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LEFT QUOTIENTS OF C^*-ALGEBRAS, II: ATOMIC PARTS OF LEFT QUOTIENTS

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Abstract. Let A be a C^*-algebra. Let z be the maximal atomic projection in A^{**}. By a theorem of Brown, an element x in A^{**} has a continuous atomic part, i.e. $zx = za$ for some a in A, whenever x is uniformly continuous on the set of pure states of A. Let L be a closed left ideal of A. Under some additional conditions, we shall show that if x is uniformly continuous on the set of pure states of A killing L, or its weak* closure, then x has a continuous atomic part modulo L^{**} in an appropriate sense.

Keywords: *C^*-algebras, continuous fields of Hilbert spaces, continuous atomic parts.*

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1. INTRODUCTION

Let A be a C^*-algebra with the Banach dual A^* and double dual A^{**}. Let

$$Q(A) = \{\varphi \in A^* : \varphi \geqslant 0, \, \|\varphi\| \leqslant 1\}$$

be the quasi-state space of A. Note that $Q(A)$ is a weak* compact convex set. The extreme boundary of $Q(A)$ is the pure state space $P(A) \cup \{0\}$ of A. In the Kadison function representation, the elements x of A^{**} are represented as bounded and affine functionals $\varphi \mapsto \varphi(x)$ of $Q(A)$ vanishing at zero. Moreover, $x \in A$ if and only if x is continuous on $Q(A)$ (see, e.g. [18]).

There is a long history ([13], [19], [1], among others) of studying closed left ideals L of a C^*-algebra A. These are in one-to-one correspondence with weak* closed faces F of the quasi-state space $Q(A)$ of A. One of our concerns in the series of this paper, together with [23], [24], [9], is the non-commutative Stone-Weierstrass problem for C^*-algebras (see e.g. [12]). After deriving a single element version of the Stone-Weierstrass Theorem, Brown ([7]) improved a result of Shultz ([20]) to classify elements in the bidual A^{**} of A having continuous atomic parts.

They are exactly those being uniformly continuous on the extreme boundary of $Q(A)$, i.e. the pure state space $P(A) \cup \{0\}$ of A. We shall obtain similar results in the context of left quotients A/L of C^*-algebras in this paper. Inspirited by Tomita ([21], [22]), we hope that our results will be helpful in solving the Stone-Weierstrass problem.

Part I ([24]) of this series of papers showed that A/L (respectively its bidual A^{**}/L^{**}) can be represented as the space of continuous affine (respectively bounded affine) sections of a continuous field of Hilbert spaces over F. In this paper, we consider the problem of recognizing continuous affine sections from their restrictions to the extreme boundary of F. This is important because F may be a huge space even when A is nice. For example, if $A = C[0,1]$ and $L = \{0\}$, then F is the set of all non-negative Borel measures μ on $[0,1]$ such that $\mu([0,1]) \leqslant 1$, whereas its extreme boundary is just the set $\{\delta_x : x \in [0,1]\} \cup \{0\}$ of point masses together with the zero functional. We shall study the theory of operators on the left quotient A/L in Part III ([9]), which answers essentially all questions raised by Tomita in [21], [22].

Let z be the maximal atomic projection in A^{**}. Note that $A^{**} = zA^{**} \oplus (1-z)A^{**}$; zA^{**} is the direct sum of type I factors and $(1-z)A^{**}$ has no type I factor direct summand of A^{**}. In particular, z is a central projection in A^{**} supporting all pure states of A. In other words, $\varphi(x) = \varphi(zx)$ for all x in A^{**} and all pure states φ of A.

We call zA^{**} the *atomic part* of A^{**}. An element x of A^{**} is said to *have a continuous atomic part* if $zx = za$ for some a in A (cf. [20]). In this case, x and a agree on $P(A) \cup \{0\}$ since $\varphi(x) = \varphi(zx) = \varphi(za) = \varphi(a)$ for all pure states φ of A. In particular, $\varphi \mapsto \varphi(x)$ is uniformly continuous on $P(A) \cup \{0\}$. The converse is also true. Shultz ([20]) showed that x in A^{**} has a continuous atomic part whenever x, x^*x and xx^* are uniformly continuous on $P(A) \cup \{0\}$. Recently, Brown ([7]) got the complete result.

THEOREM 1.1. ([7]) *Let $x \in A^{**}$. Then x has a continuous atomic part* (*i.e.* $zx \in zA$) *if and only if x is uniformly continuous on $P(A) \cup \{0\}$.*

In view of the Kadison function representation, Theorem 1.1 merely states that a uniformly continuous functional of the extreme boundary $P(A) \cup \{0\}$ of $Q(A)$ can be lifted to a continuous affine functional of the whole of $Q(A)$.

Let L be a norm closed left ideal of the C^*-algebra A. The Banach double dual L^{**} of L is a weak* closed left ideal of the w*-algebra A^{**}.

DEFINITION 1.2. Let $x \in A^{**}$. We say that *x has a continuous atomic part modulo L^{**}* if $zx + L^{**} = za + L^{**}$ for some a in A.

We are interested in the question when x in A^{**} has a continuous atomic part modulo L^{**}. By applying our tools developed in this paper to the special case $L = \{0\}$, we get new and interesting results. As a corollary of Theorems 1.1 and 1.7 below, for example, we have

THEOREM 1.3. *Let $x \in A^{**}$. Then x has a continuous atomic part (i.e. $zx \in zA$) if and only if x^*x and a^*x are uniformly continuous on $P(A) \cup \{0\}$ for all a in A.*

Theorem 1.3 supplements results of Shultz ([20]) and Brown ([7]) in the case A is non-unital. More general statements appear in succeeding sections.

We are going to give a precise meaning of our assertions. Recall that a convex subset F of a convex set Q is called a *face* of Q if we can always infer that φ and ψ belong to F whenever $\varphi, \psi \in Q$ and $\lambda\varphi + (1-\lambda)\psi \in F$ for some λ strictly between 0 and 1. Weak* closed faces of $Q(A)$ are in one-to-one correspondence with closed projections in A^{**}. In fact, a projection p in A^{**} is *closed* if and only if the face

$$F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$$

of $Q(A)$ supported by p is weak* closed ([18]). Closed projections p in A^{**} are also in one-to-one correspondence with norm closed left ideals L of A such that

$$L = A^{**}(1-p) \cap A.$$

Moreover, we have isometrical isomorphisms $a + L \mapsto ap$ and $x + L^{**} \mapsto xp$ under which

$$A/L \cong Ap \quad \text{and} \quad (A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p$$

as Banach spaces, respectively [13], [19], [1].

In the following, p is always a closed projection in A^{**}. We consider Ap as the left quotient A/L of the C^*-algebra A by the norm closed left ideal $L = A^{**}(1-p) \cap A$. Consequently, its Banach double dual $A^{**}p$ is the left quotient A^{**}/L^{**} of the w*-algebra A^{**} by the weak* closed left ideal $L^{**} = A^{**}(1-p)$.

PROPOSITION 1.4. *Let $x \in A^{**}$. Then x has a continuous atomic part modulo L^{**} if and only if $zxp = zap$ for some a in A.*

Proof. It suffices to observe the following equivalences: $zx + L^{**} = za + L^{**}$ if and only if $z(x-a) \in L^{**}$ if and only if $z(x-a)p = 0$. ∎

We also have a function representation of left quotients.

THEOREM 1.5. ([24]) *Let $xp \in A^{**}p$. Then $xp \in Ap$ if and only if the bounded and affine functionals $\varphi \mapsto \varphi(x^*x)$ and $\varphi \mapsto \varphi(a^*x)$ are continuous on $F(p)$ for all a in A.*

Note that the extreme boundary of $F(p)$ is $(P(A) \cup \{0\}) \cap F(p)$, i.e. the set of all pure states of A supported by p together with the zero functional. Motivated by Theorem 1.1, we shall attack the following

PROBLEM 1.6. Let $x \in A^{**}$. Suppose that x^*x and a^*x are uniformly continuous on $(P(A) \cup \{0\}) \cap F(p)$ for all a in A. Can we infer that x has a continuous atomic part modulo L^{**}, i.e. $zap \in zAp$?

Problem 1.6 does not always have an affirmative answer (see counter examples in Section 4). Our main result, proved at the end of Section 2, states

THEOREM 1.7. *Let $x \in A^{**}$. Then x has a continuous atomic part modulo L^{**} (i.e. $zxp \in zAp$) if and only if $zpx^*xp \in zpAp$ and $zpa^*xp \in zpAp$, $\forall a \in A$.*

An answer to Problem 1.6 should thus assert conditions on either x or p under which uniform continuities of x^*x and a^*x, $\forall a \in A$, on the extreme boundary of $F(p)$ will ensure $zpx^*xp \in zpAp$ and $zpa^*xp \in zpAp$, $\forall a \in A$. We shall see (via Theorem 2.4) that this amounts to looking for conditions under which uniformly continuous functionals of the extreme boundary of $F(p)$ can be lifted to continuous affine functionals of the whole of $F(p)$. In Section 3, we shall present some sufficient conditions to ensure the existence of such liftings, and thus Problem 1.6 can be solved affirmatively. One of them assumes the universal measurability of x^*x and a^*x on $F(p)$. The others assume that p has either MSQC (this holds if, in particular, p is central) or p is semi-atomic. Counter examples in Section 4 verify that our conclusions are sharp.

2. MAIN RESULTS

Recall the notion $(T, \{H_t\}, \Gamma)$ of a continuous field of Hilbert spaces ([14], [11]). The *base space* T is a Hausdorff space, the *fiber* H_t is a Hilbert space (possibly different) for each t in T and the *continuous structure* Γ is a family of vector sections $a = (a_t)_{t\in T}$ in the product space $\prod_{t\in T} H_t$ satisfying the following two conditions:

(1) The norm $t \mapsto \|a_t\|_{H_t}$ is continuous on T for each a in Γ.
(2) The set $\{a_t : a \in \Gamma\}$ is norm dense in H_t for each t in T.

A vector section $x = (x_t)_{t\in T}$ in $\prod_{t\in T} H_t$ is said to be *bounded* if the functional $t \mapsto \langle x_t, x_t\rangle_{H_t} = \|x_t\|^2_{H_t}$ is bounded on T. A bounded vector section x is said to be *weakly continuous* if the functionals $t \mapsto \langle x_t, a_t\rangle_{H_t}$ is continuous on T for every $a = (a_t)_{t\in T}$ in Γ. A weakly continuous vector section x is said to be *continuous* if, in addition, the functional $t \mapsto \langle x_t, x_t\rangle_{H_t} = \|x_t\|^2_{H_t}$ is continuous. The subspaces of bounded, weakly continuous, and continuous vector sections in the product space $\prod_{t\in T} H_t$ equipped with the sup norm $\|x\|_\infty = \sup_{t\in T} \|x_t\|_{H_t}$ are Banach spaces, respectively.

Although the following elementary result should have been known, we provide a proof here since we cannot find any in the literature.

LEMMA 2.1. *Let $(T, \{H_t\}_t, \Gamma)$ be a continuous field of Hilbert spaces. Let $(y_t)_{t\in T}$ be a weakly continuous vector section. Then the map $t \mapsto \|y_t\|_{H_t}$ is lower semicontinuous on T.*

Proof. For each t_0 in T, since $\{u_{t_0} \in H_{t_0} : u \in \Gamma\}$ is dense in H_{t_0}, we have

$$\|y_{t_0}\|_{H_{t_0}} = \sup\{|\langle y_{t_0}, u_{t_0}\rangle_{H_{t_0}}| : \|u_{t_0}\|_{H_{t_0}} < 1,\, u \in \Gamma\}.$$

For $\varepsilon > 0$, choose an a in Γ so that $\|a_{t_0}\|_{H_{t_0}} < 1$ and

$$|\, \|y_{t_0}\|_{H_{t_0}} - \langle y_{t_0}, a_{t_0}\rangle_{H_{t_0}}| < \frac{\varepsilon}{2}.$$

Since $t \mapsto \|a_t\|_{H_t}$ and $t \mapsto \langle y_t, a_t\rangle_{H_t}$ are continuous on T, for those t close enough to t_0 we have $\|a_t\|_{H_t} < 1$ and

$$|\langle y_t, a_t\rangle_{H_t} - \langle y_{t_0}, a_{t_0}\rangle_{H_{t_0}}| < \frac{\varepsilon}{2}.$$

Now, if $t_\alpha \to t_0$ in T, for α large enough,

$$\begin{aligned}\|y_{t_\alpha}\|_{H_{t_\alpha}} &= \sup\{|\langle y_{t_\alpha}, u_{t_\alpha}\rangle_{H_{t_\alpha}}| : \|u_{t_\alpha}\|_{H_{t_\alpha}} < 1,\, u \in \Gamma\} \\ &\geqslant |\langle y_{t_\alpha}, a_{t_\alpha}\rangle_{H_{t_\alpha}}| > |\langle y_{t_0}, a_{t_0}\rangle_{H_{t_0}}| - \frac{\varepsilon}{2} > \|y_{t_0}\|_{H_{t_0}} - \varepsilon.\end{aligned}$$

Consequently,

$$\|y_{t_0}\|_{H_{t_0}} \leqslant \underline{\lim} \|y_{t_\alpha}\|_{H_{t_\alpha}} + \varepsilon.$$

Since ε is arbitrary,

$$\|y_{t_0}\|_{H_{t_0}} \leqslant \underline{\lim} \|y_{t_\alpha}\|_{H_{t_\alpha}},$$

and thus $t \mapsto \|y_t\|_{H_t}$ is lower semicontinuous on T. ∎

Let A be a C^*-algebra and L a norm closed left ideal of A. Let p be the closed projection in A^{**} related to L such that $L = A^{**}(1-p) \cap A$. Moreover, $A/L \cong Ap$ and $(A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p$ as Banach spaces. We are going to construct a continuous field of Hilbert spaces associated to the left quotient Ap. The base space is $F(p)$, the weak* closed face of the quasi-state space $Q(A)$ of A supported by p. Note that

$$\begin{aligned}F(p) &= \{\varphi \in Q(A) : \varphi(1-p) = 0\} \\ &= \{\varphi \in Q(A) : \varphi(x) = \varphi(xp) = \varphi(px) = \varphi(pxp),\, \forall x \in A^{**}\}.\end{aligned}$$

Moreover, $F(p)$ is itself a weak* compact, Hausdorff and convex set.

For each φ in $F(p)$, the GNS construction yields a cyclic representation $(\pi_\varphi, H_\varphi, \omega_\varphi)$ of A. In particular, $\varphi(x) = \langle \pi_\varphi(x)\omega_\varphi, \omega_\varphi\rangle_\varphi, \forall x \in A^{**}$, where $\langle\,\cdot\,,\cdot\,\rangle_\varphi$ is the inner product of the Hilbert space H_φ. Set H_φ to be the zero dimensional Hilbert space when $\varphi = 0$. Note also that for each φ in $F(p)$, $\pi_\varphi(p)\omega_\varphi = \omega_\varphi$ since $\langle \pi_\varphi(p)\omega_\varphi, \omega_\varphi\rangle_\varphi = \varphi(p) = \varphi(1) = \|\omega_\varphi\|^2$.

NOTATION. Write $x\omega_\varphi$ for $\pi_\varphi(x)\omega_\varphi$ in H_φ, $\forall x \in A^{**}$, $\forall \varphi \in F(p)$.

In this way, there is an embedding $A^{**}p \hookrightarrow \prod_{\varphi \in F(p)} H_\varphi$ defined by associating each xp in $A^{**}p$ to the vector section $(x\omega_\varphi)_{\varphi \in F(p)}$ in $\prod_{\varphi \in F(p)} H_\varphi$. A continuous structure of $\prod_{\varphi \in F(p)} H_\varphi$ can be defined by the image of Ap under this embedding. In fact, the functional $\varphi \mapsto \|a\omega_\varphi\|^2 = \langle a\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(a^*a)$ is continuous on $F(p)$ for every ap in Ap. Moreover, $\pi_\varphi(A)\omega_\varphi = \{a\omega_\varphi : a \in A\}$ is norm dense in H_φ.

In the continuous field $(F(p), \{H_\varphi\}, Ap)$ of Hilbert spaces, a bounded vector section $(x_\varphi)_{\varphi \in F(p)}$ in the product space $\prod_{\varphi \in F(p)} H_\varphi$ is weakly continuous if $\varphi \mapsto \langle x_\varphi, a\omega_\varphi\rangle_\varphi$ is continuous on $F(p)$ for all a in A. A weakly continuous vector section $(x_\varphi)_{\varphi \in F(p)}$ is continuous if, in addition, $\varphi \mapsto \langle x_\varphi, x_\varphi\rangle_\varphi$ is continuous on $F(p)$. Since $F(p)$ is a convex set, we have an additional affine structure.

Definition 2.2. A vector section $(x_\varphi)_{\varphi\in F(p)}$ in $\prod_{\varphi\in F(p)} H_\varphi$ is said to be *affine* if the functionals $\varphi \mapsto \langle x_\varphi, a\omega_\varphi\rangle_\varphi$, $\forall a \in A$, are affine on $F(p)$. In other words,

$$\langle x_\varphi, a\omega_\varphi\rangle_\varphi = \lambda\langle x_\psi, a\omega_\psi\rangle_\psi + (1-\lambda)\langle x_\rho, a\omega_\rho\rangle_\rho.$$

whenever $\varphi = \lambda\psi + (1-\lambda)\rho$ in $F(p)$, $0 \leqslant \lambda \leqslant 1$ and $a \in A$.

In [24], we showed that every bounded and affine vector section $(x_\varphi)_{\varphi\in F(p)}$ arises from an xp in $A^{**}p$, i.e. $x_\varphi = x\omega_\varphi$, $\forall\varphi \in F(p)$. More precisely, we have

Theorem 2.3. ([24]) *Let A be a C^*-algebra and p a closed projection in A^{**}.*

(i) *$A^{**}p$ is isometrically linear isomorphic to the Banach space of all bounded and affine vector sections in $\prod_{\varphi\in F(p)} H_\varphi$ equipped with the norm $\sup_{\varphi\in F(p)} \|x\omega_\varphi\|_{H_\varphi} = \sup_{\varphi\in F(p)} \varphi(x^*x)^{1/2}$.*

(ii) *Ap is isometrically linear isomorphic to the Banach space of all continuous and affine vector sections in $\prod_{\varphi\in F(p)} H_\varphi$.*

Theorem 1.5 is a corollary of Theorem 2.3. In fact, xp in $A^{**}p$ defines a continuous vector section in $\prod_{\varphi\in F(p)} H_\varphi$ if and only if the affine functionals $\varphi \mapsto \langle x\omega_\varphi, x\omega_\varphi\rangle_\varphi = \varphi(x^*x)$ and $\varphi \mapsto \langle x\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(a^*x)$, $\forall a \in A$, are continuous on $F(p)$. The meaning of this translation can be more precise with the help of the following result of Brown, where A_{sa} (respectively A^{**}_{sa}) is the self-adjoint part of A (respectively A^{**}).

Theorem 2.4. ([6], 3.9) *$pA_{\mathrm{sa}}p$ (respectively $pA^{**}_{\mathrm{sa}}p$) is isometrically linear and order isomorphic to the Banach space of all continuous (respectively bounded) affine functionals of $F(p)$ which vanish at zero.*

Let $\mathcal{W}_p$ denote the Banach subspace of $A^{**}p$ of all weakly continuous affine vector sections in $\prod_{\varphi\in F(p)} H_\varphi$. When $p = 1$, $\mathcal{W}_p$ coincides with the set $\mathrm{RM}(A)$ of right multipliers of A. However, $\mathcal{W}_p \neq \mathrm{RM}(A)p$ in general ([24]).

In summary, Theorems 2.3 and 2.4 imply

Corollary 2.5. *Let $xp \in A^{**}$.*

(i) *The following are all equivalent:*

(a) *$xp \in \mathcal{W}_p$,*

(b) *the affine functionals $\varphi \mapsto \varphi(a^*x) = \varphi(pa^*xp)$ are continuous on $F(p)$ for all a in A,*

(c) *$pa^*xp \in pAp$, $\forall a \in A$.*

(ii) *The following are all equivalent:*

(a) *$xp \in Ap$,*

(b) *the affine functionals $\varphi \mapsto \varphi(x^*x) = \varphi(px^*xp)$ and $\varphi \mapsto \varphi(a^*x) = \varphi(pa^*xp)$ are continuous on $F(p)$ for all a in A,*

(c) *$px^*xp \in pAp$ and $pa^*xp \in pAp$, $\forall a \in A$.*

Recall that xp in $A^{**}p$ is said to *have a continuous* (respectively *a weakly continuous*) *atomic part modulo L^{**}* if $zxp = zyp$ for some yp in Ap (respectively in $\mathcal{W}_p$).

Lemma 2.6. *Let $x, y \in A^{**}$. Then $zxp = zyp$ if and only if $x\omega_\varphi = y\omega_\varphi$ in H_φ for every pure state φ in $F(p)$.*

Proof. Let $zxp = zyp$. Since $\pi_\varphi(z) = 1$ and $\pi_\varphi(p)\omega_\varphi = \omega_\varphi$, we have $x\omega_\varphi = \pi_\varphi(x)\omega_\varphi = \pi_\varphi(zxp)\omega_\varphi = \pi_\varphi(zyp)\omega_\varphi = \pi_\varphi(y)\omega_\varphi = y\omega_\varphi$ for every pure state φ in $F(p)$. For the converse, we note that for every ψ in $Q(A)$, the atomic $\psi(zp\cdot p)$ can be written as a countable sum of pure positive linear functionals in $F(p)$ ([15]). Now, $x\omega_\varphi = y\omega_\varphi$ implies $\varphi((x-y)^*(x-y)) = \langle (x-y)\omega_\varphi, (x-y)\omega_\varphi\rangle_\varphi = 0$. If this holds for all pure states φ in $F(p)$ then $\psi(zp(x-y)^*(x-y)p) = 0$ for all ψ in $Q(A)$. Consequently, $zp(x-y)^*(x-y)p = 0$ and thus $zxp = zyp$. ∎

Theorem 2.7. *Let $x \in A^{**}$. Then x has a weakly continuous atomic part modulo L^{**} (i.e. $zxp \in z\mathcal{W}_p$) if and only if $zpa^*xp \in zpAp$ for all a in A.*

Proof. For the necessity, we assume that $zxp = zyp$ for some yp in $\mathcal{W}_p$. Then $pa^*yp \in pAp$ by Corollary 2.5, and thus $zpa^*xp = pa^*(zxp) = pa^*(zyp) = z(pa^*yp) \in zpAp$, $\forall a \in A$.

For the sufficiency, let $X = F(p) \cap P(A)$, the set of all pure states of A supported by p. We want to find a yp in $\mathcal{W}_p$ such that $x\omega_\varphi = y\omega_\varphi$ for all φ in X. In this case, we shall have $zxp = zyp \in z\mathcal{W}_p$ by Lemma 2.6.

Since pure states are also supported by the central projection z, we have

$$\varphi(x) = \varphi(pxp) = \varphi(zpxp), \quad \forall x \in A^{**}, \forall \varphi \in X.$$

For each a in A, by the hypothesis, $zpa^*xp = zpv_ap$ for some v_a in A. And thus

$$\langle x\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(a^*x) = \varphi(zpa^*xp) = \varphi(zpv_ap) = \varphi(v_a), \quad \forall \varphi \in X. \tag{2.1}$$

Let $\overline{X}$ be the weak* closure of X in $F(p)$. If $\psi_\alpha \in X$ and $\psi = \lim \psi_\alpha \in \overline{X}$ then we have, by (2.1),

$$\begin{aligned} |\psi(v_a)| &= \lim |\psi_\alpha(v_a)| = \lim |\langle x\omega_{\psi_\alpha}, a\omega_{\psi_\alpha}\rangle_{\psi_\alpha}| \\ &\leqslant \|x\| \lim \|a\omega_{\psi_\alpha}\|_{\psi_\alpha} = \|x\|\,\|a\omega_\psi\|_\psi = \|x\|\psi(a^*a)^{\frac{1}{2}}. \end{aligned} \tag{2.2}$$

Note that $X \cup \{0\}$ is the extreme boundary of the compact convex set $F(p)$. Thus each continuous affine functionals of $F(p)$ assumes its maximum modules at a point in X. From Theorem 2.4 we know that there is an order-preserving linear isometry from $pA_{\mathrm{sa}}p$ into the Banach space $C_{\mathbb{R}}(\overline{X})$ of continuous real-valued functions defined on $\overline{X}$. Hence, as a positive linear functional of $pA_{\mathrm{sa}}p$, each φ in $F(p)$ has a (non-unique) Hahn-Banach positive extension m_φ in the space $M(\overline{X})$ of regular finite Borel measures on the compact Hausdorff space $\overline{X}$. Thus we can write

$$\begin{aligned} \varphi(a) = \varphi(pap) &= \int_{\overline{X}} \psi(pap)\,\mathrm{d}m_\varphi(\psi) \\ &= \int_{\overline{X}} \psi(a)\,\mathrm{d}m_\varphi(\psi), \quad \forall a \in A_{\mathrm{sa}}, \forall \varphi \in F(p). \end{aligned} \tag{2.3}$$

Motivating by (2.1), we define a vector section $(y_\varphi)_{\varphi\in F(p)}$ in $\prod_{\varphi\in F(p)} H_\varphi$ by the conditions

$$\langle y_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(v_a) = \int_{\overline{X}} \psi(v_a)\,\mathrm{d}m_\varphi(\psi), \quad \forall a\in A, \forall\varphi\in F(p).$$

Observe that

$$\begin{aligned} |\langle y_\varphi, a\omega_\varphi\rangle_\varphi| &\leqslant \int_{\overline{X}} |\psi(v_a)|\,\mathrm{d}m_\varphi(\psi) \\ &\leqslant \int_{\overline{X}} \|x\|\psi(a^*a)^{1/2}\,\mathrm{d}m_\varphi(\psi) \quad \text{(by (2.2))} \\ &\leqslant \|x\|\Big[\int_{\overline{X}} \psi(a^*a)\,\mathrm{d}m_\varphi(\psi)\Big]^{\frac{1}{2}} \quad (\text{since } m_\varphi(\overline{X}) = \|\varphi\| \leqslant 1) \\ &= \|x\|\varphi(a^*a)^{\frac{1}{2}} \quad \text{(by (2.3))} \\ &= \|x\|\,\|a\omega_\varphi\|_\varphi. \end{aligned}$$

Therefore, the definition of y_φ (as a bounded linear functional of the Hilbert space H_φ) makes sense and $\|(y_\varphi)_{\varphi\in F(p)}\|_\infty = \sup_{\varphi\in F(p)} \|y_\varphi\|_{H_\varphi} \leqslant \|x\|$. Clearly, the definition of y_φ is independent of the choice of m_φ and v_a. Since $\varphi \mapsto \langle y_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(v_a)$ is a continuous affine functional of $F(p)$ for each a in A, we see that $(y_\varphi)_{\varphi\in F(p)}$ is a bounded and weakly continuous affine vector section in $\prod_{\varphi\in F(p)} H_\varphi$. By Theorem 2.3, there is a yp in $\mathcal{W}_p$ such that $y\omega_\varphi = y_\varphi$ for each φ in $F(p)$. Finally, for each pure state φ in $F(p)$ and a in A we have $\langle y\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(v_a) = \langle x\omega_\varphi, a\omega_\varphi\rangle_\varphi$ by (2.1). Since $\{a\omega_\varphi : a\in A\}$ is norm dense in (indeed, equal to) H_φ, we have $y\omega_\varphi = x\omega_\varphi$ for every pure state φ in $F(p)$. Thus, $zxp = zyp$. ∎

Beside Theorem 1.3, the following result supplements Theorem 1.1 in another interesting way. Note that $\mathcal{W}_p = \mathrm{RM}(A)$ when $p = 1$ (Corollary 2.5).

COROLLARY 2.8. *Let $x \in A^{**}$. Then $zx = zy$ for some right multiplier y of A in A^{**} if and only if a^*y are uniformly continuous on $P(A)\cup\{0\}$ for all a in A.*

Now we are ready to present the proof of our main result, Theorem 1.7, which says that an element x of A^{**} has a continuous atomic part modulo L^{**} (i.e. $zxp \in zAp$) if and only if $zpx^*xp \in zpAp$ and $zpa^*xp \in zpAp$, $\forall a \in A$.

Proof of Theorem 1.7. Only the sufficiency demands a proof. By Theorem 2.7, x has a weakly continuous atomic part modulo L^{**}, i.e. $zxp = zyp$ for some yp in $\mathcal{W}_p$. By hypothesis, we also have $zpx^*xp = zpvp$ for some v in A. Because $pA_+p = (pAp)_+$, we can assume v to be positive. By Lemma 2.1, $\varphi \mapsto \varphi(y^*y - v) = \|y\omega_\varphi\|^2_{H_\varphi} - \varphi(v)$ is a lower semicontinuous real-valued affine functional of the compact convex set $F(p)$, which vanishes at all pure states in $F(p)$. Now the desired assertion follows from a result of Pedersen ([17], 3.8).

However, for the convenience of the readers we present below a somewhat elementary argument. We note that the scalar function $\varphi \mapsto \varphi(y^*y - v)$ attains its minimum value at an extreme point ψ of $F(p)$, i.e. a pure state ψ in $F(p)$. But $\psi(y^*y - v) = \psi(zp(y^*y - v)p) = 0$ for all pure states ψ in $F(p)$. Hence, $py^*yp \geqslant pvp$. On the other hand, it follows from the Krein-Milman Theorem that every φ in the compact convex set $F(p)$ is a weak* limit of convex combinations φ_α of pure states (and 0) in $F(p)$. Therefore, we always have $\varphi(y^*y) \leqslant \underline{\lim}\varphi_\alpha(y^*y) = \underline{\lim}\varphi_\alpha(v) = \varphi(v)$ by Lemma 2.1. Thus, $py^*yp \leqslant pvp$. Consequently, $py^*yp = pvp \in pAp$. Hence, $yp \in Ap$ by Corollary 2.5. Finally, $zxp = zyp \in zAp$. ∎

In Section 3, we shall investigate conditions under which our main result, Theorem 1.7, can be translated into an answer of Problem 1.6.

3. SOME APPROACHES TO THE PROBLEM

3.1. UNIVERSALLY MEASURABLE ELEMENTS. Let A be a C^*-algebra and p a closed projection in A^{**}. Recall that $A_{\rm sa}^m$ consists of all limits in $A_{\rm sa}^{**}$ of monotone increasing nets in $A_{\rm sa}$ and $(A_{\rm sa})_m = -A_{\rm sa}^m$. While $A_{\rm sa}$ consists of continuous affine real-valued functionals of $Q(A)$ (the Kadison function representation), the norm closure $\underline{(A_{\rm sa}^m)^-}$ of $A_{\rm sa}^m$ consists of *lower semicontinuous elements* and the norm closure $\overline{(A_{\rm sa})_m}$ of $(A_{\rm sa})_m$ consists of *upper semicontinuous elements* in A^{**}. Accordingly, an element x of $A_{\rm sa}^{**}$ is said to be *universally measurable* if for each φ in $Q(A)$ and $\varepsilon > 0$ there exist a lower semicontinuous element l and an upper semicontinuous element u in A^{**} such that $u \leqslant x \leqslant l$ and $\varphi(l - u) < \varepsilon$ ([17]).

We note that $pA_{\rm sa}p$ consists of continuous affine real-valued functionals of the weak* closed face $F(p)$ of $Q(A)$ supported by p (Theorem 2.4). Analogously, pxp in $pA_{\rm sa}^{**}p$ is said to be *universally measurable on* $F(p)$ if for each φ in $F(p)$ and $\varepsilon > 0$, a lower semicontinuous element l in $\underline{(A_{\rm sa}^m)^-}$ and an upper semicontinuous element u in $\overline{(A_{\rm sa})_m}$ exist such that $pup \leqslant pxp \leqslant plp$ and $\varphi(l - u) < \varepsilon$. An element pxp in $pA^{**}p$ is said to be universally measurable on $F(p)$ if both the real and imaginary parts of pxp are.

LEMMA 3.1. *Let $x \in A_{\rm sa}^{**}$ and $\overline{X}$ the weak* closure of $X = F(p) \cap P(A)$ in $F(p)$. If pxp is universally measurable on $F(p)$ and continuous on $\overline{X}$ then $pxp \in pAp$.*

Proof. First, we note that the continuity of pxp on $\overline{X}$ means that whenever $\varphi_\lambda \to \varphi$ weak* in $\overline{X}$ we have $\varphi_\lambda(x) \to \varphi(x)$. In view of Theorem 2.4, we need to verify these convergences for nets arising from the whole of $F(p)$.

Suppose there were φ_λ, φ in $F(p)$ such that $\varphi_\lambda \to \varphi$ weak* but $\varphi_\lambda(x)$ did not converge to $\varphi(x)$. Without loss of generality, we can assume there exists a $\delta > 0$ such that $|\varphi(x) - \varphi_\lambda(x)| > \delta$ for all λ. As in the proof of Theorem 2.7, we can embed $pA_{\rm sa}p$ as a closed subspace into the Banach space $C_{\mathbb{R}}(\overline{X})$ of continuous real-valued functions defined on $\overline{X}$. Let m_λ be any positive extension of φ_λ from $pA_{\rm sa}p$ to $C_{\mathbb{R}}(\overline{X})$ and $\|m_\lambda\| = \|\varphi_\lambda\| \leqslant 1$. Hence, $\{m_\lambda\}_\lambda$ is a bounded net in $M(\overline{X})$, the Banach dual space of $C_{\mathbb{R}}(\overline{X})$, consisting of regular finite Borel measures on the compact Hausdorff space $\overline{X}$. Then, by passing to a subnet if necessary, we have $m_\lambda \to m$ for some regular finite Borel measure m in the weak* topology of $M(\overline{X})$. Clearly, $m \geqslant 0$ and $m|pA_{\mathrm{sa}}p = \varphi$. On the other hand, pxp is assumed to be universally measurable on $F(p)$. So for each ϕ in $F(p)$ and $\varepsilon > 0$ there exist a u in $(A_{\mathrm{sa}})_m^-$ and an l in $(A_{\mathrm{sa}}^m)^-$ such that

$$pup \leqslant pxp \leqslant plp \quad \text{and} \quad \phi(l-u) < \varepsilon.$$

It follows from the semicontinuity and the affine property of u and l (cf. [5], p. 19, or by a direct argument) that u and l satisfy the barycenter formula of ϕ in $F(p)$, i.e.

$$\phi(u) = \int_{\overline{X}} \psi(u)\,\mathrm{d}m_\phi(\psi) \quad \text{and} \quad \phi(l) = \int_{\overline{X}} \psi(l)\,\mathrm{d}m_\phi(\psi),$$

where m_ϕ in $M(\overline{X})$ is a positive Hahn-Banach extension of ϕ to $C_{\mathbb{R}}(\overline{X})$. Since $pup \leqslant pxp \leqslant plp$, we have

$$\phi(u) = \int_{\overline{X}} \psi(u)\,\mathrm{d}m_\phi(\psi) \leqslant \int_{\overline{X}} \psi(x)\,\mathrm{d}m_\phi(\psi) \leqslant \int_{\overline{X}} \psi(l)\,\mathrm{d}m_\phi(\psi) = \phi(l), \quad \forall \phi \in F(p).$$

As $\phi(u) \leqslant \phi(x) \leqslant \phi(l)$, we have

$$\left|\phi(x) - \int_{\overline{X}} \psi(x)\ \mathrm{d}m_\phi(\psi)\right| \leqslant \phi(l-u) < \varepsilon, \quad \forall \phi \in F(p).$$

Because ε is arbitrary, x satisfies the barycenter formula of ϕ in $F(p)$ as well, i.e.

$$\phi(x) = \int_{\overline{X}} \psi(x)\,\mathrm{d}m_\phi(\psi), \quad \forall \phi \in F(p).$$

Note again that $\psi(x) = \psi(pxp)$ for each ψ in $F(p)$. Now $pxp \in C_{\mathbb{R}}(\overline{X})$ implies

$$\int_{\overline{X}} \psi(x)\,\mathrm{d}m_\lambda(\psi) = \int_{\overline{X}} \psi(pxp)\,\mathrm{d}m_\lambda(\psi) \to \int_{\overline{X}} \psi(pxp)\,\mathrm{d}m(\psi) = \int_{\overline{X}} \psi(x)\,\mathrm{d}m(\psi).$$

Consequently, $\varphi_\lambda(x) \to \varphi(x)$, a contradiction. ∎

In the proof above, what we actually need on x is that pxp satisfies barycenter formulas of elements of $F(p)$ and are continuous on $\overline{X} = \overline{P(A) \cap F(p)}$. Indeed, we have proved

PROPOSITION 3.2. *For every pxp in $pA^{**}p$ satisfying barycenter formulas of elements of $F(p)$, $pxp \in pAp$ if and only if pxp is continuous on $\overline{X}$.*

THEOREM 3.3. *Let $x \in A^{**}$ and $X = F(p) \cap P(A)$ with weak* closure $\overline{X}$ in $F(p)$.*

(i) *$xp \in \mathcal{W}_p$ if and only if pa^*xp are universally measurable (or satisfy barycenter formulas) on $F(p)$ and continuous on $\overline{X}$ for all a in A.*

(ii) *$xp \in Ap$ if and only if px^*xp and pa^*xp are universally measurable (or satisfy barycenter formulas) on $F(p)$ and continuous on $\overline{X}$ for all a in A.*

Proof. We note that the real and imaginary parts of pa^*xp both satisfy the assumptions of Lemma 3.1 for any a in A. Hence, the assertions follow from Lemma 3.1 and Corollary 2.5. ∎

3.2. THE PROJECTION p HAS MSQC. Let A be a C^*-algebra. Recall that a projection p in A^{**} is closed if the face $F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$ of $Q(A)$ supported by p is weak* closed. An element p is said to be *compact* if $F(p) \cap S(A)$ is closed ([6]), where $S(A) = \{\varphi \in Q(A) : \|\varphi\| = 1\}$ is the state space of A. Let p be a closed projection in A^{**}. An element h of $pA^{**}_{\mathrm{sa}}p$ is said to be *q-continuous* on p ([2]) if the spectral projection $E_F(h)$ (computed in $pA^{**}p$) is closed for every closed subset F of $\mathbb{R}$. Moreover, h is said to be *strongly q-continuous* on p ([6]) if, in addition, $E_F(h)$ is compact whenever F is closed and $0 \notin F$. It is known from [6], 3.43, that h is strongly q-continuous on p if and only if $h = pa = ap$ for some a in A_{sa}. In general, h in $pA^{**}p$ is said to be *strongly q-continuous* if both $\operatorname{Re} h$ and $\operatorname{Im} h$ are.

Denote by $\mathrm{SQC}(p)$ the C^*-algebra of all strongly q-continuous elements on p. We say that p has MSQC ("*many strongly q-continuous elements*") if $\mathrm{SQC}(p)$ is σ-weakly dense in $pA^{**}p$. Brown ([8]) showed that p has MSQC if and only if $pAp = \mathrm{SQC}(p)$ if and only if pAp is an algebra. In particular, every central projection p (especially, $p = 1$) has MSQC. We provide a partial answer to Problem 1.6 by the following

THEOREM 3.4. *Let p have* MSQC *and $xp \in A^{**}p$. Let $X_0 = (F(p) \cap P(A)) \cup \{0\}$ be the extreme boundary of $F(p)$.*

(i) *x has a weakly continuous atomic part modulo L^{**} (i.e. $zxp \in z\mathcal{W}_p$) if and only if pa^*xp are uniformly continuous on X_0 for all a in A.*

(ii) *x has a continuous atomic part modulo L^{**} (i.e. $zxp \in zAp$) if and only if px^*xp and pa^*xp are uniformly continuous on X_0 for all a in A.*

Proof. The necessities are obvious and we check the sufficiencies. Note that pAp is now a C^*-algebra with the pure state space $P(pAp) = F(p) \cap P(A)$. The maximal atomic projection of pAp is zp. By Theorem 1.1, $zpa^*xp \in zpAp$ for all a in A. In case px^*xp is uniformly continuous on X_0, we have $zpx^*xp \in zpAp$ as well. The assertion follows from Theorems 2.7 and 1.7. ∎

COROLLARY 3.5. *Let p have* MSQC *and $xp \in A^{**}p$. If pa^*xp is continuous on $\overline{X} = \overline{F(p) \cap P(A)}$ for all a in A then $zxp \in z\mathcal{W}_p$. If, in addition, px^*xp is continuous on $\overline{X}$ then $zxp \in zAp$.*

Proof. We simply note that either 0 belongs to $\overline{X}$ or 0 is isolated from $X = F(p) \cap P(A)$ in $F(p)$. Consequently, continuity on the compact set $\overline{X}$ ensures uniform continuity on $X_0 = (F(p) \cap P(A)) \cup \{0\}$. Thus, Theorem 3.4 applies. ∎

3.3. THE PROJECTION $\boldsymbol{p}$ IS SEMIATOMIC. Let A be a C^*-algebra and p a closed projection in A^{**}. Recall that A is said to be scattered if $Q(A) \subseteq zQ(A)$ ([15], [16]) and p is said to be atomic if $F(p) \subseteq zF(p)$ ([8]). If A is scattered then every closed projection in A^{**} is atomic. Moreover, A is said to be semiscattered ([3]) if $\overline{P(A)} \subseteq zQ(A)$.

Analogously, we say that a closed projection p is *semiatomic* if the weak* closure of $F(p) \cap P(A)$ contains only atomic positive linear functionals of A, i.e. $\overline{F(p) \cap P(A)} \subseteq zF(p)$. It is easy to see that if A is semiscattered then every closed projection in A^{**} is semiatomic. We provide another partial answer to Problem 1.6 by the following

THEOREM 3.6. *Let p be semiatomic and $xp \in A^{**}p$. Let $\overline{X} = \overline{F(p) \cap P(A)}$.*

(i) *x has a weakly continuous atomic part modulo L^{**} (i.e. $zxp \in z\mathcal{W}_p$) if and only if pa^*xp are continuous on $\overline{X}$ for all a in A.*

(ii) *x has a continuous atomic part modulo L^{**} (i.e. $zxp \in zAp$) if and only if px^*xp and pa^*xp are continuous on $\overline{X}$ for all a in A.*

To prove Theorem 3.6, we need the following generalization of [7], Theorem 6, in which $p = 1$.

LEMMA 3.7. *Let x in $zpA^{**}p$ be uniformly continuous on $X_0 = (F(p) \cap P(A)) \cup \{0\}$. Then x is in the C^*-algebra B generated by $zpAp$. In particular, $x = zy$ for some universally measurable element y of $pA^{**}p$.*

Proof. Utilizing a similar argument in [7], we can prove that $x \in B$. Moreover, the last assertion can be deduced from [4], 2.1 and the fact that $\widetilde{B}_0$ is a Jordan algebra ([10]). ∎

Lemma 3.7 merely says that if $x \in A^{**}$ such that x is uniformly continuous on the extreme boundary $(F(p) \cap P(A)) \cup \{0\}$ of $F(p)$ then x has a universally measurable atomic part modulo L^{**}.

Proof of Theorem 3.6. We prove the sufficiencies only. Let xp in $A^{**}p$ satisfy the stated conditions. For each a in A, it follows that zpa^*xp is uniformly continuous on $X_0 = (P(A) \cap F(p)) \cup \{0\}$. By Lemma 3.7, there is a universally measurable element y of $pA^{**}p$ such that $zpa^*xp = zy$. Since p is assumed to be semiatomic, each φ in $\overline{X} = \overline{P(A) \cap F(p)}$ is atomic and thus $\varphi(a^*x) = \varphi(zpa^*xp) = \varphi(zy) = \varphi(y)$. In particular, the universally measurable element y is continuous on $\overline{X}$. It follows from Lemma 3.1 that $y \in pAp$. As a consequence, $zpa^*xp \in zpAp$ for each a in A. It should be clear that we can also deduce $zpx^*xp \in zpAp$ in the same manner if px^*xp is assumed to be continuous on $\overline{X}$. Now, Theorems 2.7 and 1.7 apply. ∎

4. SOME COUNTER EXAMPLES

In this final section, we shall provide some counter examples to Problem 1.6 to show that the conclusions in Section 3 are sharp. Recall that $X_0 = (P(A) \cap F(p)) \cup \{0\}$ and $\overline{X} = \overline{F(p) \cap P(A)}$.

First of all, Example 4.2 below tells us that the continuity assumptions in Lemma 3.1 on $\overline{X}$ cannot be replaced by the uniform continuity on X_0. Moreover, we shall see in Example 4.1 that the conditions on universal measurability is also necessary. One may notice that uniform continuity of an element pxp in $pA^{**}p$ on X_0 ensures that pxp is universally measurable on X_0 (Lemma 3.7). Unfortunately, even in this case (i.e. pxp is continuous on $\overline{X}$ *and thus* universally measurable on X_0) we can have $zpxp \notin zpAp$ as shown in Example 4.3. Consequently, without assuming the universal measurability of px^*xp or pa^*xp on the *whole* of $F(p)$, Theorem 3.3 fails to give us any new result about the atomic part of x.

We now turn our attention to the MSQC assumption on the closed projection p. If p does not have MSQC then the conclusion of Theorem 3.4 may not hold. In fact, without assuming p has MSQC we have counter-examples; in one of them

px^*xp and pa^*xp are universally measurable on $F(p)$ and uniformly continuous on X_0 for all a in A (Example 4.2), and in the other one px^*xp and pa^*xp are continuous on $\overline{X}$ for all a in A (Example 4.3). But $zxp \notin z\mathcal{W}_p$ in both cases.

Finally, we shall see in Example 4.2 that the continuity assumption on $\overline{X}$ in Theorem 3.6 cannot be replaced by the uniform continuity on X_0 even when the C^*-algebra A is scattered. On the other hand, Example 4.3 tells us that the conclusion of Theorem 3.6 can be wrong if p is not semiatomic.

EXAMPLE 4.1. This example tells us that measurability conditions in Theorem 3.3 are necessary.

Let $A = C(Y)$, the abelian C^*-algebra of continuous (complex-valued) functions defined on a compact Hausdorff space Y. Then its Banach dual space $A^* \cong \bigoplus_1\{L^1(\mu) : \mu \in \mathcal{C}\} \oplus_1 \ell^1(Y)$, where $\mathcal{C}$ is a maximal family of mutually singular continuous measures on Y of total variation one. Accordingly, $A^{**} = \bigoplus_\infty\{L^\infty(\mu) : \mu \in \mathcal{C}\} \oplus_\infty \ell^\infty(Y)$. Let $p = 1$ in A^{**}. Pure states of A are exactly evaluations at points in Y. The maximal atomic projection $z = 0 \oplus 1$ in the above direct sum decomposition of A^{**}. The embedding of A into A^{**} is given by $g \mapsto (\bigoplus_{\mu\in\mathcal{C}} g_\mu) \oplus g_a$, where $g_\mu = g_a$, μ almost everywhere in Y, for all μ in $\mathcal{C}$, and $f_{\mathrm{a}} = f$ everywhere in Y. Write $f = f_{\mathrm{d}} + f_{\mathrm{a}} \in A^{**}$, where f_{d} is the diffuse part of f which comes from $\oplus_\infty\{L^\infty(\mu) : \mu \in \mathcal{C}\}$ and f_{a} is the atomic part of f which comes from $\ell^\infty(Y)$. Set $f_{\mathrm{d}} = 0$ and f_{a} to coincide with any nonzero continuous function on Y.Then f is *not* universally measurable (and neither satisfies barycenter formulas of elements of $Q(A)$). Although f is continuous on $\overline{P(A)} \cong Y$, f does not belong to A.

EXAMPLE 4.2. This example tells us that p having MSQC is necessary in Theorem 3.4 and continuity on $\overline{X}$ is necessary in Theorem 3.6.

Let A be the scattered C^*-algebra of sequences of 2×2 matrices $x = (x_n)_{n=1}^\infty$ such that

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \to x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ entrywise,}$$

and equipped with the ℓ^∞-norm. Note that the maximal atomic projection $z = 1$ in this case. Let

$$p_n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, n = 1, 2, \ldots, \quad \text{and} \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $p = (p_n)_{n=1}^\infty$ is a closed projection in A^{**}. An element $xp = (x_n p_n)_{n=1}^\infty$ of $A^{**}p$ with

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

belongs to Ap if and only if $a_n + b_n \to a$ and $c_n + d_n \to d$. Moreover, $\mathcal{W}_p = Ap$ in this case.

We claim that p does *not* have MSQC. In fact, suppose $x = (x_n)_{n=1}^\infty$ in A is given by

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

such that $x_n \to x_\infty$. Then $(pxp)_n = \lambda_n p_n$, $n = 1, 2, \ldots$, and $(pxp)_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $\lambda_n = \frac{a_n+b_n+c_n+d_n}{2} \to \frac{a+d}{2}$. Consequently, $(pxp)_n^2 = \lambda_n^2 p_n$, $n = 1, 2, \ldots$, and $(pxp)_\infty^2 = \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}$. If $(pxp)^2 \in pAp$ we must have $\lambda_n^2 \to \frac{a^2+d^2}{2}$. This occurs exactly when $a = d$. In particular, pAp is not an algebra and thus p does *not* have MSQC.

On the other hand, the set $X = P(A) \cap F(p)$ of all pure states in $F(p)$ consists exactly of φ_n, ψ_1 and ψ_2 which are given by

$$\varphi_n(x) = \operatorname{tr}(x_n p_n), \quad n = 1, 2, \ldots,$$

and

$$\psi_1(x) = a, \quad \psi_2(x) = d,$$

where $x = (x_n)_{n=1}^\infty \in A$ and $x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Since $\varphi_n \to \frac{1}{2}(\psi_1 + \psi_2) \neq 0$, the extreme boundary $X_0 = X \cup \{0\}$ of $F(p)$ is discrete. Consider $yp = (y_n p_n)_{n=1}^\infty$ in $A^{**}p$ given by

$$y_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ n = 1, 2, \ldots, \quad \text{and} \quad y_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, the universally measurable elements pa^*yp and px^*xp are uniformly continuous on X_0 for all a in A but $yp \notin Ap = \mathcal{W}_p$. ∎

EXAMPLE 4.3. This example is based on the ones in [20] and [3]. It tells us that without assuming p has MSQC or p is semiatomic, continuity on $\overline{X}$ fails to ensure $zxp \in zAp$.

Let A be a C^*-algebra given by the exact sequence

$$0 \to \oplus_{c_0} M_n \to A \to C[0,1] \to 0,$$

where M_n is the C^*-algebra of $n \times n$ matrices, $n = 1, 2, \ldots$. More precisely, if

$$h_n = \begin{pmatrix} \frac{1}{n} & & & 0 \\ & \frac{2}{n} & & \\ & & \ddots & \\ 0 & & & \frac{n}{n} \end{pmatrix}_{n\times n}, \quad n = 1, 2, \ldots,$$

then we can implement A as the family of bounded sequences $a = (a_n)_{n=1}^\infty$ such that $a_\infty = f \in C[0,1]$, $a_n \in M_n$, $n = 1, 2, \ldots$, and $\|a_n - f(h_n)\| \to 0$. Moreover, $A^{**} \cong \left(\bigoplus_{\ell^\infty} M_n\right) \oplus_{\ell^\infty} C[0,1]^{**}$.

Let p_n be the projection $\frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n}$ in M_n, $n = 1, 2, \ldots$, and $p_\infty = 1$ in $C[0,1]^{**}$. It is not difficult to see that $p = (p_n)_{n=1}^\infty$ is a closed projection in A^{**} but p does *not* have MSQC. Moreover, $X = P(A) \cap F(p) = \{\varphi_n : n =$

$1,2,\ldots\} \cup \{\chi_t : t \in [0,1]\}$. Here $\varphi_n(a) = \mathrm{tr}(a_n p_n)$, $n = 1,2,\ldots$, and $\chi_t(a) = f(t)$, $t \in [0,1]$, for every $a = (a_n)_{n=1}^\infty$ in A with $a_\infty = f$ in $C[0,1]$. Since

$$\mathrm{tr}(f(h_n)p_n) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \to \int_0^1 f(t)\,\mathrm{d}t$$

for every f in $C[0,1]$, we have

$$\varphi_n(a) \to \int_0^1 f(t)\,\mathrm{d}t$$

for every $a = (a_n)_{n=1}^\infty$ in A with $a_\infty = f$ in $C[0,1]$. Let $\varphi_\infty = \lim \varphi_n$. Note that for $x = (x_n)_{n=1}^\infty$ in A^{**}, $\varphi_\infty(x) = \int_0^1 g_m(t)\,\mathrm{d}t$ where $x_\infty \in C[0,1]^{**}$ and g_m is the component of x_∞ in $L^\infty([0,1],m)$ for the Lebesgue measure m on $[0,1]$ (cf. Example 4.1). It is then obvious that $\overline{X} = \overline{P(A) \cap F(p)} = X \cup \{\varphi_\infty\}$. Since φ_∞ is diffuse, p is *not* semiatomic.

Consider the element $x = (x_n)_{n=1}^\infty$ of A^{**} in which $x_n = 1$ in M_n, $n = 1,2,\ldots$, and x_∞ in $C[0,1]^{**}$ is such that the atomic part of x_∞ is 0 and the diffuse part of x_∞ is 1. It is easy to see that for every $a = (a_n)_{n=1}^\infty$ in A, $(pa^*xp)_n = (pa^*p)_n$, $n = 1,2,\ldots$, the atomic part of $(pa^*xp)_\infty$ is 0 and the diffuse part of $(pa^*xp)_\infty$ is the same as $(pa^*p)_\infty$. In particular, pa^*xp defines a continuous function on $\overline{X}$ for each a in A and so does px^*xp. But x does *not* even have a weakly continuous atomic part modulo L^{**}.

In fact, if there were a yp in $\mathcal{W}_p$ such that $zxp = zyp$. It follows that the atomic part of y_∞ is 0. For every continuous function f in $C[0,1]$, we define an $a = (a_n)_{n=1}^\infty$ in A by setting $a_\infty = f$ in $C[0,1]$ and $a_n = f(h_n)$ in M_n, $n = 1,2,\ldots$. Since yp is weakly continuous on $F(p)$, $pa^*yp = pbp$ for some $b = (b_n)_{n=1}^\infty$ in A. In particular, $b_\infty = 0$ in $C[0,1]$ since $p_\infty = 1$ in $C[0,1]^{**}$ and the atomic part of a^*y is 0. Hence

$$\varphi_n(pa^*yp) = \varphi_n(b) \to \varphi_\infty(b) = 0.$$

However,

$$\varphi_n(pa^*yp) = \varphi_n(pa^*xp) = \varphi_n(a^*x) = \varphi_n(a^*)$$

since each φ_n is a pure state in $F(p)$ and $x_n = 1$ in M_n, $n = 1,2,\ldots$. Now, the last term of the equalities approaches to $\int_0^1 \overline{f(t)}\,\mathrm{d}t$. Consequently, $\int_0^1 \overline{f(t)}\,\mathrm{d}t = 0$ for every continuous function f in $C[0,1]$ with complex conjugate $\overline{f(t)}$, an absurdity! Therefore, xp does *not* have a weakly continuous atomic part. ∎

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