)w_i$ for every $a \in M(A)$.*

Proof. Choose $a \in M(A)$. Then there exists a bounded net $(a_k)_{k\in K}$ in A such that $(a_k)_{k\in K}$ converges strictly to a. Then $(a_ku_i)_{k\in K}$ converges strictly to au_i. By the previous result, we have for every $k \in K$ that a_ku_i belongs to $\mathcal{N}_\varphi$ and $\Lambda(a_ku_i) = \pi(a_k)w_i$. This implies that $(\Lambda(a_ku_i))_{k\in K}$ converges strongly to $\pi(a)w_i$. The strictly-strongly closedness of $\overline{\Lambda}$ implies that au_i belongs to $\overline{\mathcal{N}}_\varphi$ and $\Lambda(au_i) = \pi(a)w_i$ (Definition 4.8). ∎

COROLLARY 7.3. *Consider $i \in I$. Then u_i belongs to $\overline{\mathcal{N}}_\varphi$ and $\Lambda(u_i) = w_i$. Therefore, $\varphi(u_i^*u_i) = w_i^*w_i = v_i^*v_i$.*

Proposition 4.12 implies the following one.

PROPOSITION 7.4. *Consider $i \in I$. For every $a \in \overline{\mathcal{N}}_\varphi$, it follows that $T_i^{\frac{1}{2}}\Lambda(a) = \pi(a)v_i$ and $S_i\Lambda(a) = \pi(a)w_i = \Lambda(au_i)$. We have for every $a_1, a_2 \in \overline{\mathcal{N}}_\varphi$ that $\rho_i(a_2^*a_1) = \Lambda(a_2)^*T_i\Lambda(a_1)$.*

The next proposition is a generalization of Proposition 2.1.15 of [10] for C^*-valued weights.

PROPOSITION 7.5. (i) *Consider $a \in M(A)^+$. Then a belongs to $\overline{\mathcal{M}}_\varphi^+$ if and only if the net $(b^*\rho_i(a)b)_{i\in I}$ is convergent in B for every $b \in B$.*

(ii) *Consider $a \in \overline{\mathcal{M}}_\varphi$. Then $(\rho_i(a))_{i\in I}$ converges strictly to $\varphi(a)$.*

Proof. (i) Choose $a \in M(A)^+$ such that $(b^*\rho_i(a)b)_{i\in I}$ is convergent in B for every $b \in B$. Proposition 6.7 implies that $(b^*\rho(a)b)_{\rho\in\mathcal{G}_\varphi}$ is convergent for every $b \in B$. By Definition 4.3, we get that a belongs to $\overline{\mathcal{M}}_\varphi^+$.

(ii) Choose $a_1, a_2 \in \overline{\mathcal{N}}_\varphi$. By the previous result, we have for every $i \in I$ that $\rho_i(a_2^*a_1) = \Lambda(a_2)^*T_i\Lambda(a_1)$. This implies that $(\rho_i(a_2^*a_1))_{i\in I}$ converges strictly to $\Lambda(a_2)^*\Lambda(a_1) = \varphi(a_2^*a_1)$.

These two results imply the proposition. ∎

COROLLARY 7.6. *Consider $a \in A^+$. Then a belongs to $\mathcal{M}_\varphi^+$ if and only if the net $(b^*\rho_i(a)b)_{i\in I}$ is convergent in B for every $b \in B$.*

The proof of the following proposition is similar to the proof of Proposition 4.16.

PROPOSITION 7.7. *Let a be an element in $\overline{\mathcal{N}}_\varphi$ such that $\varphi(a^*a)$ belongs to B. Then we have for every $i \in I$ that $\rho_i(a^*a)$ belongs to B and the net $(\rho_i(a^*a))_{i\in I}$ converges to $\varphi(a^*a)$.*

COROLLARY 7.8. *Let a be an element in $\overline{\mathcal{N}}_\varphi$. Then $\varphi(a^*a)$ belongs to B if and only if for every $i \in I$, $\rho_i(a^*a)$ belongs to B and $(\rho_i(a^*a))_{i\in I}$ is convergent in B.*

In many cases, we do not really work with $\mathcal{N}_\varphi$, but rather with a core of Λ. The following propositions give certain possible cores for Λ. We show that they even behave better than just being a core.

PROPOSITION 7.9. *Consider a dense subspace K of A and define the set $L = \langle au_i \mid a \in K, i \in I\rangle$. Let $a \in \mathcal{N}_\varphi$ such that $\Lambda(a) \neq 0$. Then there exists a net $(a_j)_{j\in J}$ in L such that:*

(i) *$\|a_j\| \leqslant \|a\|$ and $\|\Lambda(a_j)\| \leqslant \|\Lambda(a)\|$ for every $j \in J$;*

(ii) *$(a_j)_{j\in J}$ converges to a and $(\Lambda(a_j))_{j\in J}$ converges strongly* to $\Lambda(a)$.*

Proof. From the beginning of this section, we know for every $i \in I$ that $\pi(a)w_i = \Lambda(au_i) = S_i\Lambda(a)$, which implies that $(\pi(a)w_i)_{i\in I}$ converges strongly to $\Lambda(a)$. Because $\Lambda(a) \neq 0$, this implies the existence of $i_0 \in I$ such that $\pi(a)w_i \neq 0$ for every $i \in I$ with $i \geqslant i_0$. Define $I_0 = \{i \in I \mid i \geqslant i_0\}$.

For the moment, fix $i \in I_0$. There exists a sequence $(c_n)_{n=1}^\infty$ in K such that $(c_n)_{n=1}^\infty$ converges to a. Because $a \neq 0$ and $\pi(a)w_i \neq 0$, there exists $n_0 \in \mathbb{N}$ such that we have for all $n \in \mathbb{N}$ with $n \geqslant n_0$ that $c_n \neq 0$ and $\pi(c_n)w_i \neq 0$. For every $n \in \mathbb{N}$ with $n \geqslant n_0$, we define $\lambda_n = \frac{\|a\|}{\|c_n\|}$ and $\mu_n = \frac{\|\pi(a)w_i\|}{\|\pi(c_n)w_i\|}$. Then we have that the sequences $(\lambda_n)_{n=n_0}^\infty$ and $(\mu_n)_{n=n_0}^\infty$ converge to 1. For every $n \in \mathbb{N}$ with $n \geqslant n_0$, define $d_n = \min(\lambda_n, \mu_n)c_n$, so d_n belongs to K. We get immediately that $(d_n)_{n=n_0}^\infty$ converges to a. Moreover, we have for every $n \in \mathbb{N}$ with $n \geqslant n_0$ that

$$\|d_n\| \leqslant \lambda_n\|c_n\| = \|a\|$$

and

$$\|\pi(d_n)w_i\| \leqslant \mu_n\|\pi(c_n)w_i\| = \|\pi(a)w_i\| = \|S_i\Lambda(a)\| \leqslant \|\Lambda(a)\|.$$

This implies for every $i \in I_0$ and $m \in \mathbb{N}$ the existence of an element $b_{m,i} \in K$ such that $\|b_{m,i}\| \leqslant \|a\|$, $\|\pi(b_{m,i})w_i\| \leqslant \|\Lambda(a)\|$ and $\|b_{m,i} - a\| \leqslant \frac{1}{m}\frac{1}{\|w_i\|+1}$. Define $J = \mathbb{N} \times I_0$ and put on J the product ordering. For every $j = (m, i) \in J$, we put $a_j = b_{m,i}u_i \in L$. Then we get for every $j = (m, i) \in J$ that

$$\|a_j\| \leqslant \|b_{m,i}\|\,\|u_i\| \leqslant \|a\|$$

and

$$\|\Lambda(a_j)\| = \|\Lambda(b_{m,i}u_i)\| = \|\pi(b_{m,i})w_i\| \leqslant \|\Lambda(a)\|.$$

Now, we are going to prove the convergence properties.

STEP 1. Choose $\varepsilon > 0$. Then

(a) there exists $i_1 \in I_0$ such that $\|au_i - a\| \leqslant \frac{\varepsilon}{2}$ for every $i \in I_0$ with $i \geqslant i_1$;

(b) there exists an element $m_1 \in \mathbb{N}$ such that $\frac{1}{m_1} \leqslant \frac{\varepsilon}{2}$.

Therefore, we have for every $j = (m,i) \in J$ with $j \geqslant (m_1, i_1)$ that

$$\begin{aligned}\|a_j - a\| &= \|b_{m,i}u_i - a\| \leqslant \|b_{m,i}u_i - au_i\| + \|au_i - a\| \\ &\leqslant \|b_{m,i} - a\|\,\|u_i\| + \frac{\varepsilon}{2} \leqslant \frac{1}{m} + \frac{\varepsilon}{2} \leqslant \frac{1}{m_1} + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Hence, we see that $(a_j)_{j\in J}$ converges to a.

Step 2. Choose $c \in B$. Take $\varepsilon > 0$. Then

(a) there exists $i_1 \in I_0$ such that $\|S_i\Lambda(a)c - \Lambda(a)c\| \leqslant \frac{\varepsilon}{2}$ for every $i \in I_0$ with $i \geqslant i_1$;

(b) there exists an element $m_1 \in \mathbb{N}$ such that $\frac{1}{m_1} \leqslant \frac{\varepsilon}{2}\frac{1}{\|c\|+1}$.

Therefore, we have for every $j = (m,i) \in J$ with $j \geqslant (m_1, i_1)$ that

$$\begin{aligned}\|\Lambda(a_j)c - \Lambda(a)c\| &= \|\Lambda(b_{m,i}u_i)c - \Lambda(a)c\| \\ &\leqslant \|\Lambda(b_{m,i}u_i)c - \Lambda(au_i)c\| + \|\Lambda(au_i)c - \Lambda(a)c\| \\ &= \|\pi(b_{m,i})w_ic - \pi(a)w_ic\| + \|S_i\Lambda(a)c - \Lambda(a)c\| \\ &\leqslant \|b_{m,i} - a\|\,\|w_i\|\,\|c\| + \frac{\varepsilon}{2} \\ &\leqslant \frac{1}{m}\frac{1}{\|w_i\|+1}\|w_i\|\,\|c\| + \frac{\varepsilon}{2} \leqslant \frac{1}{m_1}\|c\| + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

This implies that $(\Lambda(a_j))_{j\in J}$ converges strongly to $\Lambda(a)$.

Step 3. Because $(\Lambda(a_j))_{j\in J}$ is bounded and $(a_j)_{j\in J}$ converges to a, Lemma 4.14 implies that $(\Lambda(a_j))_{j\in J}$ converges strongly* to $\Lambda(a)$. ∎

In the next proposition, we prove a similar property for elements in $\overline{\mathcal{N}}_\varphi$.

Proposition 7.10. *Consider a dense subspace of K of A and define the set $L = \langle au_i \mid a \in K, i \in I\rangle$. Let $a \in \overline{\mathcal{N}}_\varphi$ such that $\Lambda(a) \neq 0$. Then there exists a net $(a_j)_{j\in J}$ in L such that:*

(i) *$\|a_j\| \leqslant \|a\|$ and $\|\Lambda(a_j)\| \leqslant \|\Lambda(a)\|$ for every $j \in J$;*

(ii) *$(a_j)_{j\in J}$ converges strictly to a and $(\Lambda(a_j))_{j\in J}$ converges strongly* to $\Lambda(a)$.*

Proof. Take an approximate unit $(e_k)_{k\in K}$ for A. We have for every $k \in K$ that e_ka belongs to $\mathcal{N}_\varphi$ and $\Lambda(e_ka) = \pi(e_k)\Lambda(a)$. This implies that $(\Lambda(e_ka))_{k\in K}$ converges strongly to $\Lambda(a)$. Because $\Lambda(a) \neq 0$, we get the existence of an element k_0 in K such that $\Lambda(e_ka) \neq 0$ for every $k \in K$ with $k \geqslant k_0$. Put $K_0 = \{k \in K \mid k \geqslant k_0\}$. Define $J = \{(k,n,F) \mid k \in K_0, n \in \mathbb{N}, F \text{ a finite subset of } B\}$. Put on J an order $\leqslant$ such that we have for every $j_1 = (k_1,n_1,F_1) \in K$ and $j_2 = (k_2,n_2,F_2) \in K$ that

$$j_1 \leqslant j_2 \quad\Longleftrightarrow\quad k_1 \leqslant k_2, n_1 \leqslant n_2 \text{ and } F_1 \subseteq F_2.$$

In this way, J becomes a directed set. Choose $j = (k,n,F) \in J$. By Proposition 7.9, there exists an element $a_j \in L$ such that:

(1) $\|a_j\| \leqslant \|e_ka\|$ and $\|\Lambda(a_j)\| \leqslant \|\Lambda(e_ka)\|$;

(2) $\|a_j - ae_k\| \leqslant \frac{1}{n}$;

(3) $\|\Lambda(a_j)b - \Lambda(ae_k)b\| \leqslant \frac{1}{n}$ for every $b \in F$.

Then we immediately have that

$$\|a_j\| \leqslant \|a\| \quad\text{ and }\quad \|\Lambda(a_j)\| \leqslant \|\pi(e_k)\Lambda(a)\| \leqslant \|\Lambda(a)\|.$$

Now we turn to the convergence properties.

STEP 1. Choose $c \in A$. Take $\varepsilon > 0$. Then

(a) there exists $k_1 \in K_0$ such that $\|e_k ac - ac\| \leqslant \frac{\varepsilon}{2}$ for every $k \in K_0$ with $k \geqslant k_1$;

(b) there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} \leqslant \frac{\varepsilon}{2(\|c\|+1)}$.

Put $j_1 = (k_1, n_1, \emptyset) \in J$. Then we have for every $j = (k, n, F) \in J$ with $j \geqslant j_1$ that $k \geqslant k_1$ and $n \geqslant n_1$, therefore

$$\begin{aligned}\|a_j c - ac\| &\leqslant \|a_j c - e_k ac\| + \|e_k ac - ac\| \leqslant \|a_j - e_k a\| \, \|c\| + \frac{\varepsilon}{2} \\ &\leqslant \frac{1}{n}\|c\| + \frac{\varepsilon}{2} \leqslant \frac{1}{n_1}\|c\| + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Hence we see that $(a_j c)_{j \in J}$ converges to ac. Similarly, one proves that $(c a_j)_{j \in J}$ converges to ca.

STEP 2. Choose $b \in B$. Take $\varepsilon > 0$. Then

(a) there exists $k_1 \in K_0$ such that $\|\pi(e_k)\Lambda(a)b - \Lambda(a)b\| \leqslant \frac{\varepsilon}{2}$ for every $k \in K_0$ with $k \geqslant k_1$;

(b) there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} \leqslant \frac{\varepsilon}{2}$.

Put $j_1 = (k_1, n_1, \{b\}) \in J$. Then we have for every $j = (k, n, F) \in J$ with $j \geqslant j_1$ that $k \geqslant k_1$, $n_1 \geqslant n$ and $b \in F$, therefore

$$\begin{aligned}\|\Lambda(a_j)b - \Lambda(a)b\| &\leqslant \|\Lambda(a_j)b - \Lambda(e_k a)b\| + \|\Lambda(e_k a)b - \Lambda(a)b\| \\ &\leqslant \frac{1}{n} + \|\pi(e_k)\Lambda(a)b - \Lambda(a)b\| \leqslant \frac{1}{n_1} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

From this, we get that $(\Lambda(a_j))_{j \in J}$ converges strongly to $\Lambda(a)$.

STEP 3. Because $(\Lambda(a_j))_{j \in J}$ is bounded and $(a_j)_{j \in J}$ converges strictly to a, Lemma 4.14 implies that $(\Lambda(a_j))_{j \in J}$ converges strongly* to $\Lambda(a)$. ∎

So far, we have extended our weight to $\overline{\varphi}$ in such a way that $\overline{\varphi}$ takes values in $M(B)$. However, in some cases it is interesting to extend φ even further and let it take values in the set of elements affiliated to B. Now, we will make a first step in this direction.

LEMMA 7.11. *Consider $a \in M(A)^+$ and define the set*

$$D = \{ b \in B \mid (b^* \rho(a) b)_{\rho \in \mathcal{G}_\varphi} \text{ is convergent in } B \}.$$

Then D is a right ideal in $M(B)$.

Proof. It is immediately clear that 0 belongs to D, that D is closed under scalar multiplication and that D is closed under multiplication from the right with elements of $M(B)$. The addition requires a little bit of explanation.

Choose $b, c \in D$. As usual, we get for any $\rho, \eta \in \mathcal{G}_\varphi$ with $\rho \leqslant \eta$ that

$$(b + c)^*(\eta - \rho)(a)(b + c) \leqslant 2b^*(\eta - \rho)(a)b + 2c^*(\eta - \rho)(a)c.$$

This implies that $((b + c)^* \rho(a)(b + c))_{\rho \in \mathcal{G}_\varphi}$ is Cauchy and hence convergent in B. Therefore, $b + c$ belongs to D. ∎

DEFINITION 7.12. We define $\widetilde{\mathcal{N}}_\varphi$ to be the set of elements a in $M(A)$ such that the set

$$\{b \in B \mid (b^* \rho(a^* a) b)_{\rho \in \mathcal{G}_\varphi} \text{ is convergent in } B\}$$

is dense in B.

REMARK 7.13. It is easy to see that $\overline{\mathcal{N}}_\varphi$ is a subset of $\widetilde{\mathcal{N}}_\varphi$.

Choose $a \in \overline{\mathcal{N}}_\varphi$. Take $b \in B$. Then we have for every $\rho \in \mathcal{G}_\varphi$ that $\pi(a) v_\rho b = T_\rho^{\frac{1}{2}} \Lambda(a) b$ (Corollary 4.13). This implies that $(\pi(a) v_\rho b)_{\rho \in \mathcal{G}_\varphi}$ converges to $\Lambda(a) b$.

Next we want to define for every $a \in \widetilde{\mathcal{N}}_\varphi$ an operator $\Lambda(a)$ from B into E (which is not necessarily everywhere defined). The previous discussion will justify the following definition.

DEFINITION 7.14. Consider $a \in \widetilde{\mathcal{N}}_\varphi$. Then we define the mapping $\Lambda(a)$ from B into E such that:

(i) the domain of $\Lambda(a)$ is equal to $\{b \in B \mid$ the net $(\pi(a) v_\rho b)_{\rho \in \mathcal{G}_\varphi}$ is convergent in $E\}$;

(ii) $(\pi(a) v_\rho b)_{\rho \in \mathcal{G}_\varphi}$ converges to $\Lambda(a)(b)$ for every $b \in D(\Lambda(a))$.

It is not difficult to check that $\Lambda(a)$ is a B-linear map from B into E.

The following result is rather nice and depends on the results of the previous section.

PROPOSITION 7.15. *Consider $a \in \widetilde{\mathcal{N}}_\varphi$. Then:*

(i) *The domain of $\Lambda(a)$ is equal to $\{b \in B \mid (b^* \rho(a^* a) b)_{\rho \in \mathcal{G}_\varphi}$ is convergent in $B\}$;*

(ii) *Let $\rho \in \mathcal{F}_\varphi$ and $S \in \pi(A)' \cap \mathcal{L}(E)$ such that $S^* S = T_\rho$. Define v to be the unique element in $\mathcal{L}(B, E)$ such that $S \Lambda(c) = \pi(c) v$ for every $c \in \mathcal{N}_\varphi$. Then $S \Lambda(a) b = \pi(a) v b$ for every $b \in D(\Lambda(a))$.*

Proof. We define the set $D = \{b \in B \mid (b^* \rho(a^* a) b)_{\rho \in \mathcal{G}_\varphi}$ is convergent in $B\}$. Because a belongs to $\widetilde{\mathcal{N}}_\varphi$, we have that D is a dense right ideal in B. Therefore, we can use the construction of $M(A) \dot{\otimes} D$ from the previous section (and which begins after Proposition 6.7). We will use the notation of that section. The mapping U is introduced after Lemma 6.11. By definition, a belongs to $\mathcal{N}$.

(a) Choose $b \in D(\Lambda(a))$. By Notation 2.6, we have for every $\rho \in \mathcal{G}_\varphi$ that $b^* \rho(a^* a) b = \langle \pi(a) v_\rho b, \pi(a) v_\rho b \rangle$. This implies immediately that $(b^* \rho(a^* a) b)_{\rho \in \mathcal{G}_\varphi}$ is convergent in B. Therefore, b belongs to D.

(b) Choose $b \in D$. Then we can look at $a \dot{\otimes} b \in M(A) \dot{\otimes} D$. By Lemma 6.26, we know that $\pi(a) v_\rho b = T_\rho^{\frac{1}{2}} U^* (a \dot{\otimes} b)$ for every $\rho \in \mathcal{G}_\varphi$. This implies that $(\pi(a) v_\rho b)_{\rho \in \mathcal{G}_\varphi}$ converges to $U^*(a \dot{\otimes} b)$. By definition, we get that b belongs to $D(\Lambda(a))$ and $\Lambda(a) b = U^*(a \dot{\otimes} b)$.

So we get that $D(\Lambda(a)) = D$ and $\Lambda(a) b = U^*(a \dot{\otimes} b)$ for every $b \in D$. Therefore (ii) follows from Lemma 6.25. ∎

REMARK 7.16. By Proposition 7.15 (ii), we have for every $a \in \widetilde{\mathcal{N}}_\varphi$ that the mapping $D(\Lambda(a)) \to E : b \mapsto S\Lambda(a)b$ is continuous.

COROLLARY 7.17. *Consider $a \in \widetilde{\mathcal{N}}_\varphi$. Then we have for every $\rho \in \mathcal{G}_\varphi$ and $b \in D(\Lambda(a))$ that $T_\rho^{\frac{1}{2}}\Lambda(a)b = \pi(a)v_\rho b$.*

We have also the following result.

COROLLARY 7.18. *Consider $a \in M(A)$. Then a belongs to $\widetilde{\mathcal{N}}_\varphi$ if and only if the set $\{b \in B \mid$ the net $(\pi(a)v_\rho b)_{\rho\in\mathcal{G}_\varphi}$ is convergent in $E\}$ is dense in B.*

Proof. The implication from the left to the right follows from Proposition 7.15.
On the other hand, suppose that the set above is dense. Let b be an element in B such that $(\pi(a)v_\rho b)_{\rho\in\mathcal{G}_\varphi}$ is convergent in E. Because $b^*\rho(a^*a)b = \langle\pi(a)v_\rho b, \pi(a)v_\rho b\rangle$ for every $\rho \in \mathcal{G}_\varphi$ (see Notation 2.6), this implies that the net $(b^*\rho(a^*a)b)_{\rho\in\mathcal{G}_\varphi}$ is convergent in B. Using this result, the density of the set above implies that a is an element of $\widetilde{\mathcal{N}}_\varphi$. ∎

PROPOSITION 7.19. *Consider $a \in \widetilde{\mathcal{N}}_\varphi$. Then $\Lambda(a)$ is a closed densely defined B-linear map from B in E.*

Proof. It follows from the previous result that $\Lambda(a)$ is densely defined. We turn to the closedness.
Take a sequence $(b_n)_{n=1}^\infty$ in $D(\Lambda(a))$ and elements $b \in B$, $x \in E$ such that $(b_n)_{n=1}^\infty$ converges to b and $(\Lambda(a)b_n)_{n=1}^\infty$ converges to x. Choose $\eta \in \mathcal{G}_\varphi$. By Notation 2.6, we have for every $n \in \mathbb{N}$ that

$$\left\langle T_\eta^{\frac{1}{2}}\Lambda(a)b_n, T_\eta^{\frac{1}{2}}\Lambda(a)b_n\right\rangle = \langle\pi(a)v_\eta b_n, \pi(a)v_\eta b_n\rangle.$$

From this, we easily get that

$$\left\langle T_\eta^{\frac{1}{2}}x, T_\eta^{\frac{1}{2}}x\right\rangle = \langle\pi(a)v_\eta b, \pi(a)v_\eta b\rangle = b^*\eta(a^*a)b.$$

Therefore, the net $(b^*\rho(a^*a)b)_{\rho\in\mathcal{G}_\varphi}$ converges to $\langle x, x\rangle$. By Proposition 7.15, we see that b belongs to $D(\Lambda(a))$.

Choose $\eta \in \mathcal{G}_\varphi$. It is clear that $\left(T_\eta^{\frac{1}{2}}\Lambda(a)b_n\right)_{n=1}^\infty$ converges to $T_\eta^{\frac{1}{2}}x$. Because the mapping $D(\Lambda(a)) \to E : c \mapsto T_\eta^{\frac{1}{2}}\Lambda(a)c$ is continuous, we have also that $\left(T_\eta^{\frac{1}{2}}\Lambda(a)b_n\right)_{n=1}^\infty$ converges to $T_\eta^{\frac{1}{2}}\Lambda(a)b$. Hence, $T_\eta^{\frac{1}{2}}x = T_\eta^{\frac{1}{2}}\Lambda(a)b$.
The fact that $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ converges strongly to 1, gives us that $\Lambda(a)b = x$. Consequently, we have proven that $\Lambda(a)$ is closed. ∎

Another consequence of Proposition 7.15 is the following result.

RESULT 7.20. *Consider $a \in \widetilde{\mathcal{N}}_\varphi$, $\rho \in \mathcal{F}_\varphi$ and let S be an element in $\mathcal{L}(E) \cap \pi(A)'$ such that $S^*S = T_\rho$. Define v to be the element in $\mathcal{L}(B, E)$ such that $S\Lambda(c) = \pi(c)v$ for every $c \in \mathcal{N}_\varphi$. We have for every $x \in E$ that S^*x belongs to $D(\Lambda(a)^*)$ and $\Lambda(a)^*(S^*x) = v^*\pi(a^*)x$.*

Proof. By Proposition 7.15, we have for every $b \in D(\Lambda(a))$ that

$$\langle\Lambda(a)b, S^*x\rangle = \langle S\Lambda(a)b, x\rangle = \langle\pi(a)vb, x\rangle = \langle b, v^*\pi(a^*)x\rangle.$$

The result follows immediately. ∎

A special case hereof is the following result.

Result 7.21. *Consider* $a \in \widetilde{\mathcal{N}}_\varphi$, $\rho \in \mathcal{F}_\varphi$ *and* $x \in E$. *Then* $T_\rho^{\frac{1}{2}} x$ *belongs to* $D(\Lambda(a)^*)$ *and* $\Lambda(a)^*\big(T_\rho^{\frac{1}{2}} x\big) = v_\rho^* \pi(a^*) x$.

So we get the following density result.

Corollary 7.22. *Consider* $a \in \widetilde{\mathcal{N}}_\varphi$. *Then* $\Lambda(a)^*$ *is densely defined.*

Using Proposition 7.15 once more, we find:

Result 7.23. *Consider* $a_1, a_2 \in \widetilde{\mathcal{N}}_\varphi$ *and* $b_1 \in D(\Lambda(a_1))$, $b_2 \in D(\Lambda(a_2))$. *Then* $\langle T_\rho \Lambda(a_1) b_1, \Lambda(a_2) b_2 \rangle = b_2^* \rho(a_2^* a_1) b_1$ *for every* $\rho \in \mathcal{F}_\varphi$.

Proof. By Corollary 7.17, we have that

$$\begin{aligned}\langle T_\rho \Lambda(a_1) b_1, \Lambda(a_2) b_2 \rangle &= \big\langle T_\rho^{\frac{1}{2}} \Lambda(a_1) b_1, T_\rho^{\frac{1}{2}} \Lambda(a_2) b_2 \big\rangle = \langle \pi(a_1) v_\rho b_1, \pi(a_2) v_\rho b_2 \rangle \\ &= b_2^* v_\rho^* \pi(a_2^* a_1) v_\rho b_1 = b_2^* \rho(a_2^* a_1) b_1\end{aligned}$$

where we used Notation 2.6 in the last equality. ∎

This implies immediately the following convergence result.

Result 7.24. *Consider* $a_1, a_2 \in \widetilde{\mathcal{N}}_\varphi$ *and* $b_1 \in D(\Lambda(a_1))$, $b_2 \in D(\Lambda(a_2))$. *Then the net* $(b_2^* \rho(a_2^* a_1) b_1)_{\rho \in \mathcal{G}_\varphi}$ *converges to* $\langle \Lambda(a_1) b_1, \Lambda(a_2) b_2 \rangle$.

Remark 7.25. By using Result 7.23, it is easy to see that the following holds. Consider $a \in \widetilde{\mathcal{N}}_\varphi$. We have for every $c \in \widetilde{\mathcal{N}}_\varphi$, $b \in D(\Lambda(c))$ and $\rho \in \mathcal{F}_\varphi$ that $T_\rho \Lambda(c) b$ belongs to $D(\Lambda(a)^*)$ and $\Lambda(a)^*(T_\rho \Lambda(c) b) = \rho(a^* c) b$.

Remark 7.26. Consider $a \in \widetilde{\mathcal{N}}_\varphi$. We see that $\Lambda(a)$ is a densely defined closed B-linear operator from B into E such that $\Lambda(a)^*$ is densely defined. So, it behaves rather well. The question remains open whether it behaves perfect, i.e. whether $\Lambda(a)$ is a regular operator in the sense of [5] (which is true if B is commutative). Therefore, we will give the following definition.

Definition 7.27. We define the set $\widehat{\mathcal{N}}_\varphi = \{a \in \widetilde{\mathcal{N}}_\varphi \mid \Lambda(a) \text{ is regular}\}$.

Remark 7.28. By the results proven so far, we have that

$$\widehat{\mathcal{N}}_\varphi = \{a \in \widetilde{\mathcal{N}}_\varphi \mid 1 + \Lambda(a)^* \Lambda(a) \text{ has dense range in } B\}.$$

It seems to be an interesting question to look for simpler conditions on elements of $\widetilde{\mathcal{N}}_\varphi$ in order for them to belong to $\widehat{\mathcal{N}}_\varphi$.

Of course, the following is true. Consider $a \in \widetilde{\mathcal{N}}_\varphi$ such that there exists a positive element δ afilliated with B such that $\delta \subseteq \Lambda(a)^* \Lambda(a)$. Then a belongs to $\widehat{\mathcal{N}}_\varphi$ and $\Lambda(a)^* \Lambda(a) = \delta$.

Proposition 7.29. *We have the following properties:*

(i) *Consider* $a, b \in M(A)$ *such that* $a^* a = b^* b$. *Then* a *belongs to* $\widetilde{\mathcal{N}}_\varphi$ *if and only if* b *belongs to* $\widetilde{\mathcal{N}}_\varphi$.

(ii) *Consider* $a, b \in \widetilde{\mathcal{N}}_\varphi$ *such that* $a^* a = b^* b$. *Then* $D(\Lambda(a)) = D(\Lambda(b))$ *and* $\langle \Lambda(a) x, \Lambda(a) y \rangle = \langle \Lambda(b) x, \Lambda(b) y \rangle$ *for every* $x, y \in D(\Lambda(a))$.

(iii) *Consider* $a, b \in \widetilde{\mathcal{N}}_\varphi$ *such that* $a^*a = b^*b$. *Then* $\Lambda(a)^*\Lambda(a) = \Lambda(b)^*\Lambda(b)$.

(iv) *Consider* $a, b \in M(A)$ *such that* $a^*a = b^*b$. *Then* a *belongs to* $\widehat{\mathcal{N}}_\varphi$ *if and only if* b *belongs to* $\widehat{\mathcal{N}}_\varphi$.

Proof. (i) This follows immediately from the definition of $\widetilde{\mathcal{N}}_\varphi$.

(ii) Proposition 7.15 (i) implies immediately that $D(\Lambda(a)) = D(\Lambda(b))$. Choose $x, y \in D(\Lambda(a))$. Result 7.23 implies that

$$\langle T_\rho \Lambda(a)x, \Lambda(a)y\rangle = y^*\rho(a^*a)x = y^*\rho(b^*b)x = \langle T_\rho \Lambda(b)x, \Lambda(b)x\rangle$$

for every $\rho \in \mathcal{G}_\varphi$. Because $(T_\rho)_{\rho\in\mathcal{G}_\varphi}$ converges strongly to 1, this implies that $\langle \Lambda(a)x, \Lambda(a)y\rangle = \langle \Lambda(b)x, \Lambda(b)y\rangle$.

(iii) Choose $x \in D(\Lambda(a)^*\Lambda(a))$. It is clear that x belongs to $D(\Lambda(a))$. So, by (ii), we have that x belongs to $D(\Lambda(b))$. By (ii) we also have for every $y \in D(\Lambda(b))$ that y belongs to $D(\Lambda(a))$ and

$$\langle \Lambda(b)x, \Lambda(b)y\rangle = \langle \Lambda(a)x, \Lambda(a)y\rangle = \langle \Lambda(a)^*(\Lambda(a)x), y\rangle.$$

This implies by definition that $\Lambda(b)x$ belongs to $D(\Lambda(b)^*)$ and $\Lambda(b)^*(\Lambda(b)x) = \Lambda(a)^*(\Lambda(a)x)$.

From this discussion, we infer that $\Lambda(a)^*\Lambda(a) \subseteq \Lambda(b)^*\Lambda(b)$. Similarly, we have that $\Lambda(b)^*\Lambda(b) \subseteq \Lambda(a)^*\Lambda(a)$.

(iv) This follows from (i) and the fact that we have for every $c \in \widetilde{\mathcal{N}}_\varphi$ that $\Lambda(c)$ is regular if and only if $1 + \Lambda(c)^*\Lambda(c)$ has dense range in B. ∎

DEFINITION 7.30. We define the set $\widehat{\mathcal{M}}_\varphi^+ = \{a \in M(A)^+ \mid a^{\frac{1}{2}}$ belongs to $\widehat{\mathcal{N}}_\varphi\}$. For every $a \in \widehat{\mathcal{M}}_\varphi^+$, we put $\varphi(a) = \Lambda(a^{\frac{1}{2}})^*\Lambda(a^{\frac{1}{2}})$, so $\varphi(a)$ is a positive element affiliated with B.

It is clear that $\overline{\mathcal{M}}_\varphi^+ \subseteq \widehat{\mathcal{M}}_\varphi^+$ and that for every $a \in \overline{\mathcal{M}}_\varphi^+$, the notation $\varphi(a)$ is consistent with the notation introduced before.

The previous proposition implies the following one.

PROPOSITION 7.31. *Consider* $a \in M(A)$. *Then:*

(i) a *belongs to* $\widehat{\mathcal{N}}_\varphi$ *if and only if* a^*a *belongs to* $\widehat{\mathcal{M}}_\varphi^+$;

(ii) *if* a *belongs to* $\widehat{\mathcal{N}}_\varphi$, *then* $\varphi(a^*a) = \Lambda(a)^*\Lambda(a)$.

We have also the following generalizations of Proposition 7.5.

PROPOSITION 7.32. *Consider* $a \in \widetilde{\mathcal{N}}_\varphi$.

(i) *Let* b *be an element in* B. *Then* b *belongs to the domain of* $\Lambda(a)$ *if one of the following equivalent conditions is fulfilled:*

(a) *the net* $(b^*\rho_i(a^*a)b)_{i\in I}$ *is convergent in* B;

(b) *the net* $(\pi(a)v_i b)_{i\in I}$ *is convergent in* B;

(c) *the net* $(\pi(a)w_i b)_{i\in I}$ *is convergent in* B.

(ii) *Let* b *be an element in* $D(\Lambda(a))$. *Then* $(\pi(a)v_i b)_{i\in I}$ *and* $(\pi(a)w_i b)_{i\in I}$ *converge both to* $\Lambda(a)b$.

Proof. We prove everything with respect to the elements $v_i (i\in I)$. The case with the elements $w_i (i\in I)$ is completely similar.

Suppose that b belongs to $D(\Lambda(a))$. We know by Corollary 7.17 that $\pi(a)v_i b = T_i^{\frac{1}{2}}\Lambda(a)b$ for every $i \in I$. Therefore, the net $(\pi(a)v_i b)_{i\in I}$ converges to $\Lambda(a)b$. Suppose that $(\pi(a)v_i b)_{i\in I}$ converges. We have for every $i \in I$ that $b^*\rho_i(a^*a)b = \langle \pi(a)v_i b, \pi(a)v_i b\rangle$. This implies that $(b^*\rho_i(a^*a)b)_{i\in I}$ converges.

Suppose that $(b^*\rho_i(a^*a)b)_{i\in I}$ converges. By Proposition 6.7, we know that $(b^*\rho(a^*a)b)_{\rho\in\mathcal{G}_\varphi}$ converges. Therefore, b belongs to $D(\Lambda(a))$ (Proposition 7.15).

Combining these results, the proposition follows. ∎

Similarly, we have that

Proposition 7.33. *Consider $a \in M(A)$. Then a belongs to $\widetilde{\mathcal{N}}_\varphi$ if one of the following equivalent conditions is fulfilled:*

(a) *the set $\{b \in B \mid (b^*\rho_i(a^*a)b)_{i\in I}$ is convergent in $B\}$ is dense in B;*

(b) *the set $\{b \in B \mid (\pi(a)v_i b)_{i\in I}$ is convergent in $E\}$ is dense in B;*

(c) *the set $\{b \in B \mid (\pi(a)w_i b)_{i\in I}$ is convergent in $E\}$ is dense in B.*

8. The tensor product of regular C^*-valued weights

In this section, we are going to define and prove some properties about the tensor product of two regular C^*-valued weights (cf. [10] for weights). This procedure can be easily adapted to the case of the tensor product of more regular C^*-valued weights.

For the rest of this section, we will fix C^*-algebras A_1, A_2, B_1, B_2 and regular C^*-valued weights φ_1 from A_1 into $M(B_1)$ and φ_2 from A_2 into $M(B_2)$. Let us also take KSGNS-constructions (E_1, Λ_1, π_1) for φ_1 and (E_2, Λ_2, π_2) for φ_2. We will also fix a truncating net $(u_i^1)_{i\in I_1}$ for φ_1 and a truncating net $(u_i^2)_{i\in I_2}$ for φ_2.

Take $k \in \{1, 2\}$ and $i \in I_k$. We define S_i^k to be the unique element in $\mathcal{L}(E_k)$ such that $S_i^k\Lambda_k(a) = \Lambda_k(au_i^k)$ for every $a \in \mathcal{N}_{\varphi_k}$. Furthermore, we put $T_i^k = (S_i^k)^*(S_i^k)$. Moreover, ρ_i^k will denote the strict completely positive mapping from A_k into $M(B_k)$ such that $\rho_i^k(a_2^*a_1) = \Lambda_k(a_2)^*T_i^k\Lambda_k(a_1)$ for every $a_1, a_2 \in \mathcal{N}_{\varphi_k}$.

We have introduced the necessary notation, so we can start with the construction of the tensor product.

(1) The set $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ is a dense subalgebra of $A_1 \otimes A_2$. Furthermore, $\Lambda_1 \odot \Lambda_2$ is a linear mapping from $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ into $\mathcal{L}(B_1 \otimes B_2, E_1 \otimes E_2)$ such that the set $\langle(\Lambda_1 \odot \Lambda_2)(c)d \mid c \in \mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}, d \in B_1 \otimes B_2\rangle$ is dense in $E_1 \otimes E_2$.

(2) Choose $i \in I_1 \times I_2$. Define $T_i = T_{i_1}^1 \otimes T_{i_2}^2 \in \mathcal{L}(E_1 \otimes E_2)$. Then $0 \leqslant T_i \leqslant 1$. Furthermore, we put $\rho_i = \rho_{i_1}^1 \otimes \rho_{i_2}^2$ which is a strict completely positive mapping from $A_1 \otimes A_2$ into $M(B_1 \otimes B_2)$. It is easy to check that $\langle T_i(\Lambda_1 \odot \Lambda_2)(c_1)d_1, (\Lambda_1 \odot \Lambda_2)(c_2)d_2\rangle = d_2^*\rho_i(c_2^*c_1)d_1$ for every $c_1, c_2 \in \mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ and $d_1, d_2 \in B_1 \odot B_2$. This implies that $\langle T_i(\Lambda_1 \odot \Lambda_2)(c_1)d_1, (\Lambda_1 \odot \Lambda_2)(c_2)d_2\rangle = d_2^*\rho_i(c_2^*c_1)d_1$ for every $c_1, c_2 \in \mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ and $d_1, d_2 \in B_1 \otimes B_2$.

(3) It is also clear that $(T_i)_{i\in I_1\times I_2}$ converges strongly to 1.

The previous discussion implies that the elements $A_1 \otimes A_2$, $B_1 \otimes B_2$, $E_1 \otimes E_2$, $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$, $\Lambda_1 \odot \Lambda_2$ satisfy the conditions of the beginning of Section 5. Therefore, $\Lambda_1 \odot \Lambda_2$ is closable for the norm topology on $A_1 \otimes A_2$ and the strong topology on $L(B_1 \otimes B_2, E_1 \otimes E_2)$ (Lemma 5.1). We define $\Lambda_1 \otimes \Lambda_2$ to be this closure of $\Lambda_1 \odot \Lambda_2$. We denote the domain of $\Lambda_1 \otimes \Lambda_2$ by N. Then N is a dense subspace of

$A_1 \otimes A_2$ which is a left ideal of $M(A_1 \otimes A_2)$ (Proposition 5.10). The remark after Lemma 5.3 and Proposition 5.11 imply that $\Lambda_1 \otimes \Lambda_2$ is a linear mapping N into $\mathcal{L}(B_1 \otimes B_2, E_1 \otimes E_2)$ which is closed for the strict topology on $A_1 \otimes A_2$ and the strong topology on $L(B_1 \otimes B_2, E_1 \otimes E_2)$.

It is easy to check that $(\pi_1 \otimes \pi_2)(x)(\Lambda_1 \odot \Lambda_2)(c) = (\Lambda_1 \odot \Lambda_2)(xc)$ for every $x, c \in \mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$. Using Proposition 5.10, we see that $(\pi_1 \otimes \pi_2)(x)(\Lambda_1 \otimes \Lambda_2)(c) = (\Lambda_1 \otimes \Lambda_2)(xc)$ for every $x \in M(A_1 \otimes A_2)$ and $c \in N$.

Now we want to go a little bit further in our construction procedure.

(4) Choose $i \in I_1 \times I_2$. Then we define $u_i = u^1_{i_1} \otimes u^2_{i_2} \in M(A_1 \otimes A_2)$, so we have clearly that $\|u_i\| \leqslant 1$. Moreover, define $S_i = S^1_{i_1} \otimes S^2_{i_2}$, then $S_i \in \mathcal{L}(E_1 \otimes E_2)$ and $\|S_i\| \leqslant 1$. We have also that $T_i = S_i^* S_i$. It is straightforward to check for every $c \in \mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_1}$ that cu_i belongs to $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ and $S_i(\Lambda_1 \odot \Lambda_2)(c) = (\Lambda_1 \odot \Lambda_2)(cu_i)$. Hence, Lemma 5.15 implies that $Nu_i \subseteq N$ and that $S_i(\Lambda_1 \otimes \Lambda_2)(c) = (\Lambda_1 \otimes \Lambda_2)(cu_i)$ for every $c \in N$.

(5) It is clear that $(S_i)_{i \in I_1 \times I_2}$ converges strongly to 1 and that $(u_i)_{i \in I_1 \times I_2}$ converges strictly to 1.

We are now in a position to define the C^*-valued weight $\varphi_1 \otimes \varphi_2$.

Definition 8.1. The elements $A_1 \otimes A_2$, $B_1 \otimes B_2$, $E_1 \otimes E_2$, N and $\Lambda_1 \otimes \Lambda_2$ satisfy the conditions of the beginning of Section 6. Therefore, we can use Definition 6.28 to define *the tensor product* $\varphi_1 \otimes \varphi_2$, using these ingredients.

By Corollary 6.32, we have the following proposition.

Proposition 8.2. *We have that $\varphi_1 \otimes \varphi_2$ is a regular C^*-valued weight from $A_1 \otimes A_2$ into $M(B_1 \otimes B_2)$.*

Corollary 6.30 implies the following proposition.

Proposition 8.3. *We have that $(E_1 \otimes E_2, \Lambda_1 \otimes \Lambda_2, \pi_1 \otimes \pi_2)$ is a* KSGNS-*construction for $\varphi_1 \otimes \varphi_2$.*

Using this proposition, it is not difficult to check that our definition of $\varphi_1 \otimes \varphi_2$ is independent of the used construction procedure.

This last proposition determines $\varphi_1 \otimes \varphi_2$ completely and Definition 8.1 becomes in a certain sense irrelevant. In fact, we want to forget the discussion before the definition. To keep things clear, we gather the useful results of this discussion.

Proposition 8.4. *The mapping $\Lambda_1 \otimes \Lambda_2$ is a linear mapping from $\mathcal{N}_{\varphi_1 \otimes \varphi_2}$ into $\mathcal{L}(B_1 \otimes B_2, E_1 \otimes E_2)$ which is closed for the strict topology on $A_1 \otimes A_2$ and the strong topology on $L(B_1 \otimes B_2, E_1 \otimes E_2)$. Moreover, $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ is a norm-strong core for $\Lambda_1 \otimes \Lambda_2$ and $(\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2) = \Lambda_1(a_1) \otimes \Lambda_2(a_2)$ for every $a_1 \in \mathcal{N}_{\varphi_1}$ and $a_2 \in \mathcal{N}_{\varphi_2}$.*

For every $i \in I_1 \times I_2$, we have defined the element $u_i = u^1_{i_1} \otimes u^2_{i_2} \in M(A_1 \otimes A_2)$. Furthermore, we have defined the elements $S_i = S^1_{i_1} \otimes S^2_{i_2}$ and $T_i = T^1_{i_1} \otimes T^2_{i_2}$ in $\mathcal{L}(E_1 \otimes E_2)$. So $T_i = S_i^* S_i$.

We also defined the element $\rho_i = \rho^1_{i_1} \otimes \rho^2_{i_2}$, which is a strict completely positive mapping from $A_1 \otimes A_2$ into $M(B_1 \otimes B_2)$.

Concerning this elements, we have the following properties.

PROPOSITION 8.5. *The net $(u_i)_{i\in I_1\times I_2}$ is truncating for $\varphi_1\otimes\varphi_2$.*

From this, we get immediately that $\varphi_1\otimes\varphi_2$ is strongly regular if φ_1 and φ_2 are strongly regular.

PROPOSITION 8.6. *Consider $i\in I_1\times I_2$. Then we have the following properties:*

(i) *We have for every $c\in\mathcal{N}_{\varphi_1\otimes\varphi_2}$ that $S_i(\Lambda_1\otimes\Lambda_2)(c)=(\Lambda_1\otimes\Lambda_2)(cu_i)$.*

(ii) *For every $c_1,c_2\in\mathcal{N}_{\varphi_1\otimes\varphi_2}$, we have the equality $\rho_i(c_2^*c_1)=(\Lambda_1\otimes\Lambda_2)(c_2)^*T_i(\Lambda_1\otimes\Lambda_2)(c_1)$.*

(iii) *We have the inclusion $(\mathcal{N}_{\varphi_1}\odot\mathcal{N}_{\varphi_2})u_i\subseteq\mathcal{N}_{\varphi_1}\odot\mathcal{N}_{\varphi_2}$.*

Take $k\in\{1,2\}$ and $i\in I_k$. Then v_i^k and w_i^k will denote the unique elements in $\mathcal{L}(B_k,E_k)$ such that $(T_i^k)^{\frac{1}{2}}\Lambda_k(a)=\pi_k(a)v_i^k$ and $S_i^k\Lambda_k(a)=\pi_k(a)w_i^k$ for every $a\in\mathcal{N}_{\varphi_k}$.

For every $i\in I_1\times I_2$, we define the elements v_i and w_i in $\mathcal{L}(B_1\otimes B_2,E_1\otimes E_2)$ such that $v_i=v_{i_1}^1\otimes v_{i_2}^2$ and $w_i=w_{i_1}^1\otimes w_{i_2}^2$. It will not be a great surprise that these elements will be the corresponding objects for $\varphi_1\otimes\varphi_2$. More precisely:

RESULT 8.7. *Consider $i\in I_1\times I_2$. Then we have that $T_i^{\frac{1}{2}}\Lambda(c)=(\pi_1\otimes\pi_2)(c)v_i$ and $S_i\Lambda(c)=(\pi_1\otimes\pi_2)(c)w_i$ for every $c\in\mathcal{N}_{\varphi_1\otimes\varphi_2}$.*

RESULT 8.8. *Consider $\omega_1\in\mathcal{F}_{\varphi_1}$ and $\omega_2\in\mathcal{F}_{\varphi_2}$. Then $\omega_1\otimes\omega_2$ belongs to $\mathcal{F}_{\varphi_1\otimes\varphi_2}$ and $T_{\omega_1\otimes\omega_2}=T_{\omega_1}\otimes T_{\omega_2}$.*

Both results follow by checking equalities for elements in $\mathcal{N}_{\varphi_1}\odot\mathcal{N}_{\varphi_2}$ and then using the fact that $\mathcal{N}_{\varphi_1}\odot\mathcal{N}_{\varphi_2}$ is a norm-strong core for $\Lambda_1\otimes\Lambda_2$ (like in the proof of Lemma 5.15).

COROLLARY 8.9. *Consider $\omega_1\in\mathcal{G}_{\varphi_1}$ and $\omega_2\in\mathcal{G}_{\varphi_2}$. Then $\omega_1\otimes\omega_2$ belongs to $\mathcal{G}_{\varphi_1\otimes\varphi_2}$.*

The following lemma will be very useful to us.

LEMMA 8.10. *Consider $c\in M(A_1\otimes A_2)^+$, $d\in B_1\otimes B_2$ and $x\in B_1\otimes B_2$ such that the net $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ converges to x. Then $(d^*\omega(c)d)_{\omega\in\mathcal{G}_{\varphi_1\otimes\varphi_2}}$ converges to x.*

Proof. Because $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ is an increasing net in B which converges to x, we get that $d^*(\omega_1\otimes\omega_2)(c)d\leqslant x$ for every $\omega_1\in\mathcal{G}_{\varphi_1}$ and $\omega_2\in\mathcal{G}_{\varphi_2}$. From this, it is easy to conclude that $d^*(\omega_1\otimes\omega_2)(c)d\leqslant x$ for every $\omega_1\in\mathcal{F}_{\varphi_1}$ and $\omega_2\in\mathcal{F}_{\varphi_2}$. In particular, we see that $d^*\rho_i(c)d\leqslant x$ for every $i\in I_1\times I_2$. From Proposition 6.6, we infer that $d^*\rho(c)d\leqslant x$ for every $\rho\in\mathcal{F}_{\varphi_1\otimes\varphi_2}$.

Because we also have that $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ converges to x, Lemma 2.14 implies that $(d^*\omega(c)d)_{\rho\in\mathcal{G}_{\varphi_1\otimes\varphi_2}}$ converges to x. ∎

We will find a first application of this lemma in the next theorem, which gives a nice characterization of $\varphi_1\otimes\varphi_2$.

THEOREM 8.11. *We have the following properties:*

(i) *Consider* $c \in M(A_1 \otimes A_2)^+$. *Then* c *belongs to* $\overline{\mathcal{M}}^+_{\varphi_1\otimes\varphi_2}$ *if and only if the net* $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ *is convergent in* $B_1\otimes B_2$ *for every* $d \in B_1\otimes B_2$;

(ii) *Let* c *be an element in* $\overline{\mathcal{M}}_{\varphi_1\otimes\varphi_2}$. *Then the net* $((\omega_1\otimes\omega_2)(c))_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ *converges strictly to* $(\varphi_1\otimes\varphi_2)(c)$.

Proof. Suppose that the net $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ is convergent in $B_1\otimes B_2$ for every $d \in B_1\otimes B_2$. By the previous lemma, we see that $(d^*\omega(c)d)_{\omega\in\mathcal{G}_{\varphi_1\otimes\varphi_2}}$ is convergent for every $d\in B_1\otimes B_2$. By definition, this implies that c belongs to $\overline{\mathcal{M}}^+_{\varphi_1\otimes\varphi_2}$. Choose $c_1, c_2 \in \mathcal{N}_{\varphi_1\otimes\varphi_2}$. We know by Result 8.8 that

$$(\omega_1\otimes\omega_2)(c_2^*c_1) = (\Lambda_1\otimes\Lambda_2)(c_2)^*(T_{\omega_1}\otimes T_{\omega_2})(\Lambda_1\otimes\Lambda_2)(c_1)$$

for every $\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}$.

Because the net $(T_{\omega_1}\otimes T_{\omega_2})_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ converges strongly to 1, this implies that the net $((\omega_1\otimes\omega_2)(c_2^*c_1))_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ converges strictly to $(\Lambda_1\otimes\Lambda_2)(c_2)^*(\Lambda_1\otimes\Lambda_2)(c_1)$ which is equal to $(\varphi_1\otimes\varphi_2)(c_2^*c_1)$.

The proposition follows easily from these results. ∎

COROLLARY 8.12. *Consider* $c\in(A_1\otimes A_2)^+$. *Then* c *belongs to* $\mathcal{M}^+_{\varphi_1\otimes\varphi_2}$ *if and only if the net* $(d^*(\omega_1\otimes\omega_2)(c)d)_{\omega\in\mathcal{G}_{\varphi_1}\times\mathcal{G}_{\varphi_2}}$ *is convergent in* $B_1\otimes B_2$ *for every* $d\in B_1\otimes B_2$.

COROLLARY 8.13. *Consider* $a_1\in\overline{\mathcal{M}}_{\varphi_1}$ *and* $a_2\in\overline{\mathcal{M}}_{\varphi_2}$. *Then* $a_1\otimes a_2$ *belongs to* $\overline{\mathcal{M}}_{\varphi_1\otimes\varphi_2}$ *and*

$$(\varphi_1\otimes\varphi_2)(a_1\otimes a_2) = \varphi(a_1)\otimes\varphi(a_2).$$

In fact, this corollary follows easily from Theorem 8.11 for $a_1\in\overline{\mathcal{M}}^+_{\varphi_1}$ and $a_2\in\overline{\mathcal{M}}^+_{\varphi_2}$. Because any element of $\overline{\mathcal{M}}_{\varphi_1}$ can be written as a linear combination of elements in $\overline{\mathcal{M}}^+_{\varphi_1}$ (and similarly for $\overline{\mathcal{M}}_{\varphi_2}$), the corollary follows.

COROLLARY 8.14. *Consider* $a_1\in\mathcal{M}_{\varphi_1}$ *and* $a_2\in\mathcal{M}_{\varphi_2}$. *Then* $a_1\otimes a_2$ *belongs to* $\mathcal{M}_{\varphi_1\otimes\varphi_2}$.

A result which is very much related to this one is the following one.

RESULT 8.15. *Consider* $a_1\in\overline{\mathcal{N}}_{\varphi_1}$ *and* $a_2\in\overline{\mathcal{N}}_{\varphi_2}$. *Then* $a_1\otimes a_2$ *belongs to* $\overline{\mathcal{N}}_{\varphi_1\otimes\varphi_2}$ *and* $(\Lambda_1\otimes\Lambda_2)(a_1\otimes a_2) = \Lambda(a_1)\otimes\Lambda(a_2)$.

Proof. Because $a_k^*a_k$ belongs to $\overline{\mathcal{M}}^+_{\varphi_k}$ $(k=1,2)$, Corollary 8.13 implies that $a_1^*a_1\otimes a_2^*a_2$ belongs to $\overline{\mathcal{M}}^+_{\varphi_1\otimes\varphi_2}$. Therefore, $a_1\otimes a_2$ belongs to $\overline{\mathcal{N}}_{\varphi_1\otimes\varphi_2}$.

Choose $e_1\in A_1$ and $e_2\in A_2$. Then e_ka_k belongs to $\mathcal{N}_{\varphi_k}$ and $\Lambda_k(e_ka_k) = \pi(e_k)\Lambda_k(a_k)$ $(k=1,2)$. By definition, we get that $e_1a_1\otimes e_2a_2$ belongs to $\mathcal{N}_{\varphi_1\otimes\varphi_2}$ and

$$(\Lambda_1\otimes\Lambda_2)(e_1a_1\otimes e_2a_2) = \Lambda_1(e_1a_1)\otimes\Lambda(e_2a_2) = (\pi_1(e_1)\otimes\pi_2(e_2))(\Lambda_1(a_1)\otimes\Lambda(a_2)).$$

This last equality implies that

$$(\pi_1(e_1)\otimes\pi_2(e_2))(\Lambda_1\otimes\Lambda_2)(a_1\otimes a_2) = (\pi_1(e_1)\otimes\pi_2(e_2))(\Lambda_1(a_1)\otimes\Lambda(a_2)).$$

Using the nondegeneracy of π_1 and π_2, we get that $(\Lambda_1\otimes\Lambda_2)(a_1\otimes a_2) = \Lambda_1(a_1)\otimes\Lambda_2(a_2)$. ∎

Using Proposition 7.9 (and the subsequent results) and Proposition 8.6 (iii), we get the following results.

PROPOSITION 8.16. *Consider* $x \in \mathcal{N}_{\varphi_1 \otimes \varphi_2}$ *such that* $(\Lambda_1 \otimes \Lambda_2)(x) \neq 0$. *Then there exists a net* $(x_j)_{j \in J}$ *in* $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ *such that:*

(i) $\|x_j\| \leqslant \|x\|$ *and* $\|(\Lambda_1 \otimes \Lambda_2)(x_j)\| \leqslant \|(\Lambda_1 \otimes \Lambda_2)(x)\|$ *for every* $j \in J$;

(ii) $(x_j)_{j \in J}$ *converges to* x *and* $((\Lambda_1 \otimes \Lambda_2)(x_j))_{j \in J}$ *converges strongly* to* $(\Lambda_1 \otimes \Lambda_2)(x)$.

PROPOSITION 8.17. *Consider* $x \in \overline{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ *such that* $(\Lambda_1 \otimes \Lambda_2)(x) \neq 0$. *Then there exists a net* $(x_j)_{j \in J}$ *in* $\mathcal{N}_{\varphi_1} \odot \mathcal{N}_{\varphi_2}$ *such that:*

(i) $\|x_j\| \leqslant \|x\|$ *and* $\|(\Lambda_1 \otimes \Lambda_2)(x_j)\| \leqslant \|(\Lambda_1 \otimes \Lambda_2)(x)\|$ *for every* $j \in J$;

(ii) $(x_j)_{j \in J}$ *converges strictly to* x *and* $((\Lambda_1 \otimes \Lambda_2)(x_j))_{j \in J}$ *converges strongly* to* $(\Lambda_1 \otimes \Lambda_2)(x)$.

In the last part of this section, we want to prove some results about elements in $\widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$.

PROPOSITION 8.18. *We have the following properties:*

(i) *Consider* $c \in \widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ *and* $d \in B_1 \otimes B_2$. *Then* d *belongs to* $D((\Lambda_1 \otimes \Lambda_2)(c))$ *if and only if the net* $(d^*(\omega_1 \otimes \omega_2)(c^*c)d)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}}$ *is convergent in* $B_1 \otimes B_2$;

(ii) *Consider* $c_1, c_2 \in \widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ *and* $d_1 \in D((\Lambda_1 \otimes \Lambda_2)(c_1))$, $d_2 \in D((\Lambda_1 \otimes \Lambda_2)(c_2))$. *Then we have that the net* $(d_2^*(\omega_1 \otimes \omega_2)(c_2^*c_1)d_1)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}}$ *converges to* $\langle (\Lambda_1 \otimes \Lambda_2)(c_1)d_1, (\Lambda_1 \otimes \Lambda_2)(c_2)d_2 \rangle$.

Proof. (a) Choose $c \in \widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$. Suppose that d is an element in $B_1 \otimes B_2$ such that the net $(d^*(\omega_1 \otimes \omega_2)(c^*c)d)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}}$ is convergent in $B_1 \otimes B_2$. By Lemma 8.10, we know that the net $(d^*\omega(c^*c)d)_{\omega \in \mathcal{G}_{\varphi_1 \otimes \varphi_2}}$ is convergent in $B_1 \otimes B_2$. Using Proposition 7.15, we see that d belongs to $D((\Lambda_1 \otimes \Lambda_2)(c))$.

(b) Choose $c_1, c_2 \in \widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ and $d_1 \in D((\Lambda_1 \otimes \Lambda_2)(c_1))$, $d_2 \in D((\Lambda_1 \otimes \Lambda_2)(c_2))$. By Result 7.23 and Result 8.8, we have for every $\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}$ that

$$d_2^*(\omega_1 \otimes \omega_2)(c_2^*c_1)d_1 = \langle (T_{\omega_1} \otimes T_{\omega_2})(\Lambda_1 \otimes \Lambda_2)(c_1)d_1, (\Lambda_1 \otimes \Lambda_2)(c_2)d_2 \rangle.$$

This implies immediately that the net $(d_2^*(\omega_1 \otimes \omega_2)(c_2^*c_1)d_1)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}}$ converges to $\langle (\Lambda_1 \otimes \Lambda_2)(c_1)d_1, (\Lambda_1 \otimes \Lambda_2)(c_2)d_2 \rangle$.

Combining (a) and (b), the proposition follows. ∎

PROPOSITION 8.19. *Consider* $c \in M(A_1 \otimes A_2)$. *Then* c *belongs to* $\widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ *if and only if the set*

$$\{d \in B_1 \otimes B_2 \mid \text{ the net } (d^*(\omega_1 \otimes \omega_2)(c^*c)d)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}} \text{ is convergent in } B_1 \otimes B_2\}$$

is dense in $B_1 \otimes B_2$.

One implication of this proposition follows from the previous proposition. The other follows from Lemma 8.10.

PROPOSITION 8.20. *Consider $a_1 \in \widetilde{\mathcal{N}}_{\varphi_1}$ and $a_2 \in \widetilde{\mathcal{N}}_{\varphi_2}$. Then $a_1 \otimes a_2$ belongs to $\widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$, $\Lambda_1(a_1) \odot \Lambda_2(a_2)$ is closable and its closure is a restriction of $(\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)$.*

Proof. STEP 1. Choose $d_1 \in D(\Lambda_1(a_1))$, $d_2 \in D(\Lambda_2(a_2))$. We have for every $\omega_1 \in \mathcal{G}_{\varphi_1}$ and $\omega_2 \in \mathcal{G}_{\varphi_2}$ that

$$(d_1 \otimes d_2)^*(\omega_1 \otimes \omega_2)((a_1 \otimes a_2)^*(a_1 \otimes a_2))(d_1 \otimes d_2) = d_1^*\omega_1(a_1^* a_1)d_1 \otimes d_2^*\omega_2(a_2^* a_2)d_2.$$

Using Result 7.24, this implies that the net

$$\big((d_1 \otimes d_2)^*(\omega_1 \otimes \omega_2)((a_1 \otimes a_2)^*(a_1 \otimes a_2))(d_1 \otimes d_2)\big)_{\omega \in \mathcal{G}_{\varphi_1} \otimes \mathcal{G}_{\varphi_2}}$$

converges to $\langle \Lambda_1(a_1)d_1, \Lambda_1(a_1)d_1\rangle \otimes \langle \Lambda_2(a_2)d_2, \Lambda_2(a_2)d_2\rangle$.

Therefore, we conclude from the two previous propositions that $a_1 \otimes a_2$ belongs to $\widetilde{\mathcal{N}}_{\varphi_1 \otimes \varphi_2}$ and that $D(\Lambda_1(a_1)) \odot D(\Lambda_2(a_2))$ is a subset of $D((\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2))$.

STEP 2. $d_1 \in D(\Lambda_1(a_1))$ and $d_2 \in D(\Lambda_2(a_2))$. From Step 1 we already know that $d_1 \otimes d_2$ belongs to $D((\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2))$.

Choose $b_1 \in \mathcal{N}_{\varphi_1}$, $b_2 \in \mathcal{N}_{\varphi_2}$, $c_1 \in B_1$ and $c_2 \in B_2$. Then we have for every $\omega_1 \in \mathcal{G}_{\varphi_1}$ and $\omega_2 \in \mathcal{G}_{\varphi_2}$ that

$$(c_1 \otimes c_2)^*(\omega_1 \otimes \omega_2)((b_1 \otimes b_2)^*(a_1 \otimes a_2))(d_1 \otimes d_2) = c_1^*\omega_1(b_1^* a_1)d_1 \otimes c_2^*\omega_2(b_2^* a_2)d_2.$$

Using Result 7.24, this implies that the net

$$\big((c_1 \otimes c_2)^*(\omega_1 \otimes \omega_2)((b_1 \otimes b_2)^*(a_1 \otimes a_2))(d_1 \otimes d_2)\big)_{\omega \in \mathcal{G}_{\varphi_1} \otimes \mathcal{G}_{\varphi_2}}$$

converges to $\langle \Lambda_1(a_1)d_1, \Lambda_1(b_1)c_1\rangle \otimes \langle \Lambda_2(a_2)d_2, \Lambda_2(b_2)c_2\rangle$ which is equal to

$$\langle \Lambda_1(a_1)d_1 \otimes \Lambda_2(a_2)d_2, \Lambda_1(b_1)c_1 \otimes \Lambda_2(b_2)c_2\rangle. \tag{8.1}$$

On the other hand, Proposition 8.18 guarantees that the net

$$\big((c_1 \otimes c_2)^*(\omega_1 \otimes \omega_2)((b_1 \otimes b_2)^*(a_1 \otimes a_2))(d_1 \otimes d_2)\big)_{\omega \in \mathcal{G}_{\varphi_1} \times \mathcal{G}_{\varphi_2}}$$

converges to $\langle (\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)(d_1 \otimes d_2), (\Lambda_1 \otimes \Lambda_2)(b_1 \otimes b_2)(c_1 \otimes c_2)\rangle$ which by definition equals

$$\langle (\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)(d_1 \otimes d_2), \Lambda_1(b_1)c_1 \otimes \Lambda_2(b_2)c_2\rangle. \tag{8.2}$$

Combining (8.1) and (8.2), we get that

$$\begin{aligned}&\langle \Lambda_1(a_1)d_1 \otimes \Lambda_2(a_2)d_2, \Lambda_1(b_1)c_1 \otimes \Lambda_2(b_2)c_2\rangle \\ &\qquad = \langle (\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)(d_1 \otimes d_2), \Lambda_1(b_1)c_1 \otimes \Lambda_2(b_2)c_2\rangle.\end{aligned}$$

From this, we infer that $(\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)(d_1 \otimes d_2) = \Lambda_1(a_1)d_1 \otimes \Lambda_2(a_2)d_2$.

Hence, we have proven that $\Lambda_1(a_1) \odot \Lambda_2(a_2) \subseteq (\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)$. Because $(\Lambda_1 \otimes \Lambda_2)(a_1 \otimes a_2)$ is closed, the proposition follows. ∎

9. APPENDIX 1: MISCELLANEOUS RESULTS

In this appendix, we prove some general results, which were used several times in this paper.

LEMMA 9.1. *Consider a Hilbert C^*-module E over a C^*-algebra A. Let T be a positive element in $\mathcal{L}(E)$. Then we have that $\|Tv\|^2 \leqslant \|T\| \, \|\langle Tv, v\rangle\|$.*

The proof of this lemma is very simple because $\langle Tv, Tv\rangle = \langle T^2 v, v\rangle \leqslant \|T\|\langle Tv, v\rangle$, where we used the fact that $T^2 \leqslant \|T\| T$.

We will apply this little result in two situations.

LEMMA 9.2. *Consider a Hilbert C^*-module E over a C^*-algebra A. Let $(T_i)_{i\in I}$ be a net in $\mathcal{L}(E)^+$ and T an element in $\mathcal{L}(E)^+$ such that $T_i \leqslant T$ for every $i \in I$. Then $(T_i)_{i\in I}$ converges strongly to T if and only if $(\langle T_i v, v\rangle)_{i\in I}$ converges to $\langle Tv, v\rangle$ for every $v \in E$.*

Proof. We have for every $i \in I$ that $\|T_i\| \leqslant \|T\|$. Using the previous lemma, we get for every $i \in I$ and $v \in E$ that

$$\|Tv - T_i v\|^2 \leqslant \|T - T_i\| \, \|\langle Tv - T_i v, v\rangle\| \leqslant 2\|T\| \, \|\langle Tv, v\rangle - \langle T_i v, v\rangle\|.$$

Now, the lemma easily follows . ∎

LEMMA 9.3. *Consider a Hilbert C^*-module E over a C^*-algebra A. Let $(T_i)_{i\in I}$ be an increasing net in $\mathcal{L}(E)^+$. Then $(T_i)_{i\in I}$ is strongly convergent in $\mathcal{L}(E)^+$ if and only if the net $(\langle T_i v, v\rangle)_{i\in I}$ is convergent for every $v \in E$.*

Proof. One implication is trivial, we prove the other one. For this, suppose that $(\langle T_i v, v\rangle)_{i\in I}$ is convergent for every $v \in E$.

First, we prove that $(T_i)_{i\in I}$ is bounded. Indeed, choose $u \in E$. Because $(\langle T_i u, u\rangle)_{i\in I}$ is convergent, there exist a positive number N_u and an element i_0 such that $\|\langle T_i u, u\rangle\| \leqslant N_u$ for every $i \in I$ with $i \geqslant i_0$. For every $j \in I$ there exists an element $i \in I$ with $i \geqslant i_0$ and $i \geqslant j$, implying that

$$\|\langle T_j u, u\rangle\| \leqslant \|\langle T_i u, u\rangle\| \leqslant N_u.$$

So we get that the net $(\langle T_i u, u\rangle)_{i\in I}$ is bounded.

Let us fix $w \in E$. By polarisation and the previous result, we have for every $v \in E$ that the net $(\langle T_i w, v\rangle)_{i\in I}$ is bounded. Using the uniform boundedness principle, we get that the net $(T_i w)_{i\in I}$ is bounded.

Applying the uniform boundedness principle once again, we see that the net $(T_i)_{i\in I}$ is bounded. Hence there exists a strictly positive number M such that $\|T_i\| \leqslant M$ for every $i \in I$.

Choose now $u \in E$. Take $\varepsilon > 0$. Then there exists an element $i_1 \in I$ such that $\|\langle T_i u, u\rangle - \langle T_{i_1} u, u\rangle\| \leqslant \frac{\varepsilon}{2M}$ for every $i \in I$ with $i \geqslant i_1$. Using Lemma 9.1, we have for every $i \in I$ with $i \geqslant i_1$ that

$$\|T_i u - T_{i_1} u\| \leqslant \|T_i - T_{i_1}\| \, \|\langle (T_i - T_{i_1})u, u\rangle\| \leqslant 2M \frac{\varepsilon}{2M} = \varepsilon.$$

So we see that $(T_i u)_{i\in I}$ is Cauchy and hence convergent in E.

From this all, we infer the existence of a map T from E into E such that $(T_i(w))_{i\in I}$ converges to $T(w)$ for every $w \in E$. It follows immediately that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for every $v, w \in E$. This implies that T belongs to $\mathcal{L}(E)$ and $T^* = T$. Moreover, it is also clear that $\langle Tv, v\rangle \geqslant 0$ for every $v \in E$, which implies that $T \geqslant 0$. ∎

The following lemma is due to Jan Verding (see Lemma A.1.2 of [10]).

LEMMA 9.4. *Let E be a normed space, H a Hilbert space and Λ a linear mapping from E into H. Let $(x_i)_{i\in I}$ be a net in $D(\Lambda)$ and x an element in E such that $(x_i)_{i\in I}$ converges to x and $(\Lambda(x_i))_{i\in I}$ is bounded. Then there exists a sequence $(y_n)_{n=1}^{\infty}$ in the convex hull of $\{x_i \mid i \in I\}$ and an element $v \in H$ such that $(y_n)_{n=1}^{\infty}$ converges to y and $(\Lambda(y_n))_{n=1}^{\infty}$ converges to v.*

Proof. By the Banach-Alaoglu theorem, there exists a subnet $(x_{i_j})_{j\in J}$ of $(x_i)_{i\in I}$ and $v \in H$ such that $(\Lambda(x_{i_j}))_{j\in J}$ converges to v in the weak topology on H. (For this, we need H to be a Hilbert space.)

Fix $n \in \mathbb{N}$. Then there exists $j_n \in J$ such that $\|x_{i_j} - x\| \leqslant \frac{1}{n}$ for all $j \in J$ with $j \geqslant j_n$. Now v belongs to the weak-closed convex hull of the set $\{\Lambda(x_{i_j}) \mid j \in J \text{ such that } j \geqslant j_n\}$, which is the same as the norm-closed convex hull.

Therefore, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^+$ with $\sum\limits_{k=1}^{m} \lambda_k = 1$ and elements $\alpha_1, \ldots, \alpha_m \in J$ with $\alpha_1, \ldots, \alpha_m \geqslant j_n$ such that

$$\Big\| v - \sum_{k=1}^{m} \lambda_k \Lambda(x_{i_{\alpha_k}}) \Big\| \leqslant \frac{1}{n}.$$

Put $y_n = \sum\limits_{k=1}^{l} \lambda_k x_{i_{\alpha_k}}$. Then $y_n \in D(\Lambda)$, and $\Lambda(y_n) = \sum\limits_{k=1}^{m} \lambda_k \Lambda(x_{i_{\alpha_k}})$. Therefore, we have immediately that $\|v - \Lambda(y_n)\| \leqslant \frac{1}{n}$. Furthermore,

$$\|x - y_n\| = \Big\| \sum_{k=1}^{m} \lambda_k (x - x_{i_{\alpha_k}}) \Big\| \leqslant \sum_{k=1}^{m} \lambda_k \frac{1}{n} = \frac{1}{n}.$$

Therefore, we find that $(y_n)_{n=1}^{\infty}$ converges to y and that $(\Lambda(y_n))_{n=1}^{\infty}$ converges to v. ∎

LEMMA 9.5. *Consider a C^*-algebra A and a dense left ideal N in A. Let s be a positive sesquilinear form on N such that $s(ab_1, b_2) = s(b_1, a^* b_2)$ for all $a \in A$ and all $b_1, b_2 \in N$. Moreover, suppose that there exists a positive linear functional θ on A such that $s(b, b) \leqslant \theta(b^* b)$ for every $b \in N$.*

Then there exists a unique positive linear functional ω on A with $\omega \leqslant \theta$ such that $\omega(b_2^* b_1) = s(b_1, b_2)$ for every $b_1, b_2 \in N$.

Proof. Let (π, H, v) be GNS-object for θ (v is a cyclic vector). Because s is a positive sesquilinear form on N, we can use the Cauchy-Schwarz inequality for s. So we have for every $b_1, b_2 \in N$ that

$$|s(b_1, b_2)|^2 \leqslant s(b_1, b_1) s(b_2, b_2) \leqslant \theta(b_1^* b_1) \theta(b_2^* b_2) = \|\pi(b_1) v\|^2 \|\pi(b_2) v\|^2.$$

Therefore we can define a continuous positive sesquilinear form t on H such that $t(\pi(b_1)v, \pi(b_2)v) = s(b_1, b_2)$ for every $b_1, b_2 \in N$. It is clear that t is positive and $\|t\| \leqslant 1$. So there exists an element $T \in B(H)$ with $0 \leqslant T \leqslant 1$ such that $t(x, y) = \langle Tx, y \rangle$ for every $x, y \in H$. This implies that $\langle T\pi(b_1)v, \pi(b_2)v \rangle = s(b_1, b_2)$ for every $b_1, b_2 \in N$.

Next we show that T belongs to $\pi(A)'$. For this, choose $a \in N$. We have for every $b_1, b_2 \in N$ that

$$\begin{aligned}\langle T\pi(a)\pi(b_1)v, \pi(b_2)v\rangle &= \langle T\pi(ab_1)v, \pi(b_2)v\rangle = t(\pi(ab_1)v, \pi(b_2)v) = s(ab_1, b_2)\\ &= s(b_1, a^*b_2) = t(\pi(b_1)v, \pi(a^*b_2)v) = \langle T\pi(b_1)v, \pi(a^*b_2)v\rangle\\ &= \langle T\pi(b_1)v, \pi(a^*)\pi(b_2)v\rangle = \langle \pi(a)T\pi(b_1)v, \pi(b_2)v\rangle.\end{aligned}$$

This implies that $T\pi(a) = \pi(a)T$.

Now we define the continuous linear functional ω on A such $\omega(x) = \langle T\pi(x)v, v\rangle$ for every $x \in A$. Using the fact that T belongs to $\pi(A)'$, we have for every $b_1, b_2 \in N$ that

$$\omega(b_2^*b_1) = \langle T\pi(b_2^*b_1)v, v\rangle = \langle T\pi(b_1)v, \pi(b_2)v\rangle = s(b_1, b_2).$$

Consequently, we have for every $b \in N$ that $\omega(b^*b) = s(b, b)$, implying that $0 \leqslant \omega(b^*b) \leqslant \theta(b^*b)$. This implies easily that $0 \leqslant \omega \leqslant \theta$. ∎

We have even proven a stronger result where N is not assumed to be dense (of course the uniqueness is not longer valid in this case). This proof can be found in Lemma A.1.3 of [10].

10. APPENDIX 2 : A SMALL TECHNICAL RESULT

In this appendix, we will prove a technical result which was used in several sections.

Consider a Hilbert C^*-module E over a C^*-algebra B and let D be a subset of E such that its linear span is dense in E. Let $(T_i)_{i\in I}$ be a net in $\mathcal{L}(E)$ such that $(T_i)_{i\in I}$ converges strongly* to 1. Suppose that t is a mapping from B into E such that for every $i \in I$ and every $v \in D$ there exists an element $x(v, i) \in B$ such that $\langle T_i t(b), v\rangle = x(v, i)b$ for every $b \in B$.

We want to prove that t belongs to $\mathcal{L}(B, E)$.

LEMMA 10.1. *We have that t is a continuous B-linear map from B into E.*

Proof. (a) Choose $b_1, b_2 \in B$. Fix $j \in I$. We have for every $v \in D$ that

$$\langle T_j t(b_1b_2), v\rangle = x(v, j)(b_1b_2) = (x(v, j)b_1)b_2 = \langle T_j t(b_1), v\rangle b_2 = \langle T_j(t(b_1)b_2), v\rangle.$$

Because the linear span of D is dense in E, this implies that $T_j t(b_1b_2) = T_j(t(b_1)b_2)$. Because $(T_i)_{i\in I}$ converges strongly to 1, this implies that $t(b_1b_2) = t(b_1)b_2$.

(b) The linearity of t is proven in a completely similar way.

(c) Choose a sequence $(b_n)_{n=1}^\infty$ in B, $b \in B$ and $w \in E$ such that $(b_n)_{n=1}^\infty \to b$ and $(t(b_n))_{n=1}^\infty \to w$. Fix $j \in I$. Take $v \in D$. We have for every $n \in \mathbb{N}$ that $\langle T_j t(b_n), v\rangle = x(v, j)b_n$. This implies that $(\langle T_j t(b_n), v\rangle)_{n=1}^\infty$ converges to $x(v, j)b$, which is equal to $\langle T_j t(b), v\rangle$. It is also clear that $(\langle T_j t(b_n), v\rangle)_{n=1}^\infty$ converges to $\langle T_j w, v\rangle$.

The two results in (a), (b) and (c) imply that $\langle T_j t(b), v\rangle = \langle T_j w, v\rangle$. Of course, this implies that $t(b) = w$. Therefore, we have proven that t is closed. The closed graph theorem implies that t is continuous. ∎

LEMMA 10.2. *Consider an approximate unit $(e_k)_{k\in K}$ for B. For every $k \in K$, we define the element $S_k \in \mathcal{L}(B,E)$ such that $S_k(b) = t(e_k)b$ for every $b \in B$. Then we have for every $v \in E$ that $(S_k^*(v))_{k\in K}$ is convergent in B.*

Proof. Remember from the previous lemma that t is a continuous B-linear map from B into E. In particular, we have for every k in K that $\|S_k\| \leqslant \|t\|$. Choose $w \in D$. Take $\varepsilon > 0$. Then there exists an element $j \in I$ such that $\|T_j^* w - w\| \leqslant \frac{\varepsilon}{3}\frac{1}{1+\|t\|}$. We have for every $k \in K$ that

$$S_k^*(T_j^* w) = \langle T_j^* w, t(e_k)\rangle = \langle w, T_j t(e_k)\rangle = \langle T_j t(e_k), w\rangle^* = (x(j,w)e_k)^* = e_k x(j,w)^*,$$

which implies that $(S_k^*(T_j^* w))_{k\in K}$ converges to $x(j,w)^*$. Consequently, there exists an element $k_0 \in K$ such that $\|S_{k_1}^*(T_j^* w) - S_{k_2}^*(T_j^* w)\| \leqslant \frac{\varepsilon}{3}$ for every $k_1, k_2 \in K$ with $k_1, k_2 \geqslant k_0$.

Therefore, we have for every $k_1, k_2 \in K$ with $k_1, k_2 \geqslant k_0$ that

$$\begin{aligned}
\|S_{k_1}^*(w) - S_{k_2}^*(w)\| &\leqslant \|S_{k_1}^*(w) - S_{k_1}^*(T_j^* w)\| + \|S_{k_1}^*(T_j^* w) - S_{k_2}^*(T_j^* w)\| \\
&\qquad + \|S_{k_2}^*(T_j^* w) - S_{k_2}^*(w)\| \\
&\leqslant \|S_{k_1}^*\|\,\|w - T_j^* w\| + \frac{\varepsilon}{3} + \|S_{k_2}^*\|\,\|w - T_j^* w\| \\
&\leqslant \|t\|\frac{\varepsilon}{3}\frac{1}{1+\|t\|} + \frac{\varepsilon}{3} + \|t\|\frac{\varepsilon}{3}\frac{1}{1+\|t\|} \leqslant \varepsilon.
\end{aligned}$$

Hence, we see that $(S_k^*(w))_{k\in K}$ is Cauchy and hence convergent in B. From this, immediately we get that $(S_k^*(v))_{k\in K}$ is convergent for every v in the linear span of D. Because this linear span is dense in E and $(S_k^*)_{k\in K}$ is bounded, we get that $(S_k^*(v))_{k\in K}$ is convergent for every $v \in E$. ∎

Now we can formulate the final result.

PROPOSITION 10.3. *We have that t belongs to $\mathcal{L}(B,E)$.*

Proof. Take an approximate unit $(e_k)_{k\in K}$ for B. For every $k \in K$, we define the element $S_k \in \mathcal{L}(B,E)$ such that $S_k(b) = t(e_k)b$ for every $b \in B$. Choose $c \in B$. Because t is B-linear, we have for ever $k \in K$ that $S_k(c) = t(e_k)c = t(e_k c)$. The continuity of t implies that $(S_k(c))_{k\in K}$ converges to $t(c)$.

By the previous lemma, we know that there exists a linear mapping r from E into B such that $(S_k^*(v))_{k\in K}$ converges to $r(v)$ for every $v \in E$.

Combining these two results, we get that $\langle t(b), v\rangle = \langle b, r(v)\rangle$ for every $b \in B$ and $v \in E$. This implies that t belongs to $\mathcal{L}(B,E)$. ∎

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JOHAN KUSTERMANS
Department of Mathematics
University College Cork
Western Road, Cork
IRELAND
E-mail: johank@ucc.ie

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