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A MULTIPLICITY THEORY FOR ANALYTIC FUNCTIONS THAT TAKE VALUES IN A CLASS OF BANACH ALGEBRAS

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Abstract. The theories of linearization, suspension equivalence and multiplicity, previously known for analytic operator-valued functions, are generalized to analytic functions with values in a class of Banach algebras. This consists of the quotient of the class of all bounded operators by an operator ideal.

Keywords: *Analytic operator-valued functions, multiplicity theories, linearization, Banach algebras, operator ideals.*

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1. INTRODUCTION — MULTIPLICITY THEORIES

In a number of previous papers [5], [1], [2], the authors dealt with the question of multiplicity theories for analytic operator-valued functions. An analytic operator-valued function A is an analytic map $A : D \to L(E, E)$, where $D = D(A)$ is an open subset of the complex plane $\mathbb{C}$ and $E = E(A)$ is a complex Banach space. For such a function A the singular set $\Sigma(A)$ of A is defined as the set of points $z \in D(A)$ such that $A(z)$ is not invertible.

A subset Ω of $\mathbb{C}$ is said to be *admissible* for the analytic operator-valued function $A : D \to L(E, E)$ if it is bounded, $\overline{\Omega} \subset D(A)$ and $\partial\Omega \cap \Sigma(A) = \emptyset$. We then refer to (A, Ω) as an *admissible pair.* A *multiplicity theory* for analytic operator-valued functions assigns to each admissible pair (A, Ω) an element $m(A, \Omega)$ of a fixed abelian semigroup M such that certain axioms are satisfied. These axioms were stated in [2].

It is the purpose of the present paper to extend the concept of a multiplicity theory to functions taking values in certain Banach algebras. At first sight it is not at all obvious that an extension exists to any class of Banach algebras beyond the algebra of bounded operators on a Banach space. It is therefore not surprising that we have had to restrict the class of algebras. The problem is to find a class in which the concepts appearing in the axioms make sense. Moreover, the construction of a multiplicity theory depends on the interdependent concepts of suspension-equivalence and linearization. The importance of these concepts was brought out by the paper of Gohberg, Kaashoek and Lay ([3]) (these authors use the term extension instead of suspension) and their paper provides a proof of the existence of linearizations for analytic operator-valued functions.

Our task has been to find a suitable Banach-algebra framework to which the construction given in [3], which we refer to as the GKL-process, can be adapted. For this purpose we propose a class of algebras obtained by taking the quotient of the class of all bounded Banach space operators by a fixed operator ideal. With each Banach space E is then associated a Banach algebra $\mathcal{K}(E,E)$, and we are able to replace the consideration of functions with values in $L(E,E)$ by that of functions with values in $\mathcal{K}(E,E)$. Suitable replacements are then available for all the concepts needed, and moreover the GKL-process both makes sense and can be carried out.

2. Operator ideals

The notion of an operator ideal generalizes the properties of compact operators between Banach spaces. A good source of information is the monograph of A. Pietsch ([6]). Let $\mathcal{L}$ denote the class of all bounded operators between complex Banach spaces. An operator ideal is a subclass $\mathcal{J}$ of $\mathcal{L}$ which satisfies certain axioms. Let us denote the space of all bounded linear operators from the complex Banach space E to the complex Banach space F by $\mathcal{L}(E,F)$ instead of the more usual $L(E,E)$. The set $\mathcal{J} \cap \mathcal{L}(E,F)$ is denoted by $\mathcal{J}(E,F)$ and is called the (E,F)-component of $\mathcal{J}$. We may also refer to components of $\mathcal{J}$ without mentioning E and F explicitly. We now state the axioms for an operator ideal, assuming, without further comment, that all Banach spaces are complex.

(1) $\mathcal{J}(E,F)$ is a vector-subspace of $\mathcal{L}(E,F)$ for each pair of Banach spaces E and F.

(2) If $S \in \mathcal{L}(E,F)$, $T \in \mathcal{J}(F,G)$ and $U \in \mathcal{L}(G,H)$, then $UTS \in \mathcal{J}(E,H)$.

Axiom (2) can be summarized in the neat form $\mathcal{L}\mathcal{J}\mathcal{L} \subset \mathcal{J}$ and in fact it is clear that $\mathcal{L}\mathcal{J}\mathcal{L} = \mathcal{J}$. We remark that in [6] there is an axiom which asserts that $\mathcal{J}$ contains the identity operator on the one-dimensional Banach space $\mathbb{C}$. This seems to be designed to ensure that $\mathcal{J}$ is not trivial, and is an assumption we wish to avoid.

The book [6] gives many examples of operator ideals. Perhaps the most important are $\mathcal{L}$ itself, and the class of all compact operators, although for us the trivial ideal 0 consisting of the zero operator on every Banach space will be more important than $\mathcal{L}$. Further examples will be mentioned in due course.

An operator ideal $\mathcal{J}$ is said to be closed if, for each pair of Banach spaces E and F, the component $\mathcal{J}(E,F)$ is closed in $\mathcal{L}(E,F)$ in the norm topology. Throughout the rest of this article $\mathcal{J}$ will denote a fixed, closed operator ideal.

For each pair of Banach spaces E and F we form the quotient

$$\mathcal{K}(E,F) = \mathcal{L}(E,F)/\mathcal{J}(E,F).$$

Since $\mathcal{J}$ is closed, the quotient $\mathcal{K}(E,F)$ is a Banach space, and $\mathcal{K}(E,E)$ a Banach algebra. We shall denote the canonical linear mapping from $\mathcal{L}(E,F)$ to $\mathcal{K}(E,F)$ by κ, irrespective of E and F.

LEMMA 2.1. *Consider elements of* $\mathcal{L}(E\oplus F, E'\oplus F')$ *as matrices* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A\in\mathcal{L}(E,E')$, $B\in\mathcal{L}(F,E')$, $C\in\mathcal{L}(E,F')$ *and* $D\in\mathcal{L}(F,F')$. *Then* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\in\mathcal{J}(E\oplus F, E'\oplus F')$ *if and only if* $A\in\mathcal{J}(E,E')$, $B\in\mathcal{J}(F,E')$, $C\in\mathcal{J}(E,F')$ *and* $D\in\mathcal{J}(F,F')$.

Proof. Let $I_1\in\mathcal{L}(E,E\oplus F)$, $I_2\in\mathcal{L}(F,E\oplus F)$, $P_1\in\mathcal{L}(E'\oplus F',E')$ and $P_2\in\mathcal{L}(E'\oplus F',F')$ be given by $I_1u=(u,0)$, $I_2v=(0,v)$, $P_1(u,v)=u$, $P_2(u,v)=v$. Then

$$A=P_1\begin{bmatrix} A & B \\ C & D \end{bmatrix}I_1, \qquad B=P_1\begin{bmatrix} A & B \\ C & D \end{bmatrix}I_2,$$

$$C=P_2\begin{bmatrix} A & B \\ C & D \end{bmatrix}I_1, \qquad D=P_2\begin{bmatrix} A & B \\ C & D \end{bmatrix}I_2,$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}=I_1AP_1+I_2CP_1+I_1BP_2+I_2DP_2.$$

The conclusions now follow from the axioms. ∎

The lemma implies that we may view $\mathcal{K}(E\oplus F,E'\oplus F')$ as consisting of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a\in\mathcal{K}(E,E')$, $b\in\mathcal{K}(F,E')$, $c\in\mathcal{K}(E,F')$ and $d\in\mathcal{K}(F,F')$. Similar arguments permit the use of matrices of elements of other dimensions, provided the elements are suitably compatible. Henceforth we shall usually take the matrix point of view. As an example of it, let $a\in\mathcal{K}(E,E')$ and $b\in\mathcal{K}(F,F')$. We define

$$a\oplus b=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\in\mathcal{K}(E\oplus F,E'\oplus F').$$

We denote the multiplicative identity of the Banach algebra $\mathcal{K}(E,E)$, the element $\kappa(I_E)$, by 1_E. If $a\in\mathcal{K}(E,F)$ and $b\in\mathcal{K}(F,G)$ we may unambiguously define the product $ba\in\mathcal{K}(E,G)$ by

$$ba=\kappa(BA)$$

where $A\in\mathcal{L}(E,F)$, $B\in\mathcal{L}(F,G)$, $\kappa A=a$ and $\kappa B=b$. This product is bilinear, associative, distributive over addition, and continuous. Moreover it is consistent with matrix multiplication when we view $\mathcal{K}(E\oplus F,E'\oplus F')$ as consisting of matrices. An element $a\in\mathcal{K}(E,F)$ is said to be *invertible* if there exists $b\in\mathcal{K}(F,E)$ such that $ab=1_F$ and $ba=1_E$. Then b, which is clearly unique, is called the *inverse of* a and is denoted by a^{-1}.

Let E and F be Banach spaces. We say that E and F are *$\mathcal{J}$-isomorphic* if $\mathcal{K}(E,F)$ contains an invertible element. If a is such an element we obtain an algebraic isomorphism from the Banach algebra $\mathcal{K}(E,E)$ onto the Banach algebra $\mathcal{K}(F,F)$ given by $x\mapsto axa^{-1}$. We shall refer to such an isomorphism as a *$\mathcal{J}$-algebraic isomorphism.* It seems conceivable that any algebraic isomorphism from

$\mathcal{K}(E, E)$ to $\mathcal{K}(F, F)$ is a $\mathcal{J}$-algebraic isomorphism. This is known to be the case if $\mathcal{J} = 0$, see [7].

$\mathcal{J}$-isomorphism classes form an abelian semigroup with zero. Let $[E]_{\mathcal{J}}$ denote the $\mathcal{J}$-isomorphism class of the Banach space E. Then, unambiguously, we may define

$$[E]_{\mathcal{J}} + [F]_{\mathcal{J}} = [E \oplus F]_{\mathcal{J}}.$$

In fact, let $[E]_{\mathcal{J}} = [E']_{\mathcal{J}}$ and $[F]_{\mathcal{J}} = [F']_{\mathcal{J}}$. Let $a \in \mathcal{K}(E, E')$ and $b \in \mathcal{K}(F, F')$ be invertible. Then $a \oplus b$ is an invertible element of $\mathcal{K}(E \oplus F, E' \oplus F')$. Its inverse is, of course, $a^{-1} \oplus b^{-1}$.

We shall denote the semigroup defined in the last paragraph by $M_{\mathcal{J}}$. Although it is a plausible target semigroup for a multiplicity theory, we shall, in fact, employ another semigroup $P_{\mathcal{J}}$ consisting of equivalence classes of idempotents. This we proceed to define.

Let $p \in \mathcal{K}(E, E)$ and $q \in \mathcal{K}(F, F)$ be idempotents. We define an equivalence relation whereby $p \sim q$ means that there exist $\alpha \in \mathcal{K}(E, F)$ and $\beta \in \mathcal{K}(F, E)$ such that

$$\beta\alpha = p \quad \text{and} \quad \alpha\beta = q.$$

Let us check transitivity. Let $r \in \mathcal{K}(G, G)$ and suppose that $q \sim r$ in addition to $p \sim q$. Then there exist $\gamma \in \mathcal{K}(F, G)$ and $\delta \in \mathcal{K}(G, F)$ such that $\gamma\delta = r$ and $\delta\gamma = q$. Hence we have

$$(\beta\delta)(\gamma\alpha) = \beta(\delta\gamma)\alpha = \beta q\alpha = \beta\alpha\beta\alpha = p^2 = p$$

and

$$(\gamma\alpha)(\beta\delta) = \gamma(\alpha\beta)\delta = \gamma q\delta = \gamma\delta\gamma\delta = r^2 = r.$$

We shall denote the equivalence class of the idempotent p by $[p]_0$. We proceed to make the equivalence classes into an abelian semigroup. We define

$$[p]_0 + [q]_0 = [p \oplus q]_0.$$

Let us check that this is well-defined.

Lemma 2.2. *Let $p \sim p'$ and $q \sim q'$ where p, p', q and q' are idempotents. Then $p \oplus q \sim p' \oplus q'$.*

Proof. Let $\beta\alpha = p$, $\alpha\beta = p'$, $\delta\gamma = q$, $\gamma\delta = q'$. Then

$$(\beta \oplus \delta)(\alpha \oplus \gamma) = \beta\alpha \oplus \delta\gamma = p \oplus q$$

and

$$(\alpha \oplus \gamma)(\beta \oplus \delta) = \alpha\beta \oplus \gamma\delta = p' \oplus q'. \quad ∎$$

This shows that addition is well-defined.

LEMMA 2.3. *Let $p \in \mathcal{K}(E,E)$ and $q \in \mathcal{K}(E,E)$ be idempotents such that $pq = qp = 0$. Then $p+q$ is an idempotent and $[p+q]_0 = [p]_0 + [q]_0$.*

Proof. We must show that $p+q \sim p \oplus q$. But we have, in terms of matrices

$$p+q = \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \quad \text{and} \quad p \oplus q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix}. \quad ∎$$

Now let $t \in \mathcal{K}(E,E)$ and let Ω be a bounded open set in $\mathbb{C}$ whose boundary does not meet the spectrum of t. We define

$$p_{t,\Omega} = \frac{1}{2\pi \mathrm{i}} \int_{\Omega'} (\zeta 1_E - t)^{-1} \, \mathrm{d}\zeta$$

where Ω' is a Cauchy domain enclosing the same part of the spectrum of t as does Ω. It is, of course, well known that $p_{t,\Omega}$ is an idempotent. In fact it is obtained from the operational calculus as $f(t)$, where $f = 1$ on Ω and $f = 0$ outside Ω.

Next, let Ω_1 and Ω_2 be bounded open subsets of $\mathbb{C}$, disjoint with each other and such that their boundaries do not meet the spectrum of t. Then, by Lemma 2.3

$$[p_{t,\Omega_1 \cup \Omega_2}]_0 = [p_{t,\Omega_1}]_0 + [p_{t,\Omega_2}]_0.$$

This says that $[p_{t,\Omega}]_0$ is an additive function of the set Ω.

The following lemma will be needed to prove the product theorem (Axiom 3 for multiplicity theories).

LEMMA 2.4. *Let $p \in \mathcal{K}(E,E)$ and $q \in \mathcal{K}(F,F)$ be idempotents, and let $r \in \mathcal{K}(F,E)$ be such that $\begin{bmatrix} p & r \\ 0 & q \end{bmatrix}$ is an idempotent. Then $\begin{bmatrix} p & r \\ 0 & q \end{bmatrix} \sim p \oplus q$.*

Proof. From the fact that $\begin{bmatrix} p & r \\ 0 & q \end{bmatrix}$ is an idempotent we deduce that $pr + rq = r$. We let

$$\alpha = \begin{bmatrix} 1_E & r \\ 0 & 1_F \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} p & r \\ 0 & q \end{bmatrix} \begin{bmatrix} 1_E & -r \\ 0 & 1_F \end{bmatrix}$$

and

$$\beta = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} 1_E & r \\ 0 & 1_F \end{bmatrix} = \begin{bmatrix} 1_E & -r \\ 0 & 1_F \end{bmatrix} \begin{bmatrix} p & r \\ 0 & q \end{bmatrix}.$$

Then we have

$$\beta\alpha = \beta \begin{bmatrix} p & r \\ 0 & q \end{bmatrix} \begin{bmatrix} 1_E & -r \\ 0 & 1_F \end{bmatrix} = \beta \begin{bmatrix} 1_E & -r \\ 0 & 1_F \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

and

$$\alpha\beta = \alpha \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} 1_E & r \\ 0 & 1_F \end{bmatrix} = \alpha \begin{bmatrix} 1_E & r \\ 0 & 1_F \end{bmatrix} = \begin{bmatrix} p & r \\ 0 & q \end{bmatrix}. \quad ∎$$

3. EQUIVALENCE AND SUSPENSION

In this section we consider analytic mappings defined on open subsets of $\mathbb{C}$ and taking values in the Banach spaces $\mathcal{K}(E, F)$. We refer to such a mapping as an *analytic $\mathcal{K}$-valued function.*

Let D be an open subset of $\mathbb{C}$. Let $f : D \to \mathcal{K}(E, E)$ and $g : D \to \mathcal{K}(F, F)$ be analytic. We say that f and g are *equivalent* if there exist $\varphi : D \to \mathcal{K}(E, F)$ and $\psi : D \to \mathcal{K}(E, F)$, taking invertible values only, such that

$$\varphi f = g\psi.$$

Let $f : D \to \mathcal{K}(E, E)$ be analytic and let F be a Banach space. By the *F-suspension of f* we mean the analytic mapping $f \oplus 1_F : D \to \mathcal{K}(E \oplus F, E \oplus F)$ given by

$$(f \oplus 1_F)(z) = f(z) \oplus 1_F.$$

Often we do not want to mention F explicitly; then we say that $f \oplus 1_F$ is a suspension of f.

Let $f : D \to \mathcal{K}(E, E)$ and $g : D \to \mathcal{K}(F, F)$. We say that f and g are *suspension-equivalent*, s-equivalent for short, if they have equivalent suspensions; that is, if there exist Banach spaces X and Y such that $f \oplus 1_X$ and $g \oplus 1_Y$ are equivalent. It is easy to see that suspension-equivalence is an equivalence relation.

Let E be a Banach space and let $t \in \mathcal{K}(E, E)$. The analytic mapping

$$\mathbb{C} \ni z \mapsto z1_E - t$$

will be denoted by L_t.

Let $f : D \to \mathcal{K}(E, E)$. We define *the singular set* $\Sigma(f)$ of f by

$$\Sigma(f) = \{z \in D : f(z) \text{ is not invertible}\}.$$

An open set $\Omega \subset \mathbb{C}$ is said to be *admissible* for f if it is bounded, $\overline{\Omega} \subset D$ and $\partial\Omega \cap \Sigma(f) = \emptyset$. We denote $\Omega \cap \Sigma(f)$ by $\Sigma_\Omega(f)$ and call it *the singular set of f in* Ω. The pair (f, Ω) is called an *admissible pair.*

Let (f, Ω) be an admissible pair. By a *linearization* for (f, Ω) we shall mean a mapping of the form L_t, which is s-equivalent to f on an open neighbourhood of $\overline{\Omega}$.

Theorem 3.1. *Let $f : D \to \mathcal{K}(E, E)$ and let Ω be admissible for f. Then there exists a linearization L_t for the admissible pair (f, Ω).*

Proof. We adapt the GKL-process ([3], [5]). Assume that $0 \in \Omega$; when we have dealt with this case, a simple translation argument then disposes of the general case. Choose Cauchy domains Ω' and Ω'', such that $\overline{\Omega} \subset \Omega'$, $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset D$ and $f(z)$ is invertible for all $z \in \overline{\Omega''} \setminus \Omega$. We set

$$F = C(\partial\Omega', E).$$

This is the Banach space of all continuous mappings from $\partial\Omega'$ to E equipped with the norm

$$\|v\| = \sup_{s \in \partial\Omega'} \|v(s)\|.$$

Define the projection $P \in \mathcal{L}(F,F)$ by
$$(Pv)(s) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} \frac{1}{\zeta} v(\zeta)\,\mathrm{d}\zeta$$
for each $v \in F$. As in the GKL-process, P projects F onto the subspace of all constant functions, which we identify with E. Let $Z = \ker P$. We now identify F with $E \oplus Z$. We form the suspension $f \oplus 1_Z$ and view it as a mapping from D to $\mathcal{K}(F,F)$. Let $p = \kappa(P)$. Since $\mathcal{K}(F,F)$ is identified with $\mathcal{K}(E\oplus Z, E\oplus Z)$, the idempotent p is identified with $1_E \oplus 0_Z$. Then $1_F - p = 0_E \oplus 1_Z$.

Let $W \in \mathcal{L}(F,F)$ be defined by
$$(Wv)(s) = sv(s), \quad s \in \partial\Omega'$$
for each $v \in F$. Let $w = \kappa(W)$. We define $t \in \mathcal{K}(F,F)$ by
$$t = w - \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega''} \big((1_E - f)\oplus 0_Z\big)(\sigma) w(\sigma 1_F - w)^{-1}\,\mathrm{d}\sigma.$$
Note that the spectrum of W, and hence also that of w, is a subset of $\partial\Omega'$.

Lemma 3.2.
$$PW(\sigma I_F - W)^{-1}(zI_F - W)^{-1}P = -(\sigma - z)^{-1}P$$
for all $z \in \Omega'$ *and* $\sigma \in \partial\Omega''$.

Proof. Let $v \in F$. We compute
$$\begin{aligned} &PW(\sigma I_F - W)^{-1}(zI_F - W)^{-1}Pv \\ &\quad= \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} \frac{1}{\zeta}\zeta(\sigma - \zeta)^{-1}(z-\zeta)^{-1}(Pv)\,\mathrm{d}\zeta = -(\sigma - z)^{-1}(Pv)\end{aligned}$$
thus proving the lemma. ∎

Continuing the proof of Theorem 3.1 we deduce from the lemma that
$$pw(\sigma 1_F - w)^{-1}(z1_F - w)^{-1}p = -(\sigma - z)^{-1}p$$
for all $z \in \Omega'$ and $\sigma \in \partial\Omega''$. Now we compute, for $z \in \Omega'$,
$$\begin{aligned} &(z1_F - t)(z1_F - w)^{-1} \\ &\quad= 1_F + \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega''} \big((1_E - f)\oplus 0_Z\big)(\sigma) w(\sigma 1_F - w)^{-1}(z1_F - w)^{-1}\,\mathrm{d}\sigma,\end{aligned}$$
whence, recalling that $p = 1_E \oplus 0_Z$, we have
$$\begin{aligned} &p(z1_F - t)(z1_F - w)^{-1}p \\ &\quad= p + \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega''} \big((1_E - f)\oplus 0_Z\big)(\sigma) pw(\sigma 1_F - w)^{-1}(z1_F - w)^{-1}p\,\mathrm{d}\sigma \\ &\quad= p - \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega''} \big((1_E - f)\oplus 0_Z\big)(\sigma)(\sigma - z)^{-1}p\,\mathrm{d}\sigma \\ &\quad= p - \big((1_E - f)\oplus 0_Z\big)(z) = (f \oplus 0_Z)(z).\end{aligned}$$

Again we have

$$(1_F - p)(z1_F - t)(z1_F - w)^{-1} = 1_F - p = 0_E \oplus 1_Z.$$

Assembling the results we find

$$(z1_F - t)(z1_F - w)^{-1} = (f \oplus 0_Z)(z) + (0_E \oplus 1_Z) + pk(z)(1_F - p)$$

where $k(z) = (z1_F - t)(z1_F - w)^{-1}$. So in matrix terms we have

$$(z1_F - t)(z1_F - w)^{-1} = \begin{bmatrix} f(z) & h(z) \\ 0 & 1_Z \end{bmatrix}$$

where $h : D \to \mathcal{K}(Z, E)$ is analytic. Hence

$$L_t(z)(z1_F - w)^{-1} = \begin{bmatrix} 1_E & h(z) \\ 0 & 1_Z \end{bmatrix} (f(z) \oplus 1_Z)$$

and so L_t is equivalent to $f \oplus 1_Z$ on Ω'. This concludes the proof of Theorem 3.1. ∎

4. A MULTIPLICITY THEORY

In this section we consider the concept of a multiplicity theory for analytic $\mathcal{K}$-valued functions. It will be convenient to adopt the following notation. If $f : D \to \mathcal{K}(E, E)$ is analytic, we will sometimes denote the domain D by $D(f)$ and the space E by $E(f)$.

A multiplicity theory for analytic $\mathcal{K}$-valued functions assigns to each admissible pair (f, Ω) an element $m(f, \Omega)$ of a fixed abelian semigroup M such that the following axioms are satisfied:

(1) If $f : D \to \mathcal{K}(E, E)$ and $\alpha \in \mathcal{K}(E, F)$ is invertible then

$$m(\alpha f \alpha^{-1}, \Omega) = m(f, \Omega).$$

(2) If $\Sigma_\Omega(f) = \emptyset$ then $m(f, \Omega) = 0$.

(3) (The product theorem) If $E(f) = E(g)$, $D(f) = D(g)$ and Ω is admissible for f and g then

$$m(fg, \Omega) = m(f, \Omega) + m(g, \Omega).$$

(4) If $D(f) = D(g)$ and Ω is admissible for f and g then

$$m(f \oplus g, \Omega) = m(f, \Omega) + m(g, \Omega).$$

(5) If Ω_1 and Ω_2 are admissible for f and $\Omega_1 \cap \Omega_2 = \emptyset$ then

$$m(f, \Omega_1 \cup \Omega_2) = m(f, \Omega_1) + m(f, \Omega_2).$$

(6) $m(f, \Omega)$ depends only on $f|\Omega$ and Ω.

(7) $m(f, \Omega)$ depends only on f and $\Sigma_\Omega(f)$.

We shall now construct a multiplicity theory for analytic $\mathcal{K}$-valued functions.

Let $f : D \to \mathcal{K}(E, E)$ be analytic and let $\Omega \subset \mathbb{C}$ be admissible for f. Let L_t be a linearization for (f, Ω), where $t \in \mathcal{K}(F, F)$ for some Banach space F. We define the multiplicity

$$m(f, \Omega) = [p_{t,\Omega}]_0 \in P_{\mathcal{J}}.$$

The existence of a linearization was dealt with in the last section. However we are not assuming that L_t is the linearization which was constructed in Theorem 3.1; in fact, it is intended to be any linearization whatsoever. We therefore have to show that $m(f,\Omega)$ is well-defined.

It suffices to solve the following general problem. Let E and F be Banach spaces. Let $s \in \mathcal{K}(E,E)$ and $t \in \mathcal{K}(F,F)$. Let Ω be admissible for both L_s and L_t. Suppose that there exist Banach spaces X and Y such that $L_s \oplus 1_X$ is equivalent to $L_t \oplus 1_Y$ on a neighbourhood Ω' of $\overline{\Omega}$. Then we must show that $[p_{s,\Omega}]_0 = [p_{t,\Omega}]_0$.

First we simplify the problem. Let $\varphi : \Omega' \to \mathcal{K}(E,F)$ and $\psi : \Omega' \to \mathcal{K}(E,F)$ be analytic mappings, taking invertible values only, such that

$$\varphi(z)\big((z1_E - s) \oplus 1_X\big) = \big((z1_F - t) \oplus 1_Y\big)\psi(z)$$

for all $z \in \Omega'$. Choose $z_0 \notin \Omega'$. We have

$$\begin{aligned}\varphi(z)\big(1_E \oplus (z-z_0)^{-1}1_X\big)&\big(z1_{E\oplus X} - (s \oplus z_0 1_X)\big)\\ &= \big(z1_{F\oplus Y} - (t \oplus z_0 1_Y)\big)\big(1_F \oplus (z-z_0)^{-1}1_Y\big)\psi(z).\end{aligned}$$

Moreover, it is easy to check that

$$p_{s\oplus z_0 1_X,\Omega} = p_{s,\Omega} \oplus 0_X \sim p_{s,\Omega}.$$

Hence we may simplify our problem by dropping the suspensions, and repose it as follows. Let $s \in \mathcal{K}(E,E)$, $t \in \mathcal{K}(F,F)$, and suppose that L_s and L_t are equivalent on their common admissible set Ω. Then we must show that $[p_{s,\Omega}]_0 = [p_{t,\Omega}]_0$. This is a consequence of the following theorem, which generalizes a theorem of Kaashoek, van der Mee and Rodman ([4]).

THEOREM 4.1. *Let $s \in \mathcal{K}(E,E)$, $t \in \mathcal{K}(F,F)$ and let Ω be admissible for both L_s and L_t. Let $\varphi : \Omega \to \mathcal{K}(E,F)$ and $\psi : \Omega \to \mathcal{K}(E,F)$ be analytic mappings, taking invertible values only, and assume that*

$$\varphi L_s = L_t \psi$$

on a neighbourhood of $\overline{\Omega}$. Then there exist $\alpha \in \mathcal{K}(E,F)$ and $\beta \in \mathcal{K}(F,E)$ such that

$$t\alpha = \alpha s, \qquad \beta t = s\beta,$$
$$\beta\alpha = p_{s,\Omega}, \qquad \alpha\beta = p_{t,\Omega}.$$

Proof. Choose a Cauchy domain Ω' such that $\overline{\Omega'} \subset \Omega$ and all points in $\Omega \setminus \Omega'$ belong to the resolvent sets of s and t. Then

$$p_{s,\Omega} = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} (\zeta 1_E - s)^{-1}\, \mathrm{d}\zeta \quad \text{and} \quad p_{t,\Omega} = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} (\zeta 1_F - t)^{-1}\, \mathrm{d}\zeta.$$

Choose a Cauchy domain Ω'' such that $\overline{\Omega'} \subset \Omega''$ and $\overline{\Omega''} \subset \Omega$. For each function f, locally analytic on the union of the spectra of s and t, and *scalar-valued*, we define

$$\alpha_f = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} f(\zeta)\psi(\zeta)(\zeta 1_E - s)^{-1}\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} f(\zeta)(\zeta 1_F - t)^{-1}\varphi(\zeta)\, \mathrm{d}\zeta$$

and

$$\beta_f = \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega'} f(\zeta)(\zeta 1_E - s)^{-1}\varphi(\zeta)^{-1}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega'} f(\zeta)\psi(\zeta)^{-1}(\zeta 1_F - t)^{-1}\,\mathrm{d}\zeta.$$

We define $\alpha = \alpha_1$ and $\beta = \beta_1$ (that is, α and β correspond to using the function $f = 1$). Recall the operational calculus, which enables us to define the algebra elements $f(s)$ and $f(t)$ for each such f. Note that $f(t)$ commutes with $p_{t,\Omega}$ and $f(s)$ with $p_{s,\Omega}$. Now we have

$$\begin{aligned}
\alpha_f &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int\limits_{\partial\Omega'} f(\zeta)(\zeta 1_F - t)^{-1}\left(\int\limits_{\partial\Omega''} \frac{\varphi(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right)\mathrm{d}\zeta \\
&= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int\limits_{\partial\Omega''} \left(\int\limits_{\partial\Omega'} \frac{f(\zeta)}{\sigma-\zeta}(\zeta 1_F - t)^{-1}\,\mathrm{d}\zeta\right)\varphi(\sigma)\,\mathrm{d}\sigma \\
&= \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega''} f(t)p_{t,\Omega}(\sigma 1_F - t)^{-1}\varphi(\sigma)\,\mathrm{d}\sigma = f(t)p_{t,\Omega}\alpha.
\end{aligned}$$

Again we have

$$\begin{aligned}
\alpha_f &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int\limits_{\partial\Omega'} \left(\int\limits_{\partial\Omega''} \frac{\psi(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right) f(\zeta)(\zeta 1_E - s)^{-1}\,\mathrm{d}\zeta \\
&= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int\limits_{\partial\Omega''} \psi(\sigma)\left(\int\limits_{\partial\Omega'} \frac{f(\zeta)}{\sigma-\zeta}(\zeta 1_E - s)^{-1}\,\mathrm{d}\zeta\right)\mathrm{d}\sigma \\
&= \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega''} \psi(\sigma)f(s)(\sigma 1_E - s)^{-1}p_{s,\Omega}\,\mathrm{d}\sigma = \alpha f(s)p_{s,\Omega}.
\end{aligned}$$

Similarly we deduce the formulas

$$\beta_f = p_{s,\Omega}f(s)\beta = \beta p_{t,\Omega}f(t).$$

For each σ not in the union of the spectra of s and t, let g_σ be defined by $g_\sigma(z) = (\sigma - z)^{-1}$. Now we have

$$\alpha_{g_\sigma} = \alpha g_\sigma(s)p_{s,\Omega} = \alpha(\sigma 1_E - s)^{-1}p_{s,\Omega} = (\sigma 1_F - t)^{-1}p_{t,\Omega}\alpha.$$

Again we compute

$$\begin{aligned}
p_{s,\Omega} &= \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega'} (\zeta 1_E - s)^{-1}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega'} \psi(\zeta)^{-1}\psi(\zeta)(\zeta 1_E - s)^{-1}\,\mathrm{d}\zeta \\
&= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int\limits_{\partial\Omega'}\int\limits_{\partial\Omega''} (\sigma-\zeta)^{-1}\psi(\sigma)^{-1}\psi(\zeta)(\zeta 1_E - s)^{-1}\,\mathrm{d}\sigma\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega''} \psi(\sigma)^{-1}\alpha_{g_\sigma}\,\mathrm{d}\sigma = \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial\Omega''} \psi(\sigma)^{-1}(\sigma 1_F - t)^{-1}p_{t,\Omega}\alpha\,\mathrm{d}\sigma = \beta p_{t,\Omega}\alpha.
\end{aligned}$$

Similarly we find that

$$p_{t,\Omega} = \alpha p_{s,\Omega}\beta.$$

Let us summarize our findings in a box:

$$\boxed{\begin{aligned} &p_{s,\Omega} = \beta p_{t,\Omega}\alpha \\ &p_{t,\Omega} = \alpha p_{s,\Omega}\beta \\ &\alpha_f = f(t)p_{t,\Omega}\alpha = \alpha f(s)p_{s,\Omega} \\ &\beta_f = p_{s,\Omega}f(s)\beta = \beta p_{t,\Omega}f(t). \end{aligned}}$$

We deduce, putting $f = 1$, $\alpha = p_{t,\Omega}\alpha = \alpha p_{s,\Omega}$, $\beta = p_{s,\Omega}\beta = \beta p_{t,\Omega}$, whence $p_{s,\Omega} = \beta\alpha$, $p_{t,\Omega} = \alpha\beta$. Finally, putting $f(z) = z$ we deduce $t\alpha = \alpha s$, $s\beta = \beta t$. This concludes the proof. ∎

Theorem 4.1 is much more than is needed to prove that the multiplicity theory m is well-defined. In fact it can be used to define a universal multiplicity theory for analytic $\mathcal{K}$-valued functions, using a procedure that was carried out in the authors' paper ([1]) for the case of operator-valued functions. It is also clear that the theorem holds for the case where s and t are elements of an arbitrary Banach algebra Y, and the equivalence of L_s and L_t is realized by a pair of analytic mappings $\varphi, \psi : \Omega \to Y$. Then α and β turn out to be elements of Y.

The definition of m guarantees that $m(f, \Omega)$ is an invariant of the s-equivalence class of f. It is also plain that $m(f, \Omega) = 0$ if and only if $\Sigma_\Omega(f) = \emptyset$, for it is well known that $p_{t,\Omega} = 0$ if and only if the intersection of the spectrum of t with Ω is empty, and this intersection coincides with $\Sigma_\Omega(f)$. Additivity on sets (Axiom 5) is a straightforward consequence of the argument following Lemma 2.3. Axioms 6 and 7 are obvious. Once we have Axiom 3, Axiom 4 follows from $f \oplus g = (f \oplus 1_{E(g)})(1_{E(f)} \oplus g)$. Thus the verification of the axioms can be completed by verifying Axiom 3, the product theorem.

THEOREM 4.2. (Product Theorem) *Let $f : D \to \mathcal{K}(E, E)$ and $g : D \to \mathcal{K}(E, E)$ be analytic and let Ω be admissible for both. Then*

$$m(fg, \Omega) = m(f, \Omega) + m(g, \Omega).$$

Proof. The matrix equation

$$\begin{bmatrix} 0_E & -1_E \\ 1_E & f \end{bmatrix} \begin{bmatrix} fg & 0_E \\ 0_E & 1_E \end{bmatrix} \begin{bmatrix} 1_E & 0_E \\ -g & 1_E \end{bmatrix} = \begin{bmatrix} g & -1_E \\ 0_E & f \end{bmatrix}$$

shows that fg is s-equivalent to $h = \begin{bmatrix} g & -1_E \\ 0_E & f \end{bmatrix}$. We shall apply the linearization procedure of Theorem 3.1 (the GKL-process) to h. Let $F = C(\partial\Omega', E)$ and $G = C(\partial\Omega', E \oplus E)$. We identify G with $F \oplus F$. Hence $\mathcal{K}(G, G)$ is identified with $\mathcal{K}(F \oplus F, F \oplus F)$, the elements of which are written as 2×2-matrices with entries in $\mathcal{K}(F, F)$. Now F is the space in which the linearizations of f and g are defined (see Theorem 3.1), and we may also identify $\mathcal{K}(F, F)$ with $\mathcal{K}(E \oplus Z, E \oplus Z)$, using here the notation of the proof of Theorem 3.1.

Let the linearizations of f and g be L_s and L_t respectively. By Theorem 3.1 we can construct a linearization L_u for h where u is an element of the Banach algebra $\mathcal{K}(F \oplus F, F \oplus F)$, given by

$$
\begin{aligned}
&u = w \oplus w - \\
&\frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega''} \begin{bmatrix} (1_E - g) \oplus 0_Z & 1_E \oplus 0_Z \\ 0_{E\oplus Z} & (1_E - f) \oplus 0_Z \end{bmatrix} (\sigma) \big(w(\sigma 1_F - w)^{-1} \oplus w(\sigma 1_F - w)^{-1} \big) \, \mathrm{d}\sigma \\
&= \begin{bmatrix} t & v \\ 0 & s \end{bmatrix}
\end{aligned}
$$

for some $v \in \mathcal{K}(F, F)$. It follows that $p_{u,\Omega} = \begin{bmatrix} p_{t,\Omega} & r \\ 0 & p_{s,\Omega} \end{bmatrix}$ for some $r \in \mathcal{K}(F, F)$. By Lemma 2.4, $p_{u,\Omega} \sim p_{t,\Omega} \oplus p_{s,\Omega}$ whence we obtain $m(fg, \Omega) = m(f, \Omega) + m(g, \Omega)$. ∎

5. THE LIFTING POSTULATE

Up to now we have imposed no condition on the operator ideal $\mathcal{J}$ other than that it should be closed. To give some idea of the scope of this we list some closed operator ideals, taken from [6], which the reader should consult for the definitions (we use calligraphic type instead of Pietsch's gothic for typographical convenience):

$\mathcal{L}$	arbitrary operators
$\mathcal{G}$	approximable operators
$\mathcal{K}$	compact operators (not to be confused with $\mathcal{K} = \mathcal{L}/\mathcal{J}$ in the rest of this paper)
$\mathcal{W}$	weakly compact operators
$\mathcal{V}$	completely continuous operators
$\mathcal{R}$	inessential operators
$\mathcal{U}$	unconditionally summing operators
$\mathcal{X}$	separable operators
$\mathcal{S}$	strictly singular operators
$\mathcal{T}$	strictly cosingular operators.

Moreover, given any operator ideal one may produce a closed operator ideal merely by passing to the closure.

We have already introduced the semigroup $M_{\mathcal{J}}$ of $\mathcal{J}$-isomorphism classes of Banach spaces. We can imbed $M_{\mathcal{J}}$ into $P_{\mathcal{J}}$ by means of the mapping $\gamma : [E]_{\mathcal{J}} \mapsto [1_E]_0$. This is an injective semigroup homomorphism. It seems unlikely that it is surjective in general.

An idempotent $p \in \mathcal{K}(E, E)$ is said to possess a *lifting* if there exists a projection $P \in \mathcal{L}(E, E)$ such that $\kappa(P) = p$.

LEMMA 5.1. *If an idempotent p has a lifting then $[p]_0$ is in the range of γ.*

Proof. Let $P \in \mathcal{L}(E,E)$ be a projection such that $\kappa(P) = p$. Define $S \in \mathcal{L}(E, \operatorname{ran} P)$ by $Sx = Px \in \operatorname{ran} P$ and let $T \in \mathcal{L}(\operatorname{ran} P, E)$ be the inclusion. Let $s = \kappa(S)$ and $t = \kappa(T)$. Then $ST = I_{\operatorname{ran} P}$ and $TS = P$, whence, $st = 1_{\operatorname{ran} P}$ and $ts = p$. We conclude that $p \sim 1_{\operatorname{ran} P}$ and so $[p]_0 = \gamma([\operatorname{ran} P]_{\mathcal{J}})$. ∎

We now introduce two postulates as possible restrictions on the operator ideal $\mathcal{J}$. Note that we only impose these postulates where explicitly stated.

(L) The lifting postulate: *every idempotent has a lifting.*

(S) The spectral postulate: *for every Banach space E, and $T \in \mathcal{J}(E,E)$, the only accumulation point of the spectrum of T, if it has an accumulation point, is* 0.

The following is now obvious.

LEMMA 5.2. *Under the lifting postulate, γ is an isomorphism of semigroups.*

It follows that under the lifting postulate we may view the multiplicity $m(f,\Omega)$ as a $\mathcal{J}$-isomorphism class of Banach spaces, thus generalizing in a natural way the multiplicity theory treated in [5], [2].

LEMMA 5.3. *The spectral postulate implies the lifting postulate.*

Proof. Assume the spectral postulate. Let p be an idempotent in $\mathcal{K}(E,E)$. Let $T \in \mathcal{L}(E,E)$ satisfy $\kappa(T) = p$. Of course T need not be a projection, but it is our task to replace it by a projection. We know that $T^2 - T \in \mathcal{J}(E,E)$. Let $S = T^2 - T$. By the spectral postulate and the spectral mapping theorem, the only possible accumulation points of the spectrum of T are 0 and 1. We can therefore find a bounded open set $\Omega \subset \mathbb{C}$ such that the boundary of Ω is disjoint with the spectrum of T, $1 \in \Omega$, but $0 \notin \Omega$. (Here we only need: 0 and 1 are not in the same connected component of the spectrum of T.) Let P be the projection associated with T and Ω. By the operational calculus, $P = f(T)$ where

$$f(z) = \begin{cases} 1, & \text{if } z \in \Omega \\ 0, & \text{if } z \notin \Omega. \end{cases}$$

Define

$$g(z) = \begin{cases} z-1, & \text{if } z \in \Omega \\ z, & \text{if } z \notin \Omega \end{cases} \quad \text{and} \quad h(z) = \begin{cases} \dfrac{1}{z}, & \text{if } z \in \Omega \\ \dfrac{1}{z-1}, & \text{if } z \notin \Omega. \end{cases}$$

Both g and h are locally analytic on the spectrum of T. Moreover,

$$z - f(z) = g(z) = (z^2 - z)h(z),$$

and applying the operational calculus we obtain

$$T - P = (T^2 - T)h(T) = Sh(T) \in \mathcal{J}(E,E)$$

and so $\kappa(P) = \kappa(T) = p$. This concludes the proof. ∎

According to [6], the following operator ideals, among those listed before, are known to satisfy the spectral postulate: $\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{T}, \mathcal{R}$. Therefore they also satisfy the lifting postulate.

For the rest of this section we have no need of the lifting postulate.

LEMMA 5.4. *Let $p \in \mathcal{K}(E,E)$ be an idempotent. There exists $\varepsilon > 0$ such that any idempotent $q \in \mathcal{K}(E,E)$ which satisfies $\|p-q\| < \varepsilon$ also satisfies $p \sim q$.*

Proof. Let $t_q = pq + (1_E - p)(1_E - q)$. Since $t_p = 1_E$ it follows that there exists $\varepsilon > 0$ such that t_q is invertible if $\|p-q\| < \varepsilon$. If, in addition, q is an idempotent, we have $pt_q = t_q q$, whence $p \sim q$. ∎

COROLLARY 5.5. *The set of liftable idempotents in $\mathcal{K}(E,E)$ is open in the set of all idempotents in $\mathcal{K}(E,E)$.*

A multiplicity theory is *homotopy invariant* if, whenever the continuous mapping $f : [0,1] \times D \to \mathcal{K}(E,E)$ is analytic in its second argument, and the set Ω is admissible for f_λ for each $\lambda \in [0,1]$ (where $f_\lambda(z) = f(\lambda, z)$), then $m(f_\lambda, \Omega)$ is independent of λ.

COROLLARY 5.6. *The multiplicity theory m defined in Section 3 is homotopy invariant.*

Proof. By the GKL-process (proof of Theorem 3.1), we can find a Banach space F and an element t_λ, depending continuously on λ, such that f_λ is s-equivalent to $z \mapsto z1_F - t_\lambda$ on a neighbourhood of $\overline{\Omega}$ for each λ. Then $p_{t_\lambda,\Omega}$ depends continuously on λ; and so, by Lemma 5.4, the class $m(f_\lambda, \Omega) = [p_{t_\lambda,\Omega}]_0$ is independent of λ. ∎

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