J. OPERATOR THEORY
45(2001), 39–51

LIMITS OF VECTOR FUNCTIONALS

R.J. ARCHBOLD and M.H. SHAH

Communicated by Şerban Strătilă

Abstract. For vector functionals on a C^*-algebra of operators, we prove an analogue of Glimm's vector state space theorem. We deduce that a C^*-algebra is prime and antiliminal if and only if the pure functionals are w*-dense in the unit ball of the dual. We also give a necessary and sufficient condition for a convex combination of inequivalent pure functionals to be a w*-limit of pure functionals.

Keywords: *C^*-algebra, vector functional, pure functional, antiliminal, prime.*
MSC (2000): 46L05, 46L30.

1. INTRODUCTION

Let A be a C^*-algebra of operators acting on an infinite dimensional Hilbert space H. For unit vectors ξ, η in H, let $\omega_{\xi,\eta}$ be the linear functional on $B(H)$ defined by $\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle$. The functional $\omega_{\xi,\xi}$ is usually just written as ω_ξ. The functional $\omega_{\xi,\eta}|A$ is called a vector functional of A and (if A acts non-degenerately on H) the positive functional $\omega_\xi|A$ is called a vector state of A.

The unital case of Glimm's vector state space theorem asserts that if $\varphi \in S(A)$ (the state space of A) then φ is a w*-limit of vector states of A if and only if it has the form

$$\varphi = \lambda\omega_\xi|A + (1-\lambda)\psi$$

where $\lambda \in [0,1]$, ξ is a unit vector in H and ψ is a state of A that annihilates the intersection of A with the set of compact operators $K(H)$ ([7], Theorem 2). The non-unital case is similar ([8], Lemma 9). Our first main results (Theorems 3.1 and 3.2) show that if $\varphi \in A^*$ and $\|\varphi\| \leqslant 1$ then a necessary and sufficient condition for φ to be a w*-limit of vector functionals of A is that φ should have the form

$$\varphi = \lambda\omega_{\xi,\eta}|A + (1-\lambda)\psi$$

where $\lambda \in [0,1]$, ξ and η are unit vectors in H and ψ is a contractive linear functional on A that annihilates $A \cap K(H)$. The necessity of this condition is proved by using Glimm's theorem and the polar decomposition for functionals in the Banach dual A^* of A (although a significant complication arises because if $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$ in A^* then it may happen that $|\varphi_\alpha| \underset{\alpha}{\not\longrightarrow} |\varphi|$). However, this strategy appears to fail to establish the sufficiency of the condition because the polar decomposition leads to a requirement for $\sigma(A^*, A^{**})$-convergence in a situation where only $\sigma(A^*, A)$-convergence is known. In view of this we have been forced to develop a substantial extension of Glimm's original method for states (see Theorem 3.1).

Let A be a C^*-algebra (no longer assumed to be acting on any particular Hilbert space), let $P(A)$ be the set of pure states of A and let $G(A)$ be the set of pure functionals of A (the set of extreme points of the unit ball A_1^* of A^*). A well-known combination of a theorem of Glimm ([7], Theorem 1, p. 231) and a theorem of Tomiyama and Takesaki ([11], Theorem 2) asserts that $\overline{P(A)}$ (the w*-closure of $P(A)$) contains $S(A)$ if and only if either A is prime and antiliminal or $A \cong \mathbb{C}$. By considering $G(A)$ rather than $P(A)$, we are able to remove the awkward trivial case and also replace containment by equality. To be precise, we show that a necessary and sufficient condition for A to be prime and antiliminal is that $\overline{G(A)} = A_1^*$. The sufficiency follows readily from a result of Effros (see Section 2) and the theorem of Tomiyama and Takesaki. In the separable case, the necessity of the condition is obtained by using Theorem 3.1 (i). The nonseparable case is reduced to this by using a result of Batty ([4], Proposition 5).

Finally, we consider a sequence $(\varphi_i)_{i\geqslant 1}$ of pure functionals of a C^*-algebra A and a σ-convex combination $\varphi = \sum\limits_{i=1}^{\infty} \lambda_i \varphi_i$ where each $\lambda_i > 0$. For each i let $|\varphi_i|$ be the pure state obtained from φ_i via the polar decomposition and let π_i be the GNS representation of $|\varphi_i|$. Assuming that π_i and π_j are inequivalent for $i \neq j$, we prove that a necessary and sufficient condition for φ to lie in $\overline{G(A)}$ is that there exists a net in the spectrum $\widehat{A}$ of A that is convergent to every π_i. This generalizes an earlier result of the first author for pure states ([2], Theorem 2) in which inequivalence was required only for the proof of the sufficiency of the net condition. In the present case, the lack of positivity requires the assumption of inequivalence for the proof of necessity too.

2. PRELIMINARIES

Let A be a C^*-algebra and let $\widetilde{A}$ be A if A is unital and $A + \mathbb{C}1$ otherwise. A bounded linear functional φ on A has a polar decomposition $\varphi = u|\varphi| = |\varphi|(u\,\cdot)$ where $|\varphi|$ is a positive linear functional on A, u is a partial isometry in A^{**} and $|\varphi| = u^*\varphi$. Effros ([6], Lemma 3.5) has shown that if $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$ in the w*-topology on A^* and if $\|\varphi_\alpha\| \underset{\alpha}{\longrightarrow} \|\varphi\|$ then $|\varphi_\alpha| \underset{\alpha}{\longrightarrow} |\varphi|$. We shall use this result in the proof of Theorem 4.1. However, in Theorem 3.2 we have a situation in which $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$ in A^*, $\|\varphi_\alpha\| = 1$ for all α, but possibly $\|\varphi\| < 1$. By compactness, the net $(|\varphi_\alpha|)$ has a cluster point ρ in the quasi-state space $QS(A)$. Then $|\varphi(a)|^2 \leqslant \rho(a^*a)$ for all $a \in A$ and so there exists η in the GNS Hilbert space H_ρ such that $\|\eta\| \leqslant 1$ and

$$\varphi(a) = \langle \pi_\rho(a)\xi_\rho, \eta\rangle, \quad a \in A.$$

(see [6], p. 400).

If $\varphi \in A^*$ then $\varphi \in G(A)$ if and only if $|\varphi| \in P(A)$ ([1], Theorem 2.1). It follows that $\varphi \in G(A)$ if and only if there exist $\pi \in \widehat{A}$ and unit vectors ξ, η in the Hilbert space H_π such that $\varphi = \langle \pi(\cdot)\xi, \eta\rangle$, in which case $|\varphi| = \langle \pi(\cdot)\xi, \xi\rangle$ (see for example [3], 1.1) and by Kadison's transitivity theorem there exists a unitary element $u \in \widetilde{A}$ such that $\varphi = u|\varphi|$ and $|\varphi| = u^*\varphi$ (see [10], Lemma 4, for the unital case). Although this u is not the partial isometry of the polar decomposition (unless $A \cong \mathbb{C}$) it has the important advantage of being a multiplier of A. If B is a C^*-subalgebra of A and $\pi \in \widehat{B}$ then there exists an irreducible representation σ of A such that $H_\sigma \supseteq H_\pi$ and $(\sigma|B)|_{H_\pi}$ is equivalent to π ([9], 5.5.1). It follows that if $\psi \in G(B)$ then there exists $\varphi \in G(A)$ such that $\varphi|B = \psi$.

We shall frequently use the elementary fact that if J is a closed two-sided ideal of A and if $\varphi \in A^*$ annihilates J then so does $x\varphi$ for all $x \in A^{**}$. In particular, $\varphi(J) = \{0\}$ if and only if $|\varphi|(J) = \{0\}$.

Unless stated otherwise, it should be understood that any topological statements concerning A^* refer to the w*-topology (that is, the $\sigma(A^*, A)$ topology).

3. AN ANALOGUE OF GLIMM'S VECTOR STATE SPACE THEOREM

The following result will be one part of our analogue of Glimm's vector state space theorem. It might be hoped that it could be proved using Glimm's theorem and the polar decomposition. However, this approach appears to fail because $\sigma(A^*, A)$-convergence is weaker than $\sigma(A^*, A^{**})$-convergence. Thus we have had to make a substantial extension of Glimm's method. An important ingredient is the unitary decomposition for pure functionals (see Section 2).

THEOREM 3.1. *Let A be a C^*-algebra acting on an infinite-dimensional Hilbert space H. Let $\varphi \in A^*$ with $\|\varphi\| \leqslant 1$.*

(i) *Suppose that $\varphi(A \cap K(H)) = \{0\}$. Then there exist nets of unit vectors $(\xi_\alpha), (\eta_\alpha)$ in H such that*

$$\omega_{\xi_\alpha,\eta_\alpha}|A \underset{\alpha}{\longrightarrow} \varphi$$

and $\xi_\alpha, \eta_\alpha \underset{\alpha}{\longrightarrow} 0$ (weakly).

(ii) *Suppose that $\varphi = \lambda\omega_{\xi,\eta}|A + (1-\lambda)\psi$ where $\lambda \in [0,1]$, ξ, η are unit vectors in H and $\psi \in A^*$ is such that $\|\psi\| \leqslant 1$ and $\psi(A \cap K(H)) = \{0\}$. Then there exist nets of unit vectors $(\xi_\alpha), (\eta_\alpha)$ in H such that*

$$\lim_\alpha \omega_{\xi_\alpha,\eta_\alpha}|A = \varphi.$$

Proof. (i) We first consider the case when $K(H) \subseteq A$. We will then use this special case to prove the case when $K(H) \not\subseteq A$.

CASE (I). Suppose that $K(H) \subseteq A$. Let L be a finite-dimensional subspace of H and let U be an open neighbourhood of φ of the form

$$U = \{h \in A^* : |h(x_i) - \varphi(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant s\}$$

where $\varepsilon > 0$ and $x_1, x_2, \ldots, x_s \in A$. It suffices to find unit vectors $\xi, \eta \in L^\perp$ such that $\omega_{\xi,\eta}|A \in U$ (for then we may index ξ and η by the ordered pair (L, U), with the obvious directed ordering on such pairs). Since $\varphi(K(H)) = \{0\}$, there exists $\varphi_0 \in (A/K(H))^*$ such that $\varphi = \varphi_0 \circ q$, where q is the quotient map, and $\|\varphi_0\| = \|\varphi\| \leqslant 1$. By the Krein-Milman theorem, φ_0 is a w*-limit of finite convex combinations of pure functionals of $A/K(H)$. Hence there exist pure functionals $\varphi_1, \varphi_2, \dots, \varphi_n$ of A and non-negative real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ with unit sum such that $\varphi_i(K(H)) = \{0\}$ $(1 \leqslant i \leqslant n)$ and $|\rho(x_i) - \varphi(x_i)| < \varepsilon/2$ $(1 \leqslant i \leqslant s)$ where $\rho = \sum\limits_{j=1}^{n} \lambda_j \varphi_j$. Since $\varphi_j \in G(A)$, there exists a unitary element $u_j \in \widetilde{A}$ such that $\varphi_j = u_j |\varphi_j|$ for $1 \leqslant j \leqslant n$ (see Section 2).

We will construct unit vectors $\xi_1, \xi_2, \dots, \xi_n \in H$ such that, writing $\eta_j = u_j^* \xi_j$,

$$(3.1)\qquad \begin{cases} |\varphi_j(x_i) - \omega_{\xi_j,\eta_j}(x_i)| < \varepsilon/2 & (1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant n), \\ \xi_j, \eta_j \in L^\perp & (1 \leqslant j \leqslant n) \\ \text{and such that} & \\ \langle \xi_j, \xi_k \rangle = 0,\ \langle \eta_j, \eta_k \rangle = 0,\ \langle x_i\xi_j, \eta_k \rangle = 0 & \text{for all } i \text{ and } j \neq k. \end{cases}$$

Let L_0 be a finite-dimensional subspace of H containing both L and $\bigcup\limits_{j=1}^{n} u_j(L)$. Let m be an integer such that $1 \leqslant m \leqslant n$ and suppose that we have constructed unit vectors $\xi_1, \xi_2, \dots, \xi_{m-1}$ such that all the relations in (3.1) hold when $1 \leqslant j, k < m$. If $m = 1$ define $M = L_0$ and if $m > 1$ define

$$M = \operatorname{Span}\Big(L_0 \cup \{\xi_j : 1 \leqslant j < m\} \cup \{u_m u_j^* \xi_j : 1 \leqslant j < m\} \cup \{u_m x_i \xi_j : 1 \leqslant j < m,\ 1 \leqslant i \leqslant s\} \cup \{x_i^* u_k^* \xi_k : 1 \leqslant i \leqslant s,\ 1 \leqslant k < m\}\Big).$$

Let $N = M^\perp$ and let P_M, P_N be the corresponding orthogonal projections in $B(H)$. Then $P_M + P_N = 1$ and for all $a \in A$

$$(3.2)\qquad a = P_N a P_N + (P_N a P_M + P_M a P_N + P_M a P_M).$$

Since M is finite dimensional, $P_M \in K(H) \subseteq A$ and so $P_N \in \widetilde{A}$. Also, the bracketed expression on the right of (3.2) is compact. Let $B = P_N A P_N$, a hereditary C^*-subalgebra of A. Since $\varphi_m(K(H)) = \{0\}$ we have $|\varphi_m|(K(H)) = \{0\}$ and hence by (3.2), $|\varphi_m|(B) \neq \{0\}$. Because B is hereditary in A, $|\varphi_m| \,\big|B \in P(B)$. Since the kernel of the identity representation of B on N is $\{0\}$, it follows from [5], 3.4.2 (ii) that there exists a net of unit vectors (ξ_α) in N such that

$$\lim_\alpha \omega_{\xi_\alpha}|B = |\varphi_m| \,\big|B.$$

Let $a \in A$. Then, by (3.2) and the fact that $|\varphi_m|(K(H)) = \{0\}$, we have

$$\big|\omega_{\xi_\alpha}(a) - |\varphi_m|(a)\big| = \big|\omega_{\xi_\alpha}(P_N a P_N) - |\varphi_m|(P_N a P_N)\big| \xrightarrow[\alpha]{} 0.$$

Therefore

$$\lim_\alpha \omega_{\xi_\alpha}|A = |\varphi_m|.$$

Hence there exists a unit vector $\xi_m \in N$ such that

$$\big|\,|\varphi_m|(u_m x_i) - \omega_{\xi_m}(u_m x_i)\big| < \frac{\varepsilon}{2}, \qquad 1 \leqslant i \leqslant s.$$

It follows that

$$|\varphi_m(x_i) - \omega_{\xi_m,\eta_m}(x_i)| < \frac{\varepsilon}{2}, \qquad 1 \leqslant i \leqslant s.$$

Since $\xi_m \in N$, ξ_m is orthogonal to M and so all the relations in (3.1) hold when $1 \leqslant j, k \leqslant m$. Thus, by induction, we obtain the required vectors $\xi_1, \xi_2, \ldots, \xi_n$.

Let $\xi = \sum_{j=1}^{n} \sqrt{\lambda_j}\xi_j$ and $\eta = \sum_{j=1}^{n} \sqrt{\lambda_j}\eta_j$. Then, by (3.1), ξ and η are unit vectors in $L^{\perp}$. Finally, for $1 \leqslant i \leqslant s$ we have

$$\begin{aligned}
|\rho(x_i) - \omega_{\xi,\eta}(x_i)| &= \left| \sum_{j=1}^{n} \lambda_j \varphi_j(x_i) - \sum_{j,k=1}^{n} \sqrt{\lambda_j\lambda_k}\langle x_i\xi_j, \eta_k\rangle \right| \\
&\leqslant \sum_{j=1}^{n} \lambda_j |\varphi_j(x_i) - \omega_{\xi_j,\eta_j}(x_i)| + \sum_{j\neq k} \sqrt{\lambda_j\lambda_k} |\langle x_i\xi_j, \eta_k\rangle| \\
&= \sum_{j=1}^{n} \lambda_j |\varphi_j(x_i) - \omega_{\xi_j,\eta_j}(x_i)| \quad \text{(by (3.1))} \\
&< \frac{\varepsilon}{2} \sum_{j=1}^{n} \lambda_j = \frac{\varepsilon}{2}.
\end{aligned}$$

Hence $\omega_{\xi,\eta}|A \in U$ as required.

CASE (II). Suppose that $K(H) \nsubseteq A$. Let $B = K(H) + A$. Let $q_B : B \to B/K(H)$ be the quotient map and let

$$\Phi : B/K(H) \to A/(A \cap K(H))$$

be the canonical $*$-isomorphism given by

$$\Phi(x + K(H)) = x + (A \cap K(H)) \qquad \text{for } x \in A.$$

Since $\varphi(A\cap K(H)) = \{0\}$ there exists $\varphi' \in (A/(A\cap K(H)))^*$ such that $\|\varphi'\| = \|\varphi\|$ and $\varphi'(a + (A \cap K(H))) = \varphi(a)$ for all $a \in A$.

Let $\overline{\varphi} = \varphi' \circ \Phi \circ q_B$. Then $\|\overline{\varphi}\| \leqslant 1$ and $\overline{\varphi}(K(H)) = \{0\}$. By applying the result of Case (I) to $\overline{\varphi}$ and then restricting to A, we obtain the required nets $(\xi_\alpha), (\eta_\alpha)$.

(ii) Suppose that

$$\varphi = \lambda\omega_{\xi,\eta}|A + (1 - \lambda)\psi$$

where $\lambda \in [0, 1]$, ξ, η are unit vectors in H and $\psi \in A^*$ is such that $\|\psi\| \leqslant 1$ and $\psi(A \cap K(H)) = \{0\}$. By (i) there exist nets of unit vectors $(\xi_\alpha), (\eta_\alpha)$ in H such that

$$\omega_{\xi_\alpha,\eta_\alpha}|A \underset{\alpha}{\longrightarrow} \psi$$

and $\xi_\alpha, \eta_\alpha \underset{\alpha}{\longrightarrow} 0$ (weakly).

Let

$$\zeta_\alpha = \sqrt{\lambda}\xi + \sqrt{1 - \lambda}\xi_\alpha \quad \text{and} \quad {\zeta'}_\alpha = \sqrt{\lambda}\eta + \sqrt{1 - \lambda}\eta_\alpha.$$

Since $\xi_\alpha, \eta_\alpha \xrightarrow[\alpha]{} 0$ (weakly),

$$\|\zeta_\alpha\|^2, \|\zeta'_\alpha\|^2 \xrightarrow[\alpha]{} 1.$$

So eventually

$$\|\zeta_\alpha\|, \|\zeta'_\alpha\| \neq 0$$

and we can form unit vectors

$$u_\alpha = \frac{\zeta_\alpha}{\|\zeta_\alpha\|}, \quad v_\alpha = \frac{\zeta'_\alpha}{\|\zeta'_\alpha\|}.$$

For $a \in A$ we have

$$\begin{aligned}\omega_{u_\alpha, v_\alpha}(a) = \langle a u_\alpha, v_\alpha \rangle &= \frac{1}{\|\zeta_\alpha\| \, \|\zeta'{}_\alpha\|} \langle a\zeta_\alpha, \zeta'{}_\alpha \rangle \\ &= \frac{1}{\|\zeta_\alpha\| \, \|\zeta'_\alpha\|} \{\lambda \omega_{\xi,\eta}(a) + (1-\lambda)\omega_{\xi_\alpha,\eta_\alpha}(a) + \sqrt{\lambda(1-\lambda)}(\langle a\xi, \eta_\alpha \rangle + \langle \xi_\alpha, a^*\eta \rangle)\},\end{aligned}$$

and hence

$$\omega_{u_\alpha, v_\alpha}(a) \xrightarrow[\alpha]{} \lambda \omega_{\xi,\eta}(a) + (1-\lambda)\psi(a) = \varphi(a).$$

Thus

$$\lim_\alpha \omega_{u_\alpha, v_\alpha}|A = \varphi. \quad \blacksquare$$

REMARK 3.2. Suppose that H is an infinite-dimensional Hilbert space, A is a C^*-subalgebra of $B(H)$ containing $K(H)$, φ is an element of A_1^* such that $\varphi(K(H)) = \{0\}$, and (ξ_α) and (η_α) are nets of unit vectors in H such that

$$\omega_{\xi_\alpha, \eta_\alpha}|A \xrightarrow[\alpha]{} \varphi.$$

If $\|\varphi\| = 1$ then $\omega_{\xi_\alpha}|A \xrightarrow[\alpha]{} |\varphi|$ and $\omega_{\eta_\alpha}|A \xrightarrow[\alpha]{} |\varphi^*|$ and so, since $|\varphi|(K(H)) = \{0\} = |\varphi^*|(K(H))$, it follows that ξ_α and η_α necessarily converge weakly to zero. However, this need not hold if $\|\varphi\| < 1$. For example, if ξ is a fixed unit vector in H and $\eta_\alpha \xrightarrow[\alpha]{} 0$ weakly then

$$\omega_{\xi, \eta_\alpha}|A \xrightarrow[\alpha]{} 0.$$

Nevertheless, Theorem 3.1 (i) shows that it is always possible to choose ξ_α and η_α such that $\xi_\alpha, \eta_\alpha \xrightarrow[\alpha]{} 0$ weakly, as is required for the proof of Theorem 3.1 (ii).

We now state and prove, in full, our analogue of Glimm's vector state space theorem.

THEOREM 3.3. *Let A be a C^*-algebra acting on an infinite-dimensional Hilbert space H. Let $\varphi \in A^*$ with $\|\varphi\| \leqslant 1$. Then the following are equivalent:*

(i)

$$\varphi = \lambda \omega_{\xi,\eta}|A + (1-\lambda)\psi$$

where $\lambda \in [0,1]$, ξ, η are unit vectors in H and $\psi \in A^*$ is such that $\|\psi\| \leqslant 1$ and $\psi(A \cap K(H)) = \{0\}$;

(ii) *there exist nets of unit vectors $(\xi_\alpha), (\eta_\alpha)$ in H such that*

$$\lim_\alpha \omega_{\xi_\alpha, \eta_\alpha}|A = \varphi.$$

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3.1 part (ii).

(ii) $\Rightarrow$ (i). Suppose that there exist nets of unit vectors $(\xi_\alpha), (\eta_\alpha)$ in H such that

$$\lim_\alpha \omega_{\xi_\alpha,\eta_\alpha}|A = \varphi.$$

CASE (I). Suppose that $K(H) \subseteq A$. By passing to a subnet, we may assume that $\omega_{\xi_\alpha}|A \underset{\alpha}{\longrightarrow} \rho$ for some $\rho \in QS(A)$. Then $|\varphi(a)|^2 \leqslant \rho(a^*a)$ for all $a \in A$ (see Section 2). If $\varphi = 0$ we may take $\lambda = 0$ and $\psi = 0$ so we assume from now on that $\varphi \neq 0$ and hence $\rho \neq 0$. Then

$$\varphi = \langle \pi_\rho(\cdot)\xi_\rho, v\rangle$$

for some $v \in H_\rho$ with $0 < \|v\| \leqslant 1$ (see Section 2).

By [8], Lemma 9,

$$\rho = \alpha\omega_u|A + \rho_0 \tag{3.3}$$

where $0 \leqslant \alpha = \|\rho\| - \|\rho_0\| \leqslant 1$, u is a unit vector in H and ρ_0 is a positive functional of A that annihilates $K(H)$. If $\alpha = 0$ then π_ρ (and hence φ) annihilates $K(H)$ and so we may take $\lambda = 0$ and $\psi = \varphi$. Suppose that $\alpha \neq 0$. If $\rho_0 = 0$ then we may take $\pi_\rho = \mathrm{id}$ and $\xi_\rho = \sqrt{\alpha}u$ from which it follows that

$$\varphi = \sqrt{\alpha}\langle(\cdot)u, v\rangle = \sqrt{\alpha}\|v\|\langle(\cdot)u, v/\|v\|\rangle,$$

which has the required form (taking $\lambda = \sqrt{\alpha}\|v\|$, $\xi = u$, $\eta = v/\|v\|$ and $\psi = 0$). Thus we may suppose that $\rho_0 \neq 0$ (as well as $\alpha \neq 0$).

Since id and π_{ρ_0} are disjoint, we may take $\pi_\rho = \mathrm{id} \oplus \pi_{\rho_0}$, $\xi_\rho = (\sqrt{\alpha}u, \xi_{\rho_0})$ and then $v = (v_1, v_2)$. Thus, for $a \in A$,

$$\varphi(a) = \sqrt{\alpha}\langle au, v_1\rangle + \langle \pi_{\rho_0}(a)\xi_{\rho_0}, v_2\rangle.$$

If $v_1 = 0$ then $\varphi(K(H)) = \{0\}$ and we may take $\lambda = 0$ and $\psi = \varphi$. Suppose that $v_1 \neq 0$. Then let $\lambda = \sqrt{\alpha}\|v_1\|$, $\xi = u$, $\eta = v_1/\|v_1\|$. If $\lambda = 1$ then $v_2 = 0$ and we may take $\psi = 0$. Suppose that $\lambda < 1$ and define

$$\psi = (1-\lambda)^{-1}\langle \pi_{\rho_0}(\cdot)\xi_{\rho_0}, v_2\rangle.$$

Then $\varphi = \lambda\omega_{\xi,\eta}|A + (1-\lambda)\psi$, $\psi(K(H)) = \{0\}$ and

$$\begin{aligned}\|\psi\|^2 &\leqslant (1-\lambda)^{-2}\|\xi_{\rho_0}\|^2\|v_2\|^2 = (1-\lambda)^{-2}\|\rho_0\|\,\|v_2\|^2\\ &\leqslant (1-\sqrt{\alpha}\|v_1\|)^{-2}(1-\alpha)(1-\|v_1\|^2) \leqslant 1\end{aligned}$$

since $2\sqrt{\alpha}\|v_1\| \leqslant \alpha + \|v_1\|^2$.

CASE (II). Now suppose that $K(H) \not\subseteq A$ and let $B = A + K(H)$. Consider the net $(\omega_{\xi_\alpha,\eta_\alpha}|B)$ in the unit ball B_1^* of B^*. Since B_1^* is w*-compact, we may assume by passing to a subnet if necessary that $(\omega_{\xi_\alpha,\eta_\alpha}|B)$ is convergent to some $\rho \in B_1^*$. Therefore, by Case(I),

$$\rho = \lambda\omega_{\xi,\eta}|B + (1-\lambda)\psi_0$$

where $\lambda \in [0,1]$, ξ, η are unit vectors in H and $\psi_0 \in B^*$ is such that $\|\psi_0\| \leqslant 1$ and $\psi_0(K(H)) = \{0\}$. Hence, by restricting to A, we get

$$\varphi = \lambda\omega_{\xi,\eta}|A + (1-\lambda)\psi_0|A.$$

Let $\psi = \psi_0|A$. Then $\psi(A \cap K(H)) = \{0\}$ and $\|\psi\| \leqslant 1$. ∎

4. CONDITIONS FOR THE DENSITY OF THE PURE FUNCTIONALS IN THE DUAL BALL

Glimm in [7], Theorem 1, p. 231 and Tomiyama and Takesaki in [11], Theorem 2, have proved that if A is a C^*-algebra then $\overline{P(A)} \supseteq S(A)$ if and only if either A is prime and antiliminal or $A \cong \mathbb{C}$. In the following theorem we prove an analogue of this result for pure functionals of A.

Theorem 4.1. *Let A be a C^*-algebra. Then the following are equivalent:*
(i) $A_1^* = \overline{G(A)}$;
(ii) *A is antiliminal and prime.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $A_1^* = \overline{G(A)}$ and let $\varphi \in S(A)$. Then there exists a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ in $G(A)$ such that $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$. By [6], 3.5 we get $|\varphi_\alpha| \underset{\alpha}{\longrightarrow} |\varphi| = \varphi$ and so $\varphi \in \overline{P(A)}$. Thus $S(A) \subseteq \overline{P(A)}$. Therefore by [11], Theorem 2, either A is antiliminal and prime or $A \cong \mathbb{C}$. But since $A_1^* = \overline{G(A)}$, $A \ncong \mathbb{C}$.

(ii) $\Rightarrow$ (i). Suppose that A is antiliminal and prime and let $\varphi \in A_1^*$.

Case (I). Suppose that A is separable. By [5], 3.9.1 (c), A is primitive and so we may regard A as acting faithfully and irreducibly on a Hilbert space H. Since A is antiliminal, H is infinite-dimensional and $K(H) \nsubseteq A$. Hence $A \cap K(H) = \{0\}$, by irreducibility, and so by Theorem 3.1, φ is a w*-limit of vector functionals of A. Since A is acting irreducibly, the vector functionals of A are pure functionals and so $\varphi \in \overline{G(A)}$.

Case (II). Suppose that A is inseparable. Let

$$N = \{h \in A^* : |h(a_i) - \varphi(a_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\}$$

where $a_1, a_2, \ldots, a_n$ are arbitrary elements of A and $\varepsilon > 0$. To show that $\varphi \in \overline{G(A)}$ it is enough to show that $N \cap G(A) \neq \emptyset$. For this, define

$$B_0 = C^*(a_1, a_2, \ldots, a_n),$$

a separable C^*-subalgebra of A. By [4], Proposition 5 there exists a separable, antiliminal and prime (hence primitive) C^*-subalgebra B of A such that $B \supseteq B_0$. Therefore, by Case (I), $B_1^* = \overline{G(B)}$. Hence there exists $\rho \in G(B)$ such that

$$|\rho(a_i) - \varphi|B\,(a_i)| < \varepsilon, \quad 1 \leqslant i \leqslant n.$$

There exists $\widetilde{\rho} \in G(A)$ such that $\widetilde{\rho}|B = \rho$ (see Section 2). But

$$|\widetilde{\rho}(a_i) - \varphi(a_i)| = |\rho(a_i) - \varphi|B\,(a_i)| < \varepsilon, \quad 1 \leqslant i \leqslant n.$$

Therefore $\widetilde{\rho} \in N \cap G(A)$ and so the theorem follows. ∎

5. LIMITS OF PURE FUNCTIONALS

DEFINITION 5.1. Two pure functionals φ_1 and φ_2 of a C^*-algebra A are said to be *inequivalent* if the pure states $|\varphi_1|$ and $|\varphi_2|$ are inequivalent.

The next result extends [2], Theorem 1 to the case of pure functionals rather than pure states. The proof of ((1) $\Rightarrow$ (2)) is complicated by the lack of positivity.

THEOREM 5.2. *Let A be a C^*-algebra, let $\varphi_1, \varphi_2, \dots, \varphi_n$ be pairwise inequivalent pure functionals of A and let π_i be the* GNS *representations of $|\varphi_i|$ $(1 \leqslant i \leqslant n)$. Then the following are equivalent:*

(i) *there exist positive real numbers $t_1, t_2, \dots, t_n$ with unit sum, such that*

$$\sum_{i=1}^{n} t_i \varphi_i \in \overline{G(A)};$$

(ii) *there exists a net (π_α) in $\widehat{A}$ such that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$ for each i $(1 \leqslant i \leqslant n)$;*

(iii) *whenever $s_1, s_2, \dots, s_n$ are non-negative real numbers with unit sum,*

$$\sum_{i=1}^{n} s_i \varphi_i \in \overline{G(A)}.$$

Proof. (i) $\Rightarrow$ (ii). Let $\varphi = \sum\limits_{i=1}^{n} t_i \varphi_i$. Since $\varphi \in \overline{G(A)}$ there exists a net (φ_α) in $G(A)$ such that $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$. Let $\pi_\alpha = \pi_{|\varphi_\alpha|}$ and $i \in \{1, 2, \dots, n\}$. We show that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$. By reordering, we may assume that $i = 1$. Let V be an arbitrary open neighbourhood of π_1 in $\widehat{A}$. Then there exists a closed two sided ideal J of A such that $V = \widehat{J}$. Since $\pi_1(J) \neq \{0\}$ by [5], 2.4.9, $|\varphi_1|(J) \neq \{0\}$ and so $\varphi_1(J) \neq \{0\}$. So by reordering $\varphi_2, \varphi_3, \dots, \varphi_n$ we can assume that there exists an integer r with $1 \leqslant r \leqslant n$ such that

$$\varphi_i(J) \neq \{0\} \quad \text{when } 1 \leqslant i \leqslant r \quad \text{and} \quad \varphi_i(J) = \{0\} \quad \text{if } r < i \leqslant n.$$

Hence we can write

$$\varphi|J = \sum_{i=1}^{r} t_i \varphi_i|J.$$

Since $\varphi_i \in G(A)$ there exist unit vectors $\xi_i, \eta_i \in H_{\pi_i}$ such that

$$\varphi_i = \langle \pi_i(\cdot)\xi_i, \eta_i \rangle, \quad 1 \leqslant i \leqslant r.$$

Since $\pi_1|_J, \pi_2|_J, \dots, \pi_r|_J$ are inequivalent irreducible representations of J, by Kadison's transitivity theorem there exists $a \in J$ such that

$$\pi_1(a)\xi_1 = \eta_1 \quad \text{and} \quad \pi_i(a)\xi_i = 0, \quad i = 2, 3, \dots r.$$

Therefore

$$\varphi(a) = t_1 \langle \pi_1(a)\xi_1, \eta_1 \rangle = t_1 > 0$$

and so $\varphi(J) \neq 0$. Hence there exists α_0 such that, for $\alpha \geqslant \alpha_0$, $\varphi_\alpha(J) \neq 0$. So $|\varphi_\alpha|(J) \neq \{0\}$ and hence $\pi_\alpha \in V$ for $\alpha \geqslant \alpha_0$. Thus $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_1$.

(ii) $\Rightarrow$ (iii). Suppose that there is a net (π_α) in $\widehat{A}$ such that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$ $(1 \leqslant i \leqslant n)$, and let $s_1, s_2, \ldots, s_n$ be non-negative real numbers with unit sum. Let $\aleph$ be a base of w*-open neighbourhoods of zero in A^*. As in the proof of [2], Theorem 1 ((ii) $\Rightarrow$ (iii)) we obtain a net $(\sigma_N)_{N\in\aleph}$ in $\widehat{A}$ and for each $N \in \aleph$ a set $\{\xi_1^{(N)}, \xi_2^{(N)}, \ldots, \xi_n^{(N)}\}$ of unit vectors in H_{σ_N} such that

$$\langle \sigma_N(\cdot)\xi_i^{(N)}, \xi_i^{(N)}\rangle \underset{N}{\longrightarrow} |\varphi_i|, \quad 1 \leqslant i \leqslant n. \tag{5.1}$$

For $i \neq j$, $|\varphi_i|$ and $|\varphi_j|$ are inequivalent and so by [2], Lemma 2

$$\langle \xi_i^{(N)}, \xi_j^{(N)}\rangle \underset{N}{\longrightarrow} 0, \qquad \langle \sigma_N(\cdot)\xi_i^{(N)}, \xi_j^{(N)}\rangle \underset{N}{\longrightarrow} 0. \tag{5.2}$$

For $1 \leqslant i \leqslant n$, let u_i be a unitary element in $\widetilde{A}$ such that $\varphi_i = u_i|\varphi_i|$. For each $N \in \aleph$, let $\xi^{(N)} = \sum_{i=1}^{n} \sqrt{s_i}\xi_i^{(N)}$ and $\eta^{(N)} = \sum_{i=1}^{n} \sqrt{s_i}\widetilde{\sigma}_N(u_i^*)\xi_i^{(N)}$, where $\widetilde{\sigma}_N$ is the unique irreducible representation of $\widetilde{A}$ that extends σ_N. Then by (5.2),

$$\|\xi^{(N)}\|^2 \underset{N}{\longrightarrow} 1, \quad \|\eta^{(N)}\|^2 \underset{N}{\longrightarrow} 1. \tag{5.3}$$

So eventually $\|\xi^{(N)}\|, \|\eta^{(N)}\| > 0$ and we may form unit vectors

$$\xi_0^{(N)} = \xi^{(N)}/\|\xi^{(N)}\|, \quad \eta_0^{(N)} = \eta^{(N)}/\|\eta^{(N)}\|.$$

By (5.1) and (5.2),

$$\langle \sigma_N(\cdot)\xi_0^{(N)}, \eta_0^{(N)}\rangle \underset{N}{\longrightarrow} \sum_{i=1}^{n} s_i\varphi_i$$

and so we obtain that $\sum_{i=1}^{n} s_i\varphi_i \in \overline{G(A)}$.

(iii) $\Rightarrow$ (i). This is immediate. ∎

In the above theorem if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are pure states then the implication ((i) $\Rightarrow$ (ii)) holds without the assumption of inequivalence (see [2], p. 252). However, in contrast, we give an example to show that ((i) $\Rightarrow$ (ii)) may fail if the condition "inequivalent" is removed from the hypothesis of Theorem 5.2.

EXAMPLE 5.3. Let $A := C_0(\mathbb{R})$. Let φ_1, φ_2 be two inequivalent pure states of A. Define $\varphi_3 = -\varphi_1 \in G(A)$ and $\varphi_4 = -\varphi_2 \in G(A)$. Then

$$\frac{1}{4}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = 0 \in \overline{P(A)} \quad \text{(since } A \text{ is non-unital).}$$

But, since $\widehat{A}$ is Hausdorff, there is no net (π_α) convergent to all the $\pi_{|\varphi_i|}$ where $i = 1, 2, 3, 4$, i.e., to $\pi_{\varphi_1}, \pi_{\varphi_2}$.

COROLLARY 5.4. *Let A be a C^*-algebra and let φ_1, φ_2 be inequivalent pure functionals of A. Then the following are equivalent:*

(i) *there exists $t \in (0,1)$ such that $t\varphi_1 + (1-t)\varphi_2 \in \overline{G(A)}$;*

(ii) *$\pi_{|\varphi_1|}$ and $\pi_{|\varphi_2|}$ cannot be separated by disjoint open subsets of $\widehat{A}$;*

(iii) *for all $s \in [0,1]$, $s\varphi_1 + (1-s)\varphi_2 \in \overline{G(A)}$.*

Proof. This is immediate from Theorem 5.2. ∎

Our final result extends Theorem 5.2 to the case of a countably infinite set of pure functionals. Again, the lack of positivity makes the proof of ((i) $\Rightarrow$ (ii)) rather more difficult than in the case of pure states (see [2], Theorem 2).

THEOREM 5.5. *Let A be a C^*-algebra, let $(\varphi_i)_{i\geqslant 1}$ be a sequence of pairwise inequivalent pure functionals of A and let π_i be the* GNS *representations of $|\varphi_i|$ $(i \geqslant 1)$. Then the following are equivalent:*

(i) *there exists a sequence of positive real numbers (t_i) with unit sum, such that*

$$\sum_{i=1}^{\infty} t_i\varphi_i \in \overline{G(A)};$$

(ii) *there exists a net (π_α) in $\widehat{A}$ such that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$ for each i;*

(iii) *whenever (s_i) is a sequence of non-negative real numbers with unit sum,*

$$\sum_{i=1}^{\infty} s_i\varphi_i \in \overline{G(A)}.$$

Proof. (i) $\Rightarrow$ (ii). Let $\varphi = \sum\limits_{i=1}^{\infty} t_i\varphi_i$. Since $\varphi \in \overline{G(A)}$ there exists a net (φ_α) in $G(A)$ such that $\varphi_\alpha \underset{\alpha}{\longrightarrow} \varphi$. Let $\pi_\alpha = \pi_{|\varphi_\alpha|}$ and $i \geqslant 1$. We show that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$. By reordering, we may assume that $i = 1$. Let V be an arbitrary open neighbourhood of π_1 in $\widehat{A}$. Then there exists a closed two sided ideal J of A such that $V = \widehat{J}$. Since $\pi_1(J) \neq \{0\}$ by [5], 2.4.9, $|\varphi_1|(J) \neq \{0\}$ and so $\varphi_1(J) \neq \{0\}$. Since $\sum\limits_{i=1}^{\infty} t_i = 1$ there exists an integer N such that

$$\sum_{i=N+1}^{\infty} t_i < \frac{t_1}{2}.$$

By reordering $\varphi_2, \varphi_3, \ldots, \varphi_N$ we can assume that there exists an integer r with $1 \leqslant r \leqslant N$ such that $\varphi_i(J) \neq \{0\}$ when $1 \leqslant i \leqslant r$ and $\varphi_i(J) = \{0\}$ if $r < i \leqslant N$. Hence

$$\varphi|J = \sum_{i=1}^{r} t_i\varphi_i|J + \sum_{i=N+1}^{\infty} t_i\varphi_i|J.$$

Since $\varphi_i \in G(A)$ there exist unit vectors $\xi_i, \eta_i \in H_{\pi_i}$ such that

$$\varphi_i = \langle \pi_i(\cdot)\xi_i, \eta_i\rangle, \quad 1 \leqslant i \leqslant r.$$

We seek an element $b \in J$ with $\|b\| \leqslant 1$, $\varphi_1(b) = 1$ and $\varphi_i(b) = 0$ for $i = 2, 3, \ldots, r$. Since ξ_1, η_1 are unit vectors there exists a unitary element $u \in \widetilde{A}$ such that $\widetilde{\pi}_1(u)\xi_1 = \eta_1$. Define $\psi_i = \varphi_i(u\,\cdot)$. Then $\psi_i \in G(A)$ and $\pi_{|\psi_i|} = \pi_i$. Let

$$\psi = \sum_{i=1}^{\infty} t_i\psi_i = \sum_{i=1}^{\infty} t_i\varphi_i(u\cdot).$$

Let $a \in J$. Then $ua \in J$ and therefore

$$\psi(a) = \sum_{i=1}^{r} t_i\psi_i(a) + \sum_{i=N+1}^{\infty} t_i\psi_i(a).$$

Define $T_1 = I$ and if $r > 1$ define $T_i = 0$ for $i = 2, 3, \ldots, r$. Then by Kadison's transitivity theorem there exists a self-adjoint element $a \in J$ such that

$$\pi_1(a)\xi_1 = T_1\xi_1 = \xi_1, \quad \pi_i(a)\xi_i = T_i\xi_i = 0, \quad i = 2, 3, \ldots, r.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} -1 & \text{if } t < -1, \\ t & \text{if } t \in [-1, 1], \\ 1 & \text{if } t > 1, \end{cases}$$

and let $b = f(a)$.

Since $f(0) = 0$, $b \in J$. Also, since $1 \in \mathrm{Sp}(a)$, $\|b\| = 1$. Now, since $\pi_1(b)\xi_1 = \xi_1$ and $\pi_i(b)\xi_i = 0$ for $i = 2, 3, \ldots, r$, we have

$$\begin{aligned} \psi(b) &= t_1\langle\pi_1(b)\xi_1, \widetilde{\pi}_1(u^*)\eta_1\rangle + \sum_{i=2}^{r} t_i\langle\pi_i(ub)\xi_i, \eta_i\rangle + \sum_{i=N+1}^{\infty} t_i\psi_i(b) \\ &= t_1 + \sum_{i=N+1}^{\infty} t_i\psi_i(b). \end{aligned}$$

Hence

$$|\psi(b) - t_1| \leqslant \sum_{i=N+1}^{\infty} t_i\|b\| < \frac{t_1}{2}.$$

Therefore $\varphi(ub) = \psi(b) \neq 0$ and so $\varphi(J) \neq \{0\}$. Hence there exists α_0 such that for $\alpha \geqslant \alpha_0$,

$$\varphi_\alpha(J) \neq \{0\}.$$

So $|\varphi_\alpha|(J) \neq \{0\}$ and hence $\pi_\alpha \in V$ for $\alpha \geqslant \alpha_0$. Thus $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_1$.

(ii) $\Rightarrow$ (iii). Suppose that (π_α) is a net in $\widehat{A}$ such that $\pi_\alpha \underset{\alpha}{\longrightarrow} \pi_i$ for all i. Suppose that (s_i) is a sequence of non-negative real numbers with unit sum and let $\varphi = \sum\limits_{i=1}^{\infty} s_i\varphi_i$. By truncating and scaling the series for φ, we may approximate φ in norm by functionals which themselves lie in $\overline{G(A)}$ by Theorem 5.2. Hence $\varphi \in \overline{G(A)}$.

(iii) $\Rightarrow$ (i). This is immediate. ∎

REFERENCES

1. C.A. Akemann, B. Russo, Geometry of the unit sphere of a C^*-algebra and its dual, *Pacific J. Math.* **32**(1970), 575–585.
2. R.J. Archbold, Limits of pure states, *Proc. Edinburgh Math Soc.* **32**(1989), 249–254.
3. R.J. Archbold, F.W. Shultz, Characterization of C^*-algebras with continuous trace by the properties of their pure states, *Pacific J. Math.* **136**(1989), 1–13.
4. C.J.K. Batty, Irreducible representations of non-separable C^*-algebras, *Rocky Mountain J. Math.* **13**(1984), 721–727.
5. J. Dixmier, *C^*-Algebras*, North-Holland, Amsterdam 1977.

6. E.G. EFFROS, Order ideals in a C^*-algebra and its dual, *Duke Math. J.* **30**(1963), 391–412.
7. J. GLIMM, A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.* **72**(1960), 216–244.
8. J. GLIMM, Type I C^*-algebras, *Ann. of Math.* **73**(1961), 572–612.
9. G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, London 1990.
10. A.G. ROBERTSON, Best approximation in von Neumann algebras, *Math. Proc. Cambridge Philos. Soc.* **81**(1977), 233–236.
11. J. TOMIYAMA, M. TAKESAKI, Application of fibre bundles to the certain class of C^*-algebras, *Tôhoku Math. J. (2)* **13**(1961), 498–523.

R.J. ARCHBOLD
Department of Mathematical Sciences
University of Aberdeen
King's College
Aberdeen AB24 3UE
SCOTLAND
E-mail: mth020@maths.abdn.ac.uk

M.H. SHAH
Department of Mathematical Sciences
University of Aberdeen
King's College
Aberdeen AB24 3UE
SCOTLAND
E-mail: mth124@maths.abdn.ac.uk

Received July 6, 1998.