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EXTREMAL RICHNESS OF MULTIPLIER AND CORONA ALGEBRAS OF SIMPLE C^*-ALGEBRAS WITH REAL RANK ZERO

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ABSTRACT. In this paper we investigate the extremal richness of the multiplier algebra $\mathcal{M}(A)$ and the corona algebra $\mathcal{M}(A)/A$, for a simple C^*-algebra A with real rank zero and stable rank one. We show that the space of extremal quasitraces and the scale of A contain enough information to determine whether $\mathcal{M}(A)/A$ is extremally rich. In detail, if the scale is finite, then $\mathcal{M}(A)/A$ is extremally rich. In important cases, and if the scale is not finite, extremal richness is characterized by a restrictive condition: the existence of only one infinite extremal quasitrace which is isolated in a convex sense.

KEYWORDS: *Extremal richness, real rank, stable rank, refinement monoid.*

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INTRODUCTION

The class of C^*-algebras with extremal richness was introduced by Brown and Pedersen in [9], with the objective of extending the theory and results of finite C^*-algebras to the infinite case. Examples of extremally rich C^*-algebras include stable rank one algebras, von Neumann algebras and purely infinite simple C^*-algebras. Moreover, this class is invariant under the passage to hereditary subalgebras and under natural constructions such as tensoring with $M_n(\mathbb{C})$ for all $n \in \mathbb{N}$.

Extremally rich C^*-algebras have significant similarities in their properties with the class of stable rank one C^*-algebras. For example, the presence of extremal richness gives bounds on the real ranks of the algebras considered. As is proved in [21], if A is extremally rich, then $\mathrm{RR}(A) \leqslant 1$. Also, extremally rich C^*-algebras with real rank zero are shown to satisfy the weak cancellation property of separativity (see [6]), that is, the monoid $V(A)$ of Murray-von Neumann equivalence classes of projections is separative, which means by definition that whenever $a + a = a + b = b + b$ in $V(A)$ it follows that $a = b$ (see [4]). Simple and separative C^*-algebras with real rank zero are either purely infinite simple or they have stable rank one ([4], Theorem 7.6). The same behaviour is observed for simple C^*-algebras with extremal richness ([8], Corollary 10.5). This is therefore closely related to the long standing Open Question: Is every finite simple C^*-algebra stably finite?

Our aim in this paper is to analyze extremal richness of multiplier and corona algebras for a large class of (nonunital) C^*-algebras. We will work within the class of simple separable C^*-algebras with real rank zero and stable rank one. We also assume that $V(A)$ is strictly unperforated. This class has been studied in different instances: for example, see [18], [14], [22]; it contains AF algebras, and also many examples which are not AF ([11], [18], [13]). Some of our results will use the additional hypothesis that the multiplier algebra has real rank zero. As shown by Lin in [19], Theorem 10, this occurs if $\mathrm{K}_1(A) = 0$.

Let A be a nonunital C^*-algebra lying in the above mentioned class, and such that the real rank of $\mathcal{M}(A)$ is zero. We will give a complete answer to the following problem: What conditions determine whether $\mathcal{M}(A)$ or $\mathcal{M}(A)/A$ have extremal richness? For the class of σ-unital, purely infinite simple C^*-algebras, for some stabilizations of simple unital AF algebras, and for simple AF algebras with a finite number of semi-finite extremal traces, this question has been successfully considered in [15].

Our approach to the problem is based on combining the analysis of extensions with nonstable K-theoretic methods. This involves the knowledge of properties of $V(\mathcal{M}(A))$ and their relation with the ideal lattice of $\mathcal{M}(A)$. We will benefit from results concerning this issue, that appear mainly in [26], [17], [14], [22]. Thus, the first section is devoted to summarizing the basics on monoids and their connexion with C^*-algebras that will be used in the sequel. The main objective of Section 2 is to prove the following: if A has finite scale then $\mathcal{M}(A)/A$ is extremally rich, whereas if the scale is infinite, then $\mathcal{M}(A)/A$ does not have extremal richness provided that A has at least two infinite extremal quasitraces. We also discuss basic properties of extremal richness and some results that will be needed in the last section.

The case in which A has exactly one infinite extremal quasitrace is handled in Section 3. In this particular situation, the study of extremal richness of $\mathcal{M}(A)/A$ requires a deeper analysis, based on the problem of lifting isometries, for which the index map turns out to be a useful tool. We close by proving that in our situation $\mathcal{M}(A)$ is never extremally rich. It will be clear from this and [22], Lemma 7.2 that the class of separative C^*-algebras properly contains the class of C^*-algebras with extremal richness.

1. NOTATION AND PRELIMINARIES

In this section we recall some basic definitions on monoids and C^*-algebras that will be used in the subsequent sections. We emphasize the relation between the order-ideals of the monoid and the closed ideals of a C^*-algebra.

All monoids in this paper will be abelian, and consequently we will write them additively and we will use 0 for their identity element. The operation on a monoid M defines a natural preordering by:

$$x \leqslant y \Leftrightarrow y = x + z \quad \text{for some } z \in M,$$

which is translation-invariant. This preordering is sometimes called the *algebraic preordering.* As usual, we write $x < y$ if $x \leqslant y$ and $x \neq y$.

If M is a monoid, a nonzero element $u \in M$ is called an *order-unit* if for any $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leqslant nu$. We say that M is *conical* provided that the set M^* of nonzero elements is closed under addition. For a C^*-algebra A, we denote by $V(A)$ the monoid of Murray-von Neumann equivalence classes of projections from $M_\infty(A)$. (Equivalently, if A is unital $V(A)$ can be described as the additive monoid of isomorphism classes of finitely generated projective modules over A.) Note that $V(A)$ is always conical, and that $[1_A]$ is an order-unit for $V(A)$ if A is unital.

A nonempty subset of a monoid which is a submonoid and order-hereditary will be called an *order-ideal.* We say that a monoid M is *simple* if M has precisely two order-ideals, namely the ideal generated by 0 and M. In case M is conical, then M is simple if and only if M is nonzero and every nonzero element is an order-unit. This is the case for $V(A)$, where A is a simple C^*-algebra.

Let M be a monoid and let I be an order-ideal of M. Define a congruence relation on M as follows: if $x, y \in M$ write $x \sim y$ if and only if there exist $z, w \in I$ such that $x+z = y+w$. Denote by M/I the quotient of M modulo this congruence, and by $[x]$ the congruence class of an element $x \in M$. The addition $[x]+[y] = [x+y]$ is then a well-defined operation under which M/I becomes a monoid, referred as to the *quotient monoid* of M modulo I. If A is a C^*-algebra and I is a closed ideal of A, then $V(I)$ is naturally an order-ideal of $V(A)$. Moreover, if A has real rank zero, then the quotient $V(A)/V(I)$ is isomorphic to $V(A/I)$ ([4], Proposition 1.4).

We say that a cancellative monoid M is *strictly unperforated* if whenever $nx < ny$ for some $n \in \mathbb{N}$ and $x, y \in M$, it follows that $x < y$. It is remarkable that no examples are known of simple C^*-algebras A with real rank zero and stable rank one whose $V(A)$'s are not strictly unperforated, and therefore this technical condition is quite natural. (If the real rank zero condition is dropped, then there are examples with perforation on $V(A)$, as shown in [23].)

Let K be a compact convex set. We denote by $\mathrm{LAff}(K)$ the monoid of all affine and lower semicontinuous functions on K with values on $\mathbb{R} \cup \{\infty\}$, and we shall use $\mathrm{Aff}(K)$ to denote the submonoid of elements in $\mathrm{LAff}(K)$ that are continuous. Let $\mathrm{LAff}_\sigma(K)$ be the submonoid of $\mathrm{LAff}(K)$ whose elements are pointwise suprema of increasing sequences of elements from $\mathrm{Aff}(K)$. The use of the superscript $++$ will always refer to strictly positive functions.

2. Extremal richness of multiplier and corona algebras

In this section we introduce the class of extremally rich C^*-algebras, giving some equivalent definitions, and discussing related matters about extensions that can be found in [9] and [15]. We present at the end a first result that analyzes the extremal richness of the multiplier and corona algebras, for a wide class of simple C^*-algebras with real rank zero and stable rank one. To establish this fact, we need some results concerning the ideal structure of these rings, that appear in [22], and thus they will be stated as required.

If A is a unital C^*-algebra, we use $\mathfrak{E}(A)$ to denote the set of extreme points of its closed unit ball A_1, and we refer to this set as the set of extreme points of the algebra. Recall that the elements of $\mathfrak{E}(A)$ are precisely those partial isometries $v\in A$ satisfying $(1-vv^*)A(1-v^*v)=0$ (see, for example, [20], Proposition 1.4.7). Notice that if A is prime, then the extreme points are precisely the isometries and co-isometries of the algebra. An element $x\in A$ is said to be *quasi-invertible* if $x\in A^{-1}\mathfrak{E}(A)A^{-1}$, and the set of quasi-invertible elements is denoted by A_{q}^{-1}.

Definition 2.1. ([9], Section 3) We say that a (unital) C^*-algebra A is *extremally rich* if the set A_{q}^{-1} of quasi-invertible elements is dense in A. As usual, a nonunital C^*-algebra A is extremally rich if its minimal unitization $\widetilde{A}$ is extremally rich.

An equivalent notion may be found in [9], Section 3 (see also [8], Section 6): A unital C^*-algebra A is extremally rich if and only if $A_1=\mathrm{conv}(\mathfrak{E}(A))$. At this point, it is convenient to notice that in any unital C^*-algebra A, the *closure* of the convex hull of the unitaries of A (which are extreme points) equals A_1 (see, e.g. [20], Proposition 1.1.12). We denote by $\mathcal{U}(A)$ the group of unitaries of a (unital) C^*-algebra A.

Remark 2.2. Let A be a C^*-algebra. Then:

(i) ([6], Section 1) A has stable rank one if and only if A is extremally rich and $\mathfrak{E}(\widetilde{A})=\mathcal{U}(\widetilde{A})$.

(ii) ([6], [8], Corollary 10.5 and [15], Lemma 3.3) If A is simple, then A is extremally rich if and only if A is either purely infinite or it has stable rank one.

A consequence of Remark 2.2 and [4], Theorem 7.6 is that if A is a simple C^*-algebra with real rank zero, then A has extremal richness if and only if it is separative. This contrasts with the fact that, as we will see, the class of C^*-algebras with extremal richness is strictly contained in the class of separative C^*-algebras.

As in the case of C^*-algebras with real rank zero (see [7], Theorem 3.14 and [27], Section 3.2), the behaviour of extremal richness under extensions depends not only on the extremal richness of the ideal and the quotient algebra, but also on a lifting condition on the extreme points. We now record some results in this direction:

Theorem 2.3. ([9], Theorem 6.1) *Let J be a closed ideal in a unital C^*-algebra A. Then A is extremally rich if and only if J and A/J are extremally rich, the extreme points of A/J lift to those of A and $\mathfrak{E}(A)+J\subset(A_{\mathrm{q}}^{-1})^-$.* ∎

COROLLARY 2.4. ([9], Corollary 6.3) *Let A be a C^*-algebra and let J be a closed ideal of A with stable rank one. Then A is extremally rich if and only if A/J is extremally rich and the extreme points of A/J lift to those of A.* ∎

THEOREM 2.5. ([15], Theorem 3.6) *Let A be a C^*-algebra, and let J be an essential closed ideal of A which is purely infinite simple. Then A is extremally rich if and only if A/J is extremally rich and $\mathfrak{E}(A/J)$ consists of isometries and co-isometries.* ∎

Let M and N be monoids with respective order-units u and v. A monoid morphism (that is, an additive map) $f : M \to N$ is said to be *normalized* provided that $f(u) = v$. Recall that a *state* on a monoid M with order-unit $u \in M$ is a normalized monoid morphism $s : (M, u) \to (\mathbb{R}^+, 1)$. We denote the set of states on (M, u) by $\mathrm{St}(M, u)$ or by S_u when no confusion may arise. We also denote by $\phi_u : M \to \mathrm{Aff}(\mathrm{S}_u)$ the natural map, given by evaluation on the states of M. Observe that $\mathrm{St}(M, u) = \mathrm{St}(\mathrm{G}(M), u)$, where $\mathrm{G}(M)$ is the Grothendieck group of M, and hence it is a compact convex set.

In order to analyze the extremal richness of the multiplier and corona algebra of a simple C^*-algebra A with real rank zero and stable rank one, the ideal structure of $\mathcal{M}(A)$ will play a crucial role. Let A be a σ-unital simple C^*-algebra with real rank zero and stable rank one. Fix $u \in V(A)^*$, and set $d = \sup \phi_u(D(A))$. We define:

$$W_\sigma^d(\mathrm{S}_u) = \{f \in \mathrm{LAff}(\mathrm{S}_u)^{++} : f + g = nd \text{ for some } g \in \mathrm{LAff}(\mathrm{S}_u)^{++} \text{ and } n \in \mathbb{N}\}.$$

Consider the set $V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$, where $\sqcup$ stands for disjoint union of sets. We equip this set with a monoid structure that extends the natural given addition operations of both $V(A)$ and $W_\sigma^d(\mathrm{S}_u)$, and by setting $x + f = \phi_u(x) + f$, for $x \in V(A)$ and $f \in W_\sigma^d(\mathrm{S}_u)$. It is not difficult to see that this is a well defined operation. In [22], Theorem 3.9, the following important relation between $V(\mathcal{M}(A))$ and $V(A)$ was established:

THEOREM 2.6. *Let A be a σ-unital nonunital C^*-algebra. Suppose that A is simple, with real rank zero, stable rank one and that $V(A)$ is strictly unperforated. Assume that A is nonelementary. Fix a nonzero element $u \in V(A)$. Set $D(A) = \{[p] \in V(A) : p \text{ is a projection in } A\}$, and $d = \sup \phi_u(D(A))$. Then there is a normalized monoid isomorphism*

$$\varphi : V(\mathcal{M}(A)) \to V(A) \sqcup W_\sigma^d(\mathrm{S}_u),$$

such that $\varphi([p]) = [p]$ if $p \in A$, and $\varphi([p]) = \sup\{\phi_u([q]) : [q] \in V(A) \text{ and } q \lesssim p\}$ if $p \in \mathcal{M}(A) \setminus A$. ∎

We will make an extensive use of this result in the next section; for now we content ourselves with two applications. For a compact convex set K, denote by $\partial_{\mathrm{e}} K$ the set of its extreme points. Combined with [26], Theorem 2.3, Theorem 2.6 gives an effective method to study the ideal structure of multiplier and corona algebras of C^*-algebras in the class we are considering. In detail, the map $I \mapsto \varphi(V(I))$ provides a lattice isomorphism between the lattice of closed ideals of $\mathcal{M}(A)$ and the lattice of order-ideals of $V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$. In this context, we define the *finite ideal* of $\mathcal{M}(A)$ as the unique closed ideal $I_{\mathrm{fin}}(A)$ of $\mathcal{M}(A)$ such that $\varphi(V(I_{\mathrm{fin}}(A))) = V(A) \sqcup \{f \in W_\sigma^d(\mathrm{S}_u) : f|\partial_{\mathrm{e}}\mathrm{S}_u \text{ is finite}\}$.

Let A be a simple C^*-algebra with real rank zero, and let $u \in V(A)^*$. We say that A has *finite scale* provided that the (lower semicontinuous) affine function $d := \sup \phi_u(D(A))$ is finite when restricted to $\partial_e \mathrm{S}_u$ (see [22], Definition 4.5). It should be noted that this notion does not depend on the choice of the nonzero element $u \in V(A)^*$, and that it differs from the definition of finite scale given in [16], in that the condition on d is required only for the extreme boundary of the state space. In our setting, C^*-algebras with finite scale are characterized by the following nice property:

THEOREM 2.7. (cf. [22], Theorem 4.7) *Let A be a nonunital simple and separable C^*-algebra with real rank zero and stable rank one. Assume that A is nonelementary, that $V(A)$ is strictly unperforated and that $\mathcal{M}(A)$ has real rank zero. Then A has finite scale if and only if, for every closed ideal I of $\mathcal{M}(A)$ properly containing A, we have that* $\mathrm{sr}(\mathcal{M}(A)/I) = 1$. ∎

DEFINITION 2.8. ([22], Definition 5.1) Let A be a C^*-algebra. A 1-*quasitrace* on A is a map $\tau : A_+ \to [0, \infty]$ such that $\tau(\alpha x) = \alpha\tau(x)$ if $x \in A_+$ and $\alpha \in \mathbb{R}^+$, such that $\tau(x+y) = \tau(x)+\tau(y)$, whenever x and y are commuting elements in A_+, and such that $\tau(xx^*) = \tau(x^*x)$ for all $x \in A$. A *quasitrace* on A is a 1-quasitrace τ that extends to a 1-quasitrace τ_n on $M_n(A)$ for each $n \in \mathbb{N}$.

We use the convention here that $0 \cdot \infty = 0$ so that $\tau(0) = 0$. Viewing A as the upper left hand corner subalgebra of $M_n(A)$, the extension τ_n in Definition 2.8 of τ means that $\tau(x) = \tau_n(xe_{11})$, where e_{11} is the matrix unit in $M_n(\widetilde{A})$. If τ is a quasitrace, we say that τ is *densely defined* provided that the set $F_\tau := \{x \in A_+ : \tau(x) < \infty\}$ is dense in A_+. We denote the set of densely defined quasitraces by $\mathrm{QT_d}(A)$, and we also use $\mathrm{LQT}(A)$ to denote the set of lower semicontinuous quasitraces. The notation $\mathrm{LQT_d}(A)$ will stand for the set of lower semicontinuous, densely defined, quasitraces. If $x \in K(A)_+$, where $K(A)$ is the Pedersen ideal of A, we set $Q = \{\tau \in \mathrm{LQT_d}(A) : \tau|K(A) < \infty\}$ and $Q_x = \{\tau \in Q : \tau(x) = 1\}$. If A is simple, the set Q_x is (weakly) compact. If A is moreover σ-unital with real rank zero and $u = [p] \in V(A)^*$ for a nonzero projection $p \in A$, then the natural map $\alpha : Q_p \to \mathrm{S}_u$ given by $\alpha(\tau)([q]) = \tau(q)$, for $[q] \in V(A)$, provides an affine homeomorphism ([22], Theorem 5.6). This is the σ-unital, nonunital and semi-finite version of Blackadar and Handelman's theorem ([5], Theorem III.1.3).

DEFINITION 2.9. Let A be a C^*-algebra. A lower semicontinuous and order-preserving quasitrace τ is said to be *infinite* if $\sup\limits_{\lambda} \tau(u_\lambda) = +\infty$ for some approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$.

Observe that this definition does not depend on the particular approximate unit. If A has real rank zero, and τ is infinite, then $\sup \tau(p) = \infty$, where the supremum is taken over the projections $p \in A$.

THEOREM 2.10. (cf. [22], Theorem 6.3, Proposition 6.5, Theorem 6.6) *Let A be a separable nonunital simple C^*-algebra with real rank zero, stable rank one and with $V(A)$ strictly unperforated. Suppose that A is nonelementary. Let $p \in A$ be a nonzero projection and let $\mathfrak{c}$ be the cardinal of infinite extremal quasitraces in Q_p. Then $\mathcal{M}(A)$ has at least $\mathfrak{c}$ different maximal ideals that contain $I_{\mathrm{fin}}(A)$, and the quotient of $\mathcal{M}(A)$ by any of these ideals is a purely infinite simple C^*-algebra.* ∎

In order to clarify the exposition, we state a lemma whose argument is used in [15], Theorem 4.9.

Lemma 2.11. *Let A be a unital C^*-algebra. Assume that A is prime and that there exist different maximal ideals J_i, for $i = 1, 2$, such that each projection in A/J_i is infinite. Then A is not extremally rich.*

Proof. Consider the C^*-exact sequence:

$$0 \to (J_1 \cap J_2) \to A \to A/(J_1 \cap J_2) \to 0,$$

and note that $A/(J_1 \cap J_2) \cong J_1/(J_1 \cap J_2) \oplus J_2/(J_1 \cap J_2)$ and that $J_1/(J_1 \cap J_2) \cong A/J_2$ and $J_2/(J_1 \cap J_2) \cong A/J_1$. Since each quotient A/J_i is simple, its extreme points are isometries or co-isometries. Further, as every projection in $J_i/(J_1 \cap J_2)$ is infinite, we see that $J_i/(J_1 \cap J_2)$ contain non-trivial isometries (and hence co-isometries) for $i = 1, 2$. Finally, since the set of extreme points of a direct sum equals the direct sum of the extreme points of each factor, we see that $A/(J_1 \cap J_2)$ has an extreme point which is neither an isometry nor a co-isometry, and hence it cannot be lifted to an extreme point of A, because A is prime. By Theorem 2.3, A is not extremally rich. ∎

Let us now discuss the extremal richness in the elementary case. Suppose that $A \cong \mathbb{K}$, where $\mathbb{K} = \mathbb{K}(\mathcal{H})$ is the C^*-algebra of compact operators over an infinite-dimensional, separable, Hilbert space $\mathcal{H}$. Then, if $\mathbb{B} = \mathbb{B}(\mathcal{H})$, we have that $\mathcal{M}(A)/A \cong \mathbb{B}/\mathbb{K}$ is a purely infinite simple C^*-algebra, hence extremally rich by Remark 2.2 (i). On the other hand, $\mathbb{B}$ is a von Neumann algebra, whence it is also extremally rich. Thus we shall assume from now on that all C^*-algebras are nonelementary.

In [15], Theorem 4.9, it is proved that if A is a simple separable AF algebra such that $A \otimes \mathbb{K}$ contains at least two semi-finite extremal traces, then $\mathcal{M}(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$ is not extremally rich. On the other hand, in [15], Proposition 4.13 it is established that if A is a simple and separable (nonunital) AF algebra with a finite number of semi-finite extremal traces, of which at least two are infinite, then $\mathcal{M}(A)/A$ is not extremally rich. Both situations can be handled in our setting. The next result enlarges to a great extent the number of instances in which the extremal richness of the corona algebra can be analyzed. We also answer an implicit question that is posed in [15], Remark 4.19: If A has an infinite number of extremal quasitraces, has the corona extremal richness? In case none of them are infinite, the answer is positive (see also [15], Theorem 4.1, where some stable cases outside our class are considered), whereas if at least two of them are infinite, the answer is negative. The case in which there is only one infinite extremal quasitrace will be considered in the next section.

Theorem 2.12. *Let A be a nonunital separable simple C^*-algebra with real rank zero and stable rank one. Suppose that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection.*

(i) *If A has finite scale and $\mathcal{M}(A)$ has real rank zero, then $\mathcal{M}(A)/A$ is extremally rich.*

(ii) *If A has at least two infinite extremal quasitraces in Q_p, then $\mathcal{M}(A)/A$ is not extremally rich. In particular, $\mathcal{M}(A)$ is not extremally rich.*

Proof. (i) Recall that $\mathcal{M}(A)$ has a unique closed ideal $L(A)$ that properly contains A and that is contained in every closed ideal that properly contains A (see [17], Remark 2.9, and also [22], Proposition 4.1). Notice that $L(A)/A$ is an essential closed ideal of $\mathcal{M}(A)/A$, and that it is purely infinite simple (see [25], Theorem 1.3 (a)). Since A has finite scale, we have that $\mathcal{M}(A)/L(A)$ has stable rank one, by Theorem 2.7. Using Remark 2.2 (i), we conclude that $\mathcal{M}(A)/L(A)$ is extremally rich and that $\mathfrak{E}(\mathcal{M}(A)/L(A)) = \mathcal{U}(\mathcal{M}(A)/L(A))$. By Theorem 2.5, it follows that $\mathcal{M}(A)/A$ is extremally rich.

(ii) Let $\mathfrak{c}$ be the cardinal of infinite extremal quasitraces in Q_p. By hypothesis $\mathfrak{c} \geqslant 2$. Using Theorem 2.10, we get at least $\mathfrak{c}$ different closed maximal ideals in $\mathcal{M}(A)/A$. Moreover, the quotient of $\mathcal{M}(A)/A$ by each one of these ideals is a purely infinite simple C^*-algebra. Also, since $L(A)/A$ is the minimal nonzero closed ideal of $\mathcal{M}(A)/A$, we get that the corona algebra is a prime ring. Therefore, the hypotheses of Lemma 2.11 are fulfilled, whence we conclude that $\mathcal{M}(A)/A$ is not extremally rich.

That $\mathcal{M}(A)$ is not extremally rich follows from Theorem 2.3. ∎

Remark 2.13. In the proof of the previous theorem, we used the fact that the corona algebra $\mathcal{M}(A)/A$ is a prime ring. Although it is possibly well-known, we remark that this is not true in general. Let A be a simple C^*-algebra such that $\mathcal{M}(A)$ has at least two different maximal closed ideals I_1, I_2 (such examples exist; see [22]). Let $J = I_1 I_2$. Then J is a prime C^*-algebra and $\mathcal{M}(J) = \mathcal{M}(A)$. Therefore $\mathcal{M}(J)/J$ contains two nontrivial ideals, I_i/J, $i = 1, 2$, whose product is zero.

3. EXTREMAL RICHNESS OF CORONA ALGEBRAS WITH ONLY ONE INFINITE EXTREMAL QUASITRACE

The purpose of the present section is to determine when the corona algebra of a simple C^*-algebra A with real rank zero, stable rank one, with $V(A)$ strictly unperforated, and that has precisely one infinite extremal quasitrace is extremally rich. Our approach to the solution follows the lines indicated by Theorem 2.5 and Corollary 2.4. The knowledge of significant aspects of the ideal lattice of $\mathcal{M}(A)/A$, of projections in $\mathcal{M}(A)$ and in its quotient algebras will be an important ingredient in the following. This will be reflected in some computations of the index map in various situations, for which Theorem 2.6 will be essential.

LEMMA 3.1. *Let B be a (unital) C^*-algebra, and let I be a closed two-sided ideal of B. Let w be an isometry in B/I, and denote by $\pi : B \to B/I$ the natural quotient map. Then w can be lifted to an isometry $z \in B$ if and only if there exists a partial isometry $v \in B$ such that $1 - v^*v \lesssim 1 - vv^*$ and $\pi(v) = w$.*

Proof. If w can be lifted to an isometry $z \in B$, then just take $v = z$. For the converse, assume that $w = \pi(v)$ for some partial isometry $v \in B$ such that $1 - v^*v \lesssim 1 - vv^*$. Let $p = v^*v$ and $q = vv^*$. Then there exists $r \in B$ such that $1-p = r^*r$ and $rr^* \leqslant 1-q$. Since $\pi(v) = w$, we see that $\pi(r^*r) = 0$, that is, $r \in I$. Let $z = v + r$. Then $z^*z = 1$, hence z is an isometry, and $\pi(z) = \pi(v) = w$. ∎

Recall that a monoid M is a *refinement monoid* if whenever $x_1, x_2, y_1, y_2 \in M$ satisfy $x_1 + x_2 = y_1 + y_2$, then there exist elements $z_{ij} \in M$, for $i, j = 1, 2$ such that $\sum_j z_{ij} = x_i$ and $\sum_i z_{ij} = x_j$ for each i, j. By [3], Lemma 2.3 (see also [26], Theorem 1.1), if B is a C^*-algebra with real rank zero, then $V(B)$ is a refinement monoid.

LEMMA 3.2. *Let B be a C^*-algebra with real rank zero. Then* $\mathrm{K}_0(B) = \mathrm{G}(V(B))$.

Proof. We may clearly assume that B is nonunital. First, notice that $V(B) = \varinjlim V(pBp)$, where p runs over the set of projections of B. (In general, the set of projections of B need not be directed, but because of the real rank zero condition it is easy to see that given projections $p, q \in B$, there exists a projection $r \in B$ such that $V(pBp), V(qBq) \subseteq V(rBr)$.) Taking into account that G is in fact a continuous functor from the category of monoids to the category of groups, we get that $\mathrm{G}(V(B)) = \varinjlim \mathrm{G}(V(pBp))$. Since pBp does have a unit for each projection $p \in B$, it follows that $\mathrm{G}(V(pBp)) = \mathrm{K}_0(pBp)$. Hence, it remains to prove that $\mathrm{K}_0(B) = \varinjlim \mathrm{K}_0(pBp)$.

We denote by $B^+ := B \oplus \mathbb{C}$, equipped with pointwise sum and adjoint, and with a mixed multiplication given by $(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu)$, for $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$. Then B^+ is a C^*-algebra, which is isomorphic to $\widetilde{B}$ if B is nonunital ([24], Proposition 2.1.7). Note that if $p \in M_n(B^+)$ is a projection, then there exist $r \leqslant n$ and projections $g, h \in M_\infty(B)$ such that $g \leqslant 1_r$ and $p \sim (1_r - g) \oplus h$. (Here 1_r stands for the unit of $M_r(\mathbb{C})$.) This follows from [14], Lemma 10.3 (see also [2], Lemma 3.4). Applying the refinement property to the equality $(1_r - g) + g = 1_r$, we get projections $p_1, \dots, p_r \in B$ such that $g \sim \sum_{i=1}^r \oplus p_i$, while $1_r - g \sim \sum_{i=1}^r \oplus (1 - p_i)$. We therefore obtain that $1_r - g \sim (1 - p_1) \oplus \cdots \oplus (1 - p_r) \sim 1_r - \bigoplus_{i=1}^r p_i$. Using this and the fact that $V(B) = \varinjlim V(pBp)$, we conclude that $V(B^+) = \varinjlim V((pBp)^+)$. On the other hand, if $\pi : B^+ \to \mathbb{C}$ and $\pi_p : (pBp)^+ \to \mathbb{C}$, for $p \in B$ are the natural projection maps, then (using the functoriality of V) we have $V(\pi_p) = V(\pi) \circ V(i)$, where $i : (pBp)^+ \to B^+$ is the natural inclusion.

Using again the continuity of G, we obtain that $\mathrm{K}_0(B^+) = \varinjlim \mathrm{K}_0((pBp)^+)$, and also that $\mathrm{G}(\pi_p) = \mathrm{G}(\pi) \circ \mathrm{G}(i)$. We conclude then that $\mathrm{K}_0(B) = \varinjlim \mathrm{K}_0(pBp)$, as desired. ∎

As is well known, the previous result is false in general. For example, let $A = C_0(\mathbb{R}^2)$. Then $V(A) = 0$, whereas $\mathrm{K}_0(A) \cong \mathbb{Z}$ (see [24], Section 6.2).

If B is a C^*-algebra and I is a closed ideal of B, we denote by $\delta : \mathrm{K}_1(B/I) \to \mathrm{K}_0(I)$ the index map in K-Theory.

PROPOSITION 3.3. *Let B be a (unital) C^*-algebra, and let I be a closed ideal of B. Let $w \in \mathcal{U}(B/I)$, and let $\pi : B \to B/I$ be the natural quotient map. If $\mathrm{RR}(I) = 0$ and $V(I)$ is cancellative, then w can be lifted to a unitary in B (respectively a proper isometry, a proper co-isometry) if and only if $\delta[w] = 0$ (respectively $\delta[w] < 0$, $\delta[w] > 0$).*

Proof. If w can be lifted to a partial isometry $v \in B$ via π, then it is a standard fact that the index can be computed as $\delta([w]) = [1 - v^*v] - [1 - vv^*]$ (see, for example, [24], Exercise 8C).

Assume that w can be lifted to a unitary (respectively a proper isometry, a proper co-isometry) $v \in B$. Then it follows easily from the previous observation (using Lemma 3.2 and that $V(I)$ is cancellative) that $\delta[w] = 0$ (respectively $\delta[w] < 0$, $\delta[w] > 0$).

Conversely, since $\mathrm{RR}(I) = 0$ there exists a partial isometry $v \in B$ such that v is a lift for w (see the proof of [10], Lemma 2.6 or [2], Lemma 2.1). If $\delta[w] = 0$, then $[1 - v^*v] = [1 - vv^*]$ in $\mathrm{K}_0(I)$. By Lemma 3.2, and since $V(I)$ is cancellative, we get that $1 - v^*v \sim 1 - vv^*$ in I. Therefore, there exists $r \in I$ such that $1 - p = r^*r$ and $1 - q = rr^*$, where $p = v^*v$ and $q = vv^*$. Let $z = v + r$. Then $z^*z = zz^* = 1$ and $\pi(z) = \pi(v) = w$. Hence w can be lifted to a unitary.

If $\delta[w] < 0$, then $1 - v^*v \lesssim 1 - vv^*$, so that by Lemma 3.1, w can be lifted to an isometry $z \in B$, which is proper since $\delta([w]) \neq 0$. We proceed similarly if $\delta[w] > 0$, in order to get a co-isometry. ∎

In the following lemma we need to establish a slight generalization of a known result ([1], Lemma II.7.1).

LEMMA 3.4. *Let K be a compact convex set, and let $f, g \in \mathrm{LAff}_\sigma(K)^{++}$ be functions such that $f|\partial_{\mathrm{e}}K = g|\partial_{\mathrm{e}}K$. Then $f = g$.*

Proof. Since K is compact and f is lower semicontinuous, we see that f takes its minimum value on K. Indeed, if $\alpha = \min(f)$, then there exists $x \in \partial_{\mathrm{e}}K$ such that $f(x) = \alpha$ (this follows after a standard argument — see [12], Corollary 5.19).

Write $g = \sup_n g_n$, where $g_n \in \mathrm{Aff}(K)^{++}$ for each n, and note that $\{g_n\}$ form an increasing sequence. Note that $f - g_n = g - g_n \geqslant 0$ on $\partial_{\mathrm{e}}K$ for every $n \in \mathbb{N}$. Taking into account that $f - g_n$ is affine and lower semicontinuous, we conclude from the previous paragraph that $f - g_n \geqslant 0$ globally. Hence $f \geqslant g$. A similar argument shows that $f \leqslant g$, and therefore $f = g$. ∎

The following fact is a consequence of the methods developed in [22], Section 4. Note that if K is a metrizable Choquet simplex, then $\mathrm{LAff}_\sigma(K) = \mathrm{LAff}(K)$.

LEMMA 3.5. *Let K be a metrizable Choquet simplex. Let $f, g, h, d \in \mathrm{LAff}(K)^{++}$ such that $f + g = f + h = d$. Suppose that there exists $s \in \partial_{\mathrm{e}}K$ such that $d(x) = \infty$ (for $x \in \partial_{\mathrm{e}}K$) if and only if $x = s$. Assume that $g|\partial_{\mathrm{e}}K$ is a finite function.*

(i) *If $h|\partial_{\mathrm{e}}K$ is also finite and $g(s) < h(s)$, then there exist $e \in \mathrm{LAff}(K)^{++}$ and $k \in \mathbb{N}$ such that $g + k = h + e$ and $e + d = k + d$.*

(ii) *If $h(s) = \infty$ and $\{s\}'$, the complementary face of $\{s\}$, is a closed face, then there exist $e \in \mathrm{LAff}(K)^{++}$ and $k \in \mathbb{N}$ such that $g+e = h+k$ and $e+d = k+d$.*

Proof. Since $g|\partial_{\mathrm{e}}K$ is finite, we immediately get that $f(s) = \infty$ and that $g|\partial_{\mathrm{e}}K \setminus \{s\} = h|\partial_{\mathrm{e}}K \setminus \{s\}$.

(i) Let $a = h(s) - g(s)$ and take $k \in \mathbb{N}$ such that $k > a$. By an argument similar to the one used in [22], Proposition 4.10, there exists a lower semicontinuous affine function e such that $e(s) = k - a$ and $e|\{s\}' = k$. Note that $(g + k)|\partial_{\mathrm{e}}K = (h + e)|\partial_{\mathrm{e}}K$ and that $(e + d)|\partial_{\mathrm{e}}K = (k + d)|\partial_{\mathrm{e}}K$. Therefore $g + k = h + e$ and $e + d = k + d$, by Lemma 3.4.

(ii) Let $k \in \mathbb{N}$. Since $\{s\}'$ is closed, there is (by [12], Corollary 11.27) an isomorphism

$$\mathrm{Aff}(K) \cong \mathrm{Aff}(\{s\}) \times \mathrm{Aff}(\{s\}').$$

Thus, for each $n \in \mathbb{N}$, there exists $e_n \in \mathrm{Aff}(K)^{++}$ such that $e_n(s) = n$ and $e_n|\{s\}' = k$. Let $e = \sup_n e_n$. Then $e \in \mathrm{LAff}(K)^{++}$ and satisfies $e(s) = \infty$ while $e|\{s\}' = k$. Therefore $(g + e)|\partial_{\mathrm{e}}K = (h + k)|\partial_{\mathrm{e}}K$ and $(e + d)|\partial_{\mathrm{e}}K = (k + d)|\partial_{\mathrm{e}}K$, whence $g + e = h + k$ and $e + d = k + d$, also by Lemma 3.4. ∎

PROPOSITION 3.6. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Assume that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection and suppose that A has exactly one infinite extremal quasitrace in Q_p. Suppose also that the real rank of $\mathcal{M}(A)$ is zero. Let I be any closed ideal of $\mathcal{M}(A)$ such that $A \subsetneqq I \subseteq I_{\mathrm{fin}}(A)$. Then $\delta([w])$ is either zero, or positive, or negative, for any unitary $w \in \mathcal{M}(A)/I$, where $\delta : \mathrm{K}_1(\mathcal{M}(A)/I) \to \mathrm{K}_0(I/L(A))$ is the index map. Hence w can always be lifted to an isometry or a co-isometry.*

Proof. Let $u = [p] \in V(A)$, and let $d = \sup \phi_u(D(A))$. By Theorem 2.6, there exists a normalized monoid isomorphism $\varphi : V(\mathcal{M}(A)) \to V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$ (so that $\varphi([1_{\mathcal{M}(A)}]) = d$). Since the natural map $\alpha : Q_p \to \mathrm{S}_u$ given by evaluation is an isomorphism, the fact that A has exactly one infinite extremal quasitrace means that the set $\Gamma_d := \{t \in \partial_{\mathrm{e}}\mathrm{S}_u : d(t) = \infty\}$ consists of one point, which we denote by s.

Let $\phi : \mathcal{M}(A) \to \mathcal{M}(A)/I$ and $\pi : \mathcal{M}(A) \to \mathcal{M}(A)/L(A)$ be the natural projection maps. By the proof of [10], Lemma 2.6 (or also [2], Lemma 2.1), there exists a partial isometry $v \in \mathcal{M}(A)$ such that $\phi(v) = w$. Then $1 - v^*v$ and $1 - vv^*$ belong to I. Since w is not zero, we first note that $v \notin A$. Suppose that $1-v^*v \in A$. Then $\delta[w] = -[\pi(1-vv^*)] \leqslant 0$, and a similar conclusion would result if $1-v^*v \in A$. Therefore we may assume that $1-v^*v, 1-vv^* \notin A$. Let $f = \varphi([v^*v])$, $g = \varphi([1 - v^*v])$ and $h = \varphi([1 - vv^*])$, which are functions in $W_\sigma^d(\mathrm{S}_u)$. Note that $f + g = f + h = d$, and that both $g|\partial_{\mathrm{e}}\mathrm{S}_u$ and $h|\partial_{\mathrm{e}}\mathrm{S}_u$ are finite functions, since $g, h \in I_{\mathrm{fin}} := \varphi(V(I_{\mathrm{fin}}(A)))$. If $g(s) = h(s)$, then $g = h$ and so $1 - v^*v \sim 1 - vv^*$ in I, whence $\delta[w] = 0$. Suppose that $g(s) < h(s)$. Then, by Lemma 3.5 (i), there exist

$e \in \mathrm{LAff}(\mathrm{S}_u)^{++}$ and $k \in \mathbb{N}$ such that $g+k = h+e$ and $e+d = k+d$. Thus, there exist nonzero projections $p \in M_\infty(L(A))$ and $q \in M_\infty(I)$ such that $\varphi([p]) = k$ and $\varphi([q]) = e$, and $(1-v^*v)\oplus p \sim (1-vv^*)\oplus q$. Hence $\pi(1-v^*v) \sim \pi(1-vv^*)\oplus\pi(q)$ in $I/L(A)$, and it follows that $\delta([w]) = [\pi(1-v^*v)] - [\pi(1-vv^*)] \geqslant 0$. A similar argument, if $h(s) < g(s)$, shows that $\delta([w]) \leqslant 0$.

We conclude from Proposition 3.3 that w can always be lifted either to an isometry or to a co-isometry. ∎

If X is a subset of a convex set K, we denote by $\mathrm{conv}(X)$ the convex hull of X, that is, the smallest convex subset of K that contains X.

Lemma 3.7. *Let K be a Choquet simplex, and let F be a closed face of K. Let F' be the complementary face of F. Then $\overline{F'} = \overline{\mathrm{conv}(\partial_{\mathrm{e}}K \setminus \partial_{\mathrm{e}}F)}$.*

Proof. Suppose first that F' is a closed face of K. Then, since K is compact, it follows that F' is a compact convex subset of K. By Krein-Milman's Theorem, we have that $F' = \overline{\mathrm{conv}(\partial_{\mathrm{e}}F')}$. Note now that $\partial_{\mathrm{e}}F' = F'\cap\partial_{\mathrm{e}}K$, because F' is a face of K, and by definition of complementary face we conclude that $\partial_{\mathrm{e}}F' = \partial_{\mathrm{e}}K\setminus\partial_{\mathrm{e}}F$.

If F' is not closed, set $X := \overline{\mathrm{conv}(\partial_{\mathrm{e}}K \setminus \partial_{\mathrm{e}}F)}$. By construction, X is a compact convex subset of K, and it contains all the extreme points of K, except maybe those from F. Notice now that the convex hull of X and F is closed (by [12], Proposition 5.2), and it contains all the extreme points of K. Another application of Krein-Milman Theorem shows that $K = \mathrm{conv}(X \cup F)$. Let $a \in F'$. Then there exist $\alpha \in [0,1]$, $f \in F$ and $x \in X$ such that $a = \alpha f + (1-\alpha)x$. Since $f \notin F'$, and taking into account that F' is a face, we have that $\alpha = 0$, and thus $a = x$. We therefore conclude that $F' \subseteq X$. It is clear, on the other hand, that $X \subseteq \overline{F'}$, whence it follows that $\overline{F'} = X$, as desired. ∎

Proposition 3.8. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Suppose that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection, and assume that A has exactly one infinite extremal quasitrace in Q_p, which we denote by τ. If $\mathrm{RR}(\mathcal{M}(A)) = 0$, then all proper isometries of $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ can be lifted to proper isometries of $\mathcal{M}(A)/L(A)$ if and only if the complementary face of $\{\tau\}$ in Q_p is closed.*

Proof. Let $u = [p] \in V(A)$, and set $d = \sup\phi_u(D(A))$. By Theorem 2.6, there is a monoid isomorphism $\varphi : V(\mathcal{M}(A)) \to V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$ such that $\varphi([1_{\mathcal{M}(A)}]) = d$. As in the previous result, the quasitrace τ corresponds, through the affine homeomorphism $\alpha : Q_p \to \mathrm{S}_u$, to a unique point $s \in \partial_{\mathrm{e}}\mathrm{S}_u$ such that $d(s) = \infty$.

First, suppose that $\{s\}'$, the complementary face of $\{s\}$ in the simplex S_u, is not closed and that all proper isometries of $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ can be lifted to proper isometries of $\mathcal{M}(A)/L(A)$. Since all projections in $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ are infinite (see Theorem 2.10), there exists a proper isometry $w \in \mathcal{M}(A)/I_{\mathrm{fin}}(A)$. By hypothesis, w can be lifted to a proper isometry $v \in \mathcal{M}(A)/L(A)$. On the other hand, there exists a partial isometry $z \in \mathcal{M}(A)$ such that $\pi(z) = v$, where $\pi : \mathcal{M}(A) \to \mathcal{M}(A)/L(A)$ is the natural map (again by [10], proof of Lemma 2.6 or [2], Lemma 2.1). Therefore $1-z^*z \in L(A)$ and $1-zz^* \notin I_{\mathrm{fin}}(A)$. As in Proposition 3.6, we have that $z \notin A$, and moreover in this case $1-zz^* \notin A$. We may also assume that $1 - z^*z \notin A$, since the proof would be similar otherwise. Let $f = \varphi([z^*z])$, $g = \varphi([1 - z^*z])$ and $h = \varphi([1 - zz^*])$. Then $f + g = f + h = d$, and $h(s) = \infty$. Note also that $g \in \varphi(V(L(A))) = V(A) \sqcup \mathrm{Aff}(\mathrm{S}_u)^{++}$, and thus is continuous. Since $d|\partial_{\mathrm{e}}\mathrm{S}_u$ is infinite exactly at s we have that $f(s) = \infty$. Therefore $g|\partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\} = h|\partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\}$. We also have that $F := \overline{\{s\}'}$ equals the closed convex hull of $\partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\}$, by Lemma 3.7, and that F is a compact convex subset of S_u (see [12], Proposition 5.1).

If $s \notin F$, then $\partial_{\mathrm{e}}F = \partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\}$, and since g and h are affine and lower semicontinuous we get that $g|F = h|F$ (by Lemma 3.4). Since $\{s\}'$ is not closed, there exists $x \in F \setminus \{s\}'$. Thus $g(x) = h(x)$. But $x \notin \{s\}'$, so that there exist $\alpha \in (0, 1]$ and $t \in \{s\}'$ such that $x = \alpha s + (1 - \alpha)t$, and this implies $h(x) = \infty$, a contradiction since g is continuous.

Hence $s \in F$, and so $s = \lim_n y_m$, where y_m belong (for all m) to the convex hull of $\partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\}$. Thus $g(y_m) = h(y_m)$ for all m, and it follows that $g(s) = \lim_m g(y_m) = \lim_m h(y_m) \geqslant h(s)$, a contradiction.

Conversely, assume that $\{s\}'$ is a closed face, and let $w \in \mathcal{M}(A)/I_{\mathrm{fin}}(A)$ be a proper isometry. Again, there exists a partial isometry $z \in \mathcal{M}(A) \setminus A$ such that $\phi(z) = w$, where $\phi : \mathcal{M}(A) \to \mathcal{M}(A)/I_{\mathrm{fin}}(A)$ is the natural map. Thus $1 - z^*z \in I_{\mathrm{fin}}(A)$ and $1 - zz^* \notin I_{\mathrm{fin}}(A)$. If $1 - z^*z \in A$, then $1 = \pi(z)^*\pi(z)$ and $1 \neq \pi(z)\pi(z)^*$ (where π is the quotient map modulo $L(A)$). We also have that $w = \phi(z) = \overline{\pi(z)}$, where the latter denotes the class of $\pi(z)$ modulo $I_{\mathrm{fin}}(A)/L(A)$. We then conclude that w can be lifted to a (proper) isometry of $\mathcal{M}(A)/L(A)$, by Lemma 3.1. Therefore we may assume that $1 - z^*z \notin A$. Let $f = \varphi([z^*z])$, $g = \varphi([1-z^*z])$ and $h = \varphi([1-zz^*])$. Then $f+g = f+h = d$ and also $h(s) = \infty$. By Lemma 3.5 (ii), there exist $e \in \mathrm{LAff}(\mathrm{S}_u)^{++}$ and $k \in \mathbb{N}$ such that $g + e = h + k$ and $e + d = k + d$. Hence, there exist projections $p \in M_\infty(L(A))$ and $q \in M_\infty(\mathcal{M}(A))$ such that $k = \varphi([p])$ and $e = \varphi([q])$, and $(1 - z^*z) \oplus q \sim (1 - zz^*) \oplus p$. Therefore $1 - \pi(z)^*\pi(z) \lesssim 1 - \pi(z)\pi(z)^*$, and as before $w = \phi(z) = \overline{\pi(z)}$; it follows then from Lemma 3.1 that w can be lifted to a (proper) isometry. ∎

We turn our attention now to the extreme points of $\mathcal{M}(A)/L(A)$, which, in our setting, are only isometries and co-isometries.

LEMMA 3.9. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Suppose that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection, and assume that A has exactly one infinite extremal quasitrace in Q_p. If $L(A)$ has real rank zero, then $\mathfrak{E}(\mathcal{M}(A)/L(A))$ consists only of isometries and co-isometries.*

Proof. Let $u = [p] \in V(A)$ and $d = \sup \phi_u(D(A))$. As before, we have a monoid isomorphism $\varphi : V(\mathcal{M}(A)) \to V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$ such that $\varphi([1_{\mathcal{M}(A)}]) = d$, and there is a unique state $s \in \partial_{\mathrm{e}}\mathrm{S}_u$ at which $d|\partial_{\mathrm{e}}\mathrm{S}_u$ is infinite.

Let $v \in \mathfrak{E}(\mathcal{M}(A)/L(A))$. Suppose that v is neither an isometry nor a co-isometry. Choose a partial isometry $z \in \mathcal{M}(A) \setminus A$ such that $\pi(z) = v$, where $\pi : \mathcal{M}(A) \to \mathcal{M}(A)/L(A)$ is the natural map. Then $(1 - z^*z)\mathcal{M}(A)(1 - zz^*) \subseteq L(A)$ and $1 - z^*z, 1 - zz^* \notin L(A)$. Let I and J denote the closed ideals of $\mathcal{M}(A)$ generated by $1-z^*z$ and $1-zz^*$ respectively. Then $IJ \subseteq L(A)$. On the other hand, we have that $A \subsetneq I, J$ since v is neither an isometry nor a co-isometry. Therefore $L(A) \subseteq I, J$ and we conclude that $L(A) = IJ = I \cap J$.

Clearly $1 - z^*z, 1 - zz^* \notin A$. Let $f = \varphi([z^*z])$, $g = \varphi([1 - z^*z])$ and $h = \varphi([1 - zz^*])$. Then again $f + g = f + h = d$, whence $g = h$ on $\partial_{\mathrm{e}}\mathrm{S}_u \setminus \{s\}$. Let

$$I_g = V(A) \sqcup \{f_1 \in W_\sigma^d(\mathrm{S}_u) : f_1 + f_2 = ng \text{ for some } f_2 \in W_\sigma^d(\mathrm{S}_u) \text{ and some } n \in \mathbb{N}\}$$

be the order-ideal of $V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$ generated by g. Similarly, I_h denotes the order-ideal generated by h. Note that $\varphi(V(I)) = I_g$ and $\varphi(V(J)) = I_h$. Since $I \cap J = L(A)$, we have that

$$I_g \cap I_h = \varphi(V(I) \cap V(J)) = \varphi(V(L(A))) = V(A) \sqcup \mathrm{Aff}(\mathrm{S}_u)^{++}.$$

Note that $g, h \notin \mathrm{Aff}(\mathrm{S}_u)^{++}$, since v is neither an isometry, nor a co-isometry. Therefore $g \neq h$. (Otherwise $I_g = I_g \cap I_h = V(A) \sqcup \mathrm{Aff}(\mathrm{S}_u)^{++}$, and so g would be continuous, a contradiction.) In particular, g and h are not simultaneously infinite. Suppose that $g(s) = \infty$ and $h(s) < \infty$. Then $g + h = 2g$, whence $I_h \subseteq I_g$, so that h would be continuous, a contradiction. The argument is similar if $g(s) < \infty$ and $h(s) = \infty$.

This implies that $g(s), h(s) < \infty$. Suppose that $h(s) < g(s)$. Let $n \in \mathbb{N}$ be such that $n \geqslant 2$ and $nh(s) > g(s)$. By a similar argument to the one used in [22], Proposition 4.10, there exists $f' \in \mathrm{LAff}(\mathrm{S}_u)^{++}$ such that $f'(s) = nh(s) - g(s)$ and $f'|\{s\}' = (n-1)g|\{s\}'$. Then $f' + g = nh$, and therefore $I_g \subseteq I_h$, a contradiction since g is not continuous. Again, the argument is similar if $g(s) < h(s)$.

Hence we conclude that any $v \in \mathfrak{E}(\mathcal{M}(A)/L(A))$ is necessarily an isometry or a co-isometry. ∎

Remark 3.10. The previous result would follow immediately if $\mathcal{M}(A)/L(A)$ were a prime ring. We remark that this is not true in general.

Proof. Let A be a (nonunital) simple separable AF algebra, and let $p \in A$ be a nonzero projection such that if $u = [p] \in V(A)$, then $\partial_{\mathrm{e}}\mathrm{S}_u \cong [-1, 1]$. Moreover, we take A such that its scale d equals 2 on the interval $[-1, 0)$, and such that $d(x) = 1/(1 - x)$, for $x \in [0, 1]$. (The existence of this example follows after [22], Example 4.6; see also [14], Example 7.3].) Then A has only one infinite extremal (quasi)trace. Define $f \in W_\sigma^d(\mathrm{S}_u)$ as 1 on $[-1, 1)$, and set $f(1) = 1/2$. Define also $g \in W_\sigma^d(\mathrm{S}_u)$ as 1 on $[-1, 0)$ and 1/2 on $[0, 1]$. Then, if we denote by I_f and I_g the respective order-ideals generated by f and g, it is clear that $I_f \cap I_g = V(A) \sqcup C[-1, 1]^{++}$, whereas $I_f \neq V(A) \sqcup C[-1, 1]^{++}$ and also $I_g \neq V(A) \sqcup C[-1, 1]^{++}$. Since $\partial_{\mathrm{e}}\mathrm{S}_u$ is compact, it follows from Theorem 2.6 and [26], Theorem 2.3 (see also [22], Section 6) that $\mathcal{M}(A)/L(A)$ is not a prime ring. ∎

Recall that a closed ideal I of a C^*-algebra B is *stably cofinite* provided that the quotient B/I is stably finite. This is equivalent to saying that the monoid $V(B/I)$ is stably finite, that is, the relation $x + y = y$ implies $x = 0$. When studying the stably finite quotients of $\mathcal{M}(A)$, a closed ideal called the *bounded ideal* is of some significance. In our setting, it can be described as the unique closed ideal $I_{\mathrm{b}}(A)$ of $\mathcal{M}(A)$ such that $\varphi(V(I_{\mathrm{b}}(A))) = V(A) \sqcup \{f \in W_\sigma^d(\mathrm{S}_u) : f \text{ is bounded}\}$, where φ is the isomorphism established on Theorem 2.6. If A has a finite number of infinite extremal quasitraces, it is possible to characterize when this ideal is stably cofinite, as follows.

PROPOSITION 3.11. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Suppose that A is nonelementary, that $V(A)$ is strictly unperforated and that $\mathcal{M}(A)$ has real rank zero. Let $p \in A$ be a nonzero projection, and denote by Q_∞ the set of infinite extremal quasitraces in Q_p. If Q_∞ is finite, then $I_{\mathrm{b}}(A)$ is stably cofinite if and only if for any nonempty subset $X \subseteq Q_\infty$, the complementary face of* $\mathrm{conv}(X)$ *(in Q_p) is not closed.*

Proof. Let $u = [p] \in V(A)$, and let $d = \sup \phi_u(D(A))$. Suppose that Q_∞ has cardinality n. Since the natural map $\alpha : Q_p \to \mathrm{S}_u$ is an affine homeomorphism, the fact that Q_∞ is finite means exactly that the set $\Gamma_d := \{s \in \partial_{\mathrm{e}}\mathrm{S}_u : d(s) = \infty\}$ is finite, and with cardinality n. Set $\Gamma_d = \{s_1, \dots, s_n\}$.

Suppose that $I_{\mathrm{b}}(A)$ is stably cofinite. Then the order-ideal $I_{\mathrm{b}} = \varphi(V(I_b(A)))$ of $V(A) \sqcup W_\sigma^d(\mathrm{S}_u)$ is stably cofinite, where φ is the isomorphism in Theorem 2.6. Suppose that there exists a nonempty subset $X \subseteq \Gamma_d$ such that $F := (\mathrm{conv}(X))'$, the complementary face of $\mathrm{conv}(X)$, is closed. Then we have $\mathrm{Aff}(\mathrm{S}_u) \cong \mathrm{Aff}(\mathrm{conv}(X)) \times \mathrm{Aff}(F)$ ([12], Corollary 11.27). We may define functions $f_n \in \mathrm{Aff}(\mathrm{S}_u)^{++}$ by $f_n|\mathrm{conv}(X) = n$ and $f_n|F = 1$. Then, let $f = \sup_n f_n$. We have that $f \in \mathrm{LAff}(\mathrm{S}_u)^{++}$ and that $f|X = \infty$, while $f|F = 1$. Thus $f + d = 1 + d$, so that $f \in W_\sigma^d(\mathrm{S}_u)$, and in the quotient $(V(A) \sqcup W_\sigma^d(\mathrm{S}_u))/I_{\mathrm{b}}$ we have $[f] + [d] = [d]$ with $[f] \neq 0$, which contradicts the stable finiteness of I_{b}.

Conversely, suppose that for any nonempty subset $X \subseteq \Gamma_d$, the complementary face of $\mathrm{conv}(X)$ is not closed. Suppose that there exist bounded functions $l_1, l_2 \in W_\sigma^d(\mathrm{S}_u)$, a number $n \in \mathbb{N}$ and a function $g \in W_\sigma^d(\mathrm{S}_u)$ such that $nd + l_1 = nd + g + l_2$. Then g is bounded on $\partial_{\mathrm{e}}\mathrm{S}_u \setminus \Gamma_d$, say $g|\partial_{\mathrm{e}}\mathrm{S}_u \setminus \Gamma_d \leqslant M$. Let $F = (\mathrm{conv}(\Gamma_d))'$. Since F is not closed, there exists $x \in \overline{F} \setminus F$. By Lemma 3.7, there is a sequence $\{t_k\}$ in $\mathrm{conv}(\partial_{\mathrm{e}}\mathrm{S}_u \setminus \Gamma_d)$ that converges to x. Since g is affine, we have that $g(t_k) \leqslant M$ for all k, whence $g(x) \leqslant M$, due to lower semicontinuity. On the other hand, taking into account that S_u is the direct convex sum of $\mathrm{conv}(\Gamma_d)$ and F, there exist positive numbers α_i for $i = 1, \dots, n$ (not all zero), a number $\beta \geqslant 0$ and an element $t \in F$ such that $\sum_{i=1}^n \alpha_i + \beta = 1$, and such that $x = \sum_{i=1}^n \alpha_i s_i + \beta t$. Therefore there exists $1 \leqslant i \leqslant n$ such that $g(s_i) < \infty$. Without loss of generality, we may assume that $g(s_1) < \infty$. Now, a recursive argument shows that $g(s_2), \dots, g(s_n) < \infty$, whence we conclude that g is bounded, and thus I_{b} is stably cofinite. ∎

We are now in position to characterize the extremal richness of the corona algebra for a simple C^*-algebra with exactly one infinite extremal quasitrace. This situation was considered in [15], Proposition 4.18, for separable, nonunital and simple AF algebras with finitely many extremal traces. In that case the corona algebra is always extremally rich. As we will see, the general situation is quite different.

THEOREM 3.12. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Suppose that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection, and assume that A has exactly one infinite extremal quasitrace in Q_p, which we denote by τ. If $\mathrm{RR}(\mathcal{M}(A)) = 0$, then the following conditions are equivalent:*

(i) *$\mathcal{M}(A)/A$ is extremally rich;*
(ii) *the complementary face of $\{\tau\}$ (in Q_p) is closed;*
(iii) *the ideal $I_{\mathrm{b}}(A)$ is not stably cofinite.*

Proof. Note first that $L(A)/A$ is a closed essential ideal of $\mathcal{M}(A)/A$, which is simple and purely infinite. Then by Theorem 2.5, $\mathcal{M}(A)/A$ is extremally rich if and only if $\mathcal{M}(A)/L(A)$ is extremally rich and $\mathfrak{E}(\mathcal{M}(A)/L(A))$ consists only of isometries and co-isometries. The latter condition is automatic from Lemma 3.9. Thus $\mathcal{M}(A)/A$ is extremally rich if and only if $\mathcal{M}(A)/L(A)$ is extremally rich.

Since $I_{\mathrm{fin}}(A)/L(A)$ has stable rank one (see [22], Proposition 6.1), it follows from Corollary 2.4 that $\mathcal{M}(A)/L(A)$ is extremally rich if and only if $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ is extremally rich and the extreme partial isometries of $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ can be lifted to those of $\mathcal{M}(A)/L(A)$. Note that $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ is purely infinite and simple (by [22], Theorem 6.3 and Proposition 6.5), hence extremally rich by Remark 2.2 (ii). Therefore $\mathcal{M}(A)/A$ is extremally rich if and only if the extreme partial isometries of $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ can be lifted to those of $\mathcal{M}(A)/L(A)$, and this last condition holds if and only if the complementary face of $\{\tau\}$ in Q_p is a closed face, by Proposition 3.6 and Proposition 3.8. This proves (i) $\Leftrightarrow$ (ii).

The equivalence between (ii) and (iii) follows directly from Proposition 3.11. ∎

COROLLARY 3.13. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero and stable rank one. Assume that A is nonelementary and that $V(A)$ is strictly unperforated. Let $p \in A$ be a nonzero projection, and suppose that A has exactly one infinite extremal quasitrace τ in Q_p. If $\mathrm{RR}(\mathcal{M}(A)) = 0$ and $\partial_{\mathrm{e}} Q_p$ is a compact space, then $\mathcal{M}(A)/A$ is extremally rich if and only if τ is isolated in $\partial_{\mathrm{e}} Q_p$.*

Proof. By Theorem 3.12 we have to prove that $\{\tau\}$ is isolated in $\partial_{\mathrm{e}} Q_p$ if and only if the complementary face of $\{\tau\}$ in Q_p is closed.

Let $u = [p] \in V(A)$ and $d = \sup \phi_u(D(A))$. Let $s \in \partial_{\mathrm{e}} \mathrm{S}_u$ be the unique extreme point at which $d|\partial_{\mathrm{e}} \mathrm{S}_u$ is infinite. This point exists because there is an affine homeomorphism $\alpha : Q_p \to \mathrm{S}_u$ (and in fact $s := \alpha(\tau)$). We have to prove then that $\{s\}$ is isolated in $\partial_{\mathrm{e}} \mathrm{S}_u$ if and only if $\{s\}'$ is a closed face. Note that $\{s\}$ is isolated if and only if $\partial_{\mathrm{e}} \mathrm{S}_u \setminus \{s\}$ is closed.

By [12], Corollary 11.20 (and since $\partial_{\mathrm{e}} \mathrm{S}_u$ is compact), there is an affine homeomorphism β between S_u and $M_1^+(\partial_{\mathrm{e}} \mathrm{S}_u)$ given by $\beta(t) = \varepsilon_t$, for $t \in \partial_{\mathrm{e}} \mathrm{S}_u$, and where ε_t is the point mass measure at t.

Suppose that $\{\varepsilon_s\}'$ is a closed face in $M_1^+(\partial_{\mathrm{e}} \mathrm{S}_u)$. It follows from [12], Proposition 5.25 that there exists a closed subset $X \subseteq \partial_{\mathrm{e}} \mathrm{S}_u$ such that $\{\varepsilon_s\}' = \sigma(X) := \{\mu \in M_1^+(\partial_{\mathrm{e}} \mathrm{S}_u) : \mu(X) = 1\}$. In fact, $X = \{t \in \partial_{\mathrm{e}} \mathrm{S}_u : \varepsilon_t \in \{\varepsilon_s\}'\}$, which clearly equals $\partial_{\mathrm{e}} \mathrm{S}_u \setminus \{s\}$, and therefore $\partial_{\mathrm{e}} \mathrm{S}_u \setminus \{s\}$ is closed.

Conversely, if $X = \partial_{\mathrm{e}} \mathrm{S}_u \setminus \{s\}$ is closed, letting $F = \sigma(X)$, we have as before that F is a closed face, which is easily shown to coincide with $\{\varepsilon_s\}'$. Thus we have proved that $\{s\}'$ is a closed face if and only if $\partial_{\mathrm{e}} \mathrm{S}_u \setminus \{s\}$ is closed, and the result follows. ∎

The next instance of interest is to determine when $\mathcal{M}(A)$ has extremal richness. In [15], Theorem 3.1 it is proved that if A is a separable unital simple C^*-algebra with real rank zero and with a finite trace, then $\mathcal{M}(A \otimes \mathbb{K})$ is not extremally rich. In the same paper it is noted that $\mathcal{M}(A)$ is not extremally rich when A is a simple, separable AF algebra without unit and nonelementary. Part of the argument in the following is based on [15], proof of Theorem 2.3, which we include for completeness.

Proposition 3.14. *Let A be a (nonunital) simple separable C^*-algebra with real rank zero, stable rank one and with $V(A)$ strictly unperforated. Assume also that A is nonelementary. If* $\mathrm{RR}(\mathcal{M}(A)) = 0$*, then $\mathcal{M}(A)$ is not extremally rich.*

Proof. Suppose first that the scale of A is not identically infinite. Then, as noticed in [22], Section 7.8, $\mathcal{M}(A)$ is stably finite. Notice also that $\mathcal{M}(A)$ is a prime ring and thus $\mathfrak{E}(\mathcal{M}(A))$ consists of isometries and co-isometries. We conclude that $\mathfrak{E}(\mathcal{M}(A)) = \mathcal{U}(\mathcal{M}(A))$. According to Remark 2.2 (i) we have that $\mathcal{M}(A)$ is extremally rich if and only if $\mathrm{sr}(\mathcal{M}(A)) = 1$. Since $V(\mathcal{M}(A))$ is not cancellative, we conclude that $\mathrm{sr}(\mathcal{M}(A)) \neq 1$, whence $\mathcal{M}(A)$ is not extremally rich.

Suppose now that the scale of A is identically infinite. If A has at least two extremal quasitraces (which will be infinite), then $\mathcal{M}(A)$ is not extremally rich, by Theorem 2.12 (ii). We are therefore left to consider the case when A has only one extremal quasitrace, which is infinite. This means that if $\{e_n\}$ is an (increasing) approximate unit for A consisting of projections, then $\sup \tau(e_n) = \infty$. Let $\pi : \mathcal{M}(A) \to \mathcal{M}(A)/A$ be the natural quotient map, and let $q \in L(A) \setminus A$ be a projection. Then $\pi(q)$ is an infinite projection, and therefore there exists $v \in L(A)/A$ such that $v^*v = \pi(q)$ while $vv^* < \pi(q)$.

Let $w = v + 1 - \pi(q)$, a proper isometry of $\mathcal{M}(A)/A$ which lifts (since $\mathrm{RR}(A) = 0$ and by the proof of [10], Lemma 2.6) to a partial isometry $u \in \mathcal{M}(A)$. There exists (by an argument used in [15], Theorem 2.2) a projection $p \in A$ such that $u^*u = 1 - p$. It follows that $1 - uu^* \in L(A)$ and if $\{f_n\}$ is an approximate unit of projections for the C^*-subalgebra $(1 - uu^*)A(1 - uu^*)$ of A, we have that $\sup \tau(f_n) < \infty$.

Let $p' \in A$ be a projection such that $\tau(p') > \sup \tau(f_n)$. Since $1 - p, 1 - p' \notin L(A)$, and since A has only one extremal quasitrace (that is infinite) we get that $1-p \sim 1-p'$. Thus there is $r \in \mathcal{M}(A)$ such that $1-p' = r^*r$ while $1-p = rr^*$. Let $t = ur$. Then $t^*t = 1 - p'$ and $tt^* = uu^*$, so that $\pi(t)$ is an isometry in $\mathcal{M}(A)/A$.

Denote by $\phi : \mathcal{M}(A) \to \mathcal{M}(A)/L(A)$ the natural quotient map, and note that in fact $\phi(t)$ is a unitary in $\mathcal{M}(A)/L(A)$. We claim that $\pi(t)$ cannot be lifted to any isometry of $\mathcal{M}(A)$. If there exists an isometry $s \in \mathcal{M}(A)$ such that $\pi(s) = \pi(t)$, then $\phi(s) = \phi(t)$, and therefore

$$\delta([\phi(t)]) = [1 - s^*s] - [1 - ss^*] = [1 - t^*t] - [1 - tt^*],$$

where $\delta : \mathrm{K}_1(\mathcal{M}(A)/L(A)) \to \mathrm{K}_0(L(A))$ is the index map. It follows that $[1-tt^*] = [1 - t^*t] + [1 - ss^*]$. Since $\mathcal{M}(A)$ has real rank zero, we get from Lemma 3.2 a projection $f \in M_\infty(L(A))$ such that

$$(1 - tt^*) \oplus f \sim (1 - t^*t) \oplus (1 - ss^*) \oplus f = p' \oplus (1 - ss^*) \oplus f.$$

By simplicity, we assume that $f \in L(A)$. Let $\{t_n\}$ (respectively $\{g_n\}$) be an approximate unit of projections for $(1 - ss^*)A(1 - ss^*)$ (respectively for fAf).

Then by [14], Proposition 1.7, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $p' \oplus t_n \oplus g_n \lesssim f_m \oplus g_m$. Therefore, if $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\tau(p') + \tau(t_n) + \tau(g_n) \leqslant \tau(f_m) + \tau(g_m)$. It follows that $\sup \tau(t_n) \leqslant 0$, and this implies that $1 - tt^* = 0$. Thus $1 - uu^* = 0$ and hence $ww^* = 1$. This contradicts the fact that w is a proper isometry, and therefore the claim is established.

It is easy to see that $\pi(t)$ cannot be lifted to a co-isometry either. We then conclude from Theorem 2.3 that $\mathcal{M}(A)$ is not extremally rich. ∎

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