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ADAPTED ENDOMORPHISMS WHICH GENERALIZE BOGOLJUBOV TRANSFORMATIONS

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Abstract. We discuss a class of endomorphisms of the hyperfinite II_1-factor which are adapted in a certain way to a tower $\mathbb{C}1 \subset \mathbb{C}^p \subset M_p \subset M_p \otimes \mathbb{C}^p \subset \cdots$ so that for $p = 2$ we get Bogoljubov transformations of a Clifford algebra. Results are given about surjectivity, innerness, Jones index and the shift property.

Keywords: *Adapted, Jones tower, endomorphism, Bogoljubov transformation, inner, shift.*

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0. INTRODUCTION

Related to the interest in towers of algebras there is a growing interest in endomorphisms which are in some way adapted to such towers. Some examples are given by V. Jones in [9]; see also the book ([10]) of V. Jones and V.S. Sunder for some background. In these references the authors call attention to be the question how global properties of these endomorphisms can be obtained from information restricted to various stages of the tower.

The starting point for this paper has been the observation that there is a certain class of endomorphisms of the hyperfinite II_1-factor which allow more detailed answers to these questions than are available in general. They are adapted to a tower

$$\mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_{[0,1]} \subset \mathcal{E}_{[0,2]} \subset \cdots \simeq \mathbb{C}1 \subset \mathbb{C}^p \subset M_p \subset M_p \otimes \mathbb{C}^p \subset \cdots$$

in the sense that $\alpha(\mathcal{E}_{[0,n-1]}) \subset \mathcal{E}_{[0,n]}$ for all n and that some additional commutation relations are fulfilled (see Section 1 for details). For $p = 2$ we get Bogoljubov transformations of a Clifford algebra (cf. R.J. Plymen, P.L. Robinson ([14]) for an introduction to this). For $p > 2$ there is no functor, but from known results on

Bogoljubov transformations we guess some natural hypotheses about the general case, some of which are verified in this paper.

In Section 1 we introduce from various points of view the class of endomorphisms to be discussed, we fix the notation and give elementary properties to be used later. The main results are in Sections 2–4.

In Section 2 we analyze this structure from the point of view of noncommutative stochastic processes (cf. B. Kümmerer ([12]) for basic definitions). We believe that adaptedness properties with respect to a given tower add many useful possibilities to introduce and to calculate stochastic quantities. Here we calculate prediction errors (as they would be called in classical probability) and use this to determine when the endomorphism is surjective, i.e. an automorphism.

In Section 3 we go further in this direction and determine when this automorphism is even inner. For Bogoljubov transformations this has been answered by a theorem of Blattner ([1]). Not using the Clifford functor, we feel that our approach sheds some new light even on this classical situation.

In Section 4 we discuss the non-surjective case. We calculate the Jones index $[\mathcal{A} : \alpha\mathcal{A}]$ and give a sufficient condition for the endomorphism α to be a shift in the sense of Powers. In this part some work remains to be done to obtain a more complete understanding.

Special features used throughout and not always present in more general towers are commutation relations, the independence of certain subalgebras and grading. Nevertheless, one might hope that it is possible to use the experiences made in analyzing these special endomorphisms for the study of more complicated cases.

1. A CLASS OF ENDOMORPHISMS

It is convenient to start with a setting introduced by D. Bures and H.-S. Yin in [2]: given a discrete abelian group G, a shift s in G and an s-invariant 2-cocycle w of G (with values in the circle group $\mathbb{T}$), we can form the twisted group von Neumann algebra $W^*(G,w)$ generated by unitaries $\{L_g : g \in G\}$ satisfying $L_g L_h = w(g,h) L_{g+h}$ (equivalently, $L_g L_h = b(g,h) L_h L_g$, where $b(g,h) = \frac{w(g,h)}{w(h,g)}$ is an antisymmetric bicharacter of G) and an endomorphism σ of $W^*(G,w)$ satisfying $\sigma(L_g) = L_{s(g)}$, called a *group shift*.

LEMMA 1.1. ([2]) *Let H be a subgroup of G. Then*

$$W^*(H,w)' \cap W^*(G,w) = W^*\big(\{g \in G : b(g,h) = 1 \text{ for all } h \in H\}, w\big).$$

If w is nondegenerate (i.e. $\{g \in G : b(g,h) = 1$ for all $h \in G\} = \{0\}$) and G is countable, then $W^*(G,w)$ is the hyperfinite II_1-factor.

LEMMA 1.2. *Let H, K be subgroups of G with $H \cap K = \{0\}$, w nondegenerate (see above) and normalized (i.e. $w(g,-g) = 1$ for all $g \in G$). Then $W^*(H,w)$ and $W^*(K,w)$ are independent in the sense that $\mathrm{tr}(xy) = \mathrm{tr}(x)\,\mathrm{tr}(y)$ for all $x \in W^*(H,w), y \in W^*(K,w)$ (where tr is the unique trace on $W^*(G,w)$; see [2], Proposition 1.5).*

REMARK 1.3. In this paper we shall always use this notion of independence (cf. [12]), which coincides with “orthogonality with respect to the trace” in [15].

Proof of Lemma 1.2. It suffices to prove the assertion for sums $x = \sum \lambda_h L_h$, $h \in H$, respectively $y = \sum \gamma_k L_k$, $k \in K$, having only finitely many summands. Then $\mathrm{tr}\left(\sum \lambda_h L_h\right) = \lambda_0$, $\mathrm{tr}\left(\sum \gamma_k L_k\right) = \gamma_0$ and (because $h+k=0 \Leftrightarrow h=k=0$) $\mathrm{tr}\left(\sum \lambda_h L_h \cdot \sum \gamma_k L_k\right) = \mathrm{tr}\left(\sum \lambda_h \gamma_k w(h,k) L_{h+k}\right) = \lambda_0 \gamma_0$. ∎

Let the cyclic group $\mathbb{Z}_p$ be given by $\{0,\dots,p-1\}$ and addition $\bmod p$. Here p may be any natural number ($\geqslant 2$, not necessarily prime). If $G := \bigoplus_{n=0}^{\infty} \mathbb{Z}_p^{(n)}$, the group shift corresponding to the canonical shift in G is called a *p-shift* in [2]. The simplest example is the following:

Denoting $1 \in \mathbb{Z}_p^{(n)}$ by δ_n, we use the antisymmetric bicharacter b determined by $b(\delta_m, \delta_n) = \exp(2\pi \mathrm{i}/p) =: \omega$ for $m < n$. Setting $e_n := L_{\delta_n}$ we get the relations $e_n^p = 1$, $e_m e_n = \omega e_n e_m$ for $m < n$, and $\{e_n\}_{n=0}^{\infty}$ span a von Neumann algebra $\mathcal{A}$ isomorphic to the hyperfinite II_1-factor. Denote by $\mathcal{E}_J$ the von Neumann algebra spanned by $\{e_n : n \in J\}$ (in particular we use $J = [0,n] := \{0,1,\dots,n\}$ and other selfexplaining expressions; also $\mathcal{E}_{-1} := \mathbb{C}1$). Infer from Lemma 1.2 that $\mathcal{E}_I$ and $\mathcal{E}_J$ are independent if $I \cap J = \emptyset$. Using the terminology of B. Kümmerer ([12]) this means that $\{\mathcal{E}_J : J \subset \mathbb{N}_0\}$ is a *(discrete) white noise* and that the p-shift $\sigma : e_n \mapsto e_{n+1}$ (for all n) is a *(generalized) Bernoulli shift.* Using this point of view, σ has also been examined by C. Rupp ([17]), where it is called a *Gauss shift.*

In this paper we want to consider a more general class of endomorphisms (containing σ).

Definition 1.4. An endomorphism α of $\mathcal{A}$ is called *adapted with respect to the discrete white noise* $\{\mathcal{E}_J : J \subset \mathbb{N}_0\}$ if it can be written in the form $\lim_{N\to\infty} \prod_{n=1}^{N} \mathrm{Ad}\, U_n = \lim_{N\to\infty} \mathrm{Ad}(U_1 \cdots U_N)$ (pointwise weak*), where for all $n \geqslant 1$, $U_n \in \mathcal{A}$ is a unitary

(i) which is normalizing $\mathcal{E}_{[0,n]}$ i.e. $\mathrm{Ad}\, U_n(\mathcal{E}_{[0,n]}) = \mathcal{E}_{[0,n]}$ and

(ii) $U_n \in (\mathcal{E}_{[0,n-2]\cup[n+1,\infty)})'$, i.e. $\mathrm{Ad}\, U_n$ fixes $\mathcal{E}_{[0,n-2]\cup[n+1,\infty)}$ pointwise.

Remarks 1.5. (i) From Definition 1.4 we get immediately (for all $n \geqslant 1$) that $\alpha|\mathcal{E}_{[0,n-1]} = \mathrm{Ad}(U_1 \cdots U_n)|\mathcal{E}_{[0,n-1]}$ and $\alpha(\mathcal{E}_{[0,n-1]}) \subset \mathcal{E}_{[0,n]}$. This property may be called adaptedness with respect to the tower $\{\mathcal{E}_{[0,n]}\}_{n\in\mathbb{N}_0}$, and it is introduced for very general towers by V. Jones and V.S. Sunder in [10], Example 5.1.6. Compare also [9] for further examples and results. The presence of a discrete white noise allows us to define a more restricted class of endomorphisms, and we shall show in the sequel that this simplifies the task of proving results.

(ii) The tower $\mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_{[0,1]} \subset \mathcal{E}_{[0,2]} \subset \mathcal{E}_{[0,3]} \subset \mathcal{E}_{[0,4]} \cdots$ is isomorphic to $\mathbb{C}1 \subset \mathbb{C}^p \subset M_p \subset M_p \otimes \mathbb{C}^p \subset M_p \otimes M_p \subset M_p \otimes M_p \otimes \mathbb{C}^p \cdots$ where M_p denotes the algebra of $p \times p$-matrices. In this paper the number p will always be used in this meaning.

(iii) The applications $\mathrm{Ad}\, U_n$ are in a certain way localized with respect to the noise, interacting like cog-wheels. It is natural to try to understand the endomorphism α in terms of its factors.

(iv) It is also instructive to look at Definition 1.4 as a generalization of actions of "infinite tensor product type" (cf. Y. Kawahigashi, [11]), which appear if we choose $U_n = 1$ for all even n.

The mechanism of an adapted endomorphism (always with respect to the noise above) can be described very explicitely. Define $\mu := \begin{cases} \exp(\pi \mathrm{i}/p) & \text{if } p \text{ is even,} \\ 1 & \text{if } p \text{ is odd,} \end{cases}$ and $u_n := \overline{\mu} e^*_{n-1} e_n$ for all $n \geqslant 1$ (the factor $\overline{\mu}$ ensures that $u_n^p = 1$). The notation lin denotes linear hull.

PROPOSITION 1.6. $(\mathcal{E}_{[0,n-2]\cup[n+1,\infty)})' \cap \mathcal{A} = \mathrm{lin}\{u_n^k\}_{k=0}^{p-1} \simeq \mathbb{C}^p$.

REMARK 1.7. In particular $(\mathcal{E}_{[0,n-2]\cup[n+1,\infty)})' \cap \mathcal{A} \subset \mathcal{E}_{[0,n]}$, which means that (i) in Definition 1.4 always follows from (ii).

Proof. Obviously $u_n \in (\mathcal{E}_{[0,n-2]\cup[n+1,\infty)})'$. To prove the other direction note that (by Lemma 1.2) $(\mathcal{E}_{[0,n-2]\cup[n+1,\infty)})' \cap \mathcal{A}$ is spanned by unitaries L_g, where $b(g, \delta_j) = 1$ if $j \in [0, n-2] \cup [n+1, \infty)$. If $g = \sum g^{(j)}$, $g^{(j)} \in \mathbb{Z}_p^{(j)}$ and $j_1 := \min\{j : g^{(j)} \neq 0\}$, $j_2 := \max\{j : g^{(j)} \neq 0\}$, then we find $b(g, \delta_{j_2}) \neq b(g, \delta_j)$ if $j_2 < j$, but $b(g, \delta_j) = 1$ if $j \geqslant n+1$ and thus $j_2 < n+1$; also $\overline{b(g, \delta_{j_1})} \neq b(g, \delta_{n+1}) = 1$ and thus $j_1 > n-2$. Conclude that $g = g^{(n-1)} + g^{(n)}$, i.e. $L_g = \mathrm{const} \cdot e_{n-1}^{k_1} e_n^{k_2}$ with $k_1, k_2 \in \{0, \ldots, p-1\}$. To satisfy the required commutation relations we must also have $k_2 = -k_1 =: k$, which implies that L_g differs from u_n^k only by a scalar factor. ∎

REMARK 1.8. We have $u_n u_{n+1} = \omega u_{n+1} u_n$ and $[u_m, u_n] = 0$ if $|n - m| \geqslant 2$. The restriction of σ to $\overline{\mathrm{lin}}\{u_n : n \geqslant 1\}$, $\sigma : u_n \mapsto u_{n+1}$, also defines a p-shift which is treated as a derivation of σ in [3].

To discuss this structure in more detail, fix any $n \geqslant 1$ and set $e := e_{n-1}$, $f := e_n$ and $u := u_n = \overline{\mu} e^*_{n-1} e_n$. We have the (commutation) relations

$$e^p = f^p = u^p = 1, \quad ef = \omega f e, \quad eu = \omega u e, \quad fu = \omega u f.$$

The algebra spanned by e and f is isomorphic to the matrix algebra M_p where we may give the following realization:

$$e = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}, \quad f = \begin{pmatrix} & \mu\omega & & & \\ & & \mu\omega^2 & & \\ & & & \ddots & \\ & & & & \mu\omega^{p-1} \\ \mu & & & & \end{pmatrix}, \quad u = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{p-1} \end{pmatrix}.$$

By Proposition 1.6 any element $U := U_n$ arising in the definition of an adapted endomorphism has the form

$$U = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \widehat{c}(k) u^k$$

with complex coefficients $\{\widehat{c}(k)\}_{k=0}^{p-1}$. These are the discrete Fourier transforms of the eigenvalues $\{c(j)\}_{j=0}^{p-1}$ of U, i.e. $\widehat{c}(k) = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} c(j) \overline{\omega}^{jk}$. Indeed, using the realization above, we find

$$U = \begin{pmatrix} c(0) & & & & \\ & c(1) & & & \\ & & c(2) & & \\ & & & \dots & \\ & & & & c(p-1) \end{pmatrix}.$$

So $\{\widehat{c}(k)\}_{k=0}^{p-1}$ can be chosen to be the Fourier transform of any unimodular function on $\{0, \dots, p-1\}$. Note that c or $\widehat{c}$ is specified by the action $\operatorname{Ad} U$ only up to an unimodular constant, which can be suitably chosen in applications.

PROPOSITION 1.9. $\operatorname{Ad} U(e^j f^k) = \sum_{s=0}^{p-1} \gamma_{j+k,s} e^{j-s} f^{k+s} \omega^{sk} \omega^{\frac{1}{2}s(s-1)} \overline{\mu}^s$, *where* $\gamma_{ab} = \frac{1}{p} \sum_{m=0}^{p-1} \widehat{c}(m) \overline{\widehat{c}(m-b)} \overline{\omega}^{ma} = \frac{\overline{\omega}^{ab}}{p} \sum_{m=0}^{p-1} c(m) \overline{c(m+a)} \overline{\omega}^{mb}$.

Proof. Straightforward computation using the commutation relations. ∎

In particular, we have $\operatorname{Ad} U(e^j) = \sum_{s=0}^{p-1} \widetilde{\gamma}_{js} e^{j-s} f^s$ where $|\widetilde{\gamma}_{js}| = |\gamma_{js}|$. If we want to emphasize the index n of U_n in these formulas, we shall write $c^{(n)}(j)$, $\widehat{c}^{(n)}(k)$, $\gamma_{ab}^{(n)}$ etc.

LEMMA 1.10. *The coefficients γ_{ab}, $a, b \in \mathbb{Z}_p \simeq \{0, \dots, p-1\}$ have the properties:*

(i) $\gamma_{0b} = \delta_{0b}$ *for all* b;
(ii) $\sum_b |\gamma_{ab}|^2 = 1$ *for all* a;
(iii) $\overline{\gamma_{-a,-b}} = \gamma_{ab} \omega^{ab}$;
(iv) *let* $\gamma_{ab}^{(*)}$ *be associated to* $U^* = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \overline{\widehat{c}(k)} u^{-k}$. *Then* $\gamma_{ab}^{(*)} = \overline{\gamma_{a,-b}}$.

Proof. (i) reflects $\operatorname{Ad} U(1) = 1$ and (ii) follows from the fact that $\operatorname{Ad} U$ is an isometry of $L^2(\mathcal{A}, \operatorname{tr})$. (iii) and (iv) are straightforward from Proposition 1.9. ∎

EXAMPLES 1.11. (i) If

$$\widehat{c}(j) := \begin{cases} \exp[\pi \mathrm{i} (j+1)^2] & \text{if } p \text{ is even,} \\ \exp\left[\pi \mathrm{i} \left(j + \frac{1}{2}\right)^2\right] & \text{if } p \text{ is odd,} \end{cases}$$

then a short computation yields

$$\gamma_{jk} = \begin{cases} \delta_{jk} \omega^{k - \frac{k^2}{2}} & \text{if } p \text{ is even,} \\ \delta_{jk} \omega^{\frac{1}{2}(k - k^2)} & \text{if } p \text{ is odd,} \end{cases}$$

and $\operatorname{Ad} U(e) = f$. Using the corresponding unitary U for all $n \geqslant 1$ we get an adapted presentation of the Gauss shift σ introduced above. The occurrence of Gaussian sums in dealing with the discrete Fourier transforms has been the reason for the notation "Gauss shift" in [17].

(ii) $\operatorname{Ad} e_0 = \lim_{N\to\infty} \prod_{n=1}^{N} \operatorname{Ad} U_n$ where $U_n = 1$ for n odd and $U_n = u_n^*$ for n even.

(iii) The automorphism γ defined by $\gamma(e_n) = \omega e_n$ (for all n) is called the *grading automorphism*. We have an adapted presentation $\gamma = \lim_{N\to\infty} \prod_{n=1}^{N} \operatorname{Ad} U_n$ where $U_n = u_n^*$ for n odd and $U_n = 1$ for n even. Note that by an argument analogous to that presented by P. de la Harpe and R.J. Plymen in [7], Lemma 1, one can show that γ is an outer automorphism of $\mathcal{A}$.

We shall need some facts about the grading naturally associated to our way of generating $\mathcal{A}$: for $r \in \{0,\dots,p-1\}$ define $\mathcal{A}^r := \{x \in \mathcal{A} : \gamma(x) = \omega^r x\}$, the space of homogeneous elements of degree r. For example $e_{j_1}^{r_1} \cdots e_{j_k}^{r_k} \in \mathcal{A}^r$ if and only if $r_1 + \cdots + r_k = r \,(\operatorname{mod} p)$. An endomorphism α of $\mathcal{A}$ is called *graded* if $\alpha(\mathcal{A}^r) \subset \mathcal{A}^r$ for all r or, equivalently, if α commutes with γ. If $\alpha = \operatorname{Ad} U$ is graded then for all $x \in \mathcal{A}$ we get $\gamma(U)x\gamma(U)^* = \gamma(U\gamma^{p-1}(x)U^*) = \gamma\alpha\gamma^{p-1}(x) = \alpha\gamma^p(x) = \alpha(x) = UxU^*$, which implies $\gamma(U) = cU$ for some constant c. We infer that U is homogeneous, and we may classify graded inner automorphisms by the degree of U. From Proposition 1.9 it is further evident that an adapted endomorphism is graded, so all the considerations above are applicable. All this is more well known in the case $p = 2$: cf. R.J. Plymen and P.L. Robinson ([14]), where the $\mathbb{Z}_2$-grading of Clifford algebras and some applications for Bogoljubov transformations are discussed. We show next that our setting may indeed be viewed as a generalization of Bogoljubov transformations:

PROPOSITION 1.12. *Assume $p = 2$. Then an endomorphism is adapted if and only if it is a Bogoljubov transformation α_T induced by an orthogonal transformation T of the real Hilbert space $\overline{\operatorname{lin}}_{\mathbb{R}}\{e_j\}_{j=0}^{\infty}$ (with scalar product $\langle x, y\rangle := \operatorname{tr}(y^* x)$) with the property $T(\operatorname{lin}_{\mathbb{R}}\{e_j\}_{j=0}^{n-1}) \subset \operatorname{lin}_{\mathbb{R}}\{e_j\}_{j=0}^{n}$ for all $n \geqslant 1$.*

Proof. Using e_{n-1}, e_n, u_n as above (now realized by the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$), consider the rotation T_n by an angle φ_n in the real (two-dimensional) plane, where e_{n-1} respectively e_n have to be interpreted as unit vectors pointing in the direction of the x- respectively y-axis:

$$T_n = \begin{pmatrix} \cos\varphi_n & -\sin\varphi_n \\ \sin\varphi_n & \cos\varphi_n \end{pmatrix} \qquad \text{on } \operatorname{lin}_{\mathbb{R}}\{e_{n-1}, e_n\}.$$

The corresponding Bogoljubov transformation is most easily computed by writing T_n as a product of two reflections: first at the x-axis, then at an axis which is rotated by $\frac{\varphi_n}{2}$. So we find $\alpha_{T_n} = \operatorname{Ad} U_n$, where

$$U_n = \Big(\cos\frac{\varphi_n}{2} e_{n-1} + \sin\frac{\varphi_n}{2} e_n\Big) e_{n-1} = \cos\frac{\varphi_n}{2} 1 - \mathrm{i}\sin\frac{\varphi_n}{2} u_n.$$

But this is exactly the formula $U_n = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \widehat{c}^{\,(n)}(k) u_n^k$ above in the special case $p = 2$ (with an appropiate unimodular constant). So Proposition 1.12 reduces to the fact that any orthogonal transformation T with $T(\operatorname{lin}_{\mathbb{R}}\{e_j\}_{j=0}^{n-1}) \subset \operatorname{lin}_{\mathbb{R}}\{e_j\}_{j=0}^{n}$ for all $n \geqslant 1$ may be written as a product $T = \lim_{N\to\infty} \prod_{n=1}^{N} T_n$ (in the strong operator topology). But this is a well known fact in Hilbert space theory (although we only know references where this is proved for complex Hilbert spaces, the arguments given e.g. by C. Foiaş and A.E. Frazho in [5], may be easily modified to apply here). ∎

Remark 1.13. (i) Note further that any orthogonal transformation of a separable real Hilbert space may be put in such a form by starting with a unit vector and then applying the Gram-Schmidt procedure to its orbit (and repeating this if the vector has not been cyclic).

(ii) If $p > 2$ then (in general) $\overline{\operatorname{lin}}_{\mathbb{R}}\{e_j\}_{j=0}^{\infty}$ is not invariant for an adapted endomorphism.

2. THE STOCHASTIC PROCESS $(\mathcal{A}, \alpha, \mathcal{A}_0)$

Let us study adapted endomorphisms $\alpha = \lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \operatorname{Ad} U_n$ in more detail. There is a second tower associated to α, namely $\mathcal{A}_{-1} := \mathbb{C}1$, $\mathcal{A}_0 := \mathcal{E}_0$ and $\mathcal{A}_J$ is generated by $\{\alpha^j \mathcal{A}_0\}_{j\in J}$.

Theorem 2.1. *Assume* $\max\limits_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}| < 1$ *for* $1 \leqslant n < N$. *Then* $\mathcal{A}_{[0,N-1]} = \mathcal{E}_{[0,N-1]}$ *for* $1 \leqslant n < N$. *If* $\max\limits_{j=1,\dots,p-1} |\gamma_{j0}^{(N)}| = 1$ *then* $\mathcal{A}_{[0,\infty)} = \mathcal{A}_{[0,N]} = \mathcal{A}_{[0,N-1]}$. *If* $\max\limits_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}| < 1$ *for all* $n \in \mathbb{N}$ *then* $\mathcal{A}_{[0,\infty)} = \mathcal{A}$. *For p prime, one may replace* $\max\limits_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}|$ *by* $|\gamma_{10}^{(n)}|$ *in the statements above.*

Remarks 2.2. (i) In the first case $\mathcal{A}_{[0,\infty)}$ is finite dimensional, while in the second case (using the terminology of [12]) the noncommutative stochastic process $(\mathcal{A}, \alpha, \mathcal{A}_0)$ is minimal.

(ii) Note that $\gamma_{j0} = \frac{1}{p} \sum\limits_{m=0}^{p-1} c(m)\overline{c(m+j)}$, $j = 1, \dots, p-1$, may be interpreted as autocorrelations of the eigenvalues of U.

(iii) The assertions about $\big\{\max\limits_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}|\big\}_{n\in\mathbb{N}}$ are similar to the relations between choice sequences and Hilbert space isometries (compare [5], Chapter XV).

Proof. Suppressing the index n, the nontrivial part consists in showing that if $\max\limits_{j=1,\dots,p-1} |\gamma_{j0}| < 1$ and if $\mathcal{D}$ is the maximal commutative subalgebra generated by e of the matrix algebra M_p generated by e and f, then $\mathcal{D}$ and $U\mathcal{D}U^*$ together generate M_p.

Assume they do not. Then $\mathcal{D} \cap U\mathcal{D}U^* \neq \mathbb{C}1$, i.e. there exists $x \in \mathcal{D} \setminus \mathbb{C}1$ with $\operatorname{Ad} U(x) \in \mathcal{D}$. If p is prime then for any $j \in \{1, \dots, p-1\}$ there is some $r \in \{1, \dots, p-1\}$ with $e = e^{jr}$, and if $|\gamma_{10}| < 1$ (i.e. $\operatorname{Ad} U(e) \notin \mathcal{D}$) then also $|\gamma_{j0}| < 1$ (i.e. $\operatorname{Ad} U(e^j) \notin \mathcal{D}$) for all $j \in \{1, \dots, p-1\}$. If p is not prime we have $\operatorname{Ad} U(e^j) \notin \mathcal{D}$ for all $j \in \{1, \dots, p-1\}$ by assumption.

Expanding x in powers of e and taking the grading into account leads to a contradiction to the properties of x given above. This proves the assertion. ∎

Now consider the following problem: When do we get $\mathcal{A}_{[0,\infty)} = \mathcal{A}_{[1,\infty)}$, i.e. when is α surjective on $\mathcal{A}_{[0,\infty)}$ or, in probabilistic language, when is the process $(\mathcal{A}, \alpha, \mathcal{A}_0)$ deterministic? We shall need the Hilbert space $L^2(\mathcal{A}, \mathrm{tr})$, the norm of which is denoted by $\|\cdot\|_2$. Call P_J the projection onto $\mathcal{A}_J$. Related to the problem above are the "prediction errors"

$$f_n(x) := \|(1 - P_{[0,n-1]})\alpha^n(x)\|_2 \quad \text{for } x \in \mathcal{A}_0,\ n \in \mathbb{N}.$$

For any x these form a decreasing sequence of nonnegative real numbers. Note that also $\|(1 - P_{[0,n-1]})\,\mathrm{Ad}(U_n \cdots U_1)(x)\|_2 = f_n(x)$.

To $\mathcal{A}_0 \ni x = \sum_{j=0}^{p-1} x_j e_0^j$ associate a vector $v_0 := (|x_j|^2)_{j=1}^{p-1} \in (\mathbb{C}^{p-1}, \|\cdot\|_1)$. Let us further define the (substochastic) $(p-1) \times (p-1)$-matrices

$$D_n := (|\gamma_{jk}^{(n)}|^2)_{j,k=1}^{p-1}, \quad n \in \mathbb{N}.$$

The numbers γ_{jk} are defined in Proposition 1.9.

PROPOSITION 2.3. *If $x \in \mathcal{A}_0$ then $f_N(x)^2 = \left\| v_0 \prod_{n=1}^{N} D_n \right\|_1$.*

REMARKS 2.4. (i) This states that for a unit vector $x \perp 1$ in $(\mathcal{A}_0, \|\cdot\|_2)$ the squared prediction errors are convex combinations of row sums of $\prod_{n=1}^{N} D_n$. In particular:

$$f_N := \max\{f_n(x) : x \in \mathcal{A}_0,\ \|x\|_2 = 1\} = \left\| \prod_{n=1}^{N} D_n \right\|^{\frac{1}{2}},$$

where $\|\cdot\|$ denotes the maximum of row sums. Note that one may also say that Proposition 2.3 describes the squared prediction errors as transition probabilities of a (non-stationary) Markov chain with p states, one of which is absorbing (corresponding to $1 \in \mathcal{A}$).

(ii) For $p = 2$ the matrices D_n are scalars, and Proposition 2.3 reduces to a well known formula of linear prediction theory (see [5], chapter II.5, II.6).

Proof. For some n we may write $\alpha^n(x) = \sum_{j=0}^{p-1} x_j^{(n-1)} e_n^j$ where $x_j^{(n-1)} \in \mathcal{E}_{[0,n-1]}$. If we associate the vector $v_n := (\|x_j^{(n-1)}\|_2^2)_{j=1}^{p-1} \in \mathbb{C}^{p-1}$, and then use the independence of $\mathcal{E}_{[0,n-1]}$ and $\mathcal{E}_n$ we find that $f_n(x)^2 = \|v_n\|_1$.

From the computation

$$\begin{aligned}
\alpha^{n+1}(x) = \alpha(\alpha^n(x)) &= \sum_{j=0}^{p-1} \mathrm{Ad}(U_1 \cdots U_n)[x_j^{(n-1)} \cdot \mathrm{Ad}(U_{n+1})(e_n^j)] \\
&= \sum_{j=0}^{p-1} \mathrm{Ad}(U_1 \cdots U_n) \left[x_j^{(n-1)} \cdot \sum_{k=0}^{p-1} \widetilde{\gamma}_{jk}^{(n+1)} e_n^{j-k} e_{n+1}^k \right] \\
&= \sum_{k=0}^{p-1} \left[\mathrm{Ad}(U_1 \cdots U_n) \left(\sum_{j=0}^{p-1} \widetilde{\gamma}_{jk}^{(n+1)} x_j^{(n-1)} e_n^{j-k} \right) \right] \cdot e_{n+1}^k
\end{aligned}$$

we infer that $x_k^{(n)} = \mathrm{Ad}(U_1 \cdots U_n)\Big(\sum_{j=0}^{p-1} \widetilde{\gamma}_{jk}^{(n+1)} x_j^{(n-1)} e_n^{j-k}\Big)$. Note that because of $\widetilde{\gamma}_{0k} = \delta_{0k}$, we may for $k \neq 0$ only sum from $j = 1$ to $p-1$. So for $k \neq 0$ we get

$$\|x_k^{(n)}\|_2^2 = \Big\| \sum_{j=1}^{p-1} \gamma_{jk}^{(n+1)} x_j^{(n-1)} e_n^{j-k} \Big\|_2^2 = \sum_{j=1}^{p-1} \big|\gamma_{jk}^{(n+1)}\big|^2 \|x_j^{(n-1)}\|_2^2$$

(use the grading and independence). We have found a recursion: $v_n \cdot D_{n+1} = v_{n+1}$, valid for all $n \geqslant 0$. So finally

$$f_N(x)^2 = \|v_N\|_1 = \Big\| v_0 \prod_{n=1}^{N} D_n \Big\|_1. \quad ∎$$

LEMMA 2.5. *If* $\alpha^{(*)} := \lim_{N\to\infty} \prod_{n=1}^{N} \mathrm{Ad}\, U_n^*$ *and* $f_n^{(*)}(x)$ *is the corresponding prediction error then* $f_n^{(*)}(x) = f_n(x)$.

Proof. Combine Proposition 2.3 and $\gamma_{jk}^{(*)} = \overline{\gamma_{j,-k}}$ (Lemma 1.10 (iv)). ∎

THEOREM 2.6. *For an adapted endomorphism* $\alpha = \lim_{N\to\infty} \prod_{n=1}^{N} \mathrm{Ad}\, U_n$ *the following assertions are equivalent:*

(i) $\mathcal{A}_{[0,\infty)} = \mathcal{A}_{[1,\infty)}$;

(ii) $\lim_{N\to\infty} \prod_{n=1}^{N} D_n = 0$.

Proof. $\alpha|\mathcal{E}_{[0,N-1]} = \mathrm{Ad}(U_1 \cdots U_N)|\mathcal{E}_{[0,N-1]}$ implies that

$$P_{[1,N]} = \mathrm{Ad}(U_1 \cdots U_N) P_{[0,N-1]} \,\mathrm{Ad}(U_N^* \cdots U_1^*).$$

Using Lemma 2.5 we find for $x \in \mathcal{A}_0$

$$\|P_{[1,N]}x\|_2^2 = \|P_{[0,N-1]} \,\mathrm{Ad}(U_N^* \cdots U_1^*)(x)\|_2^2 = \|x\|_2^2 - f_N^{(*)}(x)^2 = \|x\|_2^2 - f_N(x)^2.$$

Now use Proposition 2.3 to show that (ii) is satisfied if and only if for all $x \in \mathcal{A}_0$

$$\|P_{[1,\infty)}x\|_2 = \lim_{N\to\infty} \|P_{[1,N]}x\|_2 = \|x\|_2,$$

i.e. $x \in \mathcal{A}_{[1,\infty)}$. But $\mathcal{A}_0 \subset \mathcal{A}_{[1,\infty)}$ if and only if (i) is satisfied. ∎

PROPOSITION 2.7. *If p is prime and* $\mathcal{A}_{[0,\infty)} \neq \mathcal{A}_{[1,\infty)}$ *then* $\lim_{N\to\infty} f_N(x) > 0$ *for all* $x \in \mathcal{A}_0 \setminus \mathbb{C}1$ *(i.e. all row sums of* $\Big\{\prod_{n=1}^{N} D_n\Big\}_N$ *have a strict positive limit for* $N \to \infty$*).*

Proof. $\mathcal{A}_{[0,\infty)} \neq \mathcal{A}_{[1,\infty)}$ implies that the maximal row sums (say of the j-th row) of $\Big\{\prod_{n=1}^{N} D_n\Big\}_N$ have a strict positive limit. Because p is prime, $e_0^j \notin \mathcal{A}_{[1,\infty)}$ implies $e_0^k \notin \mathcal{A}_{[1,\infty)}$ for all $k \in \{1, \ldots, p-1\}$. This gives the result for all powers of e_0. For general x apply now Proposition 2.3. ∎

EXAMPLE 2.8. If $p = 3$ then $D_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$ where $0 \leqslant a_n,\ b_n$ and $a_n + b_n \leqslant 1$. Thus in this case the matrices D_n commute for different n, and we can state Theorem 2.6 using the maximal eigenvalues $a_n + b_n = 1 - |\gamma_{10}^{(n)}|^2$. The following statements are equivalent:

(i) $\mathcal{A}_{[0,\infty)} = \mathcal{A}_{[1,\infty)}$;

(ii) $\lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} (a_n + b_n) = 0$;

(iii) $|\gamma_{10}^{(N)}| = 1$ for some N or $\sum\limits_{n=1}^{\infty} |\gamma_{10}^{(n)}|^2 = \infty$.

3. THE DETERMINISTIC CASE AND BLATTNER'S THEOREM

In this section we want to consider the question: When is an adapted endomorphism $\alpha = \lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \operatorname{Ad} U_n$ actually an inner automorphism of $\mathcal{A}$? If $(\mathcal{A}, \alpha, \mathcal{A}_0)$ is a minimal and deterministic process as characterized in the second section, this gives an additional refinement of the classification. If $p = 2$ our question has been answered by a theorem of Blattner ([1]). We shall return to this later.

THEOREM 3.1. *Let* $\alpha = \lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \operatorname{Ad} U_n$ *be an adapted endomorphism and* $\operatorname{tr} U_n \geqslant 0$ *for all* $n \geqslant 1$. *The following assertions are equivalent:*

(i) $\alpha = \operatorname{Ad} U$, $U \in \mathcal{A}^0$ *(homogeneous of degree* 0*);*

(ii) $\lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} U_n = U$ *(in the strong operator topology);*

(iii) $2 \sum\limits_{n=1}^{\infty} (1 - \operatorname{tr} U_n) = \sum\limits_{n=1}^{\infty} \|U_n - 1\|_2^2 < \infty$.

REMARK 3.2. $\operatorname{tr} U_n \geqslant 0$ may always be achieved by multiplying U_n with an unimodular constant. This does not change α.

Proof. We shall use the fact that on bounded subsets the strong operator (s.o.)-topology coincides with the $\|\cdot\|_2$-topology and other related facts as presented e.g. in [7], Lemma 4.

(ii) $\Leftrightarrow$ (iii) First note that for $n > m$

$$\Big\| \prod_{k=1}^{n} U_k - \prod_{k=1}^{m} U_k \Big\|_2^2 = \|U_{m+1} \cdots U_n - 1\|_2^2 = \operatorname{tr}((U_{m+1} \cdots U_n - 1)^* (U_{m+1} \cdots U_n - 1))$$

$$= 2 - \operatorname{tr}(U_n^* \cdots U_{m+1}^* + U_{m+1} \cdots U_n) = 2 - 2 \prod_{k=m+1}^{n} \operatorname{tr} U_k,$$

where for the last equality we used independence (which follows from Lemma 1.2). We conclude that $\Big\{ \prod\limits_{n=1}^{N} U_n \Big\}_N$ is s.o.-convergent if and only if $\Big\{ \prod\limits_{n=1}^{N} \operatorname{tr} U_n \Big\}_N$ converges, i.e. if and only if $\sum\limits_{n=1}^{\infty} (1 - \operatorname{tr} U_n) < \infty$. Further, note that $\|U_n - 1\|_2^2 =$

$\operatorname{tr}((U_n^*-1)(U_n-1)) = 2(1-\operatorname{tr} U_n)$. This part of the proof uses only independence.

(ii) $\Rightarrow$ (i) Using the fact that the involution of $\mathcal{A}$ is an isometry of $L^2(\mathcal{A}, \operatorname{tr})$, we infer from (ii) that for all $x \in \mathcal{A}$ we have $\lim\limits_{N\to\infty} (U_1 \cdots U_N)x(U_1 \cdots U_N)^* = UxU^*$ (in $\|\cdot\|_2$-topology), but also $\lim\limits_{N\to\infty} \operatorname{Ad}(U_1 \cdots U_n) = \alpha$ and therefore $\alpha = \operatorname{Ad} U$. Because $U_n \in \mathcal{A}^0$ for all n we also get $U \in \mathcal{A}^0$.

To prove (i) $\Rightarrow$ (ii) we need some lemmas:

Lemma 3.3. *If v is a unitary in a finite factor $\mathcal{B}$ and $\operatorname{tr} v \geqslant 0$ then*

$$\frac{1}{2}\|v-1\|_2 \leqslant \sup_{\|x\|=1} \|\operatorname{Ad}(v)(x) - x\|_2 \leqslant 2\|v-1\|_2.$$

Proof. This is more or less implicit in Dixmier ([4], Chapter 7), but for convenience we give a proof. The second inequality follows from

$$\|\operatorname{Ad}(v)(x) - x\|_2 = \|[v,x]\|_2 = \|[v-1,x]\|_2 \leqslant 2\|v-1\|_2\|x\|.$$

To get the first inequality note that the closed convex hull $K := \overline{\operatorname{conv}}\{uvu^* : u \in \mathcal{B}$ unitary$\}$ in $L^2(\mathcal{B}, \operatorname{tr})$ contains $(\operatorname{tr} v)1$ (which is the unique element $y \in K$ with $\|y\|_2$ minimal; by uniqueness, $y \in \mathcal{B} \cap \mathcal{B}' = \mathbb{C}1$).

Choose $\sum\limits_{n=1}^{N} \lambda_n u_n v u_n^* \in K$ with $\| \sum\limits_{n=1}^{N} \lambda_n u_n v u_n^* - (\operatorname{tr} v)1\|_2 < \delta$. Then

$$\Big\|v - \sum_{n=1}^{N} \lambda_n u_n v u_n^*\Big\|_2 \leqslant \sum_{n=1}^{N} \lambda_n \big\|[v,u_n]\big\|_2 \leqslant \sup_{\|x\|=1} \|\operatorname{Ad}(v)(x) - x\|_2$$

and (using $|\operatorname{tr} v - 1| \leqslant \|v - (\operatorname{tr} v)1\|_2$), finally

$$\|v-1\|_2 \leqslant \|v-(\operatorname{tr} v)1\|_2 + |\operatorname{tr} v - 1| \leqslant 2\big(\sup_{\|x\|=1} \|\operatorname{Ad}(v)(x) - x\|_2 + \delta\big). \quad ∎$$

Lemma 3.4. *If $\beta = \lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \operatorname{Ad} U_n'$ is an adapted endomorphism, $\lim\limits_{n\to\infty} \|U_n' - 1\|_2 = 0$, $\operatorname{tr} U_n' \geqslant 0$ for all n and $\Big\{ \prod\limits_{n=1}^{N} U_n' \Big\}_N$ is not* s.o.*-convergent, then there exists $\varepsilon > 0$ and for all $m \in \mathbb{N}$ an element $x_m \in \mathcal{E}_{[m,\infty)}$, $\|x_m\| = 1$, so that $\|\beta(x_m) - x_m\|_2 > \varepsilon$.*

Proof. Because $\Big\{ \prod\limits_{n=1}^{N} U_n' \Big\}_N$ is not s.o.-convergent, there are $\delta > 0$ and for all $m \in \mathbb{N}$ a number $n > m$ so that $\delta < \Big\| \prod\limits_{k=m+1}^{n} U_n' - 1 \Big\|_2$. Applying Lemma 3.3 for $v := \prod\limits_{k=m+1}^{n} U_k'$ (note that $\operatorname{tr} v = \prod\limits_{k=m+1}^{n} \operatorname{tr} U_k' \geqslant 0$ by independence), we find an element

$x_m \in \mathcal{B} := \{u_{m+1}, \ldots, u_n\}'' \subset \mathcal{E}_{[m,\infty)}$, $\|x_m\| = 1$ so that $\Big\| \mathrm{Ad}\Big(\prod_{k=m+1}^{n} U'_k\Big)(x_m) - x_m \Big\|_2 > \frac{\delta}{2}$. But (as x_m commutes with $U'_1, \ldots, U'_{m-1}$)

$$\begin{aligned}\|\beta(x_m) - x_m\|_2 &= \big\| \mathrm{Ad}(U'_1 \cdots U'_{m-1})[\mathrm{Ad}(U'_m(U'_{m+1} \cdots U'_n)U'_{n+1})(x_m) - x_m]\big\|_2 \\ &\geqslant \| \mathrm{Ad}(U'_{m+1} \cdots U'_n)(x_m) - x_m\|_2 - 2\big(\|U'_m - 1\|_2 + \|U'_{n+1} - 1\|_2\big)\end{aligned}$$

(again using Lemma 3.2).

If m is large enough so that $\|U'_k - 1\|_2 < \frac{\delta}{16}$ for all $k \geqslant m$, then we have

$$\|\beta(x_m) - x_m\|_2 > \frac{\delta}{2} - 2\Big(\frac{\delta}{16} + \frac{\delta}{16}\Big) = \frac{\delta}{4} =: \varepsilon. \quad ∎$$

Lemma 3.5. *Assume* (i) *of Theorem* 3.1. *Then:*

(i) $\lim_{n\to\infty} \|\alpha(x_n) - x_n\|_2 = 0$ *if* $x_n \in \mathcal{E}_{[n,\infty)}$, $\|x_n\| = 1$ *for all* n.

(ii) $\lim_{n\to\infty} \min_{j=1,\ldots,p-1} |\gamma^{(n)}_{j0}| = 1$.

(iii) *There is* $q \in \mathbb{Z}_p (\simeq \{0, \ldots, p-1\})$ *with* $\lim_{n\to\infty} \| \mathrm{const}(n) \cdot U_n - u_n^{(-1)^n q}\|_2 = 0$, *where* $\mathrm{const}(n)$ *is unimodular and* $\mathrm{const}(n) \equiv 1$ *may be chosen if* $q = 0$.

Proof. For (i) check that $\mathcal{A}^0 = \{u_k, k \in \mathbb{N}\}'' \subset \Big(\bigcup_{k\in\mathbb{N}} \mathrm{lin}\{u_1, \ldots, u_k\}\Big)^{-\|\cdot\|_2}$. From $U \in \mathcal{A}^0$ and $[u_k, x_n] = 0$ if $k < n$, we infer that $0 = \lim_{n\to\infty} \|[U, x_n]\|_2 = \lim_{n\to\infty} \|\alpha(x_n) - x_n\|_2$. Note the similarity with arguments using central sequences.

(ii) follows from $\sum_{k=1}^{p-1} |\gamma^{(n+1)}_{jk}|^2 \leqslant \|\alpha(e^j_n) - e^j_n\|_2^2 \to 0$ by (i).

To prove (iii) note that since $\gamma^{(n)}_{10} = \frac{1}{p} \sum_{m=0}^{p-1} |\widehat{c}^{(n)}(m)|^2 \overline{\omega}^m$ and $\frac{1}{p} \sum_{m=0}^{p-1} |\widehat{c}^{(n)}(m)|^2 = \frac{1}{p} \sum_{m=0}^{p-1} |c^{(n)}(m)|^2 = 1$, for all $\varepsilon > 0$, there is $\delta > 0$ so that if $|\gamma^{(n)}_{10}| \geqslant 1 - \delta$ the function $\widehat{c}^{(n)}$ is almost concentrated to a single point in the sense that there is $q_n \in \{0, \ldots, p-1\}$ so that (when $\mathrm{const}(n) \cdot \widehat{c}^{(n)}(q_n)$ is chosen to be positive) we have $\| \mathrm{const}(n) \cdot U_n - u_n^{q_n}\|_2 \leqslant \varepsilon$.

Given $\varepsilon > 0$ then by using (i), (ii) and Lemma 3.3 we find that for all large enough n

$$\varepsilon \geqslant \|\alpha(e_n) - e_n\|_2 = \| \mathrm{Ad}(U_n U_{n+1})(e_n) - e_n\|_2 \geqslant \|\overline{\omega}^{(q_n + q_{n+1})} e_n - e_n\|_2 - 4\varepsilon,$$

i.e. $|\overline{\omega}^{(q_n+q_{n+1})} - 1| \leqslant 5\varepsilon$.

If $\varepsilon > 0$ is small enough this implies $q_{n+1} = -q_n$. We can then define $q := q_n$ for n even. The additional assertion for $q = 0$ reflects that $\mathrm{tr}\, U_n \geqslant 0$ for all n. ∎

Proof of (i)⇒(ii) in Theorem 3.1 completed:

Consider the adapted automorphisms $\gamma^{-q} \circ \operatorname{Ad} e_0^q = \lim_{N\to\infty} \prod_{n=1}^{N} \operatorname{Ad} U_n^{(-1)^{n+1}q}$, where q is from Lemma 3.5 (iii) and where γ is the grading automorphism, and $\beta := \gamma^{-q} \circ \operatorname{Ad} e_0^q \circ \alpha = \lim_{N\to\infty} \prod_{n=1}^{N} \operatorname{Ad} U'_n$, where

$$\prod_{n=1}^{N} \operatorname{Ad} U'_n = \operatorname{Ad}(u_1^q u_2^{-q} \cdots u_{N+1}^{(-1)^N q} U_1 \cdots U_N).$$

Use $u_{n+1}^k U_n = u_{n+1}^k \sum_{j=0}^{p-1} \widehat{c}^{(n)}(j) u_n^j = \Big(\sum_{j=0}^{p-1} \widehat{c}^{(n)}(j) \overline{\omega}^{jk} u_n^j \Big) u_{n+1}^k$ and Lemma 3.5 (iii) to conclude that (after suitably choosing unimodular constants) $\lim_{n\to\infty} \|U'_n - 1\|_2 = 0$ and $\operatorname{tr} U'_n \geqslant 0$ for all n. Now $\beta|\mathcal{E}_{[1,\infty)} = \alpha|\mathcal{E}_{[1,\infty)}$, so by using Lemma 3.5 (i) we find that also $\lim_{N\to\infty} \|\beta(x_n) - x_n\|_2 = 0$ if $x_n \in \mathcal{E}_{[n,\infty)}$, $\|x_n\| = 1$ for all n. Applying Lemma 3.4 we conclude that $\left\{ \prod_{n=1}^{N} U'_n \right\}_N$ is s.o.-convergent. We infer that β is inner (see (ii) $\Rightarrow$ (i)), while α is inner (by assumption) and γ is outer. This is compatible only for $q = 0$. Therefore $U'_n = U_n$, and indeed $\left\{ \prod_{n=1}^{N} U_n \right\}_N$ is s.o.-convergent. ∎

Corollary 3.6. *An adapted endomorphism* $\beta = \lim_{N\to\infty} \prod_{n=1}^{N} \operatorname{Ad} W_n$ *is inner if and only if there is some* $r \in \{0, \ldots, p-1\}$ *so that one of the following equivalent conditions is valid:*

(i) $\beta = \operatorname{Ad} W$, $W \in \mathcal{A}^r$;

(ii) $\beta = \operatorname{Ad} e_0^r \circ \alpha$, *where* α *is as in Theorem* 3.1;

(iii) *setting* $U_n := \operatorname{const}(n) \cdot u_n^r W_n$ *for* n *even and* $U_n := \operatorname{const}(n) \cdot W_n$ *for* n *odd with* $\operatorname{const}(n)$ *unimodular so that* $\operatorname{tr} U_n \geqslant 0$, *we have* $2 \sum_{n=1}^{\infty} (1 - \operatorname{tr} U_n) = \sum_{n=1}^{\infty} \|U_n - 1\|_2^2 < \infty$.

Remark 3.7. Recall the following theorem of Blattner ([1]): A Bogoljubov transformation α_T induced by an orthogonal transformation T of a separable real Hilbert space is inner and even (i.e. $p = 2$ and $\alpha_T = \operatorname{Ad} U$ with $U \in \mathcal{A}^0$ in our terminology) if and only if $\operatorname{Ker}(T + I)$ is even- or infinite-dimensional and $T - I$ is Hilbert-Schmidt. It is inner and odd if and only if $\operatorname{Ker}(T - I)$ is odd-dimensional and $T + I$ is Hilbert-Schmidt. We indicate briefly how this is related to Theorem 3.1 and Corollary 3.6: write $T = \text{s.o.} - \lim_{N\to\infty} \prod_{n=1}^{N} T_n$ with $T_n = \begin{pmatrix} \cos\varphi_n & -\sin\varphi_n \\ \sin\varphi_n & \cos\varphi_n \end{pmatrix}$ as in the proof of Proposition 1.12. Computing the diagonal of the corresponding infinite matrix for T gives

$$T \mp 1 \text{ Hilbert-Schmidt} \Leftrightarrow \operatorname{Trace}(1 \mp T) = (1 \mp \cos\varphi_1) + \sum_{n=1}^{\infty} (1 \mp \cos\varphi_n \cos\varphi_{n+1}) < \infty.$$

Form $\alpha_{T_n} = \mathrm{Ad}\, U_n$ with $\mathrm{tr}\, U_n = \cos\frac{\varphi_n}{2}$, and because $\cos x \sim 1 - \frac{x^2}{2}$ for $x \to 0$, our results above translate into:

a. α is inner and even $\Leftrightarrow T - 1$ is Hilbert-Schmidt and $\lim\limits_{n\to\infty} \cos\varphi_n = 1$;

b. α is inner and odd $\Leftrightarrow T + 1$ is Hilbert-Schmidt and $\begin{cases} \lim\limits_{n\to\infty} \cos\varphi_{2n} = -1, \\ \lim\limits_{n\to\infty} \cos\varphi_{2n+1} = 1. \end{cases}$

This already implies that the spectral theorem for compact operators may be applied. Thus, if one wants to complete a proof of Blattner's original formulation, one is left with the more elementary part of the presentation given by P. de la Harpe and R.J. Plymen in [7].

4. THE INDETERMINISTIC CASE

We want to examine in more detail an adapted endomorphism $\alpha = \lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \mathrm{Ad}\, U_n$ which is not surjective on $\mathcal{A}_{[0,\infty)}$. By Theorem 2.6 this is characterized by $\lim\limits_{N\to\infty} \|D_1 D_2 \cdots D_N\| > 0$. A convenient sufficient condition for this is given by

$$(\Gamma) \qquad \max_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}| < 1 \quad \text{for all } n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \Big(\max_{j=1,\dots,p-1} |\gamma_{j0}^{(n)}|^2 \Big) < \infty.$$

Indeed, this means that the product of minimal row sums of $D_1, D_2, \dots, D_N$ converges to a strict positive limit for $N \to \infty$, and if A, B are any matrices with nonnegative real entries then

$$\min\{\text{row sums of } AB\} \geqslant \min\{\text{row sums of } A\} \cdot \min\{\text{row sums of } B\}.$$

Note that for $p = 2$ and $p = 3$ condition (Γ) is also necessary. It is an interesting fact that in this case the Jones index only depends on p:

THEOREM 4.1. *If condition* (Γ) *is satisfied then* $[\mathcal{A}_{[0,\infty)} : \mathcal{A}_{[1,\infty)}] = p$.

Proof. General facts about towers of algebras applied to the tower $\mathbb{C}1 \subset \mathbb{C}^p \subset M_p \subset \cdots$ used here, show that $[\mathcal{A}_{[0,\infty)} : \mathcal{A}_{[1,\infty)}] \leqslant p$ (cf. V. Jones and V.S. Sunder ([10], Chapter 5), in particular Proposition 5.1.5 and Example 5.1.6). For the converse inequality we use a result of Pimsner and Popa ([13], Theorem 2.2), which applied to our problem asserts that

$$[\mathcal{A}_{[0,\infty)} : \mathcal{A}_{[1,\infty)}] = \sup_{0<x\in\mathcal{A}} \frac{\|x\|_2^2}{\|P_{[1,\infty)}(x)\|_2^2}.$$

Setting $\xi_m := \mathrm{Ad}\Big(\prod\limits_{k=1}^{m} U_k\Big)(e_m)$, we have for $n > m$

$$\|P_{[1,n]}\xi_m^j\|_2 = \|P_{[0,n-1]}U_n^* \cdots U_1^* \xi_m^j\|_2 = \|P_{[0,n-1]}U_n^* \cdots U_{m+1}^* e_m^j\|_2 \quad \text{for all } j.$$

From (Γ) we infer that $\lim\limits_{n>m\to\infty} \inf\{\text{row sums of } D_{m+1} \cdots D_n\} = 1$. Now apply Proposition 2.3 and Lemma 2.5 (for $\lim\limits_{N\to\infty} \prod\limits_{n=m+1}^{N} \mathrm{Ad}\, U_n$) to find that

$$\lim_{n>m\to\infty} \|P_{[1,n]}\xi_m^j\|_2 = 0 \quad \text{for all } j \in \{1,\dots,p-1\}.$$

Choosing $x_m := \frac{1}{p}\sum_{j=0}^{p-1} \xi_m^j$ (which is a projection with $\|x_m\|_2 = \frac{1}{\sqrt{p}}$), we have

$$\lim_{m\to\infty} \|P_{[1,\infty)}x_m\|_2 = \lim_{n>m\to\infty} \|P_{[1,n]}x_m\|_2 = \frac{1}{p}.$$

Inserting the x_m's into the formula of Pimsner and Popa shows that, $[\mathcal{A}_{[0,\infty)} : \mathcal{A}_{[1,\infty)}] > p - \varepsilon$ for all $\varepsilon > 0$. ∎

Finally, we shall derive a sufficient condition for an adapted endomorphism α to be a shift in the sense of Powers, i.e. $\bigcap_{n\geqslant 0} \alpha^n\mathcal{A} = \mathbb{C}1$.

If on the unit circle $\mathbb{T}$ with some finite measure μ one considers the multiplication M_z on $L^2(\mathbb{T},\mu)$, then there is an interesting sufficient condition for the nonexistence of a unitary part of M_z: a strictly positive angle between past and future (see H. Helson and G. Szegö ([8]) for details). To apply a similar reasoning to our problem above we first study a general setting of adaptedness in the framework of Hilbert spaces.

Consider a tower of Hilbert spaces

$$\{0\} = \mathcal{H}_{-1} \subset \mathcal{H}_0 \subset \mathcal{H}_{[0,1]} \subset \cdots \subset \mathcal{H}_{[0,n]} \subset \cdots \subset \mathcal{H} = \overline{\mathrm{lin}}\{\mathcal{H}_{[0,n]} : n \in \mathbb{N}\}$$

(the notation is chosen to fit with the applications to adapted endomorphisms). Define $\mathcal{H}_{m,n} := \mathcal{H}_{[0,n]} \ominus \mathcal{H}_{[0,m]}$ and write $P_{[0,n]}$, respectively $P_{m,n}$ for orthogonal projections onto corresponding spaces.

If $\{V_n\}_{n=1}^\infty$ is a family of unitaries on $\mathcal{H}$, where V_n fixes $\mathcal{H}_{[0,n-2]}$ pointwise and $\prod_{n=1}^N V_n$ leaves $\mathcal{H}_{[0,N]}$ (globally) invariant, then $V := \text{s.o.}-\lim_{N\to\infty} \prod_{n=1}^N V_n$ defines an isometry on $\mathcal{H}$ which may be called *adapted* to the tower above (cf. [6] for a more detailed discussion of this concept). Here we only need the following.

LEMMA 4.2. *Let V be adapted. From the assumptions that for all $n \geqslant 1$*

(0) *$\mathcal{H}_{[0,n]}$ is finite dimensional;*

(i) *if $x \in \mathcal{H}_{n-2,n-1}$ then $P_{n-1,n}V_n x \neq 0$;*

(ii) *there is another sequence $\{V'_n\}_{n=1}^\infty$ as above with the additional property $V'_n\mathcal{H}_{n-2,n-1} \subset \mathcal{H}_{n-1,n}$; and*

(iii) $\sum_{n=0}^{\infty}\Big(\sum_{k=n+1}^{2n+1} \|V_k - V'_k\|\Big)^2 < \infty$;

it follows that for the operators $S_n := P_{[0,n-1]}V^n|\mathcal{H}_{[0,n-1]}$ there is some $\varepsilon > 0$ so that $\|S_n\| \leqslant 1-\varepsilon$ for all $n \geqslant 1$. This further implies that V has no unitary part.

REMARKS 4.3. (i) The condition $\sum_{k=0}^{\infty} k\|V_k - V'_k\| < \infty$ is sufficient for (iii) in Lemma 4.2. Indeed:

$$(n+1)\sum_{k=n+1}^{2n+1} \|V_k - V'_k\| \leqslant \sum_{k=n+1}^{\infty} k\|V_k - V'_k\| < C$$

and

$$\sum_{n=0}^{\infty}\Big(\sum_{k=n+1}^{2n+1}\|V_k - V'_k\|\Big)^2 \leqslant \sum_{n=0}^{\infty}(\frac{C}{n+1})^2 < \infty.$$

(ii) If V is extended to a unitary on a larger Hilbert space then $\|S_n\| \leqslant 1-\varepsilon$ for all $n \geqslant 1$ means that there is a positive angle between past $\overline{\mathrm{lin}}\{V^n\mathcal{H}_0 : n \leqslant 0\}$ and future $\overline{\mathrm{lin}}\{V^n\mathcal{H}_0 : n \geqslant 1\}$.

Theorem 4.4. *Let* $\alpha = \lim_{N\to\infty}\prod_{n=1}^{N}\mathrm{Ad}\,U_n$ *be an adapted endomorphism. Assume for all* $n \geqslant 1$

(i) $\max_{j=1,\dots,p-1}|\gamma_{j0}^{(n)}| < 1$;

(ii) *there is another adapted endomorphism* $\alpha' = \lim_{N\to\infty}\prod_{n=1}^{N}\mathrm{Ad}\,U'_n$ *with* $\mathrm{Ad}\,U'_n(\mathcal{E}_{n-1}) \subset \mathcal{E}_n$ *(e.g.* $\alpha' = \sigma$*, the Gauss shift); and*

(iii) $\sum_{n=0}^{\infty}\Big(\sum_{k=n+1}^{2n+1}\|U_k - U'_k\|\Big)^2 < \infty$.

Then α *is a shift with index* p.

Proof. Check that the isometry V on $\{1\}^{\perp} \subset L^2(\mathcal{A},\mathrm{tr})$ which is induced by α fulfils the assumptions of Lemma 4.2. For (iii) note that if $\mathrm{Ad}\,U_n$ is viewed as a unitary on $L^2(\mathcal{A},\mathrm{tr})$ then $\|\mathrm{Ad}\,U_n - \mathrm{Ad}\,U'_n\| \leqslant 2\|U_n - U'_n\|$ (by an argument similar to that in Lemma 3.3). Also note that (iii) implies (Γ) of Theorem 4.1 which shows that $[\mathcal{A}_{[0,\infty)} : \mathcal{A}_{[1,\infty)}] = p$. ∎

Remarks 4.5. (i) Any cyclic isometry with one-dimensional corange on a Hilbert space is already a shift operator, as can be shown with the use of spectral theory (cf. Y.A. Rozanov, [16], Chapter II.5). This shows that for $p = 2$ much more is true and indicates that there might be improvements of the results of this section also for $p > 2$.

(ii) On the other hand, Lemma 4.2 is quite general and can be applied to other towers and corresponding adapted endomorphisms.

Proof of Lemma 4.2. Using (0) and (i) it is easy to see that for all $n \geqslant 1$ there is some $\varepsilon_n > 0$ so that $\|S_n\| \leqslant 1 - \varepsilon_n$. We have to show that there is some $\varepsilon > 0$ for all n simultaneously.

Set $\delta_n := \|V_n - V'_n\|$. Assume that N is large enough so that for all $n \geqslant N$ we have

$$1 - \prod_{k=n+1}^{2n+1}(1-\delta_k^2) < \frac{4}{3}\sum_{k=n+1}^{2n+1}\delta_k^2 \leqslant \Delta_{n+1} := \frac{4}{3}\Big(\sum_{k=n+1}^{2n+1}\delta_k\Big)^2 < \frac{1}{3};$$

to see that this is possible we have to apply (iii) and (for the first inequality) $\lim_{x\to 0}\frac{1}{x}\ln(1-x) = -1$ and $e^x > 1 + x$. This will be used at appropiate places without further mentioning.

Let us write $\|\cdot\|_2$ for the norm in $\mathcal{H}$ and assume $n \geqslant N$. If $\mathcal{H}_{[0,n]} \ni x = y \oplus z$, $y \in \mathcal{H}_{[0,n-1]}$, $z \in \mathcal{H}_{n-1,n}$, then we get

$$\|S_n y\|_2^2 =: \sigma_n \|y\|_2^2, \quad \sigma_n \leqslant (1-\varepsilon_n)^2,$$

$$V^{n+1} y = V S_n y \oplus V P_{n-1,2n-1} V^n y,$$

$$\|P_{[0,n]} V P_{n-1,2n-1} V^n y\|_2 = \|P_{[0,n]} V_{n+1} \cdots V_{2n} P_{n-1,2n-1} V^n y\|_2$$

(because $\mathcal{H}_{[0,n]}$ is invariant for $V_1 \cdots V_n$)

$$= \|P_{[0,n]}(V_{n+1} \cdots V_{2n} - V'_{n+1} \cdots V'_{2n}) P_{n-1,2n-1} V^n y\|_2 \qquad \text{using (ii)}$$

$$\leqslant \left(\sum_{k=n+1}^{2n} \delta_k \right) \|P_{n-1,2n-1} V^n y\|_2 = \left(\sum_{k=n+1}^{2n} \delta_k \right) (1-\sigma_n)^{\frac{1}{2}} \|y\|_2.$$

We conclude that

$$(*) \qquad \|S_{n+1} y\|_2^2 = \|V S_n y \oplus P_{[0,n]} V P_{n-1,2n-1} V^n y\|_2^2 =: \widetilde{\sigma}_{n+1} \|y\|_2^2$$

where $\widetilde{\sigma}_{n+1} \leqslant \sigma_n + \Big(\sum\limits_{k=n+1}^{2n} \delta_k \Big)^2 (1-\sigma_n)$. Further we have

$$\|P_{[0,n]} V_{n+1} z\|_2 = \|P_{[0,n]}(V_{n+1} - V'_{n+1}) z\|_2 \leqslant \delta_{n+1} \|z\|_2,$$

$$\|P_{2n,2n+1} V^{n+1} z\|_2^2 = \|P_{2n,2n+1} V_{2n+1} \cdots V_{n+1} z\|_2^2 \geqslant \prod_{k=n+1}^{2n+1} (1-\delta_k^2) \ \|z\|_2^2,$$

$$\|P_{[0,2n]} V^{n+1} z\|_2^2 \leqslant \left(1 - \prod_{k=n+1}^{2n+1} (1-\delta_k^2) \right) \|z\|_2^2 \leqslant \Delta_{n+1} \|z\|_2^2.$$

Putting all this together we find

$$\|S_{n+1} x\|_2^2 = \|S_{n+1} y + S_{n+1} z\|_2^2 \leqslant \|S_{n+1} y\|_2^2 + \|S_{n+1} z\|_2^2 + 2|\langle S_{n+1} y, S_{n+1} z\rangle|$$

where

$$\|S_{n+1} y\|_2^2 = \widetilde{\sigma}_{n+1} \|y\|_2^2, \quad \|S_{n+1} z\|_2^2 \leqslant \Delta_{n+1} \|z\|_2^2$$

and

$$\langle S_{n+1} y, S_{n+1} z\rangle + \langle P_{n,2n} V^{n+1} y, P_{n,2n} V^{n+1} z\rangle = \langle V^{n+1} y, V^{n+1} z\rangle = \langle y, z\rangle = 0.$$

Since

$$\begin{aligned} 2|\langle S_{n+1} y, S_{n+1} z\rangle| &\leqslant 2\|P_{n,2n} V^{n+1} y\|_2 \|P_{n,2n} V^{n+1} z\|_2 \\ &\leqslant 2(1-\widetilde{\sigma}_{n+1})^{\frac{1}{2}} \|y\|_2 \Delta_{n+1}^{\frac{1}{2}} \|z\|_2 \\ &= 3 \cdot 2 \|(1-\widetilde{\sigma}_{n+1})^{\frac{1}{2}} \Delta_{n+1}^{\frac{1}{2}} y\|_2 \left\| \frac{1}{3} z \right\|_2 \\ &\leqslant 3(1-\widetilde{\sigma}_{n+1}) \Delta_{n+1} \|y\|_2^2 + \frac{1}{3} \|z\|_2^2 \end{aligned}$$

(just use $2ab \leqslant a^2 + b^2$), we get

$$(**) \qquad \|S_{n+1} x\|_2^2 \leqslant (\widetilde{\sigma}_{n+1} + (1-\widetilde{\sigma}_{n+1}) 3\Delta_{n+1}) \|y\|_2^2 + \frac{2}{3} \|z\|_2^2 =: \sigma_{n+1} \|x\|_2^2,$$

where $\sigma_{n+1} \leqslant \max\{\widetilde{\sigma}_{n+1} + (1-\widetilde{\sigma}_{n+1}) 3\Delta_{n+1}, \frac{2}{3}\}$. Note that σ_{n+1} is related to σ_n by the two recursions $(*)$ and $(**)$. Therefore our assertion $\|S_n\| \leqslant 1-\varepsilon$ for all n follows from (iii) by an application of the following elementary

LEMMA 4.6. *Assume* $0 < r_N < 1$ *and (for all* $n \geqslant N$*)* $r_{n+1} \leqslant r_n + (1 - r_n)a_{n+1}$ *with* $0 \leqslant a_n < 1$, $\sum\limits_{n=N+1}^{\infty} a_n < \infty$. *Then there is some* $\varepsilon > 0$ *so that* $r_n \leqslant 1 - \varepsilon$ *for all* $n \geqslant N$.

Proof. For this just notice that

$$1 - r_{n+1} \geqslant (1 - r_n)(1 - a_{n+1}) \geqslant (1 - r_N) \prod_{k=N}^{n} (1 - a_{k+1}),$$

which by assumption has a strict positive limit for $n \to \infty$. ∎

We still have to show that V has no unitary part. First note that also $\|P_{[0,n-1]}V^n\| \leqslant 1 - \varepsilon$ for all $n \geqslant 1$: indeed if $y \in \mathcal{H}_{[0,m-1]}$ for some $m > n$ then

$$\|P_{[0,n-1]}V^n y\|_2 = \|P_{[0,m-1]}V^{m-n}P_{[0,n-1]}V^n y\|_2 \leqslant \|P_{[0,m-1]}V^m y\|_2 = \|S_m y\|_2.$$

Now assume $x \in \bigcap\limits_{n\geqslant 0} V^n\mathcal{H}$. For any $\delta > 0$ we find some n so that $x' \in \mathcal{H}_{[0,n-1]}$, $\|x - x'\|_2 < \delta$ and some m so that $x'' \in \mathcal{H}_{[0,m-1]}$, $\|x - V^n x''\|_2 < \delta$. But we have shown above that there is a positive angle between x' and $V^n x''$ not decreasing to 0 for $\delta \to 0$. This is possible only for $x = 0$. ∎

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