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NON-COMPACT QUANTUM GROUPS ARISING FROM HEISENBERG TYPE LIE BIALGEBRAS

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Abstract. The dual Lie bialgebra of a certain *quasitriangular* Lie bialgebra structure on the Heisenberg algebra determines a (non-compact) Poisson-Lie group G. The compatible Poisson bracket on G is non-linear, but it can still be realized as a "cocycle perturbation" of the linear Poisson bracket. We construct a certain twisted group C^*-algebra A, which is shown to be a strict deformation quantization of G. Motivated by the data at the Poisson (classical) level, we then construct on A its *locally compact quantum group* structures: comultiplication, counit, antipode and Haar weight, as well as its associated multiplicative unitary operator. We also find a quasitriangular "quantum universal R-matrix" type operator for A, which agrees well with the quasitriangularity at the Lie bialgebra level.

Keywords: *Deformation quantization, twisted group C^*-algebra, Poisson-Lie group, quantum group.*

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0. INTRODUCTION

So far, the usual method of constructing quantum groups has been the method of "generators and relations", in which one tries to *deform* the relations between the generators (i.e. "coordinate functions") of the commutative algebra of functions on a Lie group (e.g. [38]). But if we wish to study locally compact (C^*-algebraic) quantum groups, this provides a serious obstacle: In non-compact situations, the generators tend to be unbounded, which makes it difficult to treat them in the C^*-algebra framework. (For example see [39], where Woronowicz introduces the highly technical theory of "unbounded operators affiliated with C^*-algebras" in his construction of the quantum $E(2)$ group.) In addition, the method of generators and relations is at best an indirect method, in the sense that the deformation of the pointwise product on the function algebra is not explicitly obtained.

Because of this, constructing new (especially, non-compact) quantum groups has been rather difficult. Among the specific examples of non-compact quantum groups which have been constructed and studied are: [27], [39], [40], [2], [37], [35], [31], [33], [42], [20].

Recently, in [15], we defined certain (in general non-linear) Poisson brackets on dual vector spaces of Lie algebras, denoted by $\{\cdot\,,\cdot\}_\omega$, which are "cocycle perturbations" of the linear Poisson brackets. We then showed that deformation quantizations of these Poisson brackets are provided by twisted group C^*-algebras. This construction is relatively general (at least for those Poisson brackets of the aforementioned type), and there are some indications ([15]) that further generalization could be possible. In addition, it is a direct approach, where we deform the pointwise product directly at the function algebra level.

We wish to use this method to construct some C^*-algebraic quantum groups. But for constructing a quantum group from a twisted group C^*-algebra, it should be given a compatible comultiplication and other quantum group structures. If we are to reasonably expect a twisted group C^*-algebra (regarded here as a deformation quantization of our Poisson bracket $\{\cdot\,,\cdot\}_\omega$) to be also equipped with a compatible comultiplication, we need to require that $\{\cdot\,,\cdot\}_\omega$ determines a Poisson-Lie group.

Since a typical Poisson bracket we consider is defined on the dual space of a Lie algebra, this means that it is reasonable to impose a condition that the dual vector space is itself a Lie group such that it forms, together with the given Poisson bracket, a Poisson-Lie group. This suggests us to consider the following.

Suppose H is a Poisson-Lie group. Then its Lie algebra $\mathfrak{h}$ is a Lie bialgebra such that its dual vector space $\mathfrak{g} = \mathfrak{h}^*$ is also a Lie bialgebra. The Lie group G of $\mathfrak{g}$ is the *dual Poisson-Lie group* of H. (See [21], [5], or Appendix of [14] for discussion on Poisson-Lie groups.) In other words, at the level of Poisson-Lie groups, the notion of a Poisson bracket defined on the "dual" of a Lie group naturally exists. Moreover, if the dual Poisson-Lie group G is exponential solvable (so G is diffeomorphic to $\mathfrak{g}$ via the exponential map), then we may transfer via the exponential map the compatible Poisson bracket on G to a Poisson bracket on $\mathfrak{g}$. To apply the result of [15], let us assume that the resulting Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$ is of the "cocycle perturbation" type.

Then by the main theorem (Theorem 3.4) in [15], a deformation quantization of $\mathfrak{g}$ is given in terms of the twisted group C^*-algebra of H. Since $\mathfrak{g} \cong G$, this can also be regarded as a deformation quantization of the Poisson-Lie group G. In particular, if G is globally linearizable (i.e. the compatible Poisson bracket on G is Poisson isomorphic to the linear Poisson bracket on $\mathfrak{g}$), its deformation quantization is given by the ordinary group C^*-algebra $C^*(H)$.

This set-up does not automatically provide a compatible comultiplication on the twisted group C^*-algebra. But we can usually collect enough data at the Poisson-Lie group level so that the candidates for comultiplication and other quantum group structures could be obtained. We then have to provide a rigorous analytic proof for our choice of comultiplication, which is not necessarily simple. It often helps to find some useful tools like multiplicative unitary operators (in the sense of Baaj and Skandalis ([3])).

Many of the earlier known examples of non-compact quantum groups, including the ones in [37], [31], [35], are deformations of some "globally linearizable" Poisson-Lie groups. So these quantum groups essentially look like ordinary group

(C^*-)algebras. (See also Section 7 of [42], [20], [9].) Whereas our method allows us to deform a more general type of Poisson groups whose compatible Poisson brackets are in general non-linear. In fact, these early examples are special cases of our construction.

In this paper, we will follow the method outlined above to construct some specific examples of quantum groups. We will begin the first section with the study of the $2n+1$ dimensional Heisenberg group H, equipped with a certain (linear) Poisson-Lie group structure (by [35], it is actually known that all possible Poisson brackets on H are linear). In particular, we will consider the one obtained from a certain "(quasitriangular) classical r-matrix".

Then we consider the dual Poisson-Lie group G of H. The dual Poisson bracket is in general not linear. But in our case, we show that it is of the "cocycle perturbation" type mentioned earlier. So, following the method of [15], we construct (in Section 2) a C^*-algebra which is a deformation quantization of this dual Poisson bracket.

On this C^*-algebra, we construct its quantum group structures, including comultiplication and the associated multiplicative unitary operator (in Section 3), counit and antipode (in Section 4), and Haar weight (in Section 5). In the last section, we find a *quasitriangular* "quantum universal R-matrix" type operator for our C^*-algebra, and relate it with the classical r-matrix we started with.

We discuss the representation theory of our quantum groups in a separate paper ([16]). The quantum R-matrix operator plays an important role here. Discussion of more general quantum groups which can be constructed using similar techniques are also postponed to a future occasion. For instance, we could consider a more general two-step nilpotent Lie group whose center has dimension higher than one, and try to deform its dual Poisson-Lie group equipped with its non-linear Poisson bracket. See [14] for a discussion.

1. THE LIE BIALGEBRAS, POISSON-LIE GROUPS

The notion of Poisson-Lie groups is more or less equivalent to the notion of Lie bialgebras ([7], [21]), and these are the objects to be quantized to produce quantum groups. In this section, we will study these "classical" objects, to find enough data we can use to construct our specific quantum groups. The Lie bialgebras we will exclusively study are either nilpotent or exponential solvable ones, so that we are able to treat their deformation quantizations in the C^*-algebra framework (see [15], [30]).

DEFINITION 1.1. Let $\mathfrak{h}$ be the $2n+1$ dimensional (real) Lie algebra generated by $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \dots, n)$, and $\mathbf{z}$, with the following relations:

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij}\mathbf{z}, \quad [\mathbf{z}, \mathbf{x}_i] = [\mathbf{z}, \mathbf{y}_i] = 0.$$

This is actually the well-known *Heisenberg algebra*. Let us also consider the extended Heisenberg algebra $\widetilde{\mathfrak{h}}$, generated by $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \dots, n), \mathbf{z}$, and $\mathbf{d}$, with the relations:

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij}\mathbf{z}, \quad [\mathbf{d}, \mathbf{x}_i] = \mathbf{x}_i, \quad [\mathbf{d}, \mathbf{y}_i] = -\mathbf{y}_i, \quad [\mathbf{z}, \mathbf{x}_i] = [\mathbf{z}, \mathbf{y}_i] = [\mathbf{z}, \mathbf{d}] = 0.$$

DEFINITION 1.2. The (connected and simply connected) Lie group corresponding to $\mathfrak{h}$ is the *Heisenberg group*, denoted by H. The space for this Lie group is isomorphic to $\mathbb{R}^{2n+1}$, and the multiplication on it is defined by

$$(x, y, z)(x', y', z') = \big(x + x', y + y', z + z' + \beta(x, y')\big),$$

for $x, y, x', y' \in \mathbb{R}^n$ and $z, z' \in \mathbb{R}$. Here $\beta(\cdot\,, \cdot)$ is the usual inner product on $\mathbb{R}^n$. We use this notation for a possible future generalization. For the extended Heisenberg group $\widetilde{H}$ (corresponding to $\widetilde{\mathfrak{h}}$), see Example 3.6 in Appendix below.

Taking advantage of the fact that their underlying spaces coincide, let us from now on identify H with $\mathfrak{h}$ (as spaces) via the evident map:

$$(x, y, z) \mapsto \sum_{i=1}^{n} (x_i \mathbf{x}_i + y_i \mathbf{y}_i) + z\mathbf{z},$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. On this space $H \cong \mathfrak{h}$, let us fix a Lebesgue measure. This would be the Haar measure for H.

Note that this definition of the Heisenberg group is different from the one that is given by the Baker-Campbell-Hausdorff series for $\mathfrak{h}$. Thus our set-up slightly differs (though isomorphic) from the setting in Section 3 of [15]. The use of the identification map as the diffeomorphism between $\mathfrak{h}$ and H will make the subsequent calculation simpler.

To obtain a Lie bialgebra structure on $\mathfrak{h}$, consider $\mathfrak{g} = \mathfrak{h}^*$, the dual vector space of $\mathfrak{h}$, and fix a nonzero real number λ. Let us define the following Lie algebra structure on $\mathfrak{g}$:

$$[\mathbf{p}_i, \mathbf{q}_j] = 0, \quad [\mathbf{p}_i, \mathbf{r}] = \lambda \mathbf{p}_i, \quad [\mathbf{q}_i, \mathbf{r}] = \lambda \mathbf{q}_i,$$

where $\mathbf{p}_i, \mathbf{q}_i$ $(i = 1, \dots, n), \mathbf{r}$ form the dual basis for $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \dots, n), \mathbf{z}$. Then we have the following:

PROPOSITION 1.3. *The (mutually dual) Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ determine a Lie bialgebra.*

Proof. We could prove this statement directly. But let us choose an indirect method, which would give us a deeper insight (and more information) about the situation.

Consider the following element contained in $\widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}}$:

$$r = \lambda \bigg(\mathbf{z} \otimes \mathbf{d} + \mathbf{d} \otimes \mathbf{z} + 2 \sum_{i=1}^{n} (\mathbf{x}_i \otimes \mathbf{y}_i) \bigg). \tag{1.1}$$

By elementary Lie algebra calculations, we can show that r satisfies the so-called "classical Yang-Baxter equation" (CYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

The notation r_{ij} is understood as an element in $\widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}}$, and the meaning is fairly obvious (see [7], [5]). We can also show without difficulty that $r_{12} + r_{21}$ is $\mathfrak{h}$-invariant. Therefore, r is a "quasitriangular classical r-matrix" ([7], [5]).

Since we have a (quasitriangular) r-matrix, a "coboundary" Lie bialgebra structure on $\widetilde{\mathfrak{h}}$ is defined by $\widetilde{\delta} : \widetilde{\mathfrak{h}} \to \widetilde{\mathfrak{h}} \wedge \widetilde{\mathfrak{h}}$, where

$$\widetilde{\delta}(X) = \mathrm{ad}_X(r).$$

By restricting $\widetilde{\delta}$ to $\mathfrak{h}$, we then obtain the map $\delta : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$, given by

$$\delta(\mathbf{x}_i) = \lambda \mathbf{x}_i \wedge \mathbf{z}, \quad \delta(\mathbf{y}_i) = \lambda \mathbf{y}_i \wedge \mathbf{z}, \quad \delta(\mathbf{z}) = 0.$$

This map is easily shown to be a 1-cocycle with respect to the adjoint representation for $\mathfrak{h}$, and hence $(\mathfrak{h}, \delta)$ defines a Lie bialgebra.

The Lie bialgebra structure δ on $\mathfrak{h}$ determines a Lie bracket on the dual vector space $\mathfrak{h}^*$ by:

$$\big\langle [\mu, \nu], X \big\rangle = \big\langle \mu \otimes \nu, \delta(X) \big\rangle,$$

where $\mu, \nu \in \mathfrak{h}^*$, $X \in \mathfrak{h}$, and $\langle \cdot\,, \cdot \rangle$ is the dual pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. By straightforward calculation using the definition of δ, we can see that the resulting Lie bracket coincides with the one we defined above on $\mathfrak{g} = \mathfrak{h}^*$. This means that the Lie bialgebra $(\mathfrak{h}, \delta)$ is exactly the one determined by the pair $\mathfrak{h}$ and $\mathfrak{g}$. ∎

Corollary 1.4. *By means of the classical r-matrix of* (1.1) *and the Lie bialgebra $(\widetilde{\mathfrak{h}}, \widetilde{\delta})$ obtained from it, we can also find the dual Lie bialgebra $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}}^*$ of $\widetilde{\mathfrak{h}}$ which is spanned by the dual basis elements $\mathbf{p}_i, \mathbf{q}_i$ $(i = 1, \ldots, n), \mathbf{r}, \mathbf{s}$, with $\mathbf{p}_i, \mathbf{q}_i, \mathbf{r}$ satisfying the same relation as before and $\mathbf{s}$ being central. By construction,* $(\mathfrak{h}, \delta)$ *or* $(\mathfrak{h}, \mathfrak{g})$ *is a "sub-bialgebra" of* $(\widetilde{\mathfrak{h}}, \widetilde{\delta})$ *or* $(\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{g}})$.

Remark 1.5. Unlike $(\widetilde{\mathfrak{h}}, \widetilde{\delta})$, the Lie bialgebra $(\mathfrak{h}, \delta)$ cannot be obtained as a coboundary from any classical r-matrix contained in $\mathfrak{h} \otimes \mathfrak{h}$. Thus the introduction of the extended Heisenberg algebra $\widetilde{\mathfrak{h}}$ is essential. The same situation occurs in [1] and [4], where the authors find (using the same classical r-matrix as above) a quantized universal enveloping algebra (i.e. QUE algebra) deformation of the Heisenberg algebra.

The Lie group G associated with $\mathfrak{g}$ is, by definition, the *dual Poisson-Lie group* of H. To know more about G, note first that the Lie algebra $\mathfrak{g}$ is a semi-direct product of its two (abelian) subalgebras $\mathfrak{m} = \operatorname{span}(\mathbf{r})$ and $\mathfrak{q} = \operatorname{span}(\mathbf{p}_i, \mathbf{q}_i | i = 1, \ldots, n)$. This is evident from its defining relations. Therefore, the connected and simply connected Lie group G associated with $\mathfrak{g}$ should be a semi-direct product group. Since $\mathfrak{m}$ and $\mathfrak{q}$ are abelian Lie algebras, they are identified (as spaces) with their corresponding abelian Lie groups. This suggests the following definition of G:

Definition 1.6. (The dual Poisson-Lie group) Let $G = \mathfrak{q} \times \mathfrak{m} = \mathfrak{g}$ as a vector space. Define the multiplication law on it by

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r').$$

Here, $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$, $r \in \mathbb{R}$, and we are identifying $(p, q, r) \in G$ with the element $\sum_{i=1}^n (p_i \mathbf{p}_i + q_i \mathbf{q}_i) + r\mathbf{r} \in \mathfrak{g}$. This means that, in particular, G is an exponential solvable Lie group. Now on G (which is being identified as a space with $\mathfrak{g} = \mathfrak{h}^*$), let us choose the Plancherel-Lebesgue measure, dual to the fixed Haar measure on $H \cong \mathfrak{h}$ (see Remark 1.7 at the end of this section). This will be the *left invariant Haar measure on G*.

The group G will be our main object of study. Following the method of [15], we are going to find a deformation quantization of G, using the duality between $\mathfrak{h}$ and $\mathfrak{g}$ (or H and G). Before we begin our main discussion, let us make a short remark on Fourier transforms between dual spaces. This will serve the purpose of setting up the notation we will use in this paper.

REMARK 1.7. (Fourier transforms between dual spaces, Plancherel measure) Let W be a (real) vector space. Let us fix a Lebesgue measure, $\mathrm{d}x$, on W. Let W^* be the dual vector space of W. We choose on W^* the dual "Plancherel measure", denoted by $\mathrm{d}\mu$, which is also a Lebesgue measure. Then the Fourier transform from $L^2(W)$ to $L^2(W^*)$ is given by

$$(\mathcal{F}\xi)(\mu) = \int_W \bar{e}\big[\langle \mu, x\rangle\big]\xi(x)\,\mathrm{d}x.$$

Here $\langle\cdot\,,\cdot\rangle$ denotes the dual pairing between W^* and W, and $e(\cdot)$ is the function defined by $e(t) = \mathrm{e}^{2\pi \mathrm{i}t}$. So $\bar{e}(t) = \mathrm{e}^{-2\pi \mathrm{i}t}$. By our choice of measures, the Fourier transform is a unitary operator, whose inverse is the following inverse Fourier transform:

$$(\mathcal{F}^{-1}\zeta)(x) = \int_{W^*} e\big[\langle \mu, x\rangle\big]\zeta(\mu)\,\mathrm{d}\mu.$$

If Z is a subspace of W and if we fix a Lebesgue measure, $\mathrm{d}z$, on Z, there is a unique Plancherel-Lebesgue measure, $\mathrm{d}\dot{x}$, on W/Z so that $\mathrm{d}x = \mathrm{d}\dot{x}\mathrm{d}z$. Since $Z^\perp = (W/Z)^*$, we can also choose as above an appropriate Plancherel measure, $\mathrm{d}q$, on $Z^\perp \subseteq W^*$. This enables us to define the "partial" Fourier transform from $L^2(W/Z \times W^*/Z^\perp)$ to $L^2(W^*) = L^2(Z^\perp \times W^*/Z^\perp)$, given by

$$f^\wedge(q,r) = \int_{W/Z} \bar{e}\big[\langle q, \dot{x}\rangle\big] f(\dot{x}, r)\,\mathrm{d}\dot{x}.$$

Its inverse, $\varphi \mapsto \varphi^\vee$, is defined similarly as above.

Let $S(W)$ denote the space of Schwartz functions on W. Then by Fourier transform, $S(W)$ is carried onto $S(W^*)$ and vice versa. The Fourier inversion theorem (the unitarity of the Fourier transform) implies that we have: $\mathcal{F}^{-1}(\mathcal{F}f) = f$ for $f \in S(W)$ and $\mathcal{F}(\mathcal{F}^{-1}\varphi) = \varphi$ for $\varphi \in S(W^*)$. A similar assertion is true for the partial Fourier transform.

2. DEFORMATION QUANTIZATION OF G

Let us compute explicitly the compatible Poisson bracket on the dual Poisson-Lie group G. Later in this section, we are going to find a deformation quantization of G in the direction of this Poisson bracket. To compute the Poisson bracket, let us first compute the Lie bialgebra structure $(\mathfrak{g}, \theta)$ on $\mathfrak{g}$. Since θ determines the dual Lie bialgebra of $(\mathfrak{h}, \delta)$, it should be the dual map of the given Lie bracket on $\mathfrak{g}^* = \mathfrak{h}$:

LEMMA 2.1. *Let $\theta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be defined by its values on the basis vectors of $\mathfrak{g}$ as follows:*

$$\theta(\mathbf{p}_i) = 0, \quad \theta(\mathbf{q}_i) = 0, \quad \theta(\mathbf{r}) = \sum_{i=1}^n (\mathbf{p}_i \otimes \mathbf{q}_i - \mathbf{q}_i \otimes \mathbf{p}_i) = \sum_{i=1}^n (\mathbf{p}_i \wedge \mathbf{q}_i).$$

Then θ is the dual map of the Lie bracket on $\mathfrak{h}$. Hence it is the 1-cocycle giving the dual Lie bialgebra structure on $\mathfrak{g}$.

Proof. Straightforward. ∎

By using the simple connectedness of G, the Lie bialgebra structure $(\mathfrak{g}, \theta)$ determines the compatible Poisson bracket on G (see [21], [5]). The calculation and the result is given below. See [35] for a similar result. Observe also that our expression of the Poisson bracket does not depend on the p and q variables.

THEOREM 2.2. *The Poisson bracket on the dual Poisson-Lie group G is given by the following expression:*
For $\varphi, \psi \in C^\infty(G)$,

$$\{\varphi, \psi\}(p, q, r) = \left(\frac{\mathrm{e}^{2\lambda r} - 1}{2\lambda}\right)\big(\beta(x, y') - \beta(x', y)\big), \quad (p, q, r) \in G \tag{2.1}$$

where $\mathrm{d}\varphi(p, q, r) = (x, y, z)$ and $\mathrm{d}\psi(p, q, r) = (x', y', z')$, which are naturally considered as elements of $\mathfrak{h}$.

Proof. Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation of G on $\mathfrak{g}$. We have to look for a map $F : G \to \mathfrak{g} \wedge \mathfrak{g}$, which is a group 1-cocycle on G for Ad and whose derivative at the identity element, $\mathrm{d}F_e$, coincides with the map θ. Since θ depends only on the r-variable, so should F. Thus we only need to look for a map F satisfying the condition:

$$F(r_1 + r_2) = F(r_1) + \mathrm{Ad}_{(0,0,r_1)}\big(F(r_2)\big),$$

such that its derivative at the identity element is the map, $\mathrm{d}F_e(r) = \theta(r) = r\sum_{i=1}^{n}(\mathbf{p}_i \wedge \mathbf{q}_i)$. Meanwhile, note that the representation Ad sends the basis vectors of $\mathfrak{g}$ as follows:

$$\mathrm{Ad}_{(0,0,r')}(\mathbf{p}_i) = (0, 0, r')(1, 0, 0)(0, 0, -r') = (\mathrm{e}^{-\lambda r'}, 0, 0) = \mathrm{e}^{-\lambda r'}\mathbf{p}_i,$$

$$\mathrm{Ad}_{(0,0,r')}(\mathbf{q}_i) = \mathrm{e}^{-\lambda r'}\mathbf{q}_i, \quad \mathrm{Ad}_{(0,0,r')}(\mathbf{r}) = \mathbf{r}.$$

So the 1-cocycle condition for F becomes:

$$F(r_1 + r_2) = F(r_1) + \mathrm{e}^{-2\lambda r_1}F(r_2).$$

From this equation together with the condition $\mathrm{d}F_e = \theta$, we obtain:

$$F(p, q, r) = F(r) = \left(\frac{1 - \mathrm{e}^{-2\lambda r}}{2\lambda}\right)\sum_{i=1}^{n}(\mathbf{p}_i \wedge \mathbf{q}_i).$$

The Poisson bivector field is the right translation of this 1-cocycle F, given by ${R_{(p,q,r)}}_*F(p, q, r)$. Since the right translations are ${R_{(p,q,r)}}_*(\mathbf{p}_i) = \mathrm{e}^{\lambda r}\mathbf{p}_i$ and ${R_{(p,q,r)}}_*(\mathbf{q}_i) = \mathrm{e}^{\lambda r}\mathbf{q}_i$, we obtain Equation (2.1) for our Poisson bracket by:

$$\{\varphi, \psi\}(p, q, r) = \big\langle {R_{(p,q,r)}}_*F(p, q, r), \mathrm{d}\varphi(p, q, r) \wedge \mathrm{d}\psi(p, q, r)\big\rangle. \quad \blacksquare$$

Since we will use the expression $(\mathrm{e}^{2\lambda r} - 1)/2\lambda$ quite often, let us give it a special notation, $\eta_{\lambda(r)}$. This function satisfies a convenient identity, which is given in Lemma 2.4. The proof is straightforward.

DEFINITION 2.3. Let $\lambda \in \mathbb{R}$ be fixed. Let us denote by η_λ the function on $\mathbb{R}$ defined by

$$\eta_\lambda(r) = \frac{e^{2\lambda r} - 1}{2\lambda}.$$

When $\lambda = 0$, we define $\eta_0(r) = r$.

LEMMA 2.4. *For $r, r' \in \mathfrak{g}/\mathfrak{q}$, we have:*

$$(e^{-2\lambda r'})\eta_\lambda(r + r') - (e^{-2\lambda r'})\eta_\lambda(r') = \eta_\lambda(r). \tag{2.2}$$

Since we are identifying $G \cong \mathfrak{g}$ as spaces, our Poisson bracket on G may as well be regarded as a Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$. It is a non-linear Poisson bracket, but it is nevertheless of the special type studied in [15]. We summarize this observation in the next proposition. Here $\mathfrak{z}$ denotes the center of $\mathfrak{h}$, spanned by $\mathbf{z}$. Also $\mathfrak{q} = \mathfrak{z}^\perp$, in $\mathfrak{g}$. As before, we regard the vectors $x, y, x', y' \in \mathbb{R}^n$ as elements of $\mathfrak{h}/\mathfrak{z} = \operatorname{span}(\mathbf{x}_i, \mathbf{y}_i \mid i = 1, \dots, n)$, and similarly $r \in \mathbb{R}$ as an element of $\mathfrak{g}/\mathfrak{q}$.

PROPOSITION 2.5. (i) *Let $\omega : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \to C^\infty(\mathfrak{g}/\mathfrak{q})$ be the map defined by*

$$\omega\big((x, y), (x', y'); r\big) = \eta_\lambda(r)\big(\beta(x, y') - \beta(x', y)\big).$$

Then ω is a Lie algebra cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in $V = C^\infty(\mathfrak{g}/\mathfrak{q})$, regarded as a trivial $U(\mathfrak{h}/\mathfrak{z})$-module.

(ii) *The Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$ given by Equation* (2.1) *is realized as the sum of the (trivial) linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$ and ω.*

(iii) *The space $V = C^\infty(\mathfrak{g}/\mathfrak{q})$ is canonically contained in $C^\infty(\mathfrak{g})$ such that $\mathfrak{h} \cap V = \mathfrak{z}$.*

Thus we conclude that our Poisson bracket is the "cocycle perturbation" (*in the sense of* [15]) *of the linear Poisson bracket on $\mathfrak{h}^*$.*

Proof. We can see easily that ω is a skew-symmetric, bilinear map, trivially satisfying the cocycle identity since $\mathfrak{h}/\mathfrak{z}$ is abelian. Since $\mathfrak{h}/\mathfrak{z}$ is an abelian Lie algebra, it also follows that the linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$ is the trivial one. Thus the second assertion of the proposition is immediate from the definition of ω.

The functions in $V = C^\infty(\mathfrak{g}/\mathfrak{q})$ can be canonically realized as functions in $C^\infty(\mathfrak{g})$ by the "pull-back" using the natural projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{q}$. If we regard the elements of $\mathfrak{h}$ also as (linear) functions in $C^\infty(\mathfrak{g})$, we have: $\mathfrak{h} \cap V = \mathfrak{z}$. It follows that our Poisson bracket is an extension of the linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$ by the cocycle ω. We showed in [15] (see Theorems 2.2 and 2.3) that this formulation is equivalent to viewing the Poisson bracket as a "cocycle perturbation" of the linear Poisson bracket on $\mathfrak{h}^*$. ∎

REMARK 2.6. When $\lambda = 0$, we have:

$$\omega_0\big((x,y),(x',y');r\big) = r\big(\beta(x,y') - \beta(x',y)\big).$$

It is a linear function on $\mathfrak{g}/\mathfrak{q}$, so we may write it as:

$$\omega_0\big((x,y),(x',y')\big) = \big(\beta(x,y') - \beta(x',y)\big)\mathbf{z}. \tag{2.3}$$

Thus ω_0 is a cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in $\mathfrak{z}$. It is clear that the linear Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$ (see [35]) is determined by the cocycle ω_0. In other words, the "perturbation" is given by (nonzero) λ and the associated cocyle ω.

Deformation quantization of our Poisson bracket on $\mathfrak{g}$, which we will denote by $\{\cdot\,,\cdot\}_\omega$ from now on, is obtained by following the steps of [15]. First, we construct from the given Lie algebra cocycle ω the continuous family of $\mathbb{T}$-valued group cocycles for the Lie group H/Z of $\mathfrak{h}/\mathfrak{z}$. Here $Z = \operatorname{span}\{\mathbf{z}\}$ is the Lie subgroup of H corresponding to $\mathfrak{z}$.

PROPOSITION 2.7. *Consider the map* $R : H/Z \times H/Z \to V = C^\infty(\mathfrak{g}/\mathfrak{q})$ *defined by*

$$R\big((x,y),(x',y');r\big) = \eta_\lambda(r)\beta(x,y').$$

Then R is a group cocycle for H/Z having values in V, regarded as an additive abelian group. Fix now an element $r \in \mathfrak{g}/\mathfrak{q}$. Define the map $\sigma^r : H/Z \times H/Z \to \mathbb{T}$ by

$$\sigma^r\big((x,y),(x',y')\big) = \bar{e}\big[R((x,y),(x',y');r)\big] = \bar{e}\big[\eta_\lambda(r)\beta(x,y')\big].$$

Then each σ^r is a $\mathbb{T}$-valued, normalized group cocycle for H/Z. Moreover, $r \mapsto \sigma^r$ forms a continuous field of cocycles.

Proof. Let $h = (x,y)$, $h' = (x',y')$, $h'' = (x'',y'')$ be elements of H/Z. Then for $r \in \mathfrak{g}/\mathfrak{q}$, we have the cocycle identity:

$$\begin{aligned}\sigma^r(hh',h'')\sigma^r(h,h') &= \bar{e}\big[\eta_\lambda(r)(\beta(x,y'') + \beta(x',y'') + \beta(x,y'))\big]\\ &= \sigma^r(h,h'h'')\sigma^r(h',h'').\end{aligned}$$

We also have $\sigma^r(h,0) = 1 = \sigma^r(0,h)$, where $0 = (0,0)$ is the identity element of H/Z. From the definition, the continuity is also clear. ∎

Let us consider $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$, the space of Schwartz functions in the (x,y,r)-variables having compact support in the $r \in \mathfrak{g}/\mathfrak{q}$ variable. Since $\mathfrak{h}/\mathfrak{z}$ is identified with the abelian group H/Z, we can regard $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ as contained in $L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$. Using the continuous field of cocycles σ, we can define on it the following twisted convolution:

$$(f *_\sigma g)(x,y,r) = \int f(\tilde{x},\tilde{y},r)g(x-\tilde{x},y-\tilde{y},r)\bar{e}\big[\eta_\lambda(r)\beta(\tilde{x},y-\tilde{y})\big]\,\mathrm{d}\tilde{x}d\tilde{y}.$$

It is not difficult to see that $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is indeed an algebra.

To transfer this algebra structure to the level of functions on $\mathfrak{g}$, we introduce the partial Fourier transform. The partial Fourier transform, $^\wedge$, from $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ to $S(\mathfrak{g}) = S(\mathfrak{q} \times \mathfrak{g}/\mathfrak{q})$ is defined by

$$f^\wedge(p,q,r) = \int \bar{e}(p\cdot x + q\cdot y)f(x,y,r)\,\mathrm{d}x\,\mathrm{d}y,$$

where $p \cdot x + q \cdot y$ is the dual pairing between $(p, q) \in \mathfrak{q}$ and $(x, y) \in \mathfrak{h}/\mathfrak{z}$. The inverse partial Fourier transform, $^\vee$, from $S(\mathfrak{g})$ to $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is defined in a similar manner, with $\bar{e}(\cdot)$ replaced by $e(\cdot)$. We are again assuming that we have chosen appropriate Plancherel measures for $\mathfrak{h}/\mathfrak{z}$ and $\mathfrak{q} = \mathfrak{z}^\perp$, so that the Fourier inversion theorem is valid.

To define the deformed multiplication between the functions on $\mathfrak{g}$, consider the subspace $\mathcal{A} = S_{3c}(\mathfrak{g}) \subseteq S(\mathfrak{g})$, which is the image under the partial Fourier transform, $^\wedge$, of the twisted convolution algebra $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$.

PROPOSITION 2.8. *Let $\mathcal{A} = S_{3c}(\mathfrak{g})$ be the space of Schwartz functions on $\mathfrak{g}$ having compact support in the r-variable. On $\mathcal{A}$, define the deformed multiplication, $\times$, by: $\varphi \times \psi = (\varphi^\vee *_\sigma \psi^\vee)^\wedge$, for $\varphi, \psi \in \mathcal{A}$. We then obtain:*

$$(\varphi \times \psi)(p, q, r) = \int \bar{e}\big[(p - p') \cdot \tilde{x}\big] \varphi(p', q, r) \psi(p, q + \eta_\lambda(r)\tilde{x}, r) \, dp' \, d\tilde{x}.$$

Proof. Use the Fourier inversion theorem to the expression:

$$\begin{aligned}&(\varphi \times \psi)(p, q, r) = (\varphi^\vee *_\sigma \psi^\vee)^\wedge(p, q, r) \\ &= \int \bar{e}(p \cdot x + q \cdot y) \varphi^\vee(\tilde{x}, \tilde{y}, r) \psi^\vee(x - \tilde{x}, y - \tilde{y}, r) \bar{e}\big[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})\big] \, d\tilde{x} \, d\tilde{y} \, dx \, dy. \quad \blacksquare\end{aligned}$$

Note that when $\lambda = 0$, the operation $*_\sigma$ on $L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$ is given by the cocycle $\big((x, y), (x', y')\big) \mapsto \bar{e}\big[r\beta(x, y')\big]$ for H/Z. But this is essentially the ordinary convolution on $S(H)$ transferred to the functions in the (x, y, r)-variables. Compare this with our case, with the cocycle $\sigma^r\big((x, y), (x', y')\big) = \bar{e}\big[\eta_\lambda(r)\beta(x, y')\big]$. We can see that the passage from the linear Poisson bracket (when $\lambda = 0$) to our "perturbed" (non-linear) Poisson bracket corresponds to the "change of cocycles", or to the passage from ordinary convolution to twisted convolution.

The situations between linear Poisson bracket case ([30]) and our perturbed case are quite similar, and this will be exploited from time to time. However, in our more general case, the space $S(\mathfrak{g})$ is no longer closed under the deformed multiplication. We had to define the multiplication in its subspace $\mathcal{A}$.

The algebra $\mathcal{A}$ is shown to be a pre-C^*-algebra, whose involution and C^*-norm are again obtained using the partial Fourier transform between $\mathcal{A}$ and $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$, the latter being viewed as a (dense) subalgebra of the $*$-algebra $L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$. See [15], for the exact definitions of the $*$-algebra operations.

PROPOSITION 2.9. *Let $\mathcal{A}$ be as above.*

(i) *The involution on $\mathcal{A}$ is defined by: $\varphi \mapsto \big((\varphi^\vee)^*\big)^\wedge$, where $*$ denotes the involution on $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$. If we denote the involution on $\mathcal{A}$ by the same notation, $*$, then we have:*

$$\varphi^*(p, q, r) = \int \overline{\varphi(p', q', r)} \bar{e}\big[(p - p') \cdot x + (q - q') \cdot y\big] \bar{e}\big[\eta_\lambda(r)\beta(x, y)\big] \, dp' dq' dx \, dy.$$

(ii) *Via partial Fourier transform, we also define the canonical C^*-norm on $\mathcal{A}$, by transferring the canonical C^*-norm on $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \subseteq L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$.*

Proof. The involution on $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \subseteq L'(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$ is given by

$$f^*(x,y,r) = \overline{f(-x,-y,r)}\bar{e}\big[\eta_\lambda(r)\beta(x,y)\big].$$

It is easy to see that $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is closed under the involution. We transfer this operation to the $\mathcal{A}$ level by $\varphi \mapsto \big((\varphi^\vee)^*\big)^\wedge$. Use the Fourier inversion theorem to obtain the above expression.

On $L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$, one has the canonical C^*-norm, $\|\cdot\|$, such that the completion with respect to $\|\cdot\|$ of this L^1 algebra is the enveloping C^*-algebra $C^*\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$, called the twisted group C^*-algebra. By $\varphi \mapsto \|\varphi^\vee\|$, we can define on $\mathcal{A}$ its C^*-norm, still denoted by $\|\cdot\|$. ∎

Since the function space $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is dense in $L^2(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ with respect to the L^2-norm, its product $*_\sigma$ corresponds to a representation of $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ on $L^2(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ such that the functions acts as the multiplication operators. This representation is naturally extended to $C^*(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$. More precisely, we have a representation, L, of the twisted group C^*-algebra on $L^2(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ defined by

$$(L_f\xi)(x,y,r) = \int f(x',y',r)\xi(x-x',y-y',r)\bar{e}\big[\eta_\lambda(r)\beta(x',y-y')\big]\,\mathrm{d}x'\mathrm{d}y',$$

for $f \in S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ and $\xi \in L^2(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$. It is actually a (*left*) *regular representation* of $C^*(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$, induced from a (faithful) representation of $C_\infty(\mathfrak{g}/\mathfrak{q})$ on $L^2(\mathfrak{g}/\mathfrak{q})$ given by multiplication.

In what follows, we will be working with the Hilbert space $L^2(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ most of the time. So let us from now on denote this Hilbert space by $\mathcal{H}$. Via the isomorphism between $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ and $\mathcal{A}$, the representation L may as well be regarded as a representation of $\mathcal{A}$ on $\mathcal{H}$. Let us also denote this representation by L. Then for $\varphi \in \mathcal{A}$,

$$\begin{aligned}(L_\varphi\xi)(x,y,r) &= \int \varphi^\vee(x',y',r)\xi(x-x',y-y',r)\bar{e}\big[\eta_\lambda(r)\beta(x',y-y')\big]\,\mathrm{d}x'\mathrm{d}y' \\ &= \int \varphi(p,q,r)\xi(x-x',y-y',r)e(p\cdot x' + q\cdot y') \\ &\qquad \cdot \bar{e}\big[\eta_\lambda(r)\beta(x',y-y')\big]\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}x'\mathrm{d}y'.\end{aligned} \tag{2.4}$$

It is clear that L is equivalent to the representation of $\mathcal{A}$ on $L^2(\mathfrak{g})$ given by the multiplication $\times$. The partial Fourier transform is the intertwining unitary operator between the Hilbert spaces $\mathcal{H}$ and $L^2(\mathfrak{g})$.

The representation L is the regular representation induced from a faithful representation of $C^\infty(\mathfrak{g}/\mathfrak{q})$. So the corresponding C^*-norm and the completion will give us the "reduced" twisted group C^*-algebra $C^*_{\mathrm{r}}\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$. Since H/Z is abelian, the amenability condition holds in our case, i.e. $C^*_{\mathrm{r}}\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big) \cong C^*\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$. This follows from the result of Packer and Raeburn ([26]), which says that the amenability of the group implies the amenability of the twisted group C^*-algebra. Because of the amenability, we can see that for $\varphi \in \mathcal{A}$, we have $\|\varphi\| = \|L_\varphi\|$.

DEFINITION 2.10. Let $\mathcal{A}$ be defined as above and let it be equipped with the multiplication $\times$ given by Proposition 2.8 and the involution $*$ given by Proposition 2.9. Let us denote by A the C^*-completion of $\mathcal{A}$ with respect to the norm defined by $\|\varphi\| = \|L_\varphi\|$, where L_φ is regarded as an operator on $\mathcal{H}$ by Equation (2.4). This is the C^*-algebra we will be interested in throughout the rest of this paper. We have: $A \cong C^*_{\mathrm{r}}\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big) \cong C^*\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$.

Recall that we are identifying G with $\mathfrak{g}$ as spaces and the Plancherel-Lebesgue measure on $\mathfrak{g}$ we have been using coincides with the Haar measure on G. We thus have, as a (dense) subspace, $\mathcal{A} \subseteq C_\infty(G)$. Thus, the results we obtained so far about functions on $\mathfrak{g}$ hold true for functions on G. The deformed function algebra $(\mathcal{A}, \times, *)$, as well as its C^*-completion A, provides a deformation quantization (to be described below) of $C_\infty(G)$.

At each of the steps above, we could have kept the parameter $\hbar$ as in [15]. In our case, the deformed algebra would be isomorphic to the twisted group algebra of $(H/Z)_\hbar = H/Z$ with the cocycle $\sigma_\hbar$ given by $\sigma^r_\hbar\big((x,y),(x',y')\big) = \bar{e}\big[\hbar\eta_\lambda(r)\beta(x,y')\big]$. Since H/Z is abelian, the group does not have to vary and only the cocycle σ varies under the introduction of the parameter $\hbar$. See [15] for more precise formulation.

Let us denote by $(\times_\hbar, *_\hbar, \|\cdot\|_\hbar)$ the corresponding operations on $\mathcal{A}$ obtained by the introduction of the parameter $\hbar$. The above discussion means that all we have to do is to replace $\beta(\cdot\,,\cdot)$ by $\hbar\beta(\cdot\,,\cdot)$. Then define $A_\hbar$ as the C^*-completion of $\mathcal{A}$ with respect to $\|\cdot\|_\hbar$. By the main theorem (Theorem 3.4) of [15], we thus obtain a (strict) deformation quantization of our Poisson bracket $\{\cdot\,,\cdot\}_\omega$ on G.

THEOREM 2.11. *Consider the dual Poisson bracket on $G \cong \mathfrak{g}$ defined by Equation* (2.1). *Let $\mathcal{A} = S_{3c}(G)$ be the subspace of $S(G)$ defined above. For any $\hbar \in \mathbb{R}$, define a deformed multiplication and an involution on $\mathcal{A}$, and also a C^*-norm on it, by replacing $\beta(\cdot\,,\cdot)$ with $\hbar\beta(\cdot\,,\cdot)$ in Definition* 2.10. *Then $\big(\mathcal{A}, \times_\hbar, {}^{*_\hbar}, \|\cdot\|_\hbar\big)_{\hbar\in\mathbb{R}}$ provides a strict deformation quantization (in the sense of* [29], [32]*) of $\mathcal{A}$ in the direction of $(1/2\pi)\{\cdot\,,\cdot\}_\omega$. In particular, we have:*

$$\lim_{\hbar\to 0}\left\|\frac{\varphi\times_\hbar\psi-\psi\times_\hbar\varphi}{\hbar}-\frac{\mathrm{i}}{2\pi}\{\varphi,\psi\}_\omega\right\|_\hbar = 0. \tag{2.5}$$

Proof. For full proof of the theorem, refer to Theorem 3.4 in [15], of which ours is a special case. We will briefly mention here a few of the main points of the proof.

First, we have to show that the family of C^*-algebras $\{A_\hbar\}_{\hbar\in\mathbb{R}}$, where each $A_\hbar$ is the C^*-completion of $\mathcal{A}$ with respect to $\|\cdot\|_\hbar$, forms a continuous field of C^*-algebras. Since each $A_\hbar$ is essentially a twisted group C^*-algebra of an abelian group H/Z, and only the cocycle is being changed, the proof is actually simpler than in [15].

Second, to prove the deformation property, it suffices to show that on $\mathcal{A}$, the expression $(\varphi\times_\hbar\psi-\psi\times_\hbar\varphi)/\hbar-(\mathrm{i}/2\pi)\{\varphi,\psi\}_\omega$ has an L^1-bound. Then by Lebesgue's dominated convergence theorem, we would have the convergence in the L^1-norm, which in turn gives the convergence (2.5) since the L^1-norm dominates all the C^*-norms $\|\cdot\|_\hbar$. The proof crucially uses the fact that our functions are Schwartz functions having compact support in the $r \in \mathfrak{g}/\mathfrak{q}$ variable. ∎

From now on, we will fix the parameter $\hbar$ (e.g. $\hbar = 1$) and take the resulting C^*-algebra A as the candidate for our quantum group. If we want to specify the deformation process, we can always re-introduce $\hbar$, and follow the arguments above.

To summarize, the meaning of above construction and Definition 2.10 is that we are viewing the functions in $\mathcal{A} \subseteq S(G)$ as operators on $\mathcal{H}$, by the regular representation L. This naturally defines the deformed multiplication on $\mathcal{A}$, which is shown to be a deformation quantization of $\big(G, \{\cdot\,,\cdot\}_\omega\big)$ by Theorem 2.11. So from now on, we will regard φ and L_φ as the same. Viewing $\varphi \in \mathcal{A}$ as a function has an advantage when we try to establish a correspondence between our quantum setting and the classical, Poisson-Lie group level. While, viewing it as an operator $L_\varphi \in A \subseteq \mathcal{B}(\mathcal{H})$ is essential to make our discussion rigorous at the C^*-algebra level of "locally compact quantum groups".

Meanwhile, note that L_φ can be written as

$$L_\varphi = \int_H (\mathcal{F}^{-1}\varphi)(x, y, z) L_{x,y,z} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z, \tag{2.6}$$

where $\mathcal{F}^{-1}$ is the (inverse) Fourier transform from $\mathcal{A}$ into $S(\mathfrak{h})$, and $L_{a,b,c}$ for $(a, b, c) \in H$ is the operator on $\mathcal{H}$ defined by

$$(L_{a,b,c}\xi)(x, y, r) = \bar{e}(rc)\bar{e}\big[\eta_\lambda(r)\beta(a, y - b)\big]\xi(x - a, y - b, r). \tag{2.7}$$

By using the Fourier inversion theorem purely formally to this expression, $L_{a,b,c}$ can be written as:

$$\begin{aligned}(L_{a,b,c}\xi)(x, y, r) = \int \bar{e}\big[\langle (p, q, r), (a, b, c)\rangle\big] e(p \cdot \widetilde{x} + q \cdot \widetilde{y})\cdot \\ \cdot\, \bar{e}\big[\eta_\lambda(r)\beta(\widetilde{x}, y - \widetilde{y})\big]\xi(x - \widetilde{x}, y - \widetilde{y}, r) \,\mathrm{d}p \,\mathrm{d}q \,\mathrm{d}\widetilde{x} \,\mathrm{d}\widetilde{y}.\end{aligned}$$

Comparing this with Equation (2.4), we see that $L_{a,b,c}$ can be regarded as a (continuous) function on G defined by

$$L_{a,b,c}(p, q, r) = \bar{e}\big[\langle (p, q, r), (a, b, c)\rangle\big] = \bar{e}[p \cdot a + q \cdot b + rc].$$

Note however that $L_{a,b,c}$ is not contained in $\mathcal{A}$. It is not even an element of A. A more precise statement is that $L_{a,b,c}$ is a multiplier (i.e. an element of $M(A)$).

By (2.6), any representation Π of $\mathcal{A}$ or A will be written as

$$\Pi(\varphi) = \Pi(L_\varphi) = \int_H (\mathcal{F}^{-1}\varphi)(x, y, z)\Pi(L_{x,y,z}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z.$$

This means that if we have to check whether two non-degenerate representations are equal, it suffices to check that they are equal on the $L_{x,y,z}$'s. In this sense, we will call the $L_{a,b,c}$'s as "building blocks" for the regular representation, or equivalently, "building blocks" of A.

3. COMULTIPLICATION. THE MULTIPLICATIVE UNITARY OPERATOR

We have constructed our C^*-algebra A as a strict deformation quantization of the Poisson-Lie group $\big(G, \{\cdot\,,\cdot\}_\omega\big)$. We now proceed to equip A with its quantum group structures. The first step is to define an appropriate comultiplication on it. An efficient way is to associate a suitable "multiplicative unitary operator" ([3]). That is, we look for a unitary operator U defined on the Hilbert space $\mathcal{H}\otimes\mathcal{H}$, such that the "pentagon equation" holds (i.e. $U_{12}U_{13}U_{23} = U_{23}U_{12}$) and such that the comultiplication on A is given by

$$\Delta\varphi = U(\varphi\otimes 1)U^* = U(L_\varphi\otimes 1)U^*,$$

for φ contained in the dense subalgebra $\mathcal{A}$ of A.

To motivate our choice of U, let us recall the multiplicative unitary operator for the ordinary group C^*-algebra $C^*(H)$. It is the operator V defined on $L^2(H\times H)$ by

$$\begin{aligned}(V\eta)(x,y,z;x',y',z') &= \eta\big(x,y,z;(x,y,z)^{-1}(x',y',z')\big)\\ &= \eta\big(x,y,z;x'-x,y'-y,z'-z-\beta(x,y'-y)\big).\end{aligned}$$

It is well known ([3], [8]) that V describes the usual cocommutative Hopf C^*-algebra structure on $C^*(H)$. Via partial Fourier transform, it may as well be viewed as an operator on the (x,y,r) variables (still denoted by V):

$$(V\xi)(x,y,r,x',y',r') = \bar{e}\big[r'\beta(x,y'-y)\big]\xi(x,y,r+r',x'-x,y'-y,r').$$

Since A is essentially a twisted $C^*(H)$, we expect that V needs to be changed accordingly. Since the above V represents the regular representation of $C^*(H)$ ([3]), we expect that the new unitary operator should reflect the regular representation L of our twisted group C^*-algebra. So by using the trick of "changing of cocycles" that we mentioned earlier, let us first consider the following unitary operator V_σ (where σ is included to emphasize the cocycle) defined on $\mathcal{H}\otimes\mathcal{H}$:

$$(V_\sigma\xi)(x,y,r,x',y',r') = \bar{e}\big[\eta_\lambda(r')\beta(x,y'-y)\big]\xi(x,y,r+r',x'-x,y'-y,r').$$

We also have to take into account the point that A should be a quantum version of $C_\infty(G)$. We will do this by introducing a certain unitary operator W carrying the information on G. The idea is similar to the "dual cocycle" of Landstad ([20], [9]), although W is not exactly a dual cocycle and V_σ is not even multiplicative. Let us consider the following operator W on $L^2(G\times G)$, motivated by the group multiplication law on G:

$$(W\zeta)(p,q,r,p',q',r') = (\mathrm{e}^{\lambda r'})^n\,\zeta(\mathrm{e}^{\lambda r'}p,\mathrm{e}^{\lambda r'}q,r,p',q',r').$$

We may view it as an operator on $\mathcal{H}\otimes\mathcal{H}$, still denoted by W:

$$(W\xi)(x,y,r,x',y',r') = (\mathrm{e}^{-\lambda r'})^n\,\xi(\mathrm{e}^{-\lambda r'}x,\mathrm{e}^{-\lambda r'}y,r,x',y',r').$$

We then incorporate W with V_σ by defining the unitary operator $U = WV_\sigma$. We will show in what follows that U is the multiplicative unitary operator for A we are looking for. We begin by showing that U is indeed multiplicative.

PROPOSITION 3.1. *Let U be the unitary operator on $\mathcal{H}\otimes\mathcal{H}$ defined by*

$$\begin{aligned} U\xi(x,y,r,x',y',r') &= WV_\sigma\xi(x,y,r,x',y',r') \\ &= (\mathrm{e}^{-\lambda r'})^n\,\bar{e}\big[\eta_\lambda(r')\beta(\mathrm{e}^{-\lambda r'}x, y'-\mathrm{e}^{-\lambda r'}y)\big]\cdot \\ &\quad\cdot\xi(\mathrm{e}^{-\lambda r'}x, \mathrm{e}^{-\lambda r'}y, r+r', x'-\mathrm{e}^{-\lambda r'}x, y'-\mathrm{e}^{-\lambda r'}y, r'). \end{aligned}$$

Then U is multiplicative. That is, it satisfies the following "pentagon equation":

$$U_{12}U_{13}U_{23} = U_{23}U_{12}.$$

Proof. Use Lemma 2.4 and calculate:

$$\begin{aligned} &(U_{23}U_{12}\xi)(x_1,y_1,r_1,x_2,y_2,r_2,x_3,y_3,r_3) \\ &\quad= (\mathrm{e}^{-\lambda r_3})^n\,\bar{e}\big[\eta_\lambda(r_3)\beta(\mathrm{e}^{-\lambda r_3}x_2, y_3-\mathrm{e}^{-\lambda r_3}y_2)\big] \\ &\qquad\cdot(\mathrm{e}^{-\lambda r_2-\lambda r_3})^n\,\bar{e}\big[\eta_\lambda(r_2+r_3)\beta(\mathrm{e}^{-\lambda(r_2+r_3)}x_1, \mathrm{e}^{-\lambda r_3}y_2-\mathrm{e}^{-\lambda(r_2+r_3)}y_1)\big] \\ &\qquad\cdot\xi(\mathrm{e}^{-\lambda(r_2+r_3)}x_1, \mathrm{e}^{-\lambda(r_2+r_3)}y_1, r_1+r_2+r_3, \mathrm{e}^{-\lambda r_3}x_2-\mathrm{e}^{-\lambda(r_2+r_3)}x_1, \\ &\qquad\cdot\mathrm{e}^{-\lambda r_3}y_2-\mathrm{e}^{-\lambda(r_2+r_3)}y_1, r_2+r_3, x_3-\mathrm{e}^{-\lambda r_3}x_2, y_3-\mathrm{e}^{-\lambda r_3}y_2, r_3) \\ &\quad= (U_{12}U_{13}U_{23}\xi)(x_1,y_1,r_1,x_2,y_2,r_2,x_3,y_3,r_3). \quad\blacksquare \end{aligned}$$

For the building block $L_{a,b,c}$ we introduced earlier, define $\Delta L_{a,b,c}$ by

$$\Delta L_{a,b,c} = U(L_{a,b,c}\otimes 1)U^*. \tag{3.1}$$

Then as an operator on $\mathcal{H}\otimes\mathcal{H}$, we have:

$$\begin{aligned} &(\Delta L_{a,b,c}\xi)(x,y,r,x',y',r') \\ &\quad= \bar{e}\big[\eta_\lambda(r+r')\beta(a,\mathrm{e}^{-\lambda r'}y-b)+\eta_\lambda(r')\beta(a,y'-\mathrm{e}^{-\lambda r'}y)\big]\bar{e}\big[(r+r')c\big] \\ &\qquad\cdot\xi(x-\mathrm{e}^{\lambda r'}a, y-\mathrm{e}^{\lambda r'}b, r, x'-a, y'-b, r'). \end{aligned}\tag{3.2}$$

We can show that it is contained in the multiplier algebra $M(A\otimes A)$, which is rather straightforward (see also the proof of Theorem 3.2 below). Moreover, we may regard it as a (continuous) function on $G\times G$ as follows:

$$\begin{aligned} (\Delta L_{a,b,c})(p,q,r,p',q',r') &= \bar{e}\big[\langle(\mathrm{e}^{\lambda r'}p+p', \mathrm{e}^{\lambda r'}q+q', r+r'),(a,b,c)\rangle\big] \\ &= L_{a,b,c}\big((p,q,r)(p',q',r')\big). \end{aligned}\tag{3.3}$$

(Call this function $F\in C_\infty(G\times G)$ and use Equation (2.4) to compute $(L\otimes L)_F$. Using partial Fourier transform purely formally, we can show that it agrees with $\Delta L_{a,b,c}$ given by (3.2).)

In other words, at the level of the building blocks $L_{a,b,c}$, the map Δ works as the natural comultiplication on $C_\infty(G)$. In view of the fact that the Poisson structure δ on $H=G^*$ is linear (see Section 1), this is a desirable choice. Let us now extend Δ to the whole algebra A and obtain our comultiplication:

THEOREM 3.2. *For $\varphi \in \mathcal{A}$, define $\Delta\varphi$ by*

$$\Delta\varphi = U(\varphi \otimes 1)U^* = \int_H (\mathcal{F}^{-1}\varphi)(x,y,z)\Delta L_{x,y,z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

As before, φ and $\Delta\varphi$ are actually understood as the operators L_φ and $(L \otimes L)_{\Delta\varphi}$. Then Δ can be extended to a map $\Delta : A \to M(A \otimes A)$, and Δ is the comultiplication on A. That is, Δ is a nondegenerate C^*-homomorphism satisfying the coassociativity law:

$$(\Delta \otimes \mathrm{id})\Delta\varphi = (\mathrm{id} \otimes \Delta)\Delta\varphi.$$

Proof. It is clear that the formula $\Delta\varphi = U(\varphi \otimes 1)U^*$ defines a $*$-homomorphism, which can be naturally extended to a representation of A into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

To prove that Δ carries A into the multiplier algebra $M(A \otimes A)$, we intend to show that:

$$(\Delta\varphi)(1 \otimes g) \in S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}), \quad g \in S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \cong \mathcal{A}.$$

Here, $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is the space of Schwartz functions having compact support in the r and the r' variables. This is a dense subspace of $A \otimes A$ (see Remark 3.3).

Let $\xi \in \mathcal{H} \otimes \mathcal{H}$ and calculate. We use the change of variables and the Fourier inversion theorem. Also, Equation (2.2) of Lemma 2.4 is very convenient. We have:

$$\begin{aligned}
&(\Delta\varphi(1 \otimes g)\xi)(x,y,r,x',y',r') \\
&= \int (\mathcal{F}^{-1}\varphi)(a,b,c)\bar{e}\big[\eta_\lambda(r+r')\beta(a, \mathrm{e}^{-\lambda r'}y - b) + \eta_\lambda(r')\beta(a, y' - \mathrm{e}^{-\lambda r'}y)\big] \\
&\qquad \cdot \bar{e}\big[(r+r')c\big] g(\widetilde{x}, \widetilde{y}, r')\bar{e}\big[\eta_\lambda(r')\beta(\widetilde{x}, y' - b - \widetilde{y})\big] \\
&\qquad \cdot \xi(x - \mathrm{e}^{\lambda r'}a, y - \mathrm{e}^{\lambda r'}b, r, x' - a - \widetilde{x}, y' - b - \widetilde{y}, r')\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y} \\
&= (F\xi)(x,y,r,x',y',r') = (L \otimes L)_F \xi(x,y,r,x',y',r'),
\end{aligned}$$

where F is defined by

$$\begin{aligned}
F(x,y,r,x',y',r') &= (\mathrm{e}^{-2\lambda r'})^n\,\bar{e}\big[\eta_\lambda(r')\beta(\mathrm{e}^{-\lambda r'}x, y' - \mathrm{e}^{-\lambda r'}y)\big] \\
&\qquad \cdot \varphi^\vee(\mathrm{e}^{-\lambda r'}x, \mathrm{e}^{-\lambda r'}y, r + r')g(x' - \mathrm{e}^{-\lambda r'}x, y' - \mathrm{e}^{-\lambda r'}y, r').
\end{aligned}$$

It is easy to see that $F \in S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$. A similar result also holds when we multiply $\Delta\varphi$ from the right.

Since $(\Delta\varphi)(1 \otimes g) \in A \otimes A$ and $(1 \otimes g)(\Delta\varphi) \in A \otimes A$, for an arbitrary g contained in a dense subset of A, we can see that $\Delta\varphi \in M(A,A)$, where $M(A,A) = \{x \in M(A \otimes A) : x(1_{M(A)} \otimes A) + (1_{M(A)} \otimes A)x \in A \otimes A\}$. It is customary to require (see [36], [3]) that the comultiplication Δ takes values in $M(A,A)$. This is done so that one is able to discuss the notion of "left invariant" Haar weight on A.

Actually, we can improve the statement even further by observing that $(\Delta\varphi)(1 \otimes g)$'s form a total set (with respect to the L^1-norm) in the Schwartz space $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$. We may check this using the expression given above. Since the Schwartz space is in turn a dense subspace of $A \otimes A$, this is enough to show that Δ is also non-degenerate.

Finally, the coassociativity of Δ follows from the fact that U is multiplicative. For $\varphi \in \mathcal{A}$, we have:

$$U_{12}U_{13}(\varphi \otimes 1 \otimes 1)U_{13}^*U_{12}^* = U_{23}U_{12}(\varphi \otimes 1 \otimes 1)U_{12}^*U_{23}^*.$$

But, by definition of Δ, this is just:

$$(\Delta \otimes \mathrm{id})\Delta\varphi = (\mathrm{id} \otimes \Delta)\Delta\varphi. \quad \blacksquare$$

REMARK 3.3. For completing the above proof, we need to show that the Schwartz space $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is a dense subset of $A \otimes A$. So, consider the natural injection from $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, which is continuous with respect to the L^1-norm on the Schwartz space and the C^*-norm on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Under this natural injection, the algebraic tensor product $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \odot S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is sent into a dense subset of the algebraic tensor product $A \odot A$. Since any element in $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ can be approximated by elements of $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \odot S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ in the L^1-norm, we conclude that $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is mapped into a dense subset of $A \otimes A$.

Our choice of $\Delta L_{a,b,c}$ (Equation (3.2)) together with the above theorem means that the comultiplication remains the "same" while the algebra is being deformed (i.e. our deformation is a *preferred* deformation ([10], [11])). In this way, we obtained our Hopf C^*-algebra (A, Δ).

DEFINITION 3.4. ([36], [3]) By a *Hopf C^*-algebra*, we mean a pair (B, Δ), where B is a C^*-algebra and Δ is a comultiplication (satisfying the conditions given in Theorem 3.2).

It may not be evident, but our construction is closely related with Baaj and Skandalis's construction of Hopf C^*-algebras via "matched pair" (*couple assorti*) and "bicrossed product" (*biproduit croisé*) of "Kac Systems" (Section 8 of [3]; a similar work at the algebraic level was done by Majid ([22])).

To be a little more specific, the abelian groups H/Z and $(\mathfrak{g}/\mathfrak{q}, +)$ (or in terms of Hopf C^*-algebras, $C^*(H/Z)$ and $C_\infty(\mathfrak{g}/\mathfrak{q})$) form a matched pair. From this, we can form a "twisted" bicrossed product, using the notion of cocycles satisfying certain equivariance condition. The multiplicative unitary operator associated with this "matched pair with cocycle" construction is *regular* ([3]). Although our construction of (A, Δ) and Baaj and Skandalis' method are rather different, we can still show that our multiplicative unitary operator U for A coincides with the multiplicative unitary operator for this twisted bicrossed product.

We do not intend to prove the regularity of U directly. (However, the result in the proof of Theorem 3.2 that $(\Delta\varphi)(1 \otimes g)$'s form a total set in $A \otimes A$ is very much related.) Instead, let us refer to the above discussions and summarize the result in the following:

PROPOSITION 3.5. *Let U be the unitary operator defined as in Proposition* 3.1. *It is a "regular" multiplicative unitary operator, in the sense of Baaj and Skandalis.*

This result gives our construction an axiomatically sound basis: If we start from the multiplicative unitary operator U, its associated Hopf C^*-algebra is exactly (A, Δ). Also associated with the regular multiplicative unitary operator is the dual Hopf C^*-algebra $(\widehat{A}, \widehat{\Delta})$. In our case, $\widehat{A}$ is essentially the group C^*-algebra $C^*(G)$. And $\widehat{A} \cong C^*(G)$ is a deformation quantization of H, equipped with the (linear) Poisson bracket given by δ defined in Section 1. In this way, we see that the duality between H and G as Poisson-Lie groups corresponds nicely to the Hopf C^*-algebra duality between $\widehat{A}$ and A, in terms of the multiplicative unitary operator U.

APPENDIX: DEFORMATION QUANTIZATION OF $\widetilde{G}$

Recall from Section 1 that the Lie bialgebra structure δ on $\mathfrak{h}$ actually came from the Lie bialgebra structure $\widetilde{\delta}$ on the extended Heisenberg algebra $\widetilde{\mathfrak{h}}$. So far, we only considered the deformation quantization of $\left(G, \{\,\cdot\,,\cdot\}_\omega\right)$, which is the dual Poisson-Lie group of the (nilpotent) Poisson-Lie group H or Lie bialgebra $(\mathfrak{h}, \delta)$. We have been avoiding the discussion of $\widetilde{H}$ and its dual Lie group $\widetilde{G}$, because $\widetilde{H}$ is not nilpotent.

Usually, there are some technical difficulties to correctly formulate the notion of "strict" deformation quantization of $C_\infty(\widetilde{G})$, if $\widetilde{H}$ is not nilpotent. Some modifications of the "strictness condition" are necessary (see [15], [30]). But in our case, if we are willing to compromise a little on shrinking the space on which the deformed multiplication is defined, we are still able to find a quantum version of $C_\infty(\widetilde{G})$ with the aid of multiplicative unitary operators. We are going to define below a multiplicative unitary operator $\widetilde{U}$, using the trick of "changing of cocycles" as before. The multiplicative unitary we obtain will again be regular.

By [3], given a regular multiplicative unitary $\widetilde{U} \in \mathcal{B}(\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}})$, there corresponds an algebra $\mathcal{A}(\widetilde{U}) \subseteq \mathcal{B}(\widetilde{\mathcal{H}})$ such that its norm closure gives a C^*-algebra $\widetilde{A}$. Usually, $\mathcal{A}(\widetilde{U})$ is a sort of an L^1-algebra. In our case, it will be the twisted group algebra whose twisted convolution is given by the cocycle associated to the definition of $\widetilde{U}$. Since we prefer to have our multiplication defined at the level of continuous functions on $\widetilde{G}$, we will consider a certain subspace $\widetilde{\mathcal{A}}$ of $S(\widetilde{G})$, to express our multiplication.

The following construction is indeed a deformation quantization of $\widetilde{G}$. The verification of this will be left to the reader.

EXAMPLE 3.6. Let $\widetilde{H}$ be the extended Heisenberg group with the group law defined by

$$(x, y, z, w)(x', y', z', w') = \left(x + \mathrm{e}^{w} x', y + \mathrm{e}^{-w} y', z + z' + (\mathrm{e}^{-w})\beta(x, y'), w + w'\right).$$

This is clearly the Lie group corresponding to the extended Heisenberg algebra $\widetilde{\mathfrak{h}}$ defined in Section 1. We use the w variable here to express the vectors in span($\mathbf{d}$). Consider the dual Poisson-Lie group $\widetilde{G}$ of $\widetilde{H}$ defined by the multiplication:

$$(p,q,r,s)(p',q',r',s') = (\mathrm{e}^{\lambda r'}p + p', \mathrm{e}^{\lambda r'}q + q', r + r', s + s').$$

It is easy to see that the above $\widetilde{G}$ is indeed the Lie group associated with the Lie algebra $\widetilde{\mathfrak{g}}$ defined in Corollary 1.4. To describe its deformation quantization, it is convenient to work in the space of (x,y,r,w) variables, $S(\widetilde{\mathfrak{h}}/\mathfrak{z} \times \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{q}})$. Here $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}}^*$ and $\widetilde{\mathfrak{q}} = \mathfrak{z}^\perp$ in $\widetilde{\mathfrak{g}}$.

Multiplication. Consider the subspace of $S(\widetilde{\mathfrak{h}}/\mathfrak{z} \times \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{q}})$ having compact support in both the r and w variables. Let $\widetilde{\mathcal{A}}$ be its image in $S(\widetilde{G})$ under partial Fourier transform in the (x,y,w) variables, still denoted by $^\wedge$. We define on $\widetilde{\mathcal{A}}$ the deformed multiplication by

$$\begin{aligned}
&(\varphi \times \psi)(p,q,r,s)\\
&\quad= \int \varphi^\vee(\widetilde{x},\widetilde{y},r,\widetilde{w})\psi^\vee(\mathrm{e}^{-\widetilde{w}}x - \mathrm{e}^{-\widetilde{w}}\widetilde{x}, \mathrm{e}^{\widetilde{w}}y - \mathrm{e}^{\widetilde{w}}\widetilde{y}, r, w - \widetilde{w})\\
&\qquad\qquad \cdot \bar{e}\big[\eta_\lambda(r)\beta(\widetilde{x}, y - \widetilde{y})\big]\bar{e}[p\cdot x + q\cdot y + sw]\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{w}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w,
\end{aligned}$$

where $^\vee$ is the (inverse) partial Fourier transform in the (p,q,s) variables. This definition of $\times$ is motivated by the fact that the Poisson bracket on $\widetilde{G}$ is essentially the extension of the linear Poisson bracket on $(\widetilde{\mathfrak{h}}/\mathfrak{z})^*$ by a cocycle. (We follow the method of [15].) Our C^*-algebra $\widetilde{A}$ will then be defined as the enveloping C^*-algebra of $(\widetilde{\mathcal{A}}, \times)$.

Comultiplication. Define the following unitary operators on $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{H}}$ is the space of L^2-functions on the (x,y,r,w) variables:

$$\begin{aligned}
&\widetilde{W}\xi(x,y,r,w,x',y',r',w') = (\mathrm{e}^{-\lambda r'})^n\,\xi(\mathrm{e}^{-\lambda r'}x, \mathrm{e}^{-\lambda r'}y, r, w, x', y', r', w')\\
&\widetilde{V}_\sigma\xi(x,y,r,w,x',y',r',w')\\
&\quad= \bar{e}\big[\eta_\lambda(r')\beta(x, y' - y)\big]\xi(x,y,r+r',w,\mathrm{e}^{-w}x' - \mathrm{e}^{-w}x, \mathrm{e}^{w}y' - \mathrm{e}^{w}y, r', w' - w).
\end{aligned}$$

Let $\widetilde{U} = \widetilde{W}\widetilde{V}_\sigma$. Then we have:

$$\begin{aligned}
&\widetilde{U}\xi(x,y,r,w,x',y',r',w') = (\mathrm{e}^{-\lambda r'})^n\,\bar{e}\big[\eta_\lambda(r')\beta(\mathrm{e}^{-\lambda r'}x, y' - \mathrm{e}^{-\lambda r'}y)\big]\\
&\quad\cdot\xi(\mathrm{e}^{-\lambda r'}x, \mathrm{e}^{-\lambda r'}y, r + r', w, \mathrm{e}^{-w}x' - \mathrm{e}^{-\lambda r' - w}x, \mathrm{e}^{w}y' - \mathrm{e}^{-\lambda r' + w}y, r', w' - w).
\end{aligned}$$

This is again a multiplicative unitary operator. Thus, we may define the comultiplication on $\widetilde{A}$ by $\widetilde{\Delta}\varphi = \widetilde{U}(\varphi \otimes 1)\widetilde{U}^*$. Since it will be useful in later calculations, let us write down the explicit formula for the comultiplication of the building block $L_{a,b,c,d}$, for $(a,b,c,d) \in \widetilde{H}$.

For $(a,b,c,d) \in \widetilde{H}$, the building block $L_{a,b,c,d}$ is the operator on $\widetilde{\mathcal{H}}$ defined similarly as in Equation (2.7) earlier:

$$(L_{a,b,c,d}\xi)(x,y,r,w) = \bar{e}(rc)\bar{e}\big[\eta_\lambda(r)\beta(a, y - b)\big]\xi(\mathrm{e}^{-d}x - \mathrm{e}^{-d}a, \mathrm{e}^{d}y - \mathrm{e}^{d}b, r, w - d).$$

So $\widetilde{\Delta}L_{a,b,c,d} = \widetilde{U}(L_{a,b,c,d} \otimes 1)\widetilde{U}^*$ is an operator on $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$ defined by

$$
\begin{aligned}
&(\widetilde{\Delta}L_{a,b,c,d}\xi)(x, y, r, w, x', y', r', w') \\
&= \bar{e}\big[\eta_\lambda(r)\beta(\mathrm{e}^{\lambda r'}a, y - \mathrm{e}^{\lambda r'}b) + \eta_\lambda(r')\beta(a, y' - b)\big]\bar{e}\big[(r + r')c\big] \\
&\quad \cdot \xi(\mathrm{e}^{-d}x - \mathrm{e}^{\lambda r' - d}a, \mathrm{e}^{d}y - \mathrm{e}^{\lambda r' + d}b, r, w - d, \mathrm{e}^{-d}x' - \mathrm{e}^{-d}a, \mathrm{e}^{d}y' - \mathrm{e}^{d}b, r', w' - d).
\end{aligned}
$$

4. COUNIT AND ANTIPODE

We return to the construction of the remaining quantum group structures for our Hopf C^*-algebra (A, Δ). Similar results will hold for $(\widetilde{A}, \widetilde{\Delta})$, since we only need to change the groups accordingly and use the appropriate cocycles. So, in this section and the next, we will exclusively study our specific example (A, Δ). Since A is our candidate for the "quantum $C_\infty(G)$", we expect that its quantum group structures will come from the corresponding group structures on G.

First, the choice for the counit is rather obvious:

THEOREM 4.1. *There exists a unique continuous linear map $\varepsilon : A \to \mathbb{C}$ such that*

$$\varepsilon(\varphi) = \varphi(0, 0, 0),$$

for $\varphi \in \mathcal{A} = S_{3c}(G)$. Then ε is a counit for (A, Δ). That is, ε is a C^*-homomorphism from A into $\mathbb{C}$ satisfying the condition:

$$(\mathrm{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}.$$

Proof. For $\varphi \in \mathcal{A}$,

$$\varepsilon(\varphi) = \varphi(0, 0, 0) = \int (\mathcal{F}^{-1}\varphi)(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int (\mathcal{F}^{-1}\varphi)(x, y, z)\varepsilon(L_{x,y,z})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where $\varepsilon(L_{x,y,z}) \equiv 1$. In other words, ε is actually the trivial representation of $\mathcal{A}$. On the other hand, we may write $\varepsilon(\varphi)$ as:

$$\varepsilon(\varphi) = \varphi(0, 0, 0) = \int \varphi^\vee(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y,$$

which shows that ε is continuous with respect to the L^1-norm. So ε has a continuous linear extension to the L^1-algebra $L^1\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$. But, since we have already seen that ε is a $*$-representation on $\mathcal{A}$, this extension is also a $*$-representation. Therefore, it can be further extended to a $*$-representation on $A \cong C^*\big(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma\big)$.

Next, let us prove the equality for our building block $\Delta L_{x,y,z} \in M(A \otimes A)$. Using the realization of $\Delta L_{x,y,z}$ as a continuous function on $G \times G$ (Equation (3.3)), we have:

$$(\mathrm{id} \otimes \varepsilon)\Delta L_{x,y,z}(p, q, r) = \bar{e}\big[\langle (p, q, r), (x, y, z)\rangle\big] = L_{x,y,z}(p, q, r),$$

and similarly for the other half of the equality. By the definition of Δ, we have proved that:

$$(\mathrm{id} \otimes \varepsilon)(\Delta\varphi) = \varphi = (\varepsilon \otimes \mathrm{id})(\Delta\varphi), \quad \varphi \in \mathcal{A}. \quad \blacksquare$$

The antipode (or coinverse) is usually defined as an anti-automorphism ([36], [41]). Let us follow the method which has been used by several authors, beginning as early as the work of Kac and Paljutkin ([13]).

Consider the operation † on $\mathcal{A}$ defined by

$$\varphi^\dagger(p,q,r) = \overline{\varphi(-\mathrm{e}^{-\lambda r}p, -\mathrm{e}^{-\lambda r}q, -r)}.$$

Here the bar means the complex conjugation. Then define $\kappa : \mathcal{A} \to \mathcal{A}$ by

$$\kappa(\varphi) = (\varphi^*)^\dagger = (\varphi^\dagger)^*,$$

where φ^* is the C^*-involution defined in Proposition 2.9. Explicitly, we have:

$$\begin{aligned}\kappa(\varphi)(p,q,r) = \int \varphi(-\mathrm{e}^{-\lambda r}\widetilde{p}, -\mathrm{e}^{-\lambda r}\widetilde{q}, -r)\overline{e}\big[(p-\widetilde{p})\cdot x + (q-\widetilde{q})\cdot y\big] \\ \cdot\, \overline{e}\big[\eta_\lambda(r)\beta(x,y)\big]\, \mathrm{d}\widetilde{p}\,\mathrm{d}\widetilde{q}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.1}$$

In the commutative case (i.e., $\beta \equiv 0$), this is nothing but

$$\kappa(\varphi)(p,q,r) = \varphi(-\mathrm{e}^{-\lambda r}p, -\mathrm{e}^{-\lambda r}q, -r) = \varphi\big((p,q,r)^{-1}\big),$$

which is just taking the inverse in G.

Let us now try to define κ at the operator level. Motivated by the operation † above, we first define an involutive operator T on $\mathcal{H}$ by

$$T\xi(x,y,r) = (\mathrm{e}^{\lambda r})^n\, \overline{\xi(\mathrm{e}^{\lambda r}x, \mathrm{e}^{\lambda r}y, -r)}.$$

LEMMA 4.2. *Let T be the operator defined above. Then T is conjugate linear, isometric, and involutive* (i.e., $T^2 = 1$). *Moreover,*

$$T\varphi T = \varphi^\dagger,$$

where $\varphi, \varphi^\dagger \in \mathcal{A}$ are viewed as operators. We thus have $TAT = A$.

Proof. We will just verify the equation $T\varphi T = \varphi^\dagger$. The other assertions are straightforward. We have:

$$\begin{aligned}(T\varphi T\xi)(x,y,r) &= (\mathrm{e}^{\lambda r})^n\, \overline{(\varphi T\xi)(\mathrm{e}^{\lambda r}x, \mathrm{e}^{\lambda r}y, -r)} \\ &= \int \overline{\varphi(p,q,-r)}\,\overline{e}(p\cdot x' + q\cdot y')e\big[\eta_\lambda(-r)\beta(x', \mathrm{e}^{\lambda r}y - y')\big] \\ &\qquad \cdot\, \xi(x - \mathrm{e}^{-\lambda r}x', y - \mathrm{e}^{-\lambda r}y', r)\, \mathrm{d}p\,\mathrm{d}q\,\mathrm{d}x'\mathrm{d}y' \\ &= \varphi^\dagger\xi(x,y,r). \quad ∎\end{aligned}$$

PROPOSITION 4.3. *Let the map $\kappa : A \to A$ be defined by $\kappa(\varphi) = T\varphi^*T$, for $\varphi \in \mathcal{A}$. Then κ is an anti-automorphism on A. At the function level, $\kappa(\varphi)$ agrees with Equation* (4.1). *Moreover, κ satisfies the condition:*

$$(\kappa \otimes \kappa)(\Delta\varphi) = \Sigma\big(\Delta(\kappa\varphi)\big),$$

where $\Sigma : A \otimes A \to A \otimes A$ denotes the flip.

Proof. The proof that κ is an anti-automorphism follows immediately from the previous lemma. Since $\kappa(\varphi) = (\varphi^\dagger)^*$ on the functions, to prove the last condition we only need to check the following equation:

$$(T\otimes T)(\Delta\varphi)(T\otimes T)\xi = (\Sigma(\Delta\varphi^\dagger))(\Sigma\xi), \qquad \xi \in \mathcal{H}\otimes\mathcal{H}.$$

Here Σ also denotes the flip on $\mathcal{H}\otimes\mathcal{H}$. The calculation is straightforward. ∎

In this way, we showed that $(A, \Delta, \varepsilon, \kappa)$ is a *counital, coinvolutive Hopf* C^*-*algebra* in the sense of [36]. However, some more axioms are needed to make the map κ to be reasonably considered as the antipode. For instance, in the purely algebraic setting of Hopf algebra theory ([34], [25]) the requirement for the antipode is given by the following equation:

$$m(\mathrm{id} \otimes \kappa)\Delta = m(\kappa \otimes \mathrm{id})\Delta = \varepsilon(\cdot)1, \tag{4.2}$$

where $m : A \otimes A \to A$ is the multiplication.

In the operator algebra setting, the multiplication map m is not continuous for the operator norms in general. Because of this, we approach a little differently rather than just translating the above formulation. Motivated by Kac algebra theory, the antipode is usually discussed together with the notion of the Haar weight. See Proposition 5.2 in the next section.

Nevertheless, at least at the function level, the algebraic condition (4.2) can be readily verified for our $(A, \Delta, \varepsilon, \kappa)$. The calculation of this claim is as follows; this will give us some modest justification for our particular definition of κ.

Using the definition of Δ and the fact that $\Delta L_{a,b,c}$ can be regarded as a continuous function on G, we have for $\varphi \in \mathcal{A}$,

$$\begin{aligned}
&(\mathrm{id} \otimes \kappa)\Delta\varphi(p, q, r, p', q', r') \\
&\quad = \int \bar{e}\big[(p' - \widetilde{p}) \cdot x + (q' - \widetilde{q}) \cdot y\big] \bar{e}\big[\eta_\lambda(r')\beta(x, y)\big] (\mathcal{F}^{-1}\varphi)(a, b, c) \\
&\qquad \cdot \bar{e}\big[(e^{-\lambda r'} p - e^{-\lambda r'} \widetilde{p}) \cdot a + (e^{-\lambda r'} q - e^{-\lambda r'} \widetilde{q}) \cdot b + (r - r')c\big] \, d\widetilde{p} \, d\widetilde{q} \, dx \, dy \, da \, db \, dc \\
&\quad = \int (\psi^1_{a,b,c} \otimes \psi^2_{a,b,c})(p, q, r, p', q', r') \, da \, db \, dc.
\end{aligned}$$

Here for a fixed $(a, b, c) \in H$,

$$\begin{aligned}
\psi^1_{a,b,c}(p, q, r) &= \bar{e}(p \cdot a + q \cdot b + rc) \\
\psi^2_{a,b,c}(p', q', r') &= (e^{2\lambda r'}) \, e(p' \cdot a + q' \cdot b + r'c) \bar{e}\big[\eta_\lambda(r')\beta(a, b)\big] (\mathcal{F}^{-1}\varphi)(e^{\lambda r'} a, e^{\lambda r'} b, c).
\end{aligned}$$

Since we have:

$$(\psi^1_{a,b,c} \times \psi^2_{a,b,c})(p, q, r) = (e^{2\lambda r})(\mathcal{F}^{-1}\varphi)(e^{\lambda r} a, e^{\lambda r} b, c),$$

it follows that:

$$m\big((\mathrm{id} \otimes \kappa)\Delta\varphi\big)(p, q, r) = \int (\mathcal{F}^{-1}\varphi)(a, b, c) \, da \, db \, dc = \varphi(0, 0, 0) = \varepsilon(\varphi)1.$$

Similarly, we can also verify that $m\big((\kappa \otimes \mathrm{id})\Delta\varphi\big) = \varepsilon(\varphi)1$.

5. HAAR WEIGHT

Since the group law on G has been chosen such that Lebesgue measure $\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r$ on the underlying vector space is its Haar measure, we expect more or less the same in the quantum case. So, let us define the linear functional h on $\mathcal{A}$ by

$$h(\varphi) = \int \varphi(p, q, r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r. \tag{5.1}$$

We intend to show that h is the appropriate Haar weight on our Hopf C^*-algebra (A, Δ).

Ideally, the definition of locally compact quantum groups would be formulated so that the existence of Haar weights follows only from the definition. Right now, the definition of the Haar weight and its left invariance property are not completely agreed upon and the existence of the Haar weight has to be assumed in the definition of quantum groups. In particular, the definition of the antipode is closely tied to that of the Haar weight. See [23], [24], [41], [19], [18].

Because of this, instead of trying to be very rigorous, we plan to give only a reasonable justification of our choice for h. What we do in the following is imitating the theory of Kac algebras ([8]).

Since h is well-defined at the level of a dense subspace of functions (i.e., in $\mathcal{A}$), it is a densely defined weight on A. As we see in the next proposition, it is actually a faithful trace.

Proposition 5.1. *Let h be defined on $\mathcal{A}$ by Equation* (5.1). *Then h is a faithful trace.*

Proof. Let $\varphi \in \mathcal{A}$. Then, by using change of variables and Fourier inversion theorem, we have:

$$\begin{aligned} h(\varphi^* \times \varphi) &= \int \bar{e}\big[(p - p')\cdot \widetilde{x}\big]\varphi^*(p, q, r)\varphi\big(p, q + \eta_\lambda(r)\widetilde{x}, r\big)\,\mathrm{d}p'\mathrm{d}\widetilde{x} \\ &= \int \overline{\varphi(p, q, r)}\varphi(p, q, r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r = \|\varphi\|_2^2, \end{aligned}$$

and similarly,

$$h(\varphi \times \varphi^*) = \int \varphi(p, q, r)\overline{\varphi(p, q, r)}\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r = \|\varphi\|_2^2.$$

Here, φ^* is the C^*-involution given in Proposition 2.9. From these equations, we can see that h is a faithful trace. ∎

To correctly define the Haar weight, we have to further require some "lower semi-continuity condition" (corresponding to the notion of normal weights in von Neumann algebra setting, like Kac algebras) and "semi-finiteness", as well as the "left invariance property". Since this will make our discussion very technical, let us overlook the details and give only a brief discussion on the left invariance property of h.

PROPOSITION 5.2. *For $\varphi, \psi \in \mathcal{A}$, the weight h satisfies the following left invariance property:*

$$(\mathrm{id} \otimes h)\big((1 \otimes \varphi)(\Delta\psi)\big) = \kappa\big((\mathrm{id} \otimes h)((\Delta\varphi)(1 \otimes \psi))\big), \tag{5.2}$$

where κ is the (antipodal) map defined in Proposition 4.3.

Proof. Even for $\varphi, \psi \in \mathcal{A}$, the expressions $(1 \otimes \varphi)(\Delta\psi)$ and $(\Delta\varphi)(1 \otimes \psi)$ do not necessarily belong to the algebraic tensor product $\mathcal{A} \odot \mathcal{A}$. (See the proof of Theorem 3.2, where we calculated $(\Delta\varphi)(1 \otimes \psi)$.) Therefore, for the left and right sides of Equation (5.2) to make sense, $\mathrm{id} \otimes h$ has to be defined more carefully.

This extension can be done using the notion of operator valued weights ([12]). But unlike in [12], since we are dealing with C^*-algebra weights ([6]), we have to modify the definitions accordingly. In short, we regard $\mathrm{id} \otimes h$ as the tensor product of two faithful, semi-finite, lower semi-continuous operator valued weights on $A \otimes A$ having values in $A \otimes \mathbb{C} \cong A$. To be able to define this more rigorously, some efforts have been made recently to introduce a somewhat stronger condition of lower semi-continuity ([28], [17]).

In our case, since we know that $(1 \otimes \varphi)(\Delta\psi)$ and $(\Delta\varphi)(1 \otimes \psi)$ are contained in $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q} \times \mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ and since the elements in this Schwartz space can be approximated by elements in $S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \odot S_{3c}(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$, we know how to define $(\mathrm{id} \otimes h)\big((1 \otimes \varphi)(\Delta\psi)\big)$ and $(\mathrm{id} \otimes h)\big((\Delta\varphi)(1 \otimes \psi)\big)$ under the extension. So, let us set aside the aforementioned technical details and try to verify the above equation. Through long but elementary calculations, we obtain:

$$\begin{aligned}
&(\mathrm{id} \otimes h)\big((1 \otimes \varphi)(\Delta\psi)\big)(p,q,r) \\
&\quad = \int \bar{e}\big[(e^{\lambda r'} p + \widetilde{p} - \widetilde{\widetilde{p}}) \cdot a + (e^{\lambda r'} q + \widetilde{q} - \widetilde{\widetilde{q}}) \cdot b\big] e\big[\eta_\lambda(r')\beta(a,b)\big] \\
&\qquad\qquad \cdot \varphi(\widetilde{p}, \widetilde{q}, r')\psi(\widetilde{\widetilde{p}}, \widetilde{\widetilde{q}}, r + r')\, d\widetilde{p}\, d\widetilde{q}\, d\widetilde{\widetilde{p}}\, d\widetilde{\widetilde{q}}\, da\, db\, dr' \\
&\quad = \kappa\big((\mathrm{id} \otimes h)((\Delta\varphi)(1 \otimes \psi))\big)(p,q,r)
\end{aligned}$$

for φ and ψ in $\mathcal{A}$. ∎

In the commutative case, Equation (5.2) is just

$$\int_G \varphi(g')\psi(gg')\, d\mu(g') = \int_G \varphi(g^{-1}g')\psi(g')\, d\mu(g'),$$

which exactly describes the left invariance condition. Actually, Equation (5.2) is the defining condition for the Haar weight in Kac algebra theory ([36], [8]).

It is true that there are still some technical details to take care of. Saying this, we may conclude from Proposition 5.2 that h is the appropriate *Haar weight* for (A, Δ). Also from the proposition, we can say that the map κ we have been using is a legitimate *antipode* for (A, Δ).

Thus, our Hopf C^*-algebra $(A, \Delta, \varepsilon, \kappa)$ together with the Haar weight h on it can be regarded as a *locally compact quantum group*. Although we did not give the precise definition of general locally compact quantum groups, any reasonable definition should allow our specific example as a special case.

Meanwhile, since our group G is not unimodular, we expect that our Haar weight should also carry certain non-unimodularity properties. One of these is given below.

Proposition 5.3. *The Haar weight h is not invariant under the antipode κ. That is, there exists $\varphi \in \mathcal{A}$ such that*

$$h\big(\kappa(\varphi)\big) \neq h(\varphi).$$

Proof. Since

$$h\big(\kappa(\varphi)\big) = \int \varphi(-\mathrm{e}^{-\lambda r}p, -\mathrm{e}^{-\lambda r}q, -r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r,$$

it is clear that we have $h\big(\kappa(\varphi)\big) \neq h(\varphi)$, in general. ∎

It is noteworthy that we have a non-unimodular Haar weight as opposed to many other examples ([31], [35], [37], [20]). It will be interesting to study its consequences and properties more thoroughly, especially in relation to the duality theory. However, for the time being we will leave this as a future project.

As a final remark, we point out that the regular representation L we used is essentially the GNS representation with respect to h (which is a faithful trace). This observation displays the importance the Haar weight has in both theory and construction of locally compact quantum groups.

6. Quantum universal R-matrix

For the QUE algebra counterparts for our Hopf C^*-algebra (A, Δ) (for instance, $U_\hbar(\mathfrak{h})$ in [1] or $H(1)_q$ in [4]), the so-called "quantum universal R-matrix" have been successfully constructed. In our case also, once we modify the definition of the universal R-matrix so that it is consistent with our C^*-algebra language, we can do the same.

Our definition given below is essentially the same one used in the QUE algebra or more general Hopf algebra setting (see [7], [5]). Note that we require the R-matrix to be contained in a multiplier algebra. (This is consistent with the definition of the comultiplication, which is a multiplier algebra valued map.) Since any nondegenerate representation of a C^*-algebra can be uniquely extended to its multiplier algebra, any element belonging to a multiplier algebra also has an image under any representation of the C^*-algebra.

Definition 6.1. Let (B, Δ) be a Hopf C^*-algebra, where Δ is its comultiplication. We will say that B is *almost cocommutative* if there exists an invertible element $R \in M(B \otimes B)$ such that

$$(\Sigma \circ \Delta)(\varphi) = R\Delta(\varphi)R^{-1}, \quad \varphi \in B \tag{6.1}$$

where Σ is the flip. We will denote the *opposite comultiplication* by $\Delta^{\mathrm{op}} = \Sigma \circ \Delta$.

The element R above cannot be arbitrary, since the opposite comultiplication Δ^{op} should also be coassociative. The following condition, though a little stronger than is needed to assure the coassociativity of Δ^{op}, defines the quantum universal R-matrix.

DEFINITION 6.2. An almost cocommutative Hopf C^*-algebra (B, R) is said to be *quasitriangular*, if R satisfies the so-called quantum Yang-Baxter equation (QYBE): $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, and also satisfies:

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}. \tag{6.2}$$

It is called *triangular*, if it is quasitriangular and, in addition, $R_{21} = R^{-1}$. If B is quasitriangular, such an element R will be called a *quantum universal R-matrix.*

If R satisfies Equation (6.2), the QYBE for R automatically follows from the coassociativity of Δ^{op} ([5]). The QYBE is a quantum version of the classical Yang-Baxter equation (CYBE) ([7]). After we find below a quantum universal R-matrix for our (A, Δ), we will show that this R-matrix is indeed closely related with the classical r-matrix given earlier (Section 1, Equation (1.1)) at the Lie bialgebra level.

Recall that the classical r-matrix associated with our construction is an element in $\widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}}$. This suggests that we better consider the Hopf C^*-algebra $(\widetilde{A}, \widetilde{\Delta})$, instead of (A, Δ). So we need to look for our quantum R-matrix in $M(\widetilde{A} \otimes \widetilde{A})$. Motivated by the R-matrix constructed at the QUE algebra level ([4], [1]), we consider R as the following (continuous) function on $\widetilde{G} \times \widetilde{G}$:

$$R(p, q, r, s, p', q', r', s') = \bar{e}\big[\lambda(rs' + r's)\big]\bar{e}\big[2\lambda(\mathrm{e}^{-\lambda r'})p \cdot q'\big].$$

Let us try to formulate a more proper definition of R as an operator on $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$. First, let us view R as a product of two functions Φ and Φ' given by

$$\Phi(p, q, r, s, p', q', r', s') = \bar{e}\big[\lambda(rs' + r's)\big]$$
$$\Phi'(p, q, r, s, p', q', r', s') = \bar{e}\big[2\lambda(\mathrm{e}^{-\lambda r'})p \cdot q'\big].$$

By using partial Fourier transform purely formally and by using the multiplication law of $\widetilde{A}$ (see Example 3.6 in Appendix of Section 3), we may regard Φ and Φ' as operators on $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$:

$$\Phi\xi(x, y, r, w, x', y', r', w') = \xi(\mathrm{e}^{-\lambda r'}x, \mathrm{e}^{\lambda r'}y, r, w - \lambda r', \mathrm{e}^{-\lambda r}x', \mathrm{e}^{\lambda r}y', r', w' - \lambda r)$$
$$\Phi'\xi(x, y, r, w, x', y', r', w') = \int \bar{e}\big[2\lambda(\mathrm{e}^{-\lambda r'})\widetilde{p} \cdot \widetilde{q}\big]e(\widetilde{p} \cdot \widetilde{x} + \widetilde{q} \cdot \widetilde{y})\bar{e}\big[\eta_\lambda(r)\beta(\widetilde{x}, y)\big]$$
$$\cdot\, \xi(x - \widetilde{x}, y, r, w, x', y' - \widetilde{y}, r', w')\, d\widetilde{p}\, d\widetilde{q}\, d\widetilde{x}\, d\widetilde{y}.$$

DEFINITION 6.3. Define R as an operator in $\mathcal{B}(\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}})$ by $R = \Phi\Phi'$. That is,

$$R\xi(x, y, r, w, x', y', r', w') = \Phi\Phi'\xi(x, y, r, w, x', y', r', w')$$
$$= \int \bar{e}\big[2\lambda(\mathrm{e}^{-\lambda r'})\widetilde{p} \cdot \widetilde{q}\big]e(\widetilde{p} \cdot \widetilde{x} + \widetilde{q} \cdot \widetilde{y})\bar{e}\big[\eta_\lambda(r)\beta(\widetilde{x}, \mathrm{e}^{\lambda r'}y)\big]$$
$$\cdot\, \xi(\mathrm{e}^{-\lambda r'}x - \widetilde{x}, \mathrm{e}^{\lambda r'}y, r, w - \lambda r', \mathrm{e}^{-\lambda r}x', \mathrm{e}^{\lambda r}y' - \widetilde{y}, r', w' - \lambda r)\, d\widetilde{p}\, d\widetilde{q}\, d\widetilde{x}\, d\widetilde{y}.$$

PROPOSITION 6.4. *Let R be the operator defined above. Then $R \in M(\widetilde{A} \otimes \widetilde{A})$.*

Proof. It is enough to show that Φ and Φ' are both left and right multipliers. To show this, consider an arbitrary function F in the dense subalgebra $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ of

$M(\widetilde{A}\otimes\widetilde{A})$, where $\widetilde{\mathcal{A}}$ is as defined in Example 3.6. Then, by straightforward calculation, we have:

$$(\Phi F)(p,q,r,s,p',q',r',s')=\bar{e}\big[\lambda(rs'+r's)\big]F\big(e^{\lambda r'}p,e^{-\lambda r'}q,r,s,e^{\lambda r}p',e^{-\lambda r}q',r',s'\big)$$
$$(F\Phi)(p,q,r,s,p',q',r',s')=\bar{e}\big[\lambda(rs'+r's)\big]F(p,q,r,s,p',q',r',s').$$

These equations are understood to mean that $\Phi F\in\mathcal{B}(\widetilde{\mathcal{H}}\otimes\widetilde{\mathcal{H}})$ is exactly the operator realization of the function $(\Phi F)\in\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}}$ defined by the first equation, and similarly for $F\Phi$. From this, it is clear that Φ is a multiplier.

The proof that Φ' is a left multiplier follows from the following:

$$\Phi'F(p,q,r,s,p',q',r',s')=\bar{e}\big[2\lambda(e^{-\lambda r'})p\cdot q'\big]F\big(p,q+2\lambda\eta_\lambda(r)e^{-\lambda r'}q',r,s,p',q',r',s'\big),$$

which is again understood in the same way as above. To prove that Φ' is also a right multiplier, it is more convenient to consider the Schwartz function space in the (p,q,r,w) variables having compact support both in the r and w variables, which is isomorphic (via partial Fourier transform) to $\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}}$. If F is in this space, we then have:

$$\begin{aligned}&F\Phi'(p,q,r,w,p',q',r',w')\\&=\bar{e}\big[2\lambda(e^{-\lambda r'+w-w'})p\cdot q')\big]F\big(p,q,r,w,p'+2\lambda(e^{-\lambda r'+w-w'})\eta_\lambda(r')p,q',r',w'\big).\end{aligned}$$

So Φ' is also a right multiplier. Thus $R=\Phi\Phi'$ is both left and right multiplier. ∎

PROPOSITION 6.5. *Let R be as above. Then R is an invertible element in $M(\widetilde{A}\otimes\widetilde{A})$ and R satisfies:*

$$R\widetilde{\Delta}L_{a,b,c,d}R^{-1}=\widetilde{\Delta}^{\mathrm{op}}L_{a,b,c,d},\quad (a,b,c,d)\in\widetilde{H}.$$

Here $L_{a,b,c,d}$ denotes the "building block" operator defined earlier. Thus R makes $(\widetilde{A},\widetilde{\Delta})$ an almost cocommutative Hopf C^*-algebra.

Proof. It turns out that, as an operator, R is unitary. Moreover, R^* is:

$$\begin{aligned}&R^*\xi(x,y,r,w,x',y',r',w')\\&=\int e\big[2\lambda(e^{-\lambda r'})\tilde{p}\cdot\tilde{q}\big]\bar{e}(\tilde{p}\cdot\tilde{x}+\tilde{q}\cdot\tilde{y})e\big[\eta_\lambda(r)\beta(\tilde{x},y)\big]\\&\qquad\cdot\xi(e^{\lambda r'}x+e^{\lambda r'}\tilde{x},e^{-\lambda r'}y,r,w+\lambda r',e^{\lambda r}x',e^{-\lambda r}y'+e^{-\lambda r}\tilde{y},r',w'+\lambda r),\end{aligned}$$

where the integration is with respect to $(\tilde{p},\tilde{q},\tilde{x},\tilde{y})$ variables. By using the expression for $\widetilde{\Delta}L_{a,b,c,d}$ given in Example 3.6, we obtain:

$$\begin{aligned}&R\widetilde{\Delta}L_{a,b,c,d}R^*\xi(x,y,r,w,x',y',r',w')\\&=\bar{e}\big[\eta_\lambda(r)\beta(a,y-b)+\eta_\lambda(r')\beta(e^{\lambda r}a,y'-e^{\lambda r}b)\big]\bar{e}\big[(r+r')c\big]\\&\quad\cdot\xi(e^{-d}x-e^{-d}a,e^{d}y-e^{d}b,r,w-d,e^{-d}x'-e^{\lambda r-d}a,e^{d}y'-e^{\lambda r+d}b,r',w'-d)\\&=\widetilde{\Delta}^{\mathrm{op}}L_{a,b,c,d}\xi(x,y,r,w,x',y',r',w').\end{aligned}$$

Since the almost cocommutativity condition holds for the building blocks, it is true for any element of $\widetilde{A}$. ∎

THEOREM 6.6. *Let R be defined by Definition 6.3. Then R satisfies the* QYBE *and the quasitriangularity condition given in Definition 6.2. Combining this result with those of Proposition 6.4 and Proposition 6.5, we conclude that R is a "quasitriangular" quantum universal R-matrix for $(\widetilde{A}, \widetilde{\Delta})$.*

Proof. The verification of the QYBE ($R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$) is a straightforward calculation. We also have:

$$\widetilde{U}_{23}R_{12}\widetilde{U}_{23}^* = R_{13}R_{12} \quad \text{and} \quad \widetilde{U}_{12}R_{13}\widetilde{U}_{12}^* = R_{13}R_{23},$$

using the definition of $\widetilde{U}$ given in Example 3.6. Since $(\mathrm{id} \otimes \widetilde{\Delta})(R) = \widetilde{U}_{23}R_{12}\widetilde{U}_{23}^*$ and since $(\widetilde{\Delta} \otimes \mathrm{id})(R) = \widetilde{U}_{12}R_{13}\widetilde{U}_{12}^*$, the quantum R-matrix condition follows. Thus, we conclude that R is indeed a quasitriangular quantum universal R-matrix for $(\widetilde{A}, \widetilde{\Delta})$. ∎

Finally, let us try to relate our quantum R-matrix with the classical r-matrix:

$$r = \mathbf{z} \otimes \mathbf{d} + \mathbf{d} \otimes \mathbf{z} + 2\sum_{i=1}^{n} \mathbf{x}_i \otimes \mathbf{y}_i \in \widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}}.$$

We wish to indicate that r is a "classical limit" of R, with λ regarded as a deformation parameter. Since we have so far been viewing λ as a fixed constant built into the definition of G and its Poisson bracket, we have to approach this a little differently. It actually corresponds to the deformation process of the dual Hopf C^*-algebra $\widehat{A}$.

One serious problem is that as we try to let λ to vary, the algebra $\widetilde{A}$ (or $\widetilde{A} \otimes \widetilde{A}$) also changes. Because of this, we will only work on its dense function space $\widetilde{\mathcal{A}}$ (or $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$), ignoring its algebra structure. Again, as in the proof of Proposition 6.4, it is more convenient to regard $\widetilde{\mathcal{A}}$ as the Schwartz function space in the (p, q, r, w) variables having compact support in the r and w variables. The L^1-completion of $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ is isomorphic to $L^1(\widetilde{H} \otimes \widetilde{H})$, independent of the value of λ.

Recall that we could realize R as a continuous function on $\widetilde{G} \times \widetilde{G}$ by Equation (6.3). To emphasize its dependence on λ, let us denote it from now on by R_λ. Consider the operator Ψ_λ on $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ (for the time being, $\widetilde{\mathcal{A}}$ is viewed as an algebra) defined by

$$\Psi_\lambda(F) = R_\lambda F R_\lambda^*, \quad F \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}.$$

Then we have:

$$\begin{aligned} \text{(6.4)} \quad \Psi_\lambda(F)(p, q, r, w, p', q', r', w') &= \bar{e}\big[2\lambda(e^{-\lambda r})p \cdot q'\big] e\big[2\lambda(e^{w-w'-\lambda r})p \cdot q'\big] \\ \cdot F\big(e^{\lambda r'}p, e^{-\lambda r'}q + 2\lambda e^{-\lambda r - \lambda r'}\eta_\lambda(r)q', r, w, &e^{\lambda r}p' - 2\lambda e^{w-w'}\eta_\lambda(r')p, e^{-\lambda r}q', r', w'\big). \end{aligned}$$

By L^1-extension, we will define Ψ_λ as an operator on the Banach space $L^1(\widetilde{H} \otimes \widetilde{H})$, ignoring any algebra structure, via Equation (6.4). This will be our operator realization of R_λ.

Let us now consider the classical r-matrix. First, by means of the dual pairing between $\widetilde{\mathfrak{h}}^*$ and $\widetilde{\mathfrak{h}}$, we may regard above $r \in \widetilde{\mathfrak{h}} \otimes \widetilde{\mathfrak{h}}$ as a linear function on $\widetilde{\mathfrak{h}}^* \otimes \widetilde{\mathfrak{h}}^*$. Let us denote it by ψ:

$$\psi(p, q, r, s, p', q', r', s') = rs' + r's + 2p \cdot q'.$$

Next, we have to find a way to make ψ to determine an operator on $L^1(\widetilde{H} \otimes \widetilde{H})$. Since it should correspond to $\lambda = 0$ case, we will construct an (unbounded) operator such that it looks like an (unbounded) "derivation" with respect to the multiplication (for $\lambda = 0$) on $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$. That is, we consider the densely defined operator:

$$F \mapsto [\psi, F] = \psi \times_{\lambda=0} F - F \times_{\lambda=0} \psi, \quad F \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}.$$

But $\times_{\lambda=0}$ is essentially the ordinary convolution on $S(\widetilde{H})$ (or $S(\widetilde{H} \times \widetilde{H})$). So, by straightforward calculation, again formally using partial Fourier transform, we obtain:

$$\begin{aligned} &[\psi, F](p, q, r, w, p', q', r', w') \\ (6.5) \qquad &= \int \big[(r\widetilde{\widetilde{s}} + r'\widetilde{s}) + 2(p \cdot q' + rq' \cdot \widetilde{y} - \mathrm{e}^{w-w'} p \cdot q' - r'\mathrm{e}^{w-w'} p \cdot \widetilde{x})\big] \\ &\qquad \cdot e[\widetilde{s}\widetilde{w} + \widetilde{\widetilde{s}}\widetilde{\widetilde{w}}] e[\widetilde{p} \cdot \widetilde{x} + \widetilde{q} \cdot \widetilde{y}] F(\mathrm{e}^{\widetilde{w}} p, \mathrm{e}^{-\widetilde{w}} q + \widetilde{q}, r, w, \mathrm{e}^{\widetilde{\widetilde{w}}} p' + \widetilde{p}, \mathrm{e}^{-\widetilde{\widetilde{w}}} q', r', w'), \end{aligned}$$

where the integration is taken with respect to all the tilde $\widetilde{(\cdot)}$ and double tilde $\widetilde{\widetilde{(\cdot)}}$ variables. From now on, we will just use (6.5) as our defining equation for $[\psi, \cdot\,]$, an unbounded operator on the Banach space $L^1(\widetilde{H} \otimes \widetilde{H})$. This will be our operator realization of ψ.

Then, by comparing the formulas (6.4) and (6.5), we obtain the following result. Although we showed directly in Theorem 6.6 that our R satisfies the QYBE, this proposition indicates that this property is actually suggested by the CYBE (see Proposition 1.3) satisfied by the associated classical r-matrix.

PROPOSITION 6.7. *Let the notation be as above. Then:*

$$\lim_{\lambda \to 0} \left\| \frac{\Psi_\lambda(F) - F}{\lambda} - (-2\pi \mathrm{i})[\psi, F] \right\|_{L^1} = 0,$$

for $F \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$. Thus, at least in the sense of the operators on the Banach space $L^1(\widetilde{H} \times \widetilde{H})$, we may say that the "classical limit" as $\lambda \to 0$ of our quantum R-matrix R_λ is $(-2\pi\mathrm{i})\psi$, the operator realization of the classical r-matrix.

Proof. From Equation (6.4), we may express $\Psi_\lambda(F)$ as follows, taking advantage of the Fourier inversion theorem:

$$\begin{aligned} &\Psi_\lambda(F)(p, q, r, w, p', q', r', w') \\ &= \int \overline{e}\big[\lambda(r\widetilde{\widetilde{s}} + r'\widetilde{s})\big] \overline{e}\big[2\lambda(\mathrm{e}^{-\lambda r}) p \cdot q'\big] \overline{e}\big[2\lambda(\mathrm{e}^{-\lambda r - \lambda r'}) \eta_\lambda(r) q' \cdot \widetilde{y}\big] \\ &\qquad \cdot e\big[2\lambda(\mathrm{e}^{w-w'-\lambda r}) p \cdot q'\big] e\big[2\lambda(\mathrm{e}^{w-w'}) \eta_\lambda(r') p \cdot \widetilde{x}\big] e[\widetilde{s}\widetilde{w} + \widetilde{\widetilde{s}}\widetilde{\widetilde{w}}] e[\widetilde{p} \cdot \widetilde{x} + \widetilde{q} \cdot \widetilde{y}] \\ &\qquad \cdot F(\mathrm{e}^{\widetilde{w}} p, \mathrm{e}^{-\widetilde{w}} q + \widetilde{q}, r, w, \mathrm{e}^{\widetilde{\widetilde{w}}} p' + \widetilde{p}, \mathrm{e}^{-\widetilde{\widetilde{w}}} q', r', w'). \end{aligned}$$

The integration is with respect to all the tilde and double tilde variables. Comparing this expression with Equation (6.5) for $[\psi, F]$, we can see easily the pointwise convergence. The L^1 convergence is proved using again the Lebesgue's dominated convergence theorem. ∎

The quantum universal R-matrix is useful in the study of representation theory of our Hopf C^*-algebras $(\widetilde{A}, \widetilde{\Delta})$ and (A, Δ). We will study representation theory of our quantum groups elsewhere (see [16]). It turns out that the representation theory satisfies interesting *quasitriangularity* property, which is not present in the earlier examples of quantum groups corresponding to linear Poisson brackets.

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