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ASYMPTOTICS OF SUBCOERCIVE SEMIGROUPS ON NILPOTENT LIE GROUPS

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Abstract. One can associate asymptotic approximates G_∞ and H_∞ with each nilpotent Lie group G and pure m-th order weighted subcoercive operator H by a scaling limit. Then the semigroups S and $S^{(\infty)}$ generated by H and H_∞, on the spaces $L_p(G)$, $p \in [1,\infty]$, satisfy $\lim\limits_{t\to\infty} \|S_t - S_t^{(\infty)}\|_{p\to p} = 0$ if, and only if, $G = G_\infty$. If $G \neq G_\infty$ then $\lim\limits_{t\to\infty} \|M_f(S_t - S_t^{(\infty)})\|_{p\to p} = 0$ on the spaces $L_p(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G, and M_f denotes the operator of multiplication by any bounded function which vanishes at infinity.

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1. INTRODUCTION

The local structure of subelliptic semigroups acting on Lie groups is now well understood but many open questions remain concerning the global behaviour. Our aim is to analyze the asymptotic properties of the semigroup S generated by weighted subcoercive operators H acting on a nilpotent group G. There have been three approaches to the large-time behaviour. The first approach has been through bounds on the semigroup kernel. This has been particularly effective for second-order operators on groups with polynomial growth (see [10] for background and references to earlier work or [7] for recent results). The second approach is through an asymptotic expansion of the kernel. This method, proposed and developed by Nagel, Ricci and Stein ([9]), is restricted to nilpotent groups but applies to operators of arbitrary order with different weights in different directions. It identifies the leading term of the asymptotic expansion as the semigroup kernel

$K^{(\infty)}$ associated with the semigroup $S^{(\infty)}$ generated by an asymptotic approximant H_∞ of H which acts on a homogeneous group G_∞ related to G by a scaling limit. Subsequent terms in the expansion are given by derivatives of $K^{(\infty)}$. The third approach, which we analyze in the sequel, is closely related to the Nagel-Ricci-Stein method and is again restricted to nilpotent groups. It compares the asymptotic orbits of S and $S^{(\infty)}$ in a manner analogous to the comparison of the free and interacting evolutions in quantum-mechanical scattering theory. The analogy with scattering theory is, however, only superficial as the semigroups S and $S^{(\infty)}$ are asymptotically close in a uniform topology. A better analogy is given by homogenization theory of periodic systems on $\mathbb{R}^d$ (see, for example, [2]). In this theory the periodic dynamics is approximated by an homogenized dynamics which is defined by a complex non-linear averaging algorithm. The evolution semigroup of the homogenized system can then be shown to approximate the evolution semigroup of the periodic system, with respect to the norm on the $L_p(\mathbb{R}^d)$-spaces, for large times ([1]).

A simple example of semigroup comparison in a uniform topology is given by the Heisenberg group G. The Lie algebra of this group has a basis a_1, a_2, a_3 satisfying $[a_1, a_2] = a_3$ with all other commutators zero. Let $H = -A_1^2 - A_2^2 - A_3^2$ denote the Laplacian formed from the representatives A_i of the a_i in the left regular representation of G on $L_2(G)$. The asymptotic approximants are given by $G_\infty = G$ and $H_\infty = -A_1^2 - A_2^2$, i.e., the sublaplacian of the algebraic basis a_1, a_2. Then one verifies by a straightforward calculation that the corresponding semigroups satisfy $\lim_{t\to\infty} \|S_t - S_t^{(\infty)}\|_{2\to 2} = 0$. Thus the evolutions corresponding to the two distinct dynamics are asymptotically close with respect to the L_2-norm, and even with respect to the L_p-norms. (For further details see Examples 2.10 and 4.3 below.) The interest in this conclusion lies with the uniform convergence.

In this example G_∞ coincides with G but in general $G_\infty \neq G$ and this introduces an interesting dichotomy of behaviour which is a central focus of the following analysis. If $G = G_\infty$ then the semigroups S and $S^{(\infty)}$ can be compared on the spaces $L_p(G)$ but if $G \neq G_\infty$ one can still compare S and $S^{(\infty)}$ by pulling back to the L_p-spaces over the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_\infty$ because these coincide as vector spaces. One of our principal results is that the difference of the pulled back semigroups converges to zero uniformly on L_p as $t \to \infty$ if, and only if, $G = G_\infty$. Note that kernel bounds immediately imply that each of the semigroups converges strongly to zero as $t \to \infty$ and hence the difference also converges to zero. But the uniform convergence is a much stronger statement about the comparability of the asymptotic orbits of the two dynamical semigroups. Even if $G \neq G_\infty$ the asymptotic orbits are uniformly close locally, but not globally. More specifically, if the difference of the pulled back semigroups is multiplied with any bounded measurable function which vanishes at infinity, then the product tends to zero uniformly as $t \to \infty$ on any L_p-space. In order to formulate our results more precisely it is necessary to introduce some definitions and notation.

Let G be a connected, simply connected, d-dimensional nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $a_1, \ldots, a_{d'}$ an algebraic basis of $\mathfrak{g}$, i.e., a set of linearly independent elements which together with their multi-commutators span $\mathfrak{g}$. Moreover, let $w_1, \ldots, w_{d'} \in \mathbb{N}$ be weights associated with the different directions in $\mathfrak{g}$. The algebraic basis with these weights is called a *weighted algebraic basis*. For further details of these and subsequent definitions we refer to [4] and [5].

We need the following multi-index notation for commutators and products. If $N \in \mathbb{N}$ set

$$J(N) = \bigcup_{n=0}^{\infty} \{1, \ldots, N\}^n \quad \text{and} \quad J^+(N) = \bigcup_{n=1}^{\infty} \{1, \ldots, N\}^n.$$

Then for $\alpha = (i_1, \ldots, i_n) \in J(d')$ set the unweighted length $|\alpha| = n$, the weighted length $\|\alpha\| = w_{i_1} + \cdots + w_{i_n}$ and, if $n \geqslant 1$, introduce the multi-commutator

$$a_{[\alpha]} = [a_{i_1}, [\ldots [a_{i_{n-1}}, a_{i_n}] \ldots]]$$

of weighted order $\|\alpha\|$. Next for each $k \in \mathbb{N}$ let

$$\mathfrak{g}^{(k)} = \operatorname{span}\{a_{[\alpha]} : \alpha \in J^+(d'),\, \|\alpha\| \geqslant k\} \tag{1.1}$$

be the ideal spanned by all multi-commutators of order at least k. Since $\mathfrak{g}$ is nilpotent, there exists a unique $r \in \mathbb{N}$ such that $\mathfrak{g}^{(r)} \neq \{0\}$, but $\mathfrak{g}^{(r+1)} = \{0\}$. We call r the *weighted rank* of the Lie algebra $\mathfrak{g}$ given the weighted algebraic basis $a_1, \ldots, a_{d'}$.

For $k \in \mathbb{N}$, let $\mathfrak{a}_k$ be a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}^{(k)} = \mathfrak{g}^{(k+1)} \oplus \mathfrak{a}_k$ and hence $\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{a}_k$. Next, for all $t > 0$ introduce the linear maps $\gamma_t : \mathfrak{g} \to \mathfrak{g}$ such that $\gamma_t(a) = t^k a$ for all $a \in \mathfrak{a}_k$ and $k \in \mathbb{N}$. Moreover, define the Lie bracket $[\,\cdot\,,\,\cdot\,]_t : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$[a, b]_t = \gamma_t^{-1}[\gamma_t(a), \gamma_t(b)].$$

Then the Lie bracket $[\,\cdot\,,\,\cdot\,]_\infty = \lim_{t\to\infty} [\,\cdot\,,\,\cdot\,]_t$ exists and

$$[\mathfrak{a}_k, \mathfrak{a}_l]_\infty \subseteq \mathfrak{a}_{k+l}$$

for all $k, l \in \mathbb{N}$. The Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,]_\infty)$ is homogeneous with respect to the group of dilations used in the construction and the graded subspaces $\mathfrak{a}_k$ correspond to the eigenspaces of the dilations. We use the shorthand notation $\mathfrak{g}_\infty(\mathfrak{a})$ for $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,]_\infty)$ and call $\mathfrak{g}_\infty(\mathfrak{a})$ an *asymptotic Lie algebra*. The definition of $\mathfrak{g}_\infty(\mathfrak{a})$ clearly depends on the choice of the family $\mathfrak{a}$ of subspaces $\mathfrak{a}_k$, but different choices lead to isomorphic asymptotic Lie algebras. If the particular choice of $\mathfrak{g}_\infty(\mathfrak{a})$ within the set of asymptotic Lie algebras is not significant, we simplify the notation by writing $\mathfrak{g}_\infty$.

Next, let $b_1, \ldots, b_d$ be a vector space basis for $\mathfrak{g}$ passing through $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ and with order respecting the order of the $\mathfrak{a}_k$, i.e., if $d_{k_l} = \dim \mathfrak{a}_l$ then $b_1, \ldots, b_{k_1}$ is a basis of $\mathfrak{a}_1$, $b_{d_{k_1}+1}, \ldots, b_{d_{k_1+k_2}}$ a basis of $\mathfrak{a}_2$ etc. Assign weights $v_1, \ldots, v_d$ where $v_i = k$ if $b_i \in \mathfrak{a}_k$. Then, by definition,

$$[b_i, b_j]_\infty = \pi_{v_i+v_j}([b_i, b_j])$$

for all $i, j \in \{1, \ldots, d\}$, where $\pi_k : \mathfrak{g} \to \mathfrak{a}_k$ is the projection onto the k-th component of the decomposition $\mathfrak{g} = \bigoplus_{l=1}^{\infty} \mathfrak{a}_l$. We define the modulus $|\cdot|$ on $\mathfrak{g}$ by

$$\Big|\sum_{i=1}^{d} \xi_i b_i\Big|^{2v} = \sum_{i=1}^{d} |\xi_i|^{2v/v_i},$$

where $v = \mathrm{lcm}(1, \ldots, r)$. Finally, if $b = \sum_{i=1}^{d} \xi_i b_i \in \mathfrak{g}$ and $\alpha = (i_1, \ldots, i_n) \in J(d)$ set $b^\alpha = \xi_{i_1} \ldots \xi_{i_n}$.

Let U be a continuous representation of G in a Banach space $\mathcal{X}$. If $a \in \mathfrak{g}$ let $\mathrm{d}U(a)$ be the generator of the one-parameter group $t \mapsto U(\exp(-ta))$. Then, set $A_i = \mathrm{d}U(a_i)$ for $i \in \{1, \ldots, d'\}$ and use the multi-index notation $A^\alpha = A_{i_1} \cdots A_{i_n}$ for $\alpha = (i_1, \ldots, i_n) \in J(d')$. For each $n \in \mathbb{N} \setminus 0$ set $\mathcal{X}'_n = \mathcal{X}'_n(U) = \bigcap_{\|\alpha\| \leqslant n} D(A^\alpha)$ with norm

$$\|x\|'_n = \|x\|'_{U,n} = \max_{\substack{\alpha \in J(d') \\ \|\alpha\| \leqslant n}} \|A^\alpha x\|.$$

Further set $\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} \mathcal{X}'_n$. The left regular representation of G on a function space is denoted by L, or L_G, and the spaces and norms associated with the left regular representation on $L_p(G)$ are denoted by $L'_{p,n}$ and $\|\cdot\|'_{p,n}$ etc.

Let $m \in \mathbb{N}$. Then a *form of order* m is a function $C : J(d') \to \mathbb{C}$ such that $C(\alpha) = 0$ for all $\alpha \in J(d')$ with $\|\alpha\| > m$ and, moreover, there exists an α with $\|\alpha\| = m$ and $C(\alpha) \neq 0$. The form is called *homogeneous of order* m if, in addition, $C(\alpha) = 0$ for all α with $\|\alpha\| < m$.

The *adjoint form* $C^\dagger$ is defined by $C^\dagger(\alpha) = (-1)^{|\alpha|}\overline{C(\alpha_*)}$ where α_* is the reverse of α, i.e., if $\alpha = (i_1, \ldots, i_n)$ then $\alpha_* = (i_n, \ldots, i_1)$, and the bar denotes complex conjugation. Moreover, C is called *self-adjoint* if $C = C^\dagger$. In the sequel we write $c_\alpha = C(\alpha)$.

Given the representation U we consider the m-th order operator

$$\mathrm{d}U(C) = \sum_{\alpha \in J(d')} c_\alpha A^\alpha$$

with domain $D(\mathrm{d}U(C)) = \mathcal{X}'_m$. The form C is called a *G-weighted subcoercive form* and the operator $\mathrm{d}U(C)$ a *G-weighted subcoercive operator* if first $m \in 2w_i\mathbb{N}$ for all $i \in \{1, \ldots, d'\}$ and secondly there exist $\mu > 0$ and $\nu \in \mathbb{R}$ such that

$$\mathrm{Re}(\varphi, \mathrm{d}L_G(C)\varphi) \geqslant \mu(\|\varphi\|'_{2,m/2})^2 - \nu\|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(G)$, i.e., the operator $\mathrm{d}L_G(C)$ satisfies a Gårding inequality on $L_2(G)$. (For many equivalent descriptions of G-weighted subcoercive forms we refer to [5], Sections 4 and 10.) It then follows from Theorem 1.1 of [5] that the closure $\overline{\mathrm{d}U(C)}$ generates a holomorphic semigroup on $\mathcal{X}$. Moreover, this semigroup has a smooth, rapidly decreasing kernel.

Unfortunately, we need a slightly stronger condition on the coefficients of the operator. Let $\widetilde{\mathfrak{g}} = \mathfrak{g}(d', r, w_1, \ldots, w_{d'})$ be the weighted nilpotent Lie algebra with d' generators $\widetilde{a}_1, \ldots, \widetilde{a}_{d'}$ and weights $w_1, \ldots, w_{d'}$ which is free of step r, i.e., it is equal to the quotient $\mathfrak{G}/I$ where $\mathfrak{G}$ is the free Lie algebra in d' generators, with the i-th generator given the weight w_i, and I is the ideal spanned by the multi-commutators of weighted order strictly larger than r. (See also [9] and [5], Example 2.7.) Let $\widetilde{G}$ be the connected, simply connected, Lie group with Lie algebra $\widetilde{\mathfrak{g}}$. Throughout the sequel we assume that C is a homogeneous m-th order $\widetilde{G}$-weighted subcoercive form. Then, it follows from [5], Proposition 11.3, that

C is also a G-weighted subcoercive form. Let K be the kernel of the semigroup S generated by the closure of the operator $H = \mathrm{d}L_G(C)$. Then K is a smooth rapidly decreasing function on G.

Next, we need analogous concepts associated with the asymptotic Lie algebra $\mathfrak{g}_\infty (= \mathfrak{g}_\infty(\mathfrak{a}))$. Let G_∞ be the connected, simply connected, Lie group with Lie algebra $\mathfrak{g}_\infty$. Define H_∞ by

$$H_\infty = \sum_{\alpha \in J(d')} c_\alpha \mathrm{d}L_{G_\infty}(\overline{a}^\alpha)$$

where $\mathrm{d}L_{G_\infty}(\overline{a}^\alpha) = \mathrm{d}L_{G_\infty}(\overline{a}_1) \cdots \mathrm{d}L_{G_\infty}(\overline{a}_n)$ if $\alpha = (i_1, \ldots, i_n)$ and $\overline{a}_i = \pi_{w_i}(a_i)$ for all $i \in \{1, \ldots, d'\}$. The domain of H_∞ equals $D(H_\infty) = \bigcap_{\|\alpha\| \leqslant m} D(\mathrm{d}L_{G_\infty}(\overline{a}^\alpha))$. Since the $\overline{a}_i$ do not necessarily form an algebraic basis of $\mathfrak{g}_\infty$, e.g., some of the $\overline{a}_i$ could be zero, one has to exercise some caution. We shall show in Section 3 that the operator H_∞ is a G_∞-weighted subcoercive operator with respect to a different form and weighted algebraic basis in $\mathfrak{g}_\infty$. It then follows from [5] that the closure of H_∞ generates a holomorphic semigroup $S^{(\infty)}$ with a smooth kernel $K^{(\infty)}$ on G_∞.

The first theorem compares the kernels K and $K^{(\infty)}$ together with their subelliptic derivatives. For $i \in \{1, \ldots, d'\}$ set $\overline{A}_i^{(\infty)} = \mathrm{d}L_{G_\infty}(\overline{a}_i) = \mathrm{d}L_{G_\infty}(\pi_{w_i}(a_i))$. Moreover, set $D = \sum_{i=1}^{d} v_i$.

Theorem 1.1. *For all $\alpha \in J(d')$ there exist $c, \tau > 0$ such that*

$$\left|(A^\alpha K_t)(\exp a) - (\overline{A}^{(\infty)\alpha} K_t^{(\infty)})(\exp_\infty a)\right| \leqslant c t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t \geqslant 1$ and $a \in \mathfrak{g}$.

Here $\exp_\infty : \mathfrak{g}_\infty \to G_\infty$ is the exponential map and the estimates are valid for all possible choices of $\mathfrak{g}_\infty$. Moreover, we assume that the Haar measure on G (respectively G_∞) is normalized such that it is the image measure of the Lebesgue measure on $\mathfrak{g} = \mathfrak{g}_\infty$ under exp (respectively $\exp_\infty$). For a special class of $\mathfrak{g}_\infty(\mathfrak{a})$ (see Section 3) Nagel-Ricci-Stein ([9]) showed that $a \mapsto (\overline{A}^{(\infty)\alpha} K_t^{(\infty)})(\exp_\infty a)$ is the first term in the asymptotic expansion of $a \mapsto (A^\alpha K_t)(\exp a)$ in powers of $t^{-1/m}$. Theorem 1.1 establishes that the difference of these kernels is bounded by a Gaussian times $t^{-1/m}$ for all large t.

If $\mathfrak{g} = \mathfrak{g}_\infty$ this immediately implies that the semigroup S_t converges uniformly to $S^{(\infty)}$.

Theorem 1.2. *If for a particular choice of $\mathfrak{a}$ one has $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras, then the semigroup $S^{(\infty)}$ corresponding to this choice satisfies $\lim_{t\to\infty} \|S_t - S_t^{(\infty)}\|_{p\to p} = 0$ for all $p \in [1, \infty]$. More specifically, there exists a $c > 0$ such that*

$$\|S_t - S_t^{(\infty)}\|_{p\to p} \leqslant c t^{-1/m}$$

uniformly for all $t \geqslant 1$ and $p \in [1, \infty]$.

We next consider the general case in which the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_\infty(\mathfrak{a})$ are distinct. Since $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as vector spaces, for all possible choices of $\mathfrak{a}$ one can use the exponential maps to compare S and $S^{(\infty)}$. Define $\widehat{S}_t, \widehat{S}_t^{(\infty)} : L_p(\mathfrak{g}) \to L_p(\mathfrak{g})$ for each $t > 0$ and $p \in [1,\infty]$ by

$$(\widehat{S}_t\varphi)(a) = \big(S_t(\varphi \circ \log)\big)(\exp a)$$
$$(\widehat{S}_t^{(\infty)}\varphi)(a) = \big(S_t^{(\infty)}(\varphi \circ \log_\infty)\big)(\exp_\infty a)$$

for all $\varphi \in L_p(\mathfrak{g})$. Here log and $\log_\infty$ are the inverse of exp and $\exp_\infty$, respectively.

Theorem 1.3. *If $\mathfrak{g} \neq \mathfrak{g}_\infty$ as Lie algebras, then there is a $b > 0$ such that*

$$\liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}_t^{(\infty)}\|_{p\to p} \geqslant b$$

for all $p \in [1,\infty]$. Moreover, if C is self-adjoint one may choose $b = 1$.

It follows immediately from these results that

$$\lim_{t\to\infty} \|\widehat{S}_t - \widehat{S}_t^{(\infty)}\|_{p\to p} = 0$$

if, and only if, $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras. Nevertheless, the uniform convergence of S to $S^{(\infty)}$ is very nearly true. For any bounded measurable function $f : \mathfrak{g} \to \mathbb{C}$, define the multiplication operator M_f on $L_p(\mathfrak{g})$ by

$$(M_f\varphi)(a) = f(a)\varphi(a).$$

We say that f vanishes at infinity if for each $\varepsilon > 0$ there exists a compact set $\Omega \subset \mathfrak{g}$ such that $|f(a)| < \varepsilon$ for all $a \in \mathfrak{g} \setminus \Omega$.

Theorem 1.4. *If $f : \mathfrak{g} \to \mathbb{C}$ is a bounded measurable function which vanishes at infinity then*

$$\lim_{t\to\infty} \|M_f(\widehat{S}_t - \widehat{S}_t^{(\infty)})\|_{p\to p} = 0$$

uniformly for all $p \in [1,\infty]$.

We illustrate the theorems with an example.

Example 1.5. Let $\mathfrak{g}$ be the five dimensional nilpotent Lie algebra with basis $a_1, \ldots, a_5$ and commutation relations $[a_2, a_3] = a_4$, $[a_2, a_4] = a_5$ and $[a_1, a_2] = a_5$. Give all directions weight one, i.e., $w_i = 1$ for all $i \in \{1, \ldots, 5\}$. Then $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(2)} = \operatorname{span}\{a_4, a_5\}$ and $\mathfrak{g}^{(3)} = \operatorname{span}\{a_5\}$. We choose $\mathfrak{a}_1 = \operatorname{span}\{a_1, a_2, a_3\}$, $\mathfrak{a}_2 = \operatorname{span}\{a_4\}$ and $\mathfrak{a}_3 = \operatorname{span}\{a_5\}$ and basis $b_i = a_i$ for all i. Then $[b_1, b_2]_\infty = \lim_{t\to\infty} \gamma_t^{-1}[\gamma_t(b_1), \gamma_t(b_2)] = \lim_{t\to\infty} \gamma_t^{-1}(t^2 b_5) = 0 \neq [b_1, b_2]$. So $[\,\cdot\,,\,\cdot\,]_\infty \neq [\,\cdot\,,\,\cdot\,]$. According to Theorem 1.3, one has $\liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}_t^{(\infty)}\|_{p\to p} \neq 0$ for any $p \in [1,\infty]$, for any choice of the operator H. On the other hand, one has $\lim_{t\to\infty} \|M_f(\widehat{S}_t - \widehat{S}_t^{(\infty)})\|_{p\to p} = 0$ for any bounded measurable function $f : \mathfrak{g} \to \mathbb{C}$ which vanishes at infinity.

The asymptotic estimates on the semigroup S will be deduced from estimates on the kernel K. The initial kernel estimates are derived from an asymptotic expansion of K, in terms of the kernel $K^{(\infty)}$ of $S^{(\infty)}$, given by Nagel, Ricci and Stein ([9]). Their procedure is based on comparison of G and G_∞ with the larger free group $\widetilde{G}$. A similar method was used in [6] to obtain Gaussian bounds on

K and its derivatives in the unweighted case via transference. The Nagel-Ricci-Stein analysis uses a particular type of asymptotic Lie algebra which has an extra form of homogeneity. Analysis of the general situation requires examination of the isomorphism relating the various asymptotic algebras. Combination of these techniques establish Theorem 1.1. The estimates of Theorem 1.4, when $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$, go beyond simple bounds on the difference of the kernels and require more detailed analysis of the algebraic structure. The relative difficulty of the two cases is analogous to the complexity of analysis of elliptic operators with variable coefficients in contrast to operators with constant coefficients.

In Section 2 we introduce the algebraic concepts required and recall various essential results from [9]. In Section 3 we give the full definition of the operator H_∞ and derive the estimates on the associated semigroup kernels. Then in Section 4 we give the detailed proofs of Theorems 1.3 and 1.4. We also discuss some similarities with the limit $t \to 0$.

2. ALGEBRAIC STRUCTURE

In this section we first examine a special choice of the $\mathfrak{a}_k$ which gives an intrinsic description of $\mathfrak{g}_\infty(\mathfrak{a})$ particularly suited to the derivation of asymptotic Gaussian kernel bounds. The definition is given in Section 3 of Nagel, Ricci and Stein ([9]). We repeat their construction and relate these special algebras to the general asymptotic Lie algebras. For the convenience of the reader we give new proofs for some of their results. Subsequently, we discuss some properties of general asymptotic Lie algebras, their relation with the special Nagel-Ricci-Stein class and the possible equality $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ which is significant for the asymptotic behaviour of the subcoercive semigroups.

Set $\widetilde{d} = \dim \widetilde{\mathfrak{g}}$. Let $(\widetilde{\gamma}_t)_{t>0}$ be the canonical dilations on the homogeneous Lie algebra $\widetilde{\mathfrak{g}}$ and for all $k \in \mathbb{N}$ set $\widetilde{\mathfrak{a}}_k = \{\widetilde{a} \in \widetilde{\mathfrak{g}} : \widetilde{\gamma}_t(\widetilde{a}) = t^k \widetilde{a} \text{ for all } t > 0\} = \operatorname{span}\{\widetilde{a}_{[\alpha]} : \|\alpha\| = k\}$. Then, if $\widetilde{\mathfrak{g}}^{(k)}$ are defined by (1.1) relative to $\widetilde{\mathfrak{g}}$, one has $\widetilde{\mathfrak{g}}^{(k)} = \widetilde{\mathfrak{g}}^{(k+1)} \oplus \widetilde{\mathfrak{a}}_k$ for all $k \in \mathbb{N}$. Let $\widetilde{\pi}_k : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{a}}_k$ be the projection. If $\Lambda : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the Lie algebra homomorphism such that $\Lambda(\widetilde{a}_i) = a_i$ for all $i \in \{1, \ldots, d'\}$ then it is not hard to see that

$$\mathfrak{g}^{(k)} = \Lambda(\widetilde{\mathfrak{g}}^{(k)}) \tag{2.1}$$

for all $k \in \{1, \ldots, r\}$. Let $\mathfrak{i} = \Lambda^{-1}(0)$ and define

$$\mathfrak{i}_\infty = \bigoplus_{k=1}^{r} \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}).$$

Since the restriction $\widetilde{\pi}_k\big|\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}$ of $\widetilde{\pi}_k$ has kernel $\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k+1)}$ and image $\widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)})$, it follows that $\dim \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) = \dim(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) - \dim(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k+1)})$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \dim \mathfrak{i}_\infty &= \sum_{k=1}^{r} \dim \pi_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) = \sum_{k=1}^{r} \dim(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) - \dim(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k+1)}) \\ &= \dim(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(1)}) = \dim \mathfrak{i}. \end{aligned} \tag{2.2}$$

LEMMA 2.1. ([9]) *The space* $\mathfrak{i}_\infty$ *is an ideal in* $\widetilde{\mathfrak{g}}$.

Proof. Let $j,k \in \{1,\ldots,r\}$, $w \in \widetilde{\mathfrak{a}}_j$ and $v \in \mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}$. Then $[w,\widetilde{\pi}_k v] \in \mathfrak{i}_\infty$ by the following argument. Obviously $[w,\widetilde{\pi}_k v] \in \widetilde{\mathfrak{a}}^{(j+k)}$ and $[w,v] \in \mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(j+k)}$ since $\mathfrak{i}$ is an ideal in $\widetilde{\mathfrak{g}}$. Moreover, since $v - \widetilde{\pi}_k v \in \widetilde{\mathfrak{g}}^{(k+1)}$ one has $[w,v] - [w,\widetilde{\pi}_k v] \in \widetilde{\mathfrak{g}}^{(j+k+1)}$. Therefore $[w,\widetilde{\pi}_k v] = \widetilde{\pi}_{j+k}[w,\widetilde{\pi}_k v] = \widetilde{\pi}_{j+k}[w,v] \in \widetilde{\pi}_{j+k}(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(j+k)}) \subseteq \mathfrak{i}_\infty$. ∎

LEMMA 2.2. ([9]) *If* $k \in \mathbb{N}$ *then* $\Lambda \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) \subseteq \mathfrak{g}^{(k+1)}$.

Proof. Let $v \in \mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}$. Since $\Lambda v = 0$ and $v - \widetilde{\pi}_k v \in \widetilde{\mathfrak{g}}^{(k+1)}$, it follows from (2.1) that $\Lambda \widetilde{\pi}_k v = \Lambda(\widetilde{\pi}_k v - v) \in \mathfrak{g}^{(k+1)}$. ∎

Next, for all $k \in \{1,\ldots,r\}$ let $\widetilde{\mathfrak{h}}_k$ be a vector subspace of $\widetilde{\mathfrak{a}}_k$ such that

$$\widetilde{\mathfrak{a}}_k = \widetilde{\mathfrak{h}}_k \oplus \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)})$$

and set $\widetilde{\mathfrak{h}} = \bigoplus_{k=1}^{r} \widetilde{\mathfrak{h}}_k$. Then $\dim \widetilde{\mathfrak{h}} = d$ by (2.2), since obviously

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \mathfrak{i}_\infty .$$

The second statement of the next lemma states that the same decomposition is also valid for the ideal $\mathfrak{i}$ instead of $\mathfrak{i}_\infty$.

LEMMA 2.3. ([9]) (i) *The restriction* $\Lambda|\widetilde{\mathfrak{h}} : \widetilde{\mathfrak{h}} \to \mathfrak{g}$ *is a bijection.*
(ii) $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \mathfrak{i}$.
(iii) $\mathfrak{g}^{(k)} = \bigoplus_{l=k}^{r} \Lambda(\widetilde{\mathfrak{h}}_l)$ *for all* $k \in \{1,\ldots,r\}$.

Proof. We first show that

$$\mathfrak{g}^{(k)} = \Lambda\Big(\bigoplus_{l=k}^{r} \widetilde{\mathfrak{h}}_l\Big) \tag{2.3}$$

for all $k \in \mathbb{N}$. This equality is trivial if $k \geqslant r+1$. Moreover, $\Lambda\Big(\bigoplus_{l=k}^{r} \widetilde{\mathfrak{h}}_l\Big) \subseteq \Lambda(\widetilde{\mathfrak{g}}^{(k)}) = \mathfrak{g}^{(k)}$ for all $k \in \mathbb{N}$. Now let $k \in \{1,\ldots,r\}$ and suppose that $\mathfrak{g}^{(k+1)} = \Lambda\Big(\bigoplus_{l=k+1}^{r} \widetilde{\mathfrak{h}}_l\Big)$. Since $\widetilde{\mathfrak{g}}^{(k)} = \widetilde{\mathfrak{h}}_k \oplus \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) \oplus \widetilde{\mathfrak{g}}^{(k+1)}$, it follows from Lemma 2.2 and the induction hypothesis that

$$\begin{aligned}
\mathfrak{g}^{(k)} &= \Lambda(\widetilde{\mathfrak{g}}^{(k)}) = \Lambda(\widetilde{\mathfrak{h}}_k) + \Lambda(\widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)})) + \Lambda(\widetilde{\mathfrak{g}}^{(k+1)}) \\
&\subseteq \Lambda(\widetilde{\mathfrak{h}}_k) + \mathfrak{g}^{(k+1)} + \mathfrak{g}^{(k+1)} = \Lambda(\widetilde{\mathfrak{h}}_k) + \Lambda\Big(\bigoplus_{l=k+1}^{r} \widetilde{\mathfrak{h}}_l\Big) = \Lambda\Big(\bigoplus_{l=k}^{r} \widetilde{\mathfrak{h}}_l\Big)
\end{aligned}$$

and (2.3) follows by induction. Setting $k = 1$ in (2.3) gives $\Lambda(\widetilde{\mathfrak{h}}) = \mathfrak{g}$. Since $\dim \widetilde{\mathfrak{h}} = \dim \widetilde{\mathfrak{g}} - \dim \mathfrak{i}_\infty = \dim \widetilde{\mathfrak{g}} - \dim \mathfrak{i} = \dim \mathfrak{g}$, statement (i) follows.

Since Λ is injective on $\widetilde{\mathfrak{h}}$ and $\mathfrak{i} = \ker \Lambda$, one has $\widetilde{\mathfrak{h}} \cap \mathfrak{i} = \{0\}$. Therefore statement (ii) follows from a dimension consideration.

Finally, the injectivity of Λ on $\widetilde{\mathfrak{h}}$ together with (2.3) yield statement (iii). ∎

Now the appropriate choice of the $\mathfrak{a}_k$ is evident. Set $\mathfrak{a}_k^{(I)} = \Lambda(\widetilde{\mathfrak{h}}_k)$ for each $k \in \{1, \ldots, r\}$. Then $\mathfrak{g}^{(k)} = \bigoplus_{l=k}^{r} \mathfrak{a}_l^{(I)}$. Define $\gamma_t^{(I)}$, $[\,\cdot\,,\,\cdot\,]_t^{(I)}$, $[\,\cdot\,,\,\cdot\,]_\infty^{(I)}$ and $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ with respect to the family of subspaces $\mathfrak{a}_k^{(I)}$. We call $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ an *ideal asymptotic Lie algebra*.

Define the linear map $\Lambda_\infty : \widetilde{\mathfrak{g}} \to \mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ by

$$\Lambda_\infty|\widetilde{\mathfrak{h}} = \Lambda|\widetilde{\mathfrak{h}} \quad \text{and} \quad \Lambda_\infty|\mathfrak{i}_\infty = 0.$$

Next introduce the projections $\pi_k^{(I)} : \mathfrak{g} \to \mathfrak{a}_k^{(I)}$ for all $k \in \{1, \ldots, r\}$.

One has the following connection between Λ and Λ_∞.

LEMMA 2.4. *If $k \in \{1, \ldots, r\}$ and $\widetilde{a} \in \widetilde{\mathfrak{a}}_k$ then $\pi_k^{(I)} \Lambda \widetilde{a} = \Lambda_\infty \widetilde{a}$.*

Proof. If $v \in \widetilde{\mathfrak{h}}_k$ then $\Lambda v \in \Lambda(\widetilde{\mathfrak{h}}_k) = \mathfrak{a}_k^{(I)}$. Therefore $\pi_k^{(I)} \Lambda v = \Lambda v = \Lambda_\infty v$. Alternatively, if $v \in \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)})$ then $\Lambda v \in \mathfrak{g}^{(k+1)}$ by Lemma 2.2. Hence $\pi_k^{(I)} \Lambda v = 0 = \Lambda_\infty v$. Now the lemma follows by linearity. ∎

PROPOSITION 2.5. ([9]) *The map Λ_∞ is a Lie algebra homomorphism from $\widetilde{\mathfrak{g}}$ onto $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$. Hence each $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ is isomorphic to $\widetilde{\mathfrak{g}}/\mathfrak{i}_\infty$ as Lie algebras.*

Proof. Let $j, k \in \{1, \ldots, r\}$, $v \in \widetilde{\mathfrak{a}}_j$ and $w \in \widetilde{\mathfrak{a}}_k$. Then it follows from Lemma 2.4 that

$$[\Lambda_\infty v, \Lambda_\infty w]_\infty^{(I)} = [\pi_j^{(I)} \Lambda v, \pi_k^{(I)} \Lambda w]_\infty^{(I)} = \pi_{j+k}^{(I)} [\pi_j^{(I)} \Lambda v, \pi_k^{(I)} \Lambda w] = \pi_{j+k}^{(I)} [\Lambda v, \Lambda w],$$

where the last equality follows because $[\pi_j^{(I)} \Lambda v, \pi_k^{(I)} \Lambda w] - [\Lambda v, \Lambda w] \in \mathfrak{g}^{(j+k+1)}$. Since Λ is a homomorphism, it follows that

$$[\Lambda_\infty v, \Lambda_\infty w]_\infty^{(I)} = \pi_{j+k}^{(I)} \Lambda([v, w]) = \Lambda_\infty([v, w])$$

by another application of Lemma 2.4. Thus Λ_∞ is a homomorphism from $\widetilde{\mathfrak{g}}$ to $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$. But $\Lambda_\infty(\widetilde{\mathfrak{g}}) = \Lambda_\infty(\widetilde{\mathfrak{h}}) = \Lambda(\widetilde{\mathfrak{h}}) = \mathfrak{g}$ by Lemma 2.3 (i). So Λ_∞ is onto. The second statement is easy. ∎

The next result establishes that there are asymptotic Lie algebras which are not ideal. The ideal asymptotic algebras are characterized by additional homogeneity properties.

LEMMA 2.6. *Let $\mathfrak{g}_\infty(\mathfrak{a})$ be an asymptotic Lie algebra. The following conditions are equivalent:*

(i) *$\mathfrak{g}_\infty(\mathfrak{a})$ is an ideal asymptotic Lie algebra.*

(ii) $\mathfrak{a}_k \subseteq \operatorname{span}\{a_{[\alpha]} : \alpha \in J(d'),\, \|\alpha\| = k\}$ *for all $k \in \mathbb{N}$.*

Proof. First note that

$$\Lambda(\widetilde{\mathfrak{a}}_k) = \operatorname{span}\left\{a_{[\alpha]} : \alpha \in J(d'),\, \|\alpha\| = k\right\}$$

for all $k \in \mathbb{N}$. But $\mathfrak{a}_k^{(I)} = \Lambda(\widetilde{\mathfrak{h}}_k) \subseteq \Lambda(\widetilde{\mathfrak{a}}_k)$ for all k and hence condition (i) implies condition (ii).

Next assume condition (ii) holds and let $k \in \mathbb{N}$. Then $\mathfrak{a}_k \subseteq \Lambda(\widetilde{\mathfrak{a}}_k)$ by assumption. Hence there exists a subspace $\mathfrak{h}_k^\sharp \subseteq \widetilde{\mathfrak{a}}_k$ such that $\dim \mathfrak{h}_k^\sharp = \dim \mathfrak{a}_k$

and $\Lambda(\mathfrak{h}_k^\sharp) = \mathfrak{a}_k$. Then the restriction $\Lambda|\mathfrak{h}_k^\sharp$ is injective. So if we can prove that $\mathfrak{h}_k^\sharp \cap \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) = \{0\}$, then $\mathfrak{h}_k^\sharp \oplus \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) = \widetilde{\mathfrak{a}}_k$ and $\mathfrak{g}_\infty(\mathfrak{a})$ is an ideal asymptotic Lie algebra.

Let $k \in \mathbb{N}$ and $a \in \mathfrak{h}_k^\sharp \cap \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)})$. Then $\Lambda(a) \in \Lambda(\mathfrak{h}_k^\sharp) = \mathfrak{a}_k$. Moreover, $\Lambda(a) \in \Lambda\widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) \subseteq \mathfrak{g}^{(k+1)}$ by Lemma 2.2. Hence $\Lambda(a) \in \mathfrak{a}_k \cap \mathfrak{g}^{(k+1)} = \{0\}$. Since $\Lambda|\mathfrak{h}_k^\sharp$ is injective, one deduces that $a = 0$. So $\mathfrak{h}_k^\sharp \cap \widetilde{\pi}_k(\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) = \{0\}$. ∎

The general asymptotic Lie algebra $\mathfrak{g}_\infty(\mathfrak{a})$ constructed in the introduction and the ideal asymptotic Lie algebra $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ are automatically isomorphic. In particular, the linear map $\Phi : \mathfrak{g}_\infty(\mathfrak{a}^{(I)}) \to \mathfrak{g}_\infty(\mathfrak{a})$ defined such that

$$\Phi(a) = \pi_k(a)$$

for all $k \in \mathbb{N}$ and $a \in \mathfrak{a}_k^{(I)}$ is an explicit isomorphism. This is established in the next lemma.

Now introduce $\overline{a}_i^{(I)} = \pi_{w_i}^{(I)} a_i$ for all $i \in \{1, \ldots, d'\}$ and define a linear map $\Psi : \mathfrak{g} \to \mathfrak{g}$ to be *super-homogeneous* if $\Psi(\mathfrak{g}^{(k)}) \subseteq \mathfrak{g}^{(k+1)}$ for all $k \in \mathbb{N}$.

LEMMA 2.7. (i) *The map Φ is a Lie algebra isomorphism.*
(ii) *The map $a \mapsto \Phi(a) - a$ is super-homogeneous.*
(iii) $\Phi(\Lambda_\infty \widetilde{a}_i) = \Phi(\overline{a}_i^{(I)}) = \overline{a}_i$ *for all* $i \in \{1, \ldots, d'\}$.

Proof. For the super-homogeneity, it suffices to show that $(\Phi - I)(\mathfrak{a}_k^{(I)}) \subseteq \mathfrak{g}^{(k+1)}$ for each $k \in \mathbb{N}$. If $k \in \mathbb{N}$ and $v \in \mathfrak{a}_k^{(I)}$, then $\Phi(v) - v = \pi_k(v) - v \in \mathfrak{g}^{(k+1)}$ since $v \in \mathfrak{g}^{(k)}$. This proves statement (ii). Moreover, Φ is surjective.

Let $j, k \in \{1, \ldots, r\}$, $v \in \mathfrak{a}_j^{(I)}$ and $w \in \mathfrak{a}_k^{(I)}$. Then

$$[\Phi(v), \Phi(w)]_\infty = [\pi_j v, \pi_k w]_\infty = \pi_{j+k}([\pi_j v, \pi_k w]) = \pi_{j+k}([v, w])$$

where the last equality holds because $[v, w] - [\pi_j v, \pi_k w] \in \mathfrak{g}^{(j+k+1)}$. On the other hand, $[v, w]_\infty^{(I)} \in \mathfrak{a}_{j+k}^{(I)}$ and therefore

$$\Phi([v, w]_\infty^{(I)}) = \pi_{j+k}([v, w]_\infty^{(I)}) = \pi_{j+k}([v, w])$$

using the fact that $[v, w] - [v, w]_\infty^{(I)} \in \mathfrak{g}^{(j+k+1)}$. This shows that Φ is a homomorphism from $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ to $\mathfrak{g}_\infty(\mathfrak{a})$. Since Φ is surjective, it follows that Φ is an isomorphism.

Next we prove statement (iii). The first equality follows from Lemma 2.4. Let $i \in \{1, \ldots, d'\}$. Since $\overline{a}_i^{(I)} \in \mathfrak{a}_{w_i}^{(I)}$ and $a_i - \overline{a}_i^{(I)} = a_i - \pi_{w_i}^{(I)} a_i \in \mathfrak{g}^{(w_i+1)}$, one deduces that

$$\Phi(\overline{a}_i^{(I)}) = \pi_{w_i}(\overline{a}_i^{(I)}) = \pi_{w_i}(a_i) = \overline{a}_i$$

and the second equality of statement (iii) is proved. ∎

The $\overline{a}_i$ used in the definition of the limit operator H_∞ do not form an algebraic basis as they are not necessarily independent. This problem can be circumvented as follows. Since $\overline{a}_i \in \bigcup_{k=1}^{r} \mathfrak{a}_k$ for all $i \in \{1,\ldots,d'\}$, there exist $d'' \in \{1,\ldots,d'\}$ and linearly independent $a_1^{(\infty)},\ldots,a_{d''}^{(\infty)} \in \bigcup_{k=1}^{r} \mathfrak{a}_k$ such that

$$\operatorname{span}\{a_1^{(\infty)},\ldots,a_{d''}^{(\infty)}\} = \operatorname{span}\{\overline{a}_1,\ldots,\overline{a}_{d'}\}.$$

For all $i \in \{1,\ldots,d''\}$ set $w_i^{(\infty)} = v_k$ if $a_i^{(\infty)} \in \mathfrak{a}_k$. Then $\gamma_t(a_i^{(\infty)}) = t^{w_i^{(\infty)}} a_i^{(\infty)}$ for all $t > 0$.

Lemma 2.8. *$a_1^{(\infty)},\ldots,a_{d''}^{(\infty)}$ is an algebraic basis for $\mathfrak{g}_\infty(\mathfrak{a})$.*

Proof. Let $\mathfrak{h}$ be the smallest Lie subalgebra of $\mathfrak{g}_\infty(\mathfrak{a})$ which contains $a_1^{(\infty)},\ldots,a_{d''}^{(\infty)}$. Then $\Phi(\Lambda_\infty \widetilde{a}_i) = \overline{a}_i \in \mathfrak{h}$ for all $i \in \{1,\ldots,d'\}$ and hence $\mathfrak{g}_\infty(\mathfrak{a}) = \Phi(\Lambda_\infty \widetilde{\mathfrak{g}}) \subseteq \mathfrak{h}$. Therefore $a_1^{(\infty)},\ldots,a_{d''}^{(\infty)}$ is a weighted algebraic basis for $\mathfrak{g}_\infty(\mathfrak{a})$ with weights $w_1^{(\infty)},\ldots,w_{d''}^{(\infty)}$. ∎

Finally we make three remarks about the possible identification $\mathfrak{g}_\infty(\mathfrak{a}) = \mathfrak{g}$ as Lie algebras.

Lemma 2.9. *Let $\mathfrak{a} = (\mathfrak{a}_k)$ be a family of subspaces of $\mathfrak{g}$ such that $\mathfrak{g}^{(k)} = \mathfrak{g}^{(k+1)} \oplus \mathfrak{a}_k$ for $k \in \{1,\ldots,r\}$. The following are equivalent:*

(i) *The subspaces $\mathfrak{a}$ are a graded family of $\mathfrak{g}$, i.e., $[\mathfrak{a}_j,\mathfrak{a}_k] \subseteq \mathfrak{a}_{j+k}$ for all $j,k \in \mathbb{N}$.*

(ii) *$\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras.*

Proof. It follows by construction that $\mathfrak{a}$ is a graded family of $\mathfrak{g}_\infty(\mathfrak{a})$ and hence condition (ii) implies condition (i). Conversely, the grading property of $\mathfrak{a}$ implies $[a,b]_t = [a,b]$ for all $a,b \in \mathfrak{g}$ and all $t > 0$. Hence $[a,b]_\infty = [a,b]$ for all $a,b \in \mathfrak{g}$ and condition (ii) is valid. ∎

There are, however, examples for which no choice of $\mathfrak{a}$ ensures $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras.

Example 2.10. Let $\mathfrak{g}$ be the three-dimensional Heisenberg algebra with basis a_1,a_2,a_3 satisfying $[a_1,a_2] = a_3$ and all other commutators zero. Consider the algebraic basis a_1,a_2,a_3 with weights $1,1,3$. Then $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(2)} = \mathfrak{g}^{(3)} = \operatorname{span}\{a_3\}$. Hence for any possible choice of $\mathfrak{a}$ there exist $\lambda,\mu \in \mathbb{R}$ such that

$$\mathfrak{a}_1 = \operatorname{span}\{a_1 + \lambda a_3, a_2 + \mu a_3\},$$

$\mathfrak{a}_2 = \{0\}$ and $\mathfrak{a}_3 = \operatorname{span}\{a_3\}$. Then $[\mathfrak{a}_1,\mathfrak{a}_1] \not\subseteq \mathfrak{a}_2$ so no choice of $\mathfrak{a}$ is graded. Thus, there is no choice of $\mathfrak{a}$ such that $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras. Another way of verifying this is by the observation that the $\mathfrak{g}_\infty(\mathfrak{a})$ are abelian.

Even if one can choose $\mathfrak{a}$ such that $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras, it is not necessarily the case that $\mathfrak{g}_\infty(\mathfrak{a})$ is an ideal asymptotic Lie algebra.

EXAMPLE 2.11. Let $\mathfrak{g}$ be the Lie algebra of dimension five with basis $b_1, \ldots, b_5$ satisfying $[b_1, b_2] = b_3$, $[b_1, b_3] = b_4$ and all other commutators zero. Consider the algebraic basis $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3 + b_5$ with all weights equal to one. Then $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(2)} = \operatorname{span}\{b_3, b_4\}$, $\mathfrak{g}^{(3)} = \operatorname{span}\{b_4\}$ and $\mathfrak{g}^{(k)} = \{0\}$ for $k > 3$. Choosing $\mathfrak{a}_1 = \operatorname{span}\{b_1, b_2, b_5\}$, $\mathfrak{a}_2 = \operatorname{span}\{b_3\}$, $\mathfrak{a}_3 = \operatorname{span}\{b_4\}$ and $\mathfrak{a}_k = \{0\}$ for $k > 3$, one has $[\mathfrak{a}_k, \mathfrak{a}_l] \subseteq \mathfrak{a}_{k+l}$ for all $k, l \in \mathbb{N}$ and hence $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras. But $\mathfrak{g}_\infty(\mathfrak{a})$ is not an ideal asymptotic Lie algebra because $\mathfrak{a}_1 \not\subseteq \operatorname{span}\{a_{[\alpha]} : |\alpha| = 1\}$, i.e., the criterion of Lemma 2.6 is not satisfied.

Now let $\{\mathfrak{a}_k\}$ satisfy the equivalent conditions of Lemma 2.6 and consider the associated asymptotic Lie algebra $\mathfrak{g}_\infty(\mathfrak{a})$. Since $\mathfrak{a}_1 \subseteq \operatorname{span}\{a_1, a_2, a_3\}$ and $\dim \mathfrak{a}_1 = 3$ one must have $\mathfrak{a}_1 = \operatorname{span}\{a_1, a_2, a_3\}$. Then

$$[a_1, a_3]_\infty = \pi_2([a_1, a_3]) = \pi_2(b_4).$$

But $b_4 \in \mathfrak{g}^{(3)}$ so $\pi_2(b_4) = 0$. Hence

$$[a_1, a_3]_\infty = 0 \neq b_4 = [a_1, a_3]$$

and consequently $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras.

3. KERNEL ESTIMATES

In this section we derive the asymptotic estimates on the semigroup kernels, i.e., we establish Theorem 1.1. First, however, we have to give a proper definition of the operator H_∞ which implies that the semigroup $S^{(\infty)}$ and the kernel $K^{(\infty)}$ exist. We use the notation of Sections 1 and 2. Let G_∞, $G_\infty^{(I)}$ and $\widetilde{G}$ be the connected, simply connected, Lie groups with Lie algebras $\mathfrak{g}_\infty(\mathfrak{a})$, $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ and $\widetilde{\mathfrak{g}}$. We denote the exponential maps by $\exp_\infty$, $\exp_\infty^{(I)}$ and $\widetilde{\exp}$, respectively, with similar notation for the logarithms.

Introduce the unitary representation U of $\widetilde{G}$ in the Hilbert space $L_2(G_\infty)$ by $U(\widetilde{\exp}\, \widetilde{a}) = L_{G_\infty}(\exp_\infty \Phi \Lambda_\infty \widetilde{a})$. Note that $dU(\widetilde{a}_i) = dL_{G_\infty}(\overline{a}_i)$ for all $i \in \{1, \ldots, d'\}$ by Lemma 2.7 (iii). Therefore $H_\infty = dU(C)$. Let $d'', a_i^{(\infty)}$ and $w_i^{(\infty)}$ be as in Section 2. Then $a_1^{(\infty)}, \ldots, a_{d''}^{(\infty)}$ is a weighted algebraic basis for $\mathfrak{g}_\infty(\mathfrak{a})$ by Lemma 2.8. Hence $(L_2(G_\infty))_\infty(U) = L_{2,\infty}(G_\infty)$. By the construction of the $a_i^{(\infty)}$, there exist $c_\beta^{(\infty)} \in \mathbb{C}$ such that

$$dU(C)\varphi = \sum_{\substack{\beta \in J(d'') \\ \|\beta\|_{w^{(\infty)}} \leqslant m}} c_\beta^{(\infty)} A^{(\infty)\beta} \varphi$$

for all $\varphi \in L_{2,\infty}(G_\infty)$, where $A_i^{(\infty)} = dL_{G_\infty}(a_i^{(\infty)})$ and $\|\beta\|_{w^{(\infty)}} = w_{i_1}^{(\infty)} + \cdots + w_{i_n}^{(\infty)}$ if $\beta = (i_1, \ldots, i_n) \in J(d'')$. Define the m-th order form $C^{(\infty)} : J(d'') \to \mathbb{C}$ by $C^{(\infty)}(\beta) = c_\beta^{(\infty)}$. Then $dU(C)\varphi = dL_{G_\infty}(C^{(\infty)})\varphi$ for all $\varphi \in L_{2,\infty}(G_\infty)$. Since U is a unitary representation it follows from [5], Theorem 9.2.III, that there exist $\mu, \nu > 0$ such that

$$\operatorname{Re}(\varphi, dL_{G_\infty}(C^{(\infty)}\varphi)) = \operatorname{Re}(\varphi, dU(C)\varphi) \geqslant \mu\, (\|\varphi\|'_{U,m/2})^2 - \nu \|\varphi\|_2^2$$

for all $\varphi \in L_{2,\infty}(G_\infty)$. But, by inspection, one deduces that $(L_2(G_\infty))'_{m/2}(U) = L'_{2,m/2}(G_\infty)$, where the last space is with respect to the weighted algebraic basis $a_1^{(\infty)}, \ldots, a_{d''}^{(\infty)}$. Moreover, there exists a $c > 0$ such that

$$c\|\varphi\|'_{L_{G_\infty},m/2} \leqslant \|\varphi\|'_{U,m/2}$$

for all $\varphi \in L'_{2,m/2}(G_\infty)$. Hence

$$\mathrm{Re}(\varphi, \mathrm{d}L_{G_\infty}(C^{(\infty)})\varphi) \geqslant \mu c^2 (\|\varphi\|'_{L_{G_\infty},m/2})^2 - \nu\, \|\varphi\|_2^2$$

for all $\varphi \in L_{2,\infty}(G_\infty)$ and $C^{(\infty)}$ is a G_∞-weighted subcoercive form. Thus it follows from [5] that H_∞ generates a holomorphic semigroup $S^{(\infty)}$ and that $S^{(\infty)}$ has a smooth kernel $K^{(\infty)}$ on G_∞. Define in a similar manner the operator

$$H_\infty^{(I)} = \sum_{\alpha \in J(d')} c_\alpha \mathrm{d}L_{G_\infty^{(I)}}(\overline{a}^{(I)\alpha}),$$

the semigroup $S^{(I,\infty)}$ and the kernel $K^{(I,\infty)}$ on $G_\infty^{(I)}$.

At this point, the asymptotic operators and the kernels are well-defined and we start with the proof of Theorem 1.1. This is based on the splitting

$$\begin{aligned}
&\big|(A^\alpha K_t)(\exp a) - (\overline{A}^{(\infty)\alpha} K_t^{(\infty)})(\exp_\infty a)\big| \\
&\qquad \leqslant \big|(A^\alpha K_t)(\exp a) - (\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)})(\exp_\infty^{(I)} a)\big| \\
&\qquad\quad + \big|(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)})(\exp_\infty^{(I)} a) - (\overline{A}^{(\infty)\alpha} K_t^{(\infty)})(\exp_\infty a)\big|,
\end{aligned} \tag{3.1}$$

where $\overline{A}_i^{(I,\infty)} = \mathrm{d}L_{G_\infty^{(I)}}(\overline{a}_i^{(I)})$ for all $i \in \{1, \ldots, d'\}$. The two terms are estimated separately. The estimate on the first term establishes the theorem for an ideal asymptotic Lie algebra. Its proof is based on a lemma which can be extracted from [9]. The bounds on the second term are a consequence of the super-homogeneity of Φ.

Let $\widetilde{H} = \mathrm{d}L_{\widetilde{G}}(C)$ and let $\widetilde{K}$ be the kernel of the semigroup $\widetilde{S}$ generated by the closure of $\widetilde{H}$. Let $\widetilde{b}_1, \ldots, \widetilde{b}_d$ be a basis for $\mathfrak{h}$ passing through $\widetilde{\mathfrak{h}}_1, \ldots, \widetilde{\mathfrak{h}}_r$ with order respecting the order of the $\widetilde{\mathfrak{h}}_k$. Set $\widetilde{d}_k = \dim \mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}$ for each $k \in \{1, \ldots, r+1\}$. Since $\mathfrak{i} = \mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(1)}$, there exists a basis $\widehat{b}_{d+1}, \ldots, \widehat{b}_{\tilde{d}}$ for $\mathfrak{i}$ such that $\widehat{b}_{\tilde{d}-\tilde{d}_k+1}, \ldots, \widehat{b}_{\tilde{d}}$ is a basis for $\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}$ for all $k \in \{1, \ldots, r\}$. If $\widetilde{d} - \widetilde{d}_k + 1 \leqslant i \leqslant \widetilde{d} - \widetilde{d}_{k+1}$, i.e., if $\widehat{b}_i \in (\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k)}) \backslash (\mathfrak{i} \cap \widetilde{\mathfrak{g}}^{(k+1)})$, define $\widetilde{b}_i = \widetilde{\pi}_k \widehat{b}_i$. Then,

$$\widehat{b}_i - \widetilde{b}_i \in \widetilde{\mathfrak{g}}^{(k+1)}. \tag{3.2}$$

Note that $\widetilde{b}_i \neq 0$. As a result, $\widetilde{b}_{d+1}, \ldots, \widetilde{b}_{\tilde{d}}$ are independent and form a basis for $\mathfrak{i}_\infty$. Hence $\widetilde{b}_1, \ldots, \widetilde{b}_d, \ldots, \widetilde{b}_{\tilde{d}}$ is a basis for $\widetilde{\mathfrak{g}}$. Set $\widetilde{v}_i = k$ if $\widetilde{b}_i \in \widetilde{\mathfrak{a}}_k$. Since $\dim \widetilde{\mathfrak{h}}_k = \dim \mathfrak{a}_k^{(I)} = \dim \mathfrak{a}_k$ for all k, it follows from the ordering of the basis b_i and the fact that the weights of b_i depend only on the $\dim \mathfrak{g}^{(k)}$ that $\widetilde{v}_i = v_i$ for all $i \in \{1, \ldots, d\}$. Define the modulus $|\cdot|$ on $\widetilde{\mathfrak{g}}$ by

$$\Big|\sum_{i=1}^{\tilde{d}} \xi_i \widetilde{b}_i\Big|^{2v} = \sum_{i=1}^{\tilde{d}} |\xi_i|^{2v/\widetilde{v}_i}.$$

Moreover, set $\widetilde{D} = \sum_{i=1}^{\tilde{d}} \widetilde{v}_i$. Since $\widetilde{H}$ is a homogeneous operator on the homogeneous group $\widetilde{G}$ it follows from [5], Proposition 5.5, that for all $\alpha \in J(\widetilde{d})$ there exist $c, \tau > 0$ such that

$$|(\widetilde{B}^\alpha \widetilde{K}_t)(\widetilde{\exp}\,\widetilde{a})| \leqslant c t^{-\tilde{D}/m} t^{-\|\alpha\|_{\tilde{v}}/m} e^{-\tau(|\tilde{a}|^m t^{-1})^{1/(m-1)}} \tag{3.3}$$

uniformly for all $t > 0$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$, where $\widetilde{B}_i = dL_{\tilde{G}}(\widetilde{b}_i)$ and $\|\alpha\|_{\tilde{v}} = \widetilde{v}_{i_1} + \cdots + \widetilde{v}_{i_n}$ if $\alpha = (i_1, \ldots, i_n)$. Set $b_i^{(I)} = \Lambda(\widetilde{b}_i)$ for all $i \in \{1, \ldots, d\}$. Then $b_1^{(I)}, \ldots, b_d^{(I)}$ is a basis for $\mathfrak{g}_\infty^{(I)}$.

Using the bases $b_1^{(I)}, \ldots, b_d^{(I)}$ and $\widetilde{b}_1, \ldots, \widetilde{b}_{\tilde{d}}$, we fix the Lebesgue measure on the vector spaces $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ and $\widetilde{\mathfrak{g}}$. Then the Haar measures on G, $G_\infty^{(I)}$ and $\widetilde{G}$ are normalized such that the exponential maps are measure preserving. Note that the restrictions of the maps Λ and Λ_∞ to $\widetilde{\mathfrak{h}}$ have Jacobian equal to one. Define the linear map $\Psi : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ such that

$$\Psi(\widetilde{b}_i) = \begin{cases} 0 & \text{if } i \in \{1, \ldots, d\}, \\ \widehat{b}_i - \widetilde{b}_i & \text{if } i \in \{d+1, \ldots, \widetilde{d}\}. \end{cases}$$

Then $\mathfrak{i} = \{\widetilde{b} + \Psi(\widetilde{b}) : \widetilde{b} \in \mathfrak{i}_\infty\}$. Moreover, the map Ψ is super-homogeneous, by (3.2).

The basic lemma relates the kernels $\widetilde{K}, K$ and $K^{(I,\infty)}$.

Lemma 3.1. *If $t > 0$ then*

$$(A^\alpha K_t)(\exp \Lambda\widetilde{a}) = \int_{\mathfrak{i}} d\widetilde{b}\, (\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b})) = \int_{\mathfrak{i}_\infty} db\, (\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b} + \Psi(\widetilde{b}))) \tag{3.4}$$

and

$$(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)})(\exp_\infty^{(I)} \Lambda_\infty \widetilde{a}) = \int_{\mathfrak{i}_\infty} d\widetilde{b}\, (\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b})) \tag{3.5}$$

for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$, $t > 0$ and $\alpha \in J(d')$.

Proof. The result for the kernels, without derivatives, is stated in Section 1 of [9] but it is not explicitly proved although its proof is implicit in the discussion of Section 6. Note that the integrals in the lemma exist by the Gaussian bounds (3.3).

We only prove (3.4), the proof of (3.5) is similar. It follows as in [6], Lemma 3.2, that

$$\int_G dg\, \varphi(g)(A^\alpha K_t)(g) = \int_{\tilde{G}} d\widetilde{g}\, \varphi(\exp \Lambda \widetilde{\log} \widetilde{g})(\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{g})$$

for all $\varphi \in C_c(G)$, $\alpha \in J(d')$ and $t > 0$. Hence

$$\begin{aligned} \int_{\tilde{\mathfrak{h}}} d\widetilde{a}\, \psi(\Lambda\widetilde{a})(A^\alpha K_t)(\exp \Lambda\widetilde{a}) &= \int_{\tilde{\mathfrak{g}}} d\widetilde{a}\, \psi(\Lambda\widetilde{a})(\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}\,\widetilde{a}) \\ &= \int_{\tilde{\mathfrak{g}}} d\widetilde{a}\, \psi(\Lambda(\widetilde{a} + \Psi(\widetilde{a})))(\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}\,(\widetilde{a} + \Psi(\widetilde{a}))) \end{aligned}$$

for all $\psi \in C_c(\mathfrak{g})$, $t > 0$ and $\alpha \in J(d')$, since Ψ is super-homogeneous. Therefore

$$\int_{\tilde{\mathfrak{h}}} d\widetilde{a}\, \psi(\Lambda\widetilde{a})(A^\alpha K_t)(\exp \Lambda\widetilde{a})$$

$$= \int_{\tilde{\mathfrak{h}}} d\widetilde{a} \int_{\mathfrak{i}_\infty} d\widetilde{b}\, \psi(\Lambda(\widetilde{a} + \widetilde{b} + \Psi(\widetilde{a} + \widetilde{b})))(\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b} + \Psi(\widetilde{a} + \widetilde{b})))$$

$$= \int_{\tilde{\mathfrak{h}}} d\widetilde{a} \int_{\mathfrak{i}_\infty} d\widetilde{b}\, \psi(\Lambda\widetilde{a})(\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b} + \Psi(\widetilde{b})))$$

because $\widetilde{b} + \Psi(\widetilde{a} + \widetilde{b}) = \widetilde{b} + \Psi(\widetilde{b}) \in \mathfrak{i}$ for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$ and $\widetilde{b} \in \mathfrak{i}_\infty$. Now the statement of the lemma follows easily. ∎

For all $i \in \{1, \ldots, \widetilde{d}\}$ and $\varphi \in C^1(\mathfrak{g})$ define $D_i\varphi \in C(\widetilde{\mathfrak{g}})$ by $(D_i\varphi)(\widetilde{a}) = \frac{d}{dt}\varphi(\widetilde{a} + t\widetilde{b}_i)\big|_{t=0}$. Moreover, if $\widetilde{a} = \sum_{i=1}^{\tilde{d}} \xi_i \widetilde{b}_i$ and $\alpha = (i_1, \ldots, i_n) \in J(\widetilde{d})$ define $\widetilde{a}^\alpha = \xi_{i_1} \cdots \xi_{i_n}$. (Although we also use the notation $\widetilde{a}^\alpha$ for an element in the complex universal enveloping algebra, the meaning will be clear from the context.) Note that $|\widetilde{a}^\alpha| \leqslant |\widetilde{a}|^{\|\alpha\|_{\tilde{v}}}$.

To bound the first term in (3.1) we need one more lemma.

LEMMA 3.2. *For all $\alpha \in J(d')$ there exist $c, \tau > 0$ such that*

$$\big|\big(D_i((\widetilde{A}^\alpha \widetilde{K}_t) \circ \widetilde{\exp}\,)\big)(\widetilde{a})\big| \leqslant c t^{-(\widetilde{D} + \|\alpha\|)/m} t^{-\tilde{v}_i/m} e^{-\tau(|\tilde{a}|^m t^{-1})^{1/(m-1)}}$$

for all $i \in \{1, \ldots, \widetilde{d}\}$, $t > 0$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$.

Proof. It follows from the Campbell-Baker-Hausdorff formula that there exist $c_{ij\beta} \in \mathbb{R}$ such that

$$(\widetilde{B}_i\varphi)(\widetilde{\exp}\,\widetilde{a}) = -\big(D_i(\varphi \circ \widetilde{\exp}\,)\big)(\widetilde{a}) + \sum_{j=1}^{\tilde{d}} \sum_{0 < \|\beta\|_{\tilde{v}} = \tilde{v}_j - \tilde{v}_i} c_{ij\beta} \widetilde{a}^\beta \big(D_j(\varphi \circ \widetilde{\exp}\,)\big)(\widetilde{a})$$

for all $i \in \{1, \ldots, \widetilde{d}\}$, $\varphi \in C^1(\widetilde{G})$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$. If one temporarily orders the basis $\widetilde{b}_1, \ldots, \widetilde{b}_{\tilde{d}}$ such that $\widetilde{v}_1 \leqslant \cdots \leqslant \widetilde{v}_{\tilde{d}}$, then the transition matrix from the D_i to the $\widetilde{B}_i$ is triangular, with -1 entries on the diagonal. Then one can solve for the D_i and it follows that there are polynomial functions $P_{ij} : \widetilde{\mathfrak{g}} \to \mathbb{R}$ such that

$$\big(D_i(\varphi \circ \widetilde{\exp}\,)\big)(\widetilde{a}) = -(\widetilde{B}_i\varphi)(\widetilde{\exp}\,\widetilde{a}) + \sum_{j=1}^{\tilde{d}} P_{ij}(\widetilde{a})(\widetilde{B}_j\varphi)(\widetilde{\exp}\,\widetilde{a}) \tag{3.6}$$

for all $i \in \{1, \ldots, \widetilde{d}\}$, $\varphi \in C^1(\widetilde{G})$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$. Then, by scaling, it follows that the P_{ij} are homogeneous of degree $\widetilde{v}_j - \widetilde{v}_i$.

By (3.3) there exist $c, \tau > 0$ such that

$$\big|(\widetilde{B}_i \widetilde{A}^\alpha \widetilde{K}_1)(\widetilde{\exp}\,\widetilde{a})\big| \leqslant c e^{-\tau|\tilde{a}|^{m/(m-1)}}$$

for all $i \in \{1, \ldots, \widetilde{d}\}$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$. Then, by an elementary estimate, one deduces from (3.6) that there exists a $c' > 0$ such that

$$\big|\big(D_i((\widetilde{A}^\alpha \widetilde{K}_1) \circ \widetilde{\exp}\,)\big)(\widetilde{a})\big| \leqslant c' \mathrm{e}^{-2^{-1}\tau |\tilde{a}|^{m/(m-1)}}$$

for all $i \in \{1, \ldots, \widetilde{d}\}$ and $\widetilde{a} \in \widetilde{\mathfrak{g}}$. The statement of the lemma follows by scaling. ∎

Now we prove Theorem 1.1 for an ideal asymptotic Lie algebra by bounding the first term in (3.1).

Fix $\alpha \in J(d')$. By (3.4), (3.5) and the Duhamel formula one has

$$\begin{aligned}
&(A^\alpha K_t)(\exp \Lambda \widetilde{a}) - \big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)\big(\exp_\infty^{(I)} \Lambda_\infty \widetilde{a}\big) \\
&\quad = \int_{\mathfrak{i}_\infty} \mathrm{d}b\, (\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b} + \Psi(\widetilde{b}))) - (\widetilde{A}^\alpha \widetilde{K}_t)(\widetilde{\exp}(\widetilde{a} + \widetilde{b})) \\
&\quad = \int_{\mathfrak{i}_\infty} \mathrm{d}\widetilde{b} \int_0^1 \mathrm{d}\lambda \sum_{i=1}^{\tilde{d}} \sum_{\substack{j=d+1 \\ \widetilde{v}_j < \widetilde{v}_i}}^{\tilde{d}} c_{ij} [\widetilde{b}]_j \big(D_i((\widetilde{A}^\alpha \widetilde{K}_t) \circ \widetilde{\exp}\,)\big)(\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b}))
\end{aligned}$$

for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$, where the $c_{ij} \in \mathbb{R}$ are such that $\Psi(\widetilde{b}) = \sum_{i=1}^{\tilde{d}} \sum_{j=d+1}^{\tilde{d}} c_{ij} [\widetilde{b}]_j \widetilde{b}_i$ for all $\widetilde{b} \in \mathfrak{i}_\infty$ and $[\widetilde{b}]_j$ denotes the j-th coordinate of $\widetilde{b}$ with respect to the basis $\widetilde{b}_1, \ldots, \widetilde{b}_{\tilde{d}}$. Hence the bounds of Lemma 3.2 together with the estimate $\big|[\widetilde{b}]_j\big| \leqslant |\widetilde{b}|^{\tilde{v}_j}$ give

$$\begin{aligned}
&\big|(A^\alpha K_t)(\exp \Lambda \widetilde{a}) - \big(A^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)\big(\exp_\infty^{(I)} \Lambda_\infty \widetilde{a}\big)\big| \\
&\leqslant c t^{-(\tilde{D} + \|\alpha\|)/m} \sum_{i=1}^{\tilde{d}} \sum_{\substack{j=d+1 \\ \tilde{v}_j < \tilde{v}_i}}^{\tilde{d}} |c_{ij}| \int_0^1 \mathrm{d}\lambda \int_{\mathfrak{i}_\infty} \mathrm{d}\widetilde{b}\, |\widetilde{b}|^{\tilde{v}_j} t^{-\tilde{v}_i/m} \mathrm{e}^{-\tau(|\tilde{a} + \tilde{b} + \lambda \Psi(\tilde{b})|^m t^{-1})^{1/(m-1)}}
\end{aligned}$$

for all $t \geqslant 1$ and $\widetilde{a} \in \widetilde{\mathfrak{h}}$.

Since the map Ψ is super-homogeneous, by (3.2), there exists an $M \geqslant 1$ such that $|\Psi(\widetilde{b})| \leqslant 4^{-1} |\widetilde{a} + \widetilde{b}|$ for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$ and $\widetilde{b} \in \mathfrak{i}_\infty$ with $|\widetilde{a} + \widetilde{b}| \geqslant M$. Then

$$|\widetilde{a} + \widetilde{b}|^{2v} \leqslant 2^{2v} |\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b})|^{2v} + 2^{2v} |\lambda \Psi(\widetilde{b})|^{2v} \leqslant 2^{2v} |\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b})|^{2v} + 2^{-2v} |\widetilde{a} + \widetilde{b}|^{2v}$$

and

$$|\widetilde{a}|^{2v} + |\widetilde{b}|^{2v} = |\widetilde{a} + \widetilde{b}|^{2v} \leqslant 2^{2v+1} |\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b})|^{2v}$$

for all $\lambda \in [0,1]$, $\widetilde{a} \in \widetilde{\mathfrak{h}}$ and $\widetilde{b} \in \mathfrak{i}_\infty$ with $|\widetilde{a} + \widetilde{b}| \geqslant M$. Then $|\widetilde{a}|^{m/(m-1)} + |\widetilde{b}|^{m/(m-1)} \leqslant 16 |\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b})|^{m/(m-1)}$ if $|\widetilde{a} + \widetilde{b}| \geqslant M$. So

$$|\widetilde{a}|^{m/(m-1)} + |\widetilde{b}|^{m/(m-1)} \leqslant 16 |\widetilde{a} + \widetilde{b} + \lambda \Psi(\widetilde{b})|^{m/(m-1)} + 2M^2$$

for all $\lambda \in [0,1]$, $\widetilde{a} \in \widetilde{\mathfrak{h}}$ and $\widetilde{b} \in \mathfrak{i}_\infty$. Therefore

$$\begin{aligned}
&\big|(A^\alpha K_t)(\exp \Lambda\widetilde{a}) - \big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)(\exp_\infty^{(I)} \Lambda_\infty \widetilde{a})\big| \\
&\quad \leqslant c e^{\tau M^2 t^{-1/(m-1)}} t^{-(\tilde{D}+\|\alpha\|)/m} \sum_{i=1}^{\tilde{d}} \sum_{\substack{j=d+1 \\ \tilde{v}_j<\tilde{v}_i}}^{\tilde{d}} |c_{ij}| \int_0^1 d\lambda \int_{\mathfrak{i}_\infty} d\widetilde{b}\, (|\widetilde{b}| t^{-1/m})^{\tilde{v}_j} t^{-(\tilde{v}_i-\tilde{v}_j)/m} \\
&\qquad \cdot e^{-16^{-1}\tau(|\tilde{a}|^m t^{-1})^{1/(m-1)}} e^{-16^{-1}\tau(|\tilde{b}|^m t^{-1})^{1/(m-1)}} \\
&\quad \leqslant c e^{\tau M^2} t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-16^{-1}\tau(|\tilde{a}|^m t^{-1})^{1/(m-1)}} \\
&\qquad \cdot \sum_{i=1}^{\tilde{d}} \sum_{\substack{j=d+1 \\ \tilde{v}_j<\tilde{v}_i}}^{\tilde{d}} |c_{ij}| \Big(\int_{\mathfrak{i}_\infty} d\widetilde{b}\, t^{-(\tilde{D}-D)/m} (|\widetilde{b}| t^{-1/m})^{\tilde{v}_j} e^{-16^{-1}\tau(|\tilde{b}|^m t^{-1})^{1/(m-1)}} \Big)
\end{aligned}$$

for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$ and $t \geqslant 1$. Since the factor between the brackets is finite and independent of t and $\Lambda_\infty \widetilde{a} = \Lambda \widetilde{a}$ for all $\widetilde{a} \in \widetilde{\mathfrak{h}}$, there exists a $c' > 0$ such that

$$\begin{aligned}
&\big|(A^\alpha K_t)(\exp a) - (\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)})(\exp_\infty^{(I)} a)\big| \\
&\qquad \leqslant c' t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-16^{-1}\tau(|a|_{(I)}^m t^{-1})^{1/(m-1)}}
\end{aligned} \tag{3.7}$$

uniformly for all $a \in \mathfrak{g}$ and $t \geqslant 1$, where $|\cdot|_{(I)}$ is the modulus on $\mathfrak{g}$ defined by

$$\Big|\sum_{i=1}^{d} \xi_i b_i^{(I)}\Big|_{(I)}^{2v} = \sum_{i=1}^{d} |\xi_i|^{2v/v_i}.$$

So it remains to replace $|\cdot|_{(I)}$ by $|\cdot|$. The two moduli $|\cdot|_{(I)}$ and $|\cdot|$ are equivalent for large distances.

Lemma 3.3. *There exists a $C > 0$ such that $C^{-1}|a|_{(I)} \leqslant |a| \leqslant C|a|_{(I)}$ for all $a \in \mathfrak{g}$ with $|a| \geqslant 1$.*

Proof. For all $i, j \in \{1, \ldots, d\}$ there exist $c_{ij} \in \mathbb{R}$ such that $b_i^{(I)} = \sum_{\substack{j \\ v_j \geqslant v_i}} c_{ij} b_j$ for all $i \in \{1, \ldots, d\}$. Let $a = \sum_{i=1}^{d} \xi_i b_i^{(I)} \in \mathfrak{g}$ and suppose that $|a|_{(I)} \geqslant 1$. Then

$$a = \sum_{i=1}^{d} \sum_{\substack{j \\ v_j \geqslant v_i}} \xi_i c_{ij} b_j = \sum_{j=1}^{d} \Big(\sum_{\substack{i \\ v_i \leqslant v_j}} \xi_i c_{ij} \Big) b_j.$$

Therefore

$$\begin{aligned}|a|^{2v} &= \sum_{j=1}^{d} \Big| \sum_{\substack{i \\ v_i \leqslant v_j}} \xi_i c_{ij} \Big|^{2v/v_j} \leqslant \sum_{j=1}^{d} d^{2v} \max_{\substack{i \\ v_i \leqslant v_j}} |c_{ij}|^{2v/v_j} |\xi_i|^{2v/v_j} \\ &\leqslant \sum_{j=1}^{d} d^{2v} \max_{\substack{i \\ v_i \leqslant v_j}} |c_{ij}|^{2v/v_j} (1 + |\xi_i|^{2v/v_i}) \leqslant \Big(\sum_{j=1}^{d} d^{2v} \max_{\substack{i \\ v_i \leqslant v_j}} |c_{ij}|^{2v/v_j} \Big) (1 + |a|_{(I)}^{2v}) \\ &\leqslant 2 \Big(\sum_{j=1}^{d} d^{2v} \max_{\substack{i \\ v_i \leqslant v_j}} |c_{ij}|^{2v/v_j} \Big) |a|_{(I)}^{2v}.\end{aligned}$$

Hence there exists a $C > 0$ such that $|a| \leqslant c|a|_{(I)}$ for all $a \in \mathfrak{g}$ with $|a| \geqslant 1$. The other estimate follows similarly. ∎

It follows from Lemma 3.3 that there exists a $C \geqslant 1$ such that $|a| \leqslant 1 + C|a|_{(I)}$ for all $a \in \mathfrak{g}$. Then $(|a|^m t^{-1})^{1/(m-1)} \leqslant 2C^2 (|a|_{(I)}^m t^{-1})^{1/(m-1)} + 2t^{-1/(m-1)} \leqslant 2C^2 (|a|_{(I)}^m t^{-1})^{1/(m-1)} + 2$ for all $a \in \mathfrak{g}$ and $t \geqslant 1$. Hence it follows from (3.7) that

$$\begin{aligned}&\big|(A^\alpha K_t)(\exp a) - \big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)(\exp_\infty^{(I)} a)\big| \\ &\qquad\qquad \leqslant c' e^\tau t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-32^{-1} \tau C^{-2} (|a|^m t^{-1})^{1/(m-1)}}\end{aligned}$$

uniformly for all $a \in \mathfrak{g}$ and $t \geqslant 1$. This is the required estimate for the first term in (3.7).

The estimate of the second term in (3.1) requires the following lemma.

Lemma 3.4. *If $t > 0$ then*

$$\big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)(\exp_\infty^{(I)} a) = \big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty \Phi(a))$$

for all $a \in \mathfrak{g}$ and $\alpha \in J(d)$.

Proof. Since Φ is a Lie algebra isomorphism from $\mathfrak{g}_\infty(\mathfrak{a}^{(I)})$ onto $\mathfrak{g}_\infty(\mathfrak{a})$, it follows from Lemma 2.7 (iii) that $H_\infty^{(I)}(\varphi \circ \Psi) = (H_\infty \varphi) \circ \Psi$ for all $\varphi \in C_c^\infty(G_\infty)$, where $\Psi = \exp_\infty \circ \Phi \circ \log_\infty^{(I)}$ is the lifted Lie group isomorphism from $G_\infty^{(I)}$ onto G_∞. Then $S_t^{(I,\infty)}(\varphi \circ \Psi) = (S_t^{(\infty)} \varphi) \circ \Psi$ for all $t > 0$. Hence $K_t^{(I,\infty)}(g) = K_t^{(\infty)}(\Psi(g))$ for all $t > 0$ and $g \in G_\infty^{(I)}$. This proves the lemma if $|\alpha| = 0$. The lemma for general α then follows by differentiation and Lemma 2.7 (iii). ∎

Now we are prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $\alpha \in J(d')$. Arguing as in the proof of Lemma 3.2, it follows that there exist $c, \tau > 0$ such that

$$\big|\big(D_j\big(\big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big) \circ \exp_\infty\big)\big)(a)\big| \leqslant c t^{-(D+\|\alpha\|)/m} t^{-v_j/m} e^{-\tau (|a|^m t^{-1})^{1/(m-1)}}$$

for all $j \in \{1, \ldots, d\}$, $t > 0$ and $a \in \mathfrak{g}$. Since $\Phi - I$ is super-homogeneous by Lemma 2.7 (ii) there exist $c_{ij} \in \mathbb{R}$ such that $\Phi(b_i^{(I)}) = b_i^{(I)} + \sum_{\substack{j \\ v_j > v_i}} c_{ij} b_j$ for all

$i \in \{1, \ldots, d\}$. Let $a = \sum_{i=1}^{d} \xi_i b_i^{(I)} \in \mathfrak{g}$. Then it follows from Lemma 3.4 and the Duhamel formula that

$$\begin{aligned}
&\big|\big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)(\exp_\infty^{(I)} a) - \big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty a)\big| \\
&\quad = \big|\big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty \Phi(a)) - \big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty a)\big| \\
&\quad \leqslant \sum_{\substack{i,j \\ v_j > v_i}} \int_0^1 d\lambda\, |\xi_i|\, |c_{ij}|\, \big|\big(D_j\big(\big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big) \circ \exp_\infty\big)\big)(a + \lambda(\Phi(a) - a))\big| \\
&\quad \leqslant c t^{-(D+\|\alpha\|)/m} \sum_{\substack{i,j \\ v_j > v_i}} \int_0^1 d\lambda\, |a|_{(I)}^{v_i} t^{-v_j/m} |c_{ij}| e^{-\tau(|a+\lambda(\Phi(a)-a)|^m t^{-1})^{1/(m-1)}}
\end{aligned}$$

for all $t > 0$. By Lemma 2.7 (ii) there exists an $M \geqslant 1$ such that $|\Phi(a) - a| \leqslant 4^{-1}|a|$ for all $a \in \mathfrak{g}$ with $|a| \geqslant M$. Then $|a|^{m/(m-1)} \leqslant 16\, |a + \lambda(\Phi(a) - a)|^{m/(m-1)} + M^2$ for all $a \in \mathfrak{g}$ and $\lambda \in [0, 1]$. If C is as in Lemma 3.3 then

$$\begin{aligned}
&\big|\big(\overline{A}^{(I,\infty)\alpha} K_t^{(I,\infty)}\big)(\exp_\infty^{(I)} a) - \big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty a)\big| \\
&\quad \leqslant c e^{\tau M^2} t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-16^{-1}\tau(|a|^m t^{-1})^{1/(m-1)}} \sum_{\substack{i,j \\ v_j > v_i}} (1 + C|a| t^{-1/m})^{v_i} |c_{ij}| \\
&\quad \leqslant c' t^{-(D+\|\alpha\|)/m} t^{-1/m} e^{-32^{-1}\tau(|a|^m t^{-1})^{1/(m-1)}}
\end{aligned}$$

for a suitable $c' > 0$, uniformly for all $a \in \mathfrak{g}$ and $t \geqslant 1$. This bounds the second term in (3.1) and the proof of Theorem 1.1 is complete. ∎

As a consequence of Theorem 1.1, one has the following kernel bounds for K_t.

Corallary 3.5. *For all $\alpha \in J(d')$ there exist $c, \tau > 0$ such that*

$$|(A^\alpha K_t)(\exp a)| \leqslant c t^{-(D+\|\alpha\|)/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}} \tag{3.8}$$

for all $t \geqslant 1$ and $a \in \mathfrak{g}$. Hence there exists an $M \geqslant 1$ such that $\|A^\alpha S_t\|_{p\to p} \leqslant M\, t^{-\|\alpha\|/m}$ uniformly for all $t > 0$ and $p \in [1, \infty]$.

Proof. It follows from [5], Proposition 5.5, applied to the group G_∞, that there exist $c, \tau > 0$ such that

$$\big|\big(\overline{A}^{(\infty)\alpha} K_t^{(\infty)}\big)(\exp_\infty a)\big| \leqslant c t^{-D/m} t^{-\|\alpha\|/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $a \in \mathfrak{g}_\infty$. The first statement of the corollary then follows for $t \geqslant 1$ from Theorem 1.1.

Finally, one has $\|A^\alpha S_t\|_{p\to p} \leqslant \|A^\alpha K_t\|_1$ and the right hand side of (3.8) can be estimated on $L_1(\mathfrak{g}_\infty)$. Using the dilations, one sees that $\int_{\mathfrak{g}} da\, t^{-D/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$ is independent of t. Obviously $t \mapsto t^{\|\alpha\|/m} \|A^\alpha S_t\|_{p\to p}$ is bounded on $(0, 1]$ uniformly for $p \in [1, \infty]$ (see [5], Corollary 8.3.II). ∎

The kernel bounds have immediate implications for strong convergence of the semigroup.

Corallary 3.6. *If* $p \in (1,\infty)$ *then* $\lim_{t\to\infty} S_t = 0$ *strongly on* L_p.

Proof. Let $\varphi \in C_c(G)$. Then

$$\begin{aligned}\|S_t\varphi\|_2^2 &= \int_G dg \int_G dh_1 \int_G dh_2\, \overline{\varphi}(h_1)\overline{K_t}(gh_1^{-1})K_t(gh_2^{-1})\varphi(h_2)\\ &\leqslant \int_G dg \int_G dh_1 \int_G dh_2\, |\varphi(h_1)|\,|K_t(gh_1^{-1})|\,|K_t(gh_2^{-1})|\,|\varphi(h_2)|\end{aligned}$$

for all $t > 0$. Arguing as in the proof of Lemma 2.2 of [3], there exist $c', \tau' > 0$ such that

$$\begin{aligned}&\int_G dg\, |K_t(gh_1^{-1})|\,|K_t(gh_2^{-1})|\\ &\quad\leqslant c^2 t^{-2D/m}\int_{\mathfrak{g}} da\, e^{-\tau(|\log(h_1\exp(-a))|^m t^{-1})^{1/(m-1)}} e^{-\tau(|\log((\exp a)h_2)|^m t^{-1})^{1/(m-1)}}\\ &\quad\leqslant c' t^{-D/m} e^{-\tau'(|\log(h_1 h_2^{-1})|^m t^{-1})^{1/(m-1)}}\end{aligned}$$

uniformly for all $t > 0$ and $h_1, h_2 \in G$. Hence

$$\begin{aligned}\|S_t\varphi\|_2^2 &\leqslant c' t^{-D/m} \int_G dh_1 \int_G dh_2\, |\varphi(h_1)|\,|\varphi(h_2)| e^{-\tau'(|\log(h_1h_2^{-1})|^m t^{-1})^{1/(m-1)}}\\ &\leqslant c' t^{-D/m} \int_G dh_1 \int_G dh_2\, |\varphi(h_1)|\,|\varphi(h_2)|\end{aligned}$$

and $\lim_{t\to\infty} \|S_t\varphi\|_2 = 0$. Next, for all $p \in (1,2)$, one has

$$\|S_t\varphi\|_p \leqslant \|S_t\varphi\|_1^{(2-p)/p}\|S_t\varphi\|_2^{(2p-2)/p}$$

and as $\|S_t\|_{1\to1}$ is uniformly bounded, it follows that $\lim_{t\to\infty}\|S_t\varphi\|_p = 0$. Similarly, since $\|S_t\|_{\infty\to\infty}$ is uniformly bounded one deduces that $\lim_{t\to\infty}\|S_t\varphi\|_p = 0$ for all $p \in (2,\infty)$. Finally, since the $\|S_t\|_{p\to p}$ are uniformly bounded, it follows that $\lim_{t\to\infty} S_t = 0$ strongly on L_p for all $p \in (1,\infty)$. ∎

The values $p = 1$ and $p = \infty$ are truly exceptional for the strong convergence of S to zero. For example, if H is an unweighted sublaplacian then $\|S_t\varphi\|_1 = \|K_t\|_1\|\varphi\|_1 = \|\varphi\|_1$ for each positive $\varphi \in L_1$ and $\|S_t\varphi\|_\infty = \|\varphi\|_\infty$ for each constant $\varphi \in L_\infty$.

One can also give a new proof of Theorem 3.5 in [6] which deals with unweighted operators.

Define the modulus $|\cdot|'$ on G by

$$|g|' = \sup\Big\{|\psi(g)-\psi(e)| : \psi \in C_b^\infty(G),\, \sum_{i=1}^{d'} |(A_i\psi)|^2 \leqslant 1,\, \psi \text{ real}\Big\},$$

where $A_i\psi$ denotes the left derivative in the direction a_i. Moreover, for all $\rho > 0$ set $V(\rho) = |\{g \in G : |g|' < \rho\}|$, the Haar measure (volume) of the ball of radius ρ.

Corallary 3.7. *Suppose all weights w_i equal one. Then for all $\alpha \in J(d')$ there exist $c, \tau > 0$ such that*

$$\big|(A^\alpha K_t)(g)\big| \leqslant cV(t)^{-1/m} t^{-\|\alpha\|/m} \mathrm{e}^{-\tau((|g|')^m t^{-1})^{1/(m-1)}} \tag{3.9}$$

for all $t > 0$ and $g \in G$.

Proof. The estimates (3.9) are always valid for $t \leqslant 1$ for suitable constants c and τ by [5], Theorem 1.1. Therefore we have to concentrate on bounds uniform for all $t \geqslant 1$.

By the proof of Proposition IV.5.6 in [11], there exists a $\tau_1 \geqslant 1$ such that $|\exp a|' \leqslant \tau_1^{(m-1)/m} |a|$ for all $a \in \mathfrak{g}$ such that $|\exp a|' \geqslant 1$. Then it follows from Corollary 3.5 that there exist $c, \tau > 0$ such that

$$|(A^\alpha K_t)(g)| \leqslant ct^{-D/m} t^{-\|\alpha\|/m} \mathrm{e}^{-\tau\tau_1^{-1}((|g|')^m t^{-1})^{1/(m-1)}} \tag{3.10}$$

for all $t \geqslant 1$ and $g \in G$ with $|g|' \geqslant 1$. But $(|g|')^m t^{-1} \leqslant 1$ for all $t \geqslant 1$ and $g \in G$ with $|g|' \leqslant 1$. Hence, by enlarging c if necessary, one can assume that (3.10) is valid for all $t \geqslant 1$ and all $g \in G$. Therefore the estimates (3.9) are valid for all $t \geqslant 1$ since there is a $c' > 0$ such that $V(t) \leqslant c' t^D$ for all $t \geqslant 1$ (see [11], Theorem IV. 5.8). ∎

4. SEMIGROUP ESTIMATES

In the previous section we showed that the kernel K_t converges to the kernel $K_t^{(\infty)}$ as t tends to infinity. If $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$, this immediately implies that the semigroup S converges uniformly to the corresponding asymptotic semigroup $S^{(\infty)}$.

Theorem 4.1. *If $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ then there exists a $c > 0$ such that*

$$\|S_t - S_t^{(\infty)}\|_{p\to p} \leqslant ct^{-1/m}$$

uniformly for all $t \geqslant 1$ and $p \in [1, \infty]$. Hence $\lim_{t\to\infty} \|S_t - S_t^{(\infty)}\|_{p\to p} = 0$ for all $p \in [1, \infty]$.

Proof. Since one has the estimate $\|S_t - S_t^{(\infty)}\|_{p\to p} \leqslant \|K_t - K_t^{(\infty)}\|_1$, the theorem is a direct corollary of Theorem 1.1. ∎

The convergence of S to $S^{(\infty)}$ on $L_2(G\,; dg)$ immediately yields information about the corresponding semigroups in each irreducible unitary representation of $G\,(= G_\infty)$. Let U be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. Then

$$S_t^U = U(K_t) = \int_G \mathrm{d}g\, K_t(g) U(g)$$

is the strongly continuous semigroup with generator

$$H^U = \mathrm{d}U(C) = \sum_{\alpha \in J(d')} c_\alpha \mathrm{d}U(a^\alpha).$$

Similarly $S_t^{U,(\infty)} = U(K_t^{(\infty)})$ is the strongly continuous semigroup with generator

$$H_\infty^U = \mathrm{d}U(C^{(\infty)}) = \sum_{\alpha \in J(d')} c_\alpha \mathrm{d}U(\overline{a}^\alpha).$$

PROPOSITION 4.2. *If $G = G_\infty$, then in each irreducible unitary representation $(\mathcal{H}, G, U)$ one has*

$$\lim_{t\to\infty} \|S_t^U - S_t^{U,(\infty)}\|_{\mathrm{HS}} = 0$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm on the space of Hilbert-Schmidt operators on $\mathcal{H}$.

Proof. Suppose $t > 1$, then

$$\|S_t^U - S_t^{U,(\infty)}\|_{\mathrm{HS}} \leqslant \big(\|S_1^U\|_{\mathrm{HS}} + \|S_1^{U,(\infty)}\|_{\mathrm{HS}}\big)\|S_{t-1}^U - S_{t-1}^{U,(\infty)}\|$$

where $\|\cdot\|$ denotes the norm on $\mathcal{B}(\mathcal{H})$. But it follows from Theorem 1.1 that there exists a $c > 0$ such that

$$\|S_{t-1}^U - S_{t-1}^{U,(\infty)}\| \leqslant \|K_{t-1} - K_{t-1}^{(\infty)}\|_1 \leqslant c(t-1)^{-1/m}$$

uniformly for all $t > 1$. Therefore $\lim_{t\to\infty} \|S_t^U - S_t^{U,(\infty)}\|_{\mathrm{HS}} = 0$. In fact this estimate establishes the bounds

$$\|S_t^U - S_t^{U,(\infty)}\|_{\mathrm{HS}} \leqslant c_U\, t^{-1/m}$$

for all $t \geqslant 2$. ∎

EXAMPLE 4.3. Let $\mathfrak{g}$ be the Heisenberg Lie algebra of Example 2.10 and choose the algebraic basis a_1, a_2, a_3 with all weights equal to one. Then choosing $\mathfrak{a}_1 = \mathrm{span}(a_1, a_2)$ and $\mathfrak{a}_2 = \mathrm{span}(a_3)$ one has $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$. Hence if $H = -\sum_{i=1}^{3} A_i^2$ is the Laplacian in the left regular representation $H_\infty = -\sum_{i=1}^{2} A_i^2$ is the sublaplacian and the difference between the respective semigroups converges uniformly to zero on each of the L_p-spaces.

We next consider the situation for which $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras. Since $\mathfrak{g} = \mathfrak{g}_\infty(\mathfrak{a})$ as vector spaces, one can, however, compare the semigroups $\widehat{S}_t$ and $\widehat{S}_t^{(\infty)}$ on $L_p(\mathfrak{g})$. One might expect that $\lim_{t\to\infty} \|\widehat{S}_t - \widehat{S}_t^{(\infty)}\|_{p\to p} = 0$ but this is too optimistic.

THEOREM 4.4. *If $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras then there is a $b > 0$ such that*

$$\liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}_t^{(\infty)}\|_{p\to p} \geqslant b$$

for all $p \in [1, \infty]$. Moreover, if the form C defining S is self-adjoint one may choose $b = 1$.

Proof. Since $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras there exist $j, k \in \{1, \ldots, d\}$ such that

$$[b_j, b_k] \neq [b_j, b_k]_\infty.$$

Let $\sigma > 0$. For $t > 0$ define $\varphi_t \in L_2(\mathfrak{g})$ by

$$\varphi_t(a) = \|\widehat{K}^{(\infty)}_{\sigma t}\|_2^{-1} \widehat{K}^{(\infty)}_{\sigma t}(a *_\infty t^\delta b_j),$$

where $\delta = 2r^2$, $\widehat{K}^{(\infty)}_t = K^{(\infty)}_t \circ \exp_\infty$ and $a *_\infty b = \log_\infty(\exp_\infty a \exp_\infty b)$ for all $a, b \in \mathfrak{g}_\infty$. Then $\|\varphi_t\|_2 = 1$ by right invariance of the Haar measure on G_∞.

The starting point for the proof is the estimate

$$\|(\widehat{S}_t - \widehat{S}^{(\infty)}_t)\varphi_t\|_2^2 \geqslant \|\widehat{S}^{(\infty)}_t \varphi_t\|_2^2 - 2\,\mathrm{Re}(\widehat{S}_t\varphi_t, \widehat{S}^{(\infty)}_t \varphi_t)$$

for all $t > 0$.

Since

$$(\widehat{S}^{(\infty)}_t \varphi_t)(a) = \|\widehat{K}^{(\infty)}_{\sigma t}\|_2^{-1} \widehat{K}^{(\infty)}_{(1+\sigma)t}(a *_\infty t^\delta b_j) \tag{4.1}$$

for all $t > 0$ and $a \in \mathfrak{g}$, one has $\|\widehat{S}^{(\infty)}_t \varphi_t\|_2^2 = (\sigma(1+\sigma)^{-1})^{D/m}$ by scaling, uniformly for all $t > 0$. So if we can show that

$$\lim_{t\to\infty} (\widehat{S}_t \varphi_t, \widehat{S}^{(\infty)}_t \varphi_t) = 0 \tag{4.2}$$

then

$$\liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{2\to 2}^2 \geqslant (\sigma(1+\sigma)^{-1})^{D/m}$$

and the first part of the theorem follows for $p = 2$. Moreover, one may arrange that the lower bound equals one for $p = 2$. But for dual variables $p, q \in [1, \infty]$, one has

$$\|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{2\to 2} \leqslant (\|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{p\to p})^{1/2} (\|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{q\to q})^{1/2}.$$

Moreover, $\widehat{S}$ and $\widehat{S}^{(\infty)}$ are uniformly bounded on each L_p-space, by Corollary 3.5. Therefore the first statement of the theorem then follows for all $p \in [1, \infty]$. Finally, the self-adjointness of C implies

$$\|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{p\to p} = \|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{q\to q}$$

for dual exponents p, q and therefore

$$1 \leqslant \liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{2\to 2} \leqslant \liminf_{t\to\infty} \|\widehat{S}_t - \widehat{S}^{(\infty)}_t\|_{p\to p}$$

for all $p \in [1, \infty]$. Thus the proof of the theorem is reduced to establishing (4.2).

First for all $a \in \mathfrak{g}$ one has

$$\begin{aligned} \|\widehat{K}^{(\infty)}_{\sigma t}\|_2 (\widehat{S}_t \varphi_t)(a) &= \int db\, \widehat{K}_t(a * (-b)) \widehat{K}^{(\infty)}_{\sigma t}(b *_\infty t^\delta b_j) \\ &= \int db\, \widehat{K}_t(a * (t^\delta b_j *_\infty (-b))) \widehat{K}^{(\infty)}_{\sigma t}(b) \end{aligned}$$

for all $t \geqslant 1$, where $\widehat{K}_t = K_t \circ \exp$ and $a * b = \log(\exp a \exp b)$ for all $a, b \in \mathfrak{g}$. Hence by (4.1) and Corollary 3.5, there exist $C, \tau > 0$ such that

$$\begin{aligned} &\|\widehat{K}^{(\infty)}_{\sigma t}\|_2^2\, |(\widehat{S}_t \varphi_t, \widehat{S}^{(\infty)}_t \varphi_t)| \\ &\quad = \Big| \int da \int db\, \overline{\widehat{K}_t}(a * (t^\delta b_j *_\infty (-b))) \overline{\widehat{K}^{(\infty)}_{\sigma t}}(b) \widehat{K}^{(\infty)}_{(1+\sigma)t}(a *_\infty t^\delta b_j) \Big| \\ &\quad = \Big| \int da \int db\, \overline{\widehat{K}_t}((a *_\infty (-t^\delta b_j)) * (t^\delta b_j *_\infty (-b))) \overline{\widehat{K}^{(\infty)}_{\sigma t}}(b) \widehat{K}^{(\infty)}_{(1+\sigma)t}(a) \Big| \\ &\quad \leqslant \int da \int db\, G_t((a *_\infty (-t^\delta b_j)) * (t^\delta b_j *_\infty (-b))) \big|\widehat{K}^{(\infty)}_{\sigma t}(b)\big|\, \big|\widehat{K}^{(\infty)}_{(1+\sigma)t}(a)\big| \end{aligned}$$

for all $t \geqslant 1$, where

$$G_t(a) = Ct^{-D/m} \mathrm{e}^{-\tau(|a|^m t^{-1})^{1/(m-1)}} .$$

Then, using the scaling law,

$$\widehat{K}^{(\infty)}_{st}(a) = t^{-D/m} \widehat{K}^{(\infty)}_s(\gamma_{t^{-1/m}}(a))$$

and by a subsequent change of variables $a' = \gamma_{t^{-1/m}}(a)$, $b' = \gamma_{t^{-1/m}}(b)$ one finds

$$\begin{aligned}\|\widehat{K}^{(\infty)}_\sigma\|_2^2 \, |(\widehat{S}_t\varphi_t, \widehat{S}^{(\infty)}_t\varphi_t)| \leqslant \int \mathrm{d}a \int \mathrm{d}b \, \big|\widehat{K}^{(\infty)}_\sigma(b)\big| \, \big|\widehat{K}^{(\infty)}_{1+\sigma}(a)\big| \\ \cdot G_1((a *_\infty (-t^{\delta - v_j/m} b_j)) *_{t^{1/m}} (t^{\delta - v_j/m} b_j *_\infty (-b)))\end{aligned}$$

for all $t \geqslant 1$, where $a *_t b = \gamma_t^{-1}(\gamma_t(a) * \gamma_t(b))$. Therefore, if we can show that

$$\lim_{t\to\infty} |(a *_\infty (-t^{m\delta - v_j} b_j)) *_t (t^{m\delta - v_j} b_j *_\infty (-b))| = \infty$$

for almost all $(a, b) \in \mathfrak{g} \times \mathfrak{g}$, then (4.2) follows by the Lebesgue dominated convergence theorem and the proof of the theorem is complete.

Define $P : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$P(a, b; c) = (a *_\infty (-c)) * (c *_\infty (-b)).$$

Then it follows from the Campbell-Baker-Hausdorff formula that

$$\begin{aligned} P(a, b\,; c) &= \Big(a - c - \frac{1}{2}[a, c]_\infty + \cdots\Big) * \Big(c - b + \frac{1}{2}[b, c]_\infty + \cdots\Big) \\ &= a - b - \frac{1}{2}[a - b, c]_\infty + \frac{1}{2}[a - b, c] - \frac{1}{2}[a, b] + \cdots \\ &= a * (-b) - \frac{1}{2}[a - b, c]_\infty + \frac{1}{2}[a - b, c] + \cdots \end{aligned}$$

where the dots denote a sum of multi-commutators in a, b, c of order at least 3 and the multi-commutators may be mixed in $[\,\cdot\,,\,\cdot\,]$ and $[\,\cdot\,,\,\cdot\,]_\infty$. Since the weighted rank of $\mathfrak{g}$ equals r, it follows that P is a polynomial of unweighted order at most r. Hence there exist $c_{i\alpha\beta\gamma} \in \mathbb{R}$ such that

$$P(a, b\,; c) = a * (-b) - \frac{1}{2}[a - b, c]_\infty + \frac{1}{2}[a - b, c] + \sum_{i=1}^{d} \sum_{\substack{\alpha,\beta,\gamma \in J(d) \\ 3 \leqslant |\alpha|+|\beta|+|\gamma| \leqslant r}} c_{i\alpha\beta\gamma} a^\alpha b^\beta c^\gamma b_i$$

for all $a, b, c \in \mathfrak{g}$.

Next, for all $t \geqslant 1$ and $a, b, c \in \mathfrak{g}$, one has

$$\begin{aligned} (a *_\infty (-c)) *_t (c *_\infty (-b)) &= \gamma_t^{-1}\big(\gamma_t(a *_\infty (-c)) * \gamma_t(c *_\infty (-b))\big) \\ &= \gamma_t^{-1}\big((\gamma_t(a) *_\infty (-\gamma_t(c))) * (\gamma_t(c) *_\infty (-\gamma_t(b)))\big) \\ &= a *_t (-b) - \frac{1}{2}[a - b, c]_\infty + \frac{1}{2}[a - b, c]_t \\ &\quad + \sum_{i=1}^{d} \sum_{\substack{\alpha,\beta,\gamma \in J(d) \\ 3 \leqslant |\alpha|+|\beta|+|\gamma| \leqslant r}} c_{i\alpha\beta\gamma} t^{\|\alpha\|+\|\beta\|+\|\gamma\| - v_i} a^\alpha b^\beta c^\gamma b_i. \end{aligned}$$

Substitute $c = t^{m\delta - v_j} b_j$ in the previous identity. Then

$$\begin{aligned}(a{*_\infty}(-t^{m\delta-v_j}b_j)) *_t (t^{m\delta-v_j}b_j *_\infty (-b)) \\ = a *_t (-b) - \frac{1}{2} t^{m\delta - v_j}\big([a-b,b_j]_\infty - [a-b,b_j]_t\big) \\ + \sum_{i=1}^d \sum_{\substack{\alpha,\beta\in J(d)\\ p\in\mathbb{N}_0 \\ 3\leqslant|\alpha|+|\beta|+p\leqslant r}} c_{i\alpha\beta\gamma_p} t^{\|\alpha\|+\|\beta\|+mp\delta-pv_j-v_i} a^\alpha b^\beta b_i,\end{aligned} \tag{4.3}$$

where $\gamma_p = (j,\ldots,j)$ is the multi-index with p indices equal to j. Let $c_{iq} \in \mathbb{R}$ be such that

$$[b_i,b_j] = \sum_{\substack{q\\ v_q\geqslant v_i+v_j}} c_{iq} b_q$$

for all $i \in \{1,\ldots,d\}$. Then there exists an $n \in \{1,\ldots,d\}$ with $v_n > v_j + v_k$ such that $c_{kn} \neq 0$, since $[b_k,b_j] \neq [b_k,b_j]_\infty$.

For each $a,b \in \mathfrak{g}$, the right hand side of (4.3) is a Laurent polynomial in t. Consider the coefficient of

$$t^{m\delta-v_j} t^{v_j+v_k-v_n} = t^{m\delta+v_k-v_n}.$$

Since $\delta = 2r^2$, one obviously has

$$m\delta \geqslant m\delta + v_k - v_n \geqslant m\delta - r > 2r^2.$$

But if $p = 0$ and $|\alpha|+|\beta| \leqslant r$, then $\|\alpha\|+\|\beta\|+mp\delta-pv_j-v_i \leqslant 2r^2$. Alternatively, if $2 \leqslant p \leqslant r$ then $\|\alpha\|+\|\beta\|+mp\delta-pv_j-v_i \geqslant 4mr^2 - r^2 - r > 2mr^2$. Therefore the only possible contribution of the last term in the right hand side of (4.3) occurs with $p = 1$. Since in addition $3 \leqslant |\alpha|+|\beta|+p$, this implies that in each of the contributions one has $|\alpha|+|\beta| \geqslant 2$. Moreover, $\lim_{t\to\infty} a *_t (-b) = a *_\infty (-b)$ exists, so the term $a *_t (-b)$ gives no contribution to the coefficient of $t^{m\delta+v_k-v_n}$. Therefore there exist $c_{i\alpha\beta} \in \mathbb{R}$ such that the coefficient of $t^{m\delta+v_k-v_n}$ equals

$$\sum_{\substack{i,q\\ v_i-v_q=v_k-v_n\\ v_q>v_i+v_j}} \frac{1}{2}(\xi_i-\eta_i)c_{iq}b_q + \sum_{i=1}^d \sum_{2\leqslant|\alpha|+|\beta|\leqslant r} c_{i\alpha\beta}\xi^\alpha\eta^\beta b_i$$

if $a = \sum_{i=1}^d \xi_i b_i$ and $b = \sum_{i=1}^d \eta_i b_i$, where $\xi^\alpha = a^\alpha$ and $\eta^\beta = b^\beta$. This is an element of $\mathfrak{g}$ with the coefficient of b_n equal to

$$Q(a,b) = \sum_{\substack{i\\ v_i=v_k}} \frac{1}{2}(\xi_i-\eta_i)c_{in} + \sum_{2\leqslant|\alpha|+|\beta|\leqslant r} c_{n\alpha\beta}\xi^\alpha\eta^\beta.$$

Since $c_{kn} \neq 0$, one has $Q(a,b) \neq 0$ for almost all $(a,b) \in \mathfrak{g}\times\mathfrak{g}$. Thus

$$\lim_{t\to\infty} |(a *_\infty (-t^{m\delta-v_j}b_j)) *_t (t^{m\delta-v_j}b_j *_\infty (-b))| = \infty$$

for almost all $(a,b) \in \mathfrak{g}\times\mathfrak{g}$ and the proof is complete. ∎

Although Theorem 4.4 precludes the uniform convergence of $S-S^{(\infty)}$ to zero whenever $G \neq G_\infty$, the next result shows that this is very nearly true.

THEOREM 4.5. *If $f : \mathfrak{g} \to \mathbb{C}$ is a bounded measurable function which vanishes at infinity and $S^{(\infty)}$, the semigroup associated with a general asymptotic Lie algebra then*

$$\lim_{t\to\infty} \|M_f(\widehat{S}_t - \widehat{S}_t^{(\infty)})\|_{p\to p} = 0 = \lim_{t\to\infty} \|(\widehat{S}_t - \widehat{S}_t^{(\infty)})M_f\|_{p\to p}$$

uniformly for all $p \in [1,\infty]$.

Proof. For all $a, b \in \mathfrak{g}$ define $a * b = \log(\exp a \exp b) \in \mathfrak{g}$ and $a *_\infty b = \log_\infty(\exp_\infty a \exp_\infty b) \in \mathfrak{g}_\infty$. Moreover, set $\widehat{K}_t = K_t \circ \exp$ and $\widehat{K}_t^{(\infty)} = K_t^{(\infty)} \circ \exp_\infty$.

Let $\varphi, \psi \in C_c^\infty(\mathfrak{g})$ and $t \geqslant 1$. Then

$$\begin{aligned}
&|(\psi, (\widehat{S}_t - \widehat{S}_t^{(\infty)})\varphi)| \\
&\quad = \Big| \int da \int db\, \overline{\psi}(a)\varphi(b)\big(\widehat{K}_t(a*(-b)) - \widehat{K}_t^{(\infty)}(a *_\infty (-b))\big)\Big| \\
&\quad \leqslant \int da \int db\, |\psi(a)|\,|\varphi(b)|\,\big|\widehat{K}_t(a*(-b)) - \widehat{K}_t^{(\infty)}(a*(-b))\big| \\
&\qquad + \Big| \int da \int db\, \overline{\psi}(a)\varphi(b)\big(\widehat{K}_t^{(\infty)}(a*(-b)) - \widehat{K}_t^{(\infty)}(a *_\infty (-b))\big)\Big|
\end{aligned}$$

where all integrals are over $\mathfrak{g}$. We estimate the two terms separately.

For the first term we can use Theorem 1.1. Let $c, \tau > 0$ as in Theorem 1.1 for the $\alpha \in J(d')$ with $\|\alpha\| = 0$ and define $G_t : \mathfrak{g} \to \mathbb{R}$ by $G_t(a) = t^{-D/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$. One has

$$\begin{aligned}
&\int da \int db\, |\psi(a)|\,|\varphi(b)|\,\big|\widehat{K}_t(a*(-b)) - \widehat{K}_t^{(\infty)}(a*(-b))\big| \\
&\quad \leqslant c t^{-1/m} \int da \int db\, |\psi(a)|\,|\varphi(b)| G_t(a*(-b)) \\
&\quad = c t^{-1/m} (|\psi \circ \log|, (G_t \circ \log) * |\varphi \circ \log|) \\
&\quad \leqslant c t^{-1/m} \|\psi \circ \log\|_{L_q(G)} \|G_t \circ \log\|_{L_1(G)} \|\varphi \circ \log\|_{L_p(G)} = c' t^{-1/m} \|\psi\|_q \|\varphi\|_p
\end{aligned}$$

since $\|G_t \circ \log\|_{L_1(G)} = \|G_t\|_{L_1(\mathfrak{g}_\infty)}$ is independent of t by scaling. Here q is the dual exponent of p.

The second term is more elaborate. Note that it arises if $\mathfrak{g} \neq \mathfrak{g}_\infty(\mathfrak{a})$ as Lie algebras. Hence $a * b \neq a *_\infty b$ for some pair $a, b \in \mathfrak{g}$.

It follows from the Campbell-Baker-Hausdorff formula that there exist $c_{i\alpha\beta} \in \mathbb{R}$ such that

$$P(a,b) = a * b - a *_\infty b = \sum_{i=1}^{d} \sum_{\substack{\alpha,\beta \\ \|\alpha\|_v + \|\beta\|_v < v_i}} c_{i\alpha\beta} a^\alpha b^\beta b_i$$

for all $a, b \in \mathfrak{g}$ (see also [9]). Then the Duhamel formula gives

$$\int \mathrm{d}a \int \mathrm{d}b\, \overline{\psi}(a)\varphi(b)\big(\widehat{K}_t^{(\infty)}(a*(-b)) - \widehat{K}_t^{(\infty)}(a*_\infty(-b))\big)$$

$$= \sum_{i=1}^{d} \sum_{\substack{\alpha,\beta \\ \|\alpha\|_v+\|\beta\|_v<v_i}} c_{i\alpha\beta} \int_0^1 \mathrm{d}\lambda \int \mathrm{d}b \int \mathrm{d}a\, \overline{\psi}(a)\varphi(b) a^\alpha b^\beta \cdot (D_i\widehat{K}_t^{(\infty)})(a*_\infty(-b)+\lambda P(a,-b)),$$

where D_i denotes the partial derivative in the direction b_i. Next, the Campbell-Baker-Hausdorff formula establishes the existence of constants $c_{ij\alpha\beta}$ such that

$$T_{\lambda,b}(a) = a*_\infty(-b)+\lambda P(a,-b) = a - b + \sum_{i=1}^{d}\sum_{j=0}^{1} \sum_{\substack{\|\alpha\|_v+\|\beta\|_v\leqslant v_i \\ \|\alpha\|_v<v_i}} c_{ij\alpha\beta} a^\alpha b^\beta \lambda^j b_i$$

for all $a, b \in \mathfrak{g}$ and $\lambda \in [0,1]$. Then for each $\lambda \in [0,1]$ and $b \in \mathfrak{g}$, the map $T_{\lambda,b}$ is a bijection from $\mathfrak{g}$ onto $\mathfrak{g}$ with Jacobian equal to one. Moreover, by induction it follows that there exist $\widetilde{c}_{ij\alpha\beta} \in \mathbb{R}$ such that

$$T_{\lambda,b}^{-1}(a) = a + b + \sum_{i=1}^{d}\sum_{j=0}^{r^d} \sum_{\substack{\|\alpha\|_v+\|\beta\|_v\leqslant v_i \\ \|\alpha\|_v<v_i}} \widetilde{c}_{ij\alpha\beta} a^\alpha b^\beta \lambda^j b_i$$

uniformly for all $a, b \in \mathfrak{g}$ and $\lambda \in [0,1]$. Then by Leibniz' rule for all $\alpha, \beta \in J^+(d)$ with $\|\alpha\|_v + \|\beta\|_v < r$ there exist $c_{j\alpha\beta\gamma\delta} \in \mathbb{R}$ such that

$$(T_{\lambda,b}^{-1}(a))^\alpha b^\beta = \sum_{\substack{\gamma,\delta \\ \|\gamma\|_v+\|\delta\|_v\leqslant\|\alpha\|_v+\|\beta\|_v}} \sum_{j=0}^{r^{d+1}} c_{j\alpha\beta\gamma\delta} a^\gamma b^\delta \lambda^j$$

uniformly for all $a, b \in \mathfrak{g}$ and $\lambda \in [0,1]$. Arguing as in the proof of Lemma 3.2 it follows that there exist $c, \tau > 0$ such that

$$|(D_i\widehat{K}_t^{(\infty)})(a)| \leqslant c t^{-D/m} t^{-v_i/m} \mathrm{e}^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

for all $i \in \{1,\ldots,d\}$, $t > 0$ and $a \in \mathfrak{g}$. Then

$$\Big|\int \mathrm{d}a \int \mathrm{d}b\, \overline{\psi}(a)\varphi(b)\big(\widehat{K}_t^{(\infty)}(a*(-b)) - \widehat{K}_t^{(\infty)}(a*_\infty(-b))\big)\Big|$$

$$= \Big|\sum_{i=1}^{d} \sum_{\substack{\alpha,\beta \\ \|\alpha\|_v+\|\beta\|_v<v_i}} c_{i\alpha\beta} \int_0^1 \mathrm{d}\lambda \int \mathrm{d}b \int \mathrm{d}a\, \overline{\psi}(T_{\lambda,b}^{-1}(a))\varphi(b)(T_{\lambda,b}^{-1}(a))^\alpha b^\beta (D_i\widehat{K}_t^{(\infty)})(a)\Big|$$

$$\leqslant c \sum_{i=1}^{d} \sum_{\substack{\alpha,\beta \\ \|\alpha\|_v+\|\beta\|_v<v_i}} \sum_{\substack{\gamma,\delta \\ \|\gamma\|_v+\|\delta\|_v\leqslant\|\alpha\|_v+\|\beta\|_v}} \sum_{j=0}^{r^{d+1}} |c_{i\alpha\beta}|\,|c_{j\alpha\beta\gamma\delta}|$$

$$\cdot \int_0^1 \mathrm{d}\lambda \int \mathrm{d}b \int \mathrm{d}a\, |\psi(T_{\lambda,b}^{-1}(a))|\,|\varphi(b)|\,|a^\gamma|\,|b^\delta|\,|\lambda^j| t^{-D/m} t^{-v_i/m} \mathrm{e}^{-\tau(|a|^m t^{-1})^{1/(m-1)}}.$$

Note that $\|\gamma\|_v + \|\delta\|_v \leqslant \|\alpha\|_v + \|\beta\|_v < v_i$. Therefore an elementary estimate gives

$$\int_0^1 d\lambda \int db \int da\, |\psi(T_{\lambda,b}^{-1}(a))|\, |\varphi(b)|\, |a^\gamma|\, |b^\delta|\, |\lambda^j| t^{-D/m} t^{-v_i/m} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

$$\leqslant \int_0^1 d\lambda \int db \int da\, |\psi(T_{\lambda,b}^{-1}(a))|\, |\varphi(b)| t^{-D/m} t^{-(v_i - \|\gamma\|_v - \|\delta\|_v)/m}$$

$$\cdot (|a| t^{-1/m})^{\|\gamma\|_v} (|b| t^{-1/m})^{\|\delta\|_v} e^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

$$\leqslant c' t^{-1/m} \int_0^1 d\lambda \int db \int da\, |\psi(T_{\lambda,b}^{-1}(a))|\, |(N_t\varphi)(b)|\, t^{-D/m} e^{-2^{-1}\tau(|a|^m t^{-1})^{1/(m-1)}}$$

$$= c' t^{-1/m} \int_0^1 d\lambda \int db \int da\, |\psi(a)|\, |(N_t\varphi)(b)| G_t^{(\lambda)}(a,b)$$

uniformly for all $t \geqslant 1$, where $(N_t\varphi)(b) = (1 + (|b|t^{-1/m})^r)\varphi(b)$ and

$$G_t^{(\lambda)}(a,b) = t^{-D/m} e^{-2^{-1}\tau(|a *_\infty (-b) + \lambda P(a,b)|^m t^{-1})^{1/(m-1)}}$$

for all $\lambda \in [0,1]$. Using the transformation $T_{\lambda,b}$ once again it follows that

$$c_1 = \sup_{\lambda \in [0,1]} \sup_{b \in \mathfrak{g}} \sup_{t>0} \int da\, G_t^{(\lambda)}(a,b) < \infty.$$

Similarly,

$$c_2 = \sup_{\lambda \in [0,1]} \sup_{a \in \mathfrak{g}} \sup_{t>0} \int db\, G_t^{(\lambda)}(a,b) < \infty$$

and then, by interpolation,

$$\int db \int da\, |\psi(a)|\, |(N_t\varphi)(b)| G_t^{(\lambda)}(a,b) \leqslant c_1^{1/p} c_2^{1/q} \|\psi\|_q \|N_t\varphi\|_p$$

uniformly for all $\lambda \in [0,1]$ and $t > 0$. Since all sums have a finite number of terms, one deduces that there exists a $c > 0$ such that

$$\big|(\psi, (\widehat{S}_t - \widehat{S}_t^{(\infty)})\varphi)\big| \leqslant c t^{-1/m} \|\psi\|_q \|N_t\varphi\|_p$$

for all $t \geqslant 1$ and $\varphi, \psi \in C_c^\infty(\mathfrak{g})$. Hence

$$\|(\widehat{S}_t - \widehat{S}_t^{(\infty)}) N_t^{-1}\varphi\|_p \leqslant c t^{-1/m} \|\varphi\|_p$$

for all $t \geqslant 1$ and $\varphi \in L_p$.

Next, let $f : \mathfrak{g} \to \mathbb{C}$ be a bounded measurable function which vanishes at infinity. Then, with $D_t = \widehat{S}_t - \widehat{S}_t^{(\infty)}$, one has

$$\begin{aligned}
\big\|(\widehat{S}_t - \widehat{S}_t^{(\infty)}) M_f\big\|_{p\to p} &\leqslant \|D_t N_t^{-1} M_f\|_{p\to p} + \|D_t (I - N_t^{-1}) M_f\|_{p\to p} \\
&\leqslant \|D_t N_t^{-1}\|_{p\to p} \|M_f\|_{p\to p} + \|D_t\|_{p\to p} \|(I - N_t^{-1}) M_f\|_{p\to p} \\
&\leqslant c\|f\|_\infty t^{-1/m} + 2M \sup_{a \in \mathfrak{g}} \frac{(t^{-1/m}|a|)^r}{1 + (t^{-1/m}|a|)^r} \cdot |f(a)|
\end{aligned}$$

for all $t \geqslant 1$, where M is as in Corollary 3.5. But f is bounded and

$$\lim_{t\to\infty} \sup_{a\in\mathfrak{g}} \frac{(t^{-1/m}|a|)^r}{1+(t^{-1/m}|a|)^r} \cdot |f(a)| = \lim_{t\to\infty} \sup_{a\in\mathfrak{g}} \frac{|a|^r}{1+|a|^r} \cdot |f(\gamma_{t^{1/m}}(a))| = 0$$

since f is bounded and vanishes at infinity. Therefore $\|(\widehat{S}_t - \widehat{S}_t^{(\infty)})M_f\|_{p\to p} \to 0$ as $t \to \infty$ and $\|M_f(\widehat{S}_t - \widehat{S}_t^{(\infty)})\|_{p\to p} \to 0$ by duality. ∎

Finally, we note that there is an analogue of the behaviour of the semigroups as $t \to 0$ with the asymptotics for $t \to \infty$. For each weighted algebraic basis on a nilpotent Lie algebra one can construct a contraction $\mathfrak{g}_0$ of $\mathfrak{g}$ as $t \to 0$ and then proceed as before to obtain $H_0, S^{(0)}, K^{(0)}$ etc. (see [9], [8] and [5]). For small t one has good bounds on the semigroup kernels,

$$\big|(A^\alpha K_t)(\exp a) - (A^{(0)\alpha} K_t^{(0)})(\exp_0 a)\big| \leqslant c t^{-(D'+\|\alpha\|)/m} t^{1/m} \mathrm{e}^{-\tau(|a|^m t^{-1})^{1/(m-1)}}$$

for all $t \in (0,1]$ and $a \in \mathfrak{g}$, where D' is the local dimension and $|\cdot|$ the appropriate modulus on the Lie algebra adapted to $\mathfrak{g}_0$. Bounds of this type can be proved similarly to the proof of Theorem 1.1 or, alternatively, from the proof of Theorem 7.2 in [5].

Nevertheless, the uniform convergence of the semigroups as $t \to 0$ is valid only in a special case.

Theorem 4.6. *If* $p \in [1,\infty]$ *then* $\lim_{t\to 0} \big\|\widehat{S}_t - \widehat{S}_t^{(0)}\big\|_{\infty\to\infty} = 0$ *if, and only if,* $\mathfrak{g} = \mathfrak{g}_0$ *as Lie algebras.*

Proof. The proof is a repetition of the arguments in the proofs of the previous two theorems. ∎

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