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REPRESENTATIONS OF $H^\infty(\mathbb{D}^N)$

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ABSTRACT. We consider dual operator algebra properties of the range of a representation from $H^\infty(\mathbb{D}^N)$ into the bounded linear operators on Hilbert space. If the representation is of C_{00} type, properties $(\mathbb{A}_{\aleph_0})$ and $X_{0,1/M}$ coincide, where M is the bound of the representation. Specializing to the representation induced by a pair of commuting contractions, we obtain an improved result.

KEYWORDS: *Dual operator algebras, representation, dilation, compression, polynomially bounded operator, commuting contractions.*

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Let $\mathcal{H}$ denote a separable infinite dimensional complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Recall that a dual algebra is a unital subalgebra of $\mathcal{L}(\mathcal{H})$ closed in the weak*-topology. We study here representations of $H^\infty(\mathbb{D}^N)$, where $\mathbb{D}^N$ is the polydisk, and a representation is a weak*-weak*-continuous linear and multiplicative map. Our concern is chiefly with those properties, arising in the study of dual algebras and the Scott Brown technique, which the range of such a representation might have. We later specialize to the case of the representation arising from a pair of commuting contraction operators. A selection of related papers is to be found in the bibliography.

The organization of this paper is as follows. After reviewing some definitions, we turn in Section 1 to the elementary study of the property in question. In Section 2 we give conditions, based on dilations, when a representation with property $(\mathbb{A}_{\aleph_0})$ has a range with some property $X_{\theta,\gamma}$ (definitions reviewed below). In Section 3 we specialize to the case of a pair of commuting contractions, and prove a similar result.

It is well known that a dual algebra $\mathcal{A}$ is the Banach dual of a certain quotient of the trace class operators; we denote this predual of $\mathcal{A}$ by $\mathcal{Q}_{\mathcal{A}}$. We denote the elements (cosets) in $\mathcal{Q}_{\mathcal{A}}$ by $[L]_{\mathcal{A}}$ or simply $[L]$ if no confusion will result. Given vectors x and y in $\mathcal{H}$, we denote by $x \otimes y$ the usual rank one operator on $\mathcal{H}$ defined by $x \otimes y(u) = (u, y)x$ for $u \in \mathcal{H}$. The dual action of some element B of $\mathcal{A}$ on $[x \otimes y]$ is $\langle B, [x \otimes y]\rangle = (Bx, y)_{\mathcal{H}}$. The elements $[x \otimes y]_{\mathcal{A}}$ have been particularly important in the theory of dual algebras. Indeed, the Scott Brown technique rests upon a certain procedure for solving an equation $[L] = [x \otimes y]$, where $[L]$ is given. Generalized, this leads to some properties: we say that a dual algebra $\mathcal{A}$ (indeed, any weak*-closed subspace of $\mathcal{L}(\mathcal{H})$) has property $(\mathbb{A}_{m,n})$, some $1 \leqslant m, n \leqslant \aleph_0$, if, given any array $\{[L_{ij}]\}_{0 \leqslant i < m, 0 \leqslant j < n} \subseteq \mathcal{Q}_{\mathcal{A}}$, there exist sequences of vectors $\{x_i\}_{0 \leqslant i < m}$, $\{y_j\}_{0 \leqslant j < n}$ in $\mathcal{H}$ such that $[L_{ij}] = [x_i \otimes y_j]$ for all i and j. We usually shorten $(\mathbb{A}_{m,m})$ to $(\mathbb{A}_m)$.

The "classical" (Scott Brown) technique to solve equations relies on iterative approximation. In this context it is useful to define the set $\mathcal{X}_\theta$, $0 \leqslant \theta < 1$, to be the set of $[L]_{\mathcal{A}}$ for which there exist sequences $\{x_n\}$ and $\{y_n\}$ in the (closed) unit ball of $\mathcal{H}$ satisfying

(a) $\limsup \|[L] - [x_n \otimes y_n]\|_{\mathcal{A}} \leqslant \theta$, and

(b) $\|[x_n \otimes w]\|_{\mathcal{A}} + \|[w \otimes y_n]\|_{\mathcal{A}} \to 0$, $w \in \mathcal{H}$.

A dual algebra $\mathcal{A}$ has property $X_{\theta,\gamma}$, $0 \leqslant \theta < \gamma \leqslant 1$, if $\overline{\mathrm{aco}}(\mathcal{X}_\theta)$ contains $B_{0,\gamma}$, where $\overline{\mathrm{aco}}(\cdot)$ denotes the closure of the absolutely convex hull, and where $B_{0,\gamma}$ is the ball in $\mathcal{Q}_{\mathcal{A}}$ of radius γ centered at the origin. (While this is the standard definition, it was later observed, first by Azoff, that $\mathcal{X}_\theta$ is always absolutely convex and closed (see [8]), and therefore "$\overline{\mathrm{aco}}(\cdot)$" may be omitted in the definition of $X_{\theta,\gamma}$.)

Let $\mathbb{D}^N$ denote the N-polydisk, $\mathbb{T}$ the unit circle, $\mathbb{C}$ the complex numbers, and $\mathbb{N}$ the positive integers. Let $H^p(\mathbb{D}^N)$, $1 \leqslant p \leqslant \infty$, denote the usual Hardy spaces of functions. It is well known that $H^\infty(\mathbb{D}^N)$ is (in a natural way) the Banach dual of a certain quotient space, thus inheriting a weak*-topology. We will consider representations Φ from $H^\infty(\mathbb{D}^N)$ into $\mathcal{L}(\mathcal{H})$; it is well known that there is a map ϕ between the preduals such that $\phi^* = \Phi$, and that good properties of Φ are reflected in those of ϕ (for example, if Φ is a surjective isometry onto a dual algebra $\mathcal{A}$ then ϕ is a surjective isometry between the appropriate preduals). Finally, for T and T' in $\mathcal{L}(\mathcal{K})$ we write $T \cong T'$ to denote T is unitarily equivalent to T'.

1. REPRESENTATIONS

Definition 1.1. A *representation* $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is a weak*-weak*-continuous linear map satisfying in addition

$$\Phi(uv) = \Phi(u)\Phi(v), \quad u, v \in H^\infty(\mathbb{D}^N).$$

A representation Φ is *unital* if $\Phi(1) = I_{\mathcal{H}}$; it is *contractive* if it satisfies the further condition

$$\|\Phi(u)\| \leqslant \|u\|_\infty, \quad u \in H^\infty(\mathbb{D}^N).$$

We will generally abbreviate weak*-weak*-continuous to weak*-continuous. Observe that (as in [4]) a representation is necessarily bounded since the image of the unit ball is necessarily weak*-compact. Note also that some authors (e.g., [16]) use "representation" to denote what we call here "contractive representation", and others (e.g., [4]) assume that the map is unital. Finally, in [17] it is shown that if Φ

is a contractive multiplicative linear map into $\mathcal{L}(\mathcal{H})$ which is absolutely continuous (in a certain sense) then Φ is weak*-continuous.

Let us recall the following lemma from [7].

LEMMA 1.2. *Suppose X and Y are Banach spaces. Let $\Phi : X^* \to Y^*$ be a weak*-continuous linear map. If Φ is one-to-one and $\Phi(X^*)$ is norm-closed, then $\Phi(X^*)$ is weak*-closed and $\Phi : X^* \to \Phi(X^*)$ is a weak*-homeomorphism.*

The following definition is slightly generalized (by removing the assumption that Φ is unital) from [4].

DEFINITION 1.3. A weak*-continuous representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is said to be (m,n)-*elementary* if for any weak*-continuous $f_{ij} : H^\infty(\mathbb{D}^N) \to \mathbb{C}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, there exist x_i and y_j in $\mathcal{H}$ such that $f_{ij}(u) = (\Phi(u)x_i, y_j)$ for all $u \in H^\infty(\mathbb{D}^N)$.

The next proposition captures some easy facts about (m,n)-elementary representations. Recall that for each $\lambda \in \mathbb{D}^N$ there is a continuous linear functional $\mathcal{E}_\lambda$ on $H^\infty(\mathbb{D}^N)$ consisting of evaluation at λ:

$$\mathcal{E}_\lambda(v) \triangleq v(\lambda), \quad v \in H^\infty(\mathbb{D}^N).$$

PROPOSITION 1.4. *Suppose $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is a $(1,1)$-elementary representation. Then*

$$\|v\| \leqslant \|\Phi(v)\| \leqslant \|\Phi\|\,\|v\|, \quad v \in H^\infty(\mathbb{D}^N);$$

that is, Φ is bounded below by 1. Further, if Φ is $(1,1)$-elementary, then Φ is a weak-homeomorphism, where this notion means that Φ is a weak*-homeomorphism onto $\Phi(H^\infty(\mathbb{D}^N))$. In particular, if Φ is contractive and $(1,1)$-elementary then Φ is an isometry and a weak*-homeomorphism.*

Proof. For the first assertion, suppose that there exists $v \in H^\infty(\mathbb{D}^N)$ satisfying $\|v\| = 1$ and $\|\Phi(v)\| < 1$. Then there exists $\lambda \in \mathbb{D}^N$ such that

$$|\mathcal{E}_\lambda(v)| = |v(\lambda)| > \|\Phi(v)\|.$$

Clearly

$$|\mathcal{E}_\lambda(v^m)| = |v^m(\lambda)| = |v(\lambda)|^m, \quad m \in \mathbb{N}.$$

Since Φ is $(1,1)$-elementary there exist vectors x and y in $\mathcal{H}$ satisfying

$$\mathcal{E}_\lambda(f) = (\Phi(f)x, y), \quad f \in H^\infty(\mathbb{D}^N),$$

so in particular

$$\mathcal{E}_\lambda(v^m) = (\Phi(v^m)x, y), \quad m \in \mathbb{N}.$$

But for each $m \in \mathbb{N}$,

$$|v(\lambda)|^m = |\mathcal{E}_\lambda(v^m)| = |(\Phi(v^m)x, y)| \leqslant \|x\| \cdot \|y\| \cdot \|\Phi(v^m)\| \leqslant \|x\| \cdot \|y\| \cdot \|\Phi(v)\|^m.$$

Then

$$\|x\| \cdot \|y\| \geqslant |v(\lambda)|^m / \|\Phi(v)\|^m, \quad m \in \mathbb{N},$$

which yields a contradiction for m sufficiently large.

Since Φ is bounded by $\|\Phi\|$ and bounded below by 1, and the range of Φ is norm-closed, the remaining part is immediate using Lemma 1.2. ∎

The property of being (m, n)-elementary for some m and n is clearly a property of the representation; we define below a property which concerns only the range of the representation.

Definition 1.5. A representation Φ of $H^\infty(\mathbb{D}^N)$ has property $(\mathbb{A}_{m,n})$, for some $1 \leqslant m, n \leqslant \aleph_0$, if $\Phi(H^\infty(\mathbb{D}^N))^-$ has property $(\mathbb{A}_{m,n})$, where the closure is taken in the weak*-topology.

The following propositions verify the intuition that (m, n)-elementary, a property on the representation, is stronger than, but related to, the property $(\mathbb{A}_{m,n})$ on the range of the representation.

Proposition 1.6. *For any m, n, $1 \leqslant m, n \leqslant \aleph_0$, if a representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is (m, n)-elementary, then Φ has property $(\mathbb{A}_{m,n})$. If Φ is a weak*-homeomorphism onto $\Phi(H^\infty(\mathbb{D}^N))$, then the above two properties are equivalent.*

Proof. We only consider the case $m = n = 1$.

For the first assertion, let $f : \Phi(H^\infty(\mathbb{D}^N))^- \to \mathbb{C}$ be a weak*-continuous linear functional (note that $\Phi(H^\infty(\mathbb{D}^N))$ is weak*-closed by Proposition 1.4. Then there exist $x, y \in \mathcal{H}$ such that

$$(f \circ \Phi)(u) = (\Phi(u)x, y), \quad \text{for all } u \in H^\infty(\mathbb{D}^N), \tag{1.1}$$

which implies that $f = x \otimes y$ on $\Phi(H^\infty(\mathbb{D}^N))$.

To prove the second claim, let $g : H^\infty(\mathbb{D}^N) \to \mathbb{C}$ be a weak*-continuous functional. Then there exist $x, y \in \mathcal{H}$ such that

$$g \circ \Phi^{-1}(\Phi(u)) = (\Phi(u)x, y), \quad \text{for all } u \in H^\infty(\mathbb{D}^N), \tag{1.2}$$

which implies that $g(u) = (\Phi(u)x, y)$. ∎

Recall (see, e.g., [2]) that if S is the unilateral shift of multiplicity one acting on $H^2(\mathbb{D})$, and $\theta \in H^\infty(\mathbb{D})$ is an inner function, there is an operator $S(\theta)$ defined on $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$ by $S(\theta) = P_{\mathcal{H}(\theta)} S|\mathcal{H}(\theta)$. Further, $\theta(S(\theta)) = 0$.

Proposition 1.7. *Let θ be some inner function in $H^\infty(\mathbb{D})$ not identically one, and let $\Phi_{S(\theta)}$ be the usual Sz.-Nagy–Foiaş functional calculus $\Phi_{S(\theta)} : H^\infty(\mathbb{D}) \to \mathcal{A}_{S(\theta)}$. Then $\Phi_{S(\theta)}$ has property $(\mathbb{A}_{1,1})$ but is not $(1,1)$-elementary.*

Proof. It is known that $\mathcal{A}_{S(\theta)}$ has property $(\mathbb{A}_{1,1})$ (see [2], Proposition III 1.21) but observe that $\Phi_{S(\theta)}$ has θ in its kernel and so cannot be $(1,1)$-elementary using Proposition 1.4. ∎

We turn next to the standard task of showing that, for representations, the properties $(\mathbb{A}_{m,n})$ are distinct. Let T be a completely non-unitary contraction on $\mathcal{H}$. Let p_i be the i-th coordinate function on $H^\infty(\mathbb{D}^N)$. Now let us define a map Φ_T, first on the p_i, then on $H^\infty(\mathbb{D}^N)$, by

$$\Phi_T(p_i^n) = T^n, \quad i = 1, \ldots, N,\ n = 0, 1, 2, \ldots. \tag{1.3}$$

Extend Φ linearly and multiplicatively. Observe that for any polynomial $p(z_1, \ldots, z_N) \in H^\infty(\mathbb{D}^N)$, there exists a one-variable polynomial $g(x)$ such that

$$\|\Phi_T(p)\| = \|p(T, \ldots, T)\| = \|g(T)\| \leqslant \|g\|_\infty \leqslant \|p\|_\infty. \tag{1.4}$$

For any $u \in H^\infty(\mathbb{D}^N)$, choose a sequence $\{p_n\}$ weak*-convergent to u, and define a map (which we still call Φ_T) on $H^\infty(\mathbb{D}^N)$ by

$$\Phi_T(u) = \lim \Phi_T(p_n). \tag{1.5}$$

By (1.5) it is easy to see that Φ_T is "well-defined" and a contractive homomorphism. Hence by [14], Propositions 4.1 and 4.6, Φ_T is weak*-continuous and a contractive representation. Let $u \in H^\infty(\mathbb{D}^N)$ and choose a sequence $\{p_n\}$ of polynomials weak*-convergent to u. Then $\Phi(p_n)$ converges weak* to $\Phi(u)$, so that $\Phi(u) \in \mathcal{A}_T$. Hence $\Phi_T(H^\infty(\mathbb{D}^N)) \subset \mathcal{A}_T$ and it is clear that $\Phi_T(H^\infty(\mathbb{D}^N))$ is weak*-dense in $\mathcal{A}_T$. Thus if we assume that $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$, then $\Phi_T(H^\infty(\mathbb{D}^N))^-$ has property $(\mathbb{A}_{m,n})$. This yields the following proposition.

Proposition 1.8. *Let T be a completely non-unitary contraction such that $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. Then Φ_T as defined above is a weak*-continuous contractive representation with property $(\mathbb{A}_{m,n})$.*

The following theorem shows that properties $(\mathbb{A}_{m,n})$, $1 \leqslant m, n \leqslant \aleph_0$, for representations are in fact distinct. For an operator T, let $T^{(n)}$ denote the n-fold ampliation of T.

Theorem 1.9. *Suppose that $(m,n) \neq (p,q)$. Then either:*

(i) *there exists a weak*-continuous representation with property $(\mathbb{A}_{m,n})$ but not property $(\mathbb{A}_{p,q})$ or*

(ii) *there exists a weak*-continuous representation with property $(\mathbb{A}_{p,q})$ but not property $(\mathbb{A}_{m,n})$.*

Proof. Let S be the unilateral shift of multiplicity 1. Then it is obvious that $S^{(n)}$ is completely non-unitary. It is well known that $\mathcal{A}_{S^{(n)}}$ has property $(\mathbb{A}_{n,\aleph_0})$ but not property $(\mathbb{A}_{n+1,1})$. Hence $\Phi_{S^{(n)}}$, as defined above, has property $(\mathbb{A}_{n,\aleph_0})$ but not property $(\mathbb{A}_{n+1,1})$, by Proposition 1.6. Similarly, $\Phi_{S^{(n)*}}$ has property $(\mathbb{A}_{\aleph_0,n})$ but not property $(\mathbb{A}_{1,n+1})$. The result then follows using the argument in [15], Theorem 6.1. ∎

Remark that the above theorem serves as well to distinguish the various properties (m,n)-elementary, since (in the case $N = 1$) $\Phi_{S^{(n)}}$ is isometric.

We turn next to consideration of these properties under similarities and compressions of representations. A representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is *similar* to a representation $\Psi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{K})$ if there is a similarity $X : \mathcal{H} \to \mathcal{K}$ such that $X^{-1}\Psi(u)X = \Phi(u)$ for all $u \in H^\infty(\mathbb{D}^N)$. If X is a surjective isometry, we abuse language slightly and say Φ is *unitarily* equivalent to Ψ (essentially, we ignore the distinction between the isometrically isomorphic Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$).

A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be Φ-*invariant* if $\Phi(u)\mathcal{M} \subset \mathcal{M}$ for all $u \in H^\infty(\mathbb{D}^N)$. Moreover, a subspace $\mathcal{M}$ is Φ-*semi-invariant* if there exist Φ-invariant subspaces $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{U} \supset \mathcal{V}$ such that $\mathcal{M} = \mathcal{U} \ominus \mathcal{V}$. We may consider the representation $\Phi_{\mathcal{M}} = \Phi_{\mathcal{U}\ominus\mathcal{V}}$ defined by

$$\Phi_{\mathcal{M}}(u) = P_{\mathcal{M}}\Phi(u)|\mathcal{M}, \quad u \in H^\infty(\mathbb{D}^N).$$

We say that Ψ is a *compression* [respectively, *compression up to similarity*] of Φ if there exists a Φ-semi-invariant subspace $\mathcal{M}$ for Φ such that Ψ is unitarily equivalent [respectively, similar] to $\Phi_{\mathcal{M}}$.

Proposition 1.10. *Suppose Ψ is a weak*-continuous representation which is a compression of a weak*-continuous representation Φ. Then:*

(i) *if Ψ is bounded below, then Φ is bounded below, and hence a homeomorphism,*

(ii) *if Ψ is bounded below by 1 and Φ is contractive, then Φ is an isometry,*

(iii) *if Ψ has property $(\mathbb{A}_{m,n})$ and is bounded below, then Φ has property $(\mathbb{A}_{m,n})$.*

Proof. (i) Suppose that Ψ is bounded below by c; by the definition of compression, we have

$$\|\Phi(u)\| \geqslant \|\Psi(u)\| \geqslant c\|u\|_\infty, \quad \text{for all } u \in H^\infty(\mathbb{D}^N). \tag{1.6}$$

(ii) Obvious.

(iii) For brevity we assume that $m = n = 1$. Suppose Φ is a representation into $\mathcal{L}(\mathcal{H})$; without loss of generality we may assume that Ψ is a representation into $\mathcal{L}(\mathcal{K})$ where the subspace $\mathcal{K}$ is semi-invariant for Φ, and that the map X guaranteed by the definition of compression is the injection of $\mathcal{K}$ into $\mathcal{H}$. Let $[L]_\mathcal{B} \in \mathcal{Q}_\mathcal{B}$, where $\mathcal{B} = \Phi(H^\infty(\mathbb{D}^N))$. Since ϕ is homeomorphism, where $\phi^* = \Phi$, by Lemma 1.2 there exists $[L'] \in Q_\mathcal{A}$, where $\mathcal{A} = \Psi(H^\infty(\mathbb{D}^N))$, such that $\phi^{-1} \circ \psi([L']_\mathcal{A}) = [L]_\mathcal{B}$. (Note that $\mathcal{A}$ and $\mathcal{B}$ are weak*-closed.) By the assumption, there exist x and y in $\mathcal{K}$ such that $[L']_\mathcal{A} = [x \otimes y]_\mathcal{A}$. Write $\mathcal{H}$ in the obvious decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{K} \oplus \mathcal{H}_2$ with respect to the semi-invariant subspace $\mathcal{K}$. Let $\widetilde{x} = 0 \oplus x \oplus 0$ and $\widetilde{y} = 0 \oplus y \oplus 0$. Then

$$\begin{aligned}\langle \Phi(u), [L]_\mathcal{B}\rangle &= \langle u, \phi([L]_\mathcal{B})\rangle = \langle u, \psi([L']_\mathcal{A})\rangle = \langle u, \psi([x \otimes y]_\mathcal{A})\rangle = \langle \Psi(u), [x \otimes y]_\mathcal{A}\rangle \\ &= (\Psi(u)x, y)_\mathcal{K} = (\Phi(u)\widetilde{x}, \widetilde{y})_\mathcal{H} = \langle \Phi(u), [\widetilde{x} \otimes \widetilde{y}]_\mathcal{B}\rangle.\end{aligned}$$

Hence $[L]_\mathcal{B} = [\widetilde{x} \otimes \widetilde{y}]_\mathcal{B}$. ∎

A representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ has an (m,n)-*cyclic set* if there exist sequences $\{e_i\}_{1\leqslant i\leqslant m}$ and $\{f_j\}_{1\leqslant j\leqslant n}$ in $\mathcal{H}$ such that

$$\bigvee_{i=1}^{m} \Phi(H^\infty(\mathbb{D}^N))e_i = \mathcal{H} \quad \text{and} \quad \bigvee_{j=1}^{n} \Phi(H^\infty(\mathbb{D}^N))^* f_j = \mathcal{H},$$

where "$\bigvee$" denotes the closed linear span.

Proposition 1.11. *Suppose that a representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is (m,n)-elementary. Assume that a representation $\Psi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{K})$ is (m,n)-cyclic and $\dim \mathcal{K} < \infty$. Then Ψ is a similarity compression of Φ.*

Proof. See [3], Theorem 4.12 or the proof of [10], Theorem 3.2. ∎

Theorem 1.12. *Suppose that a representation Φ is (m,m)-elementary. Let $\{\lambda_i\}_{i=1}^m \subset \mathbb{D}^N$, $1 \leqslant m \leqslant \infty$. Then there exist Φ-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \supset \mathcal{N}$ such that $\Phi(u)_{\mathcal{M}\ominus\mathcal{N}} \cong \mathrm{Diag}\{u(\lambda_1), \ldots, u(\lambda_m)\}$ for all $u \in H^\infty(\mathbb{D}^N)$.*

Proof. We consider only the case m finite; the modifications for $m = \aleph_0$ are not difficult. Denote the set of distinct λ_i by $\{\lambda_{k_i}\}_{i=1}^l$. Consider the representation $\Psi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathbb{C}^l)$ defined by

$$\Psi(u)e_i = u(\lambda_{k_i})e_i, \quad 1 \leqslant i \leqslant l, \tag{1.7}$$

where $\{e_i\}_{i=1}^l$ is the standard basis for $\mathbb{C}^l$. Since the $\{\lambda_{k_i}\}_{i=1}^l$ are distinct, there is some $u_0 \in H^\infty(\mathbb{D}^N)$ such that the values $u_0(\lambda_{k_i})$, $1 \leqslant i \leqslant l$, are also distinct. Observe then that the representation Ψ has a cyclic vector.

Consider now the representation $\Psi' : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathbb{C}^{m\cdot l})$ consisting of m copies of Ψ:

$$\Psi'(u) = \Psi(u) \oplus \cdots \oplus \Psi(u), \quad u \in H^\infty(\mathbb{D}^N).$$

Clearly Ψ' has an (m,m)-cyclic set. Thus Ψ' is a compression up to similarity of Φ to some space $\mathcal{M} \ominus \mathcal{N}$ semi-invariant for Φ. Suppose S is the implementing similarity. We may now imitate the proof of [14], Theorem 1.1 to extract in $\mathcal{M}\ominus\mathcal{N}$, from the images under S of the basis vectors for $\mathbb{C}^{m\cdot l}$, vectors suitable to build a diagonal compression of Φ. ∎

2. PROPERTY $X_{\theta,\gamma}$ AND PROPERTIES $(\mathbb{A}_{m,n})$

We need some definitions: a representation Φ is said to be of $C_{0\cdot}$ type if Φ is weak*-SOT continuous on bounded sets. A representation Φ is said to be of $C_{\cdot 0}$ type if $\Phi^*(h) := \Phi(h)^*$ is of $C_{0\cdot}$ type. Also, a representation Φ is of C_{00} type if it is of both $C_{0\cdot}$ and $C_{\cdot 0}$ types.

We need as well some classes of contractions defined in [19]: a contraction T in $\mathcal{L}(\mathcal{H})$ is in the class $C_{0\cdot}$ if $\|T^n x\| \to 0$ for all $x \in \mathcal{H}$, and is in the class $C_{\cdot 0}$ if T^* is in $C_{0\cdot}$. As well, T is in $C_{1\cdot}$ if $\|T^n x\| \to 0$ implies $x = 0$, and $C_{\cdot 1}$ is again defined by duality. The classes $C_{\alpha\beta}$, α and β each 0 or 1, are defined by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$.

Recall (cf. [14]) that $E \subset \mathbb{D}^N$ is *dominating* for $H^\infty(\mathbb{D}^N)$ if

$$\|u\|_\infty = \sup_{z\in E} |u(z)|, \quad u \in H^\infty(\mathbb{D}^N). \tag{2.1}$$

Let $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be a representation. Let T_i denote $\Phi(p_i)$, where p_i is the i-th coordinate function in $H^\infty(\mathbb{D}^N)$.

Various authors have considered conditions under which a *contractive* Φ has (in our language) property $(\mathbb{A}_{\aleph_0,\aleph_0})$. Recall that as in [17] the joint left essential spectrum $\sigma_{\rm le}(T_1,\ldots,T_N)$ of a tuple of contractions $(T_1,\ldots,T_N)$ is the set of $\lambda = (\lambda_1,\ldots,\lambda_N)$ in $\mathbb{C}^N$ for which there exists an orthonormal sequence $\{x_n\}$ satisfying the condition

$$\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0, \quad 1 \leqslant i \leqslant N.$$

From [17] we have that if Φ is a representation such that $\Phi(p_i) = T_i$, $1 \leqslant i \leqslant N$, and $\sigma_{\rm le}(T_1,\ldots,T_N) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $\Phi(H^\infty(\mathbb{D}^N))$ is weak*-closed. The following is in the result [17], Theorem 4.16 and its proof.

THEOREM 2.1. *Suppose that a representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is of $C_{0\cdot}$ type, is isometric, and*

$$\sigma_{\rm le}(T_1,\ldots,T_N) \cap \mathbb{D}^N \tag{2.2}$$

is dominating for $H^\infty(\mathbb{D}^N)$. Then Φ has property $(\mathbb{A}_{\aleph_0,\aleph_0})$.

In [4] there is another condition sufficient for a representation to have property $(\mathbb{A}_{\aleph_0,\aleph_0})$. (See also Theorem 1.12 above for a result implying the converse of this result.)

THEOREM 2.2. ([4], Theorem 4.1) *Suppose* $\{\lambda_j \in \mathbb{D}^N : j \geqslant 0\}$ *is dominating for* $H^\infty(\mathbb{D}^N)$. *If* $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ *is defined by*

$$\Phi(u)e_j = u(\lambda_j)e_j, \quad \text{for all } u \in H^\infty(\mathbb{D}^N),\ j \geqslant 0, \tag{2.3}$$

then Φ *has property* $(\mathbb{A}_{\aleph_0,\aleph_0})$. *(In fact,* Φ *is* $(\mathbb{A}_{\aleph_0,\aleph_0})$*-elementary.)*

The next lemma is implicit in [17], Lemma 4.8 or see [4]; in fact, the $[C_\lambda]$ appearing is $\phi^{-1}(\mathcal{E}_\lambda)$, where ϕ is a map between preduals satisfying $\phi^* = \Phi$, and $\mathcal{E}_\lambda$ is as before Proposition 1.4.

LEMMA 2.3. *If* Φ *is a weak*-weak*-homeomorphism (in particular, if* Φ *is an isometry) and* $\lambda \in \mathbb{D}^N$, *then there exists* $[C_\lambda] \in Q_{\mathcal{A}}$, *where* $\mathcal{A} = \Phi(H^\infty(\mathbb{D}^N))$, *such that*

$$\langle [C_\lambda]_{\mathcal{A}}, \Phi(u)\rangle = \operatorname{tr}(\Phi(u)C_\lambda) = u(\lambda), \quad \text{for all } u \in H^\infty(\mathbb{D}^N). \tag{2.4}$$

The following is the main theorem of this section. It should be compared to [17], Theorem 5.2, which deals with the case of an n-tuple of commuting contractions, and also with [19], Theorem 2.14, in which is considered the representation from $H^\infty(\mathbb{D})$ associated with an absolutely continuous polynomially bounded operator. See also the results in [12], [11], and related papers.

THEOREM 2.4. *Suppose that* $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ *is a representation bounded below and of* C_{00} *type. Then* Φ *has property* $(\mathbb{A}_1)$ *if and only if* Φ *has property* $(\mathbb{A}_{\aleph_0})$.

We prove the theorem by a sequence of lemmas; from the definitions, it is clear that we need only show that property $(\mathbb{A}_1)$ yields property $(\mathbb{A}_{\aleph_0})$ under our additional assumptions. For brevity, we write $\mathcal{A} = \Phi(H^\infty(\mathbb{D}^N))$ and $T_i = \Phi(p_i)$, where p_i is the i-th coordinate function in $H^\infty(\mathbb{D}^N)$.

LEMMA 2.5. *Suppose that* Φ *is a representation bounded below. If* E *is dominating for* $H^\infty(\mathbb{D}^N)$, *then* $(1/\|\Phi\|) \cdot B_{0,1} \subseteq \overline{\operatorname{aco}}\{[C_\lambda] : \lambda \in E\}$ *in* $Q_{\mathcal{A}}$.

Proof. Since E is dominating for $H^\infty(\mathbb{D}^N)$, we have

$$\sup_{\lambda\in E} |u(\lambda)| = \|u\|_\infty, \quad u \in H^\infty(\mathbb{D}^N). \tag{2.5}$$

Then by Lemma 2.3 we have

$$\|\Phi(u)\| \leqslant \|\Phi\| \cdot \|u\|_\infty = \|\Phi\| \sup_{\lambda\in E} \big|\langle \Phi(u), [C_\lambda]\rangle\big|, \quad u \in H^\infty(\mathbb{D}^N). \tag{2.6}$$

By [9], Proposition 2.2, $\overline{\operatorname{aco}}\{[C_\lambda] : \lambda \in E\} \supset (1/\|\Phi\|) \cdot B_{0,1}$. ∎

The following result needs only minor adaptations from [3], Proposition 6.5.

LEMMA 2.6. *Let Φ be a representation bounded below and of C_{00} type. Then for any sequence $\{x_j\}_{j=1}^\infty$ converging weakly to* 0,

$$\|[x_j \otimes z]\|_{\mathcal{A}} \to 0 \quad \textit{and} \quad \|[z \otimes x_j]\|_{\mathcal{A}} \to 0, \quad z \in \mathcal{H}. \tag{2.7}$$

Proof. Since Φ is of C_{00} type, by [17], Lemma 5.1 we have that if $x_j \to 0$ weakly, then $\|[z \otimes x_j]\| \to 0,$ for all $z \in \mathcal{H}$ (cf. [14], Lemma 4.12; again, the proof is essentially unchanged if the assumption there that Φ is an isometry is replaced by Φ bounded below and above). The proof for the other limit follows by considering $\Psi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ defined by $\Psi(u) = \Phi(u)^*$ and standard techniques exchanging one predual for another (using, for example, that if $\mathcal{B} = \Psi(H^\infty(\mathbb{D}^N))$, then $\|[x_j \otimes z]\|_{\mathcal{A}} = \|[z \otimes x_j]\|_{\mathcal{B}}$). ∎

We now discuss the converse implication of Lemma 2.6 as follows.

REMARK 2.7. Let Φ be a representation with the T_i absolutely continuous on $\mathcal{H}$. Assume that for any sequence $\{x_j\}$ converging weakly to 0,

$$\|[x_j \otimes z]\|_{\mathcal{A}} + \|[z \otimes x_j]\|_{\mathcal{A}} \to 0, \quad z \in \mathcal{H}.$$

Then $T_i \in C_{00}$ for all i, $1 \leqslant i \leqslant N$. (Indeed, suppose $T_i \notin C_{\cdot 0}$ for some i, i.e., there exists $z \in \mathcal{H}$ such that $\|T_i^{*^n} z\| \not\to 0$. It is clear that T_i has no singular unitary part (since T_i is absolutely continuous); observe also that T_i is polynomially bounded since Φ is a representation. If we then consider the Mlak functional calculus (a generalization of the Sz.-Nagy–Foiaş functional calculus (see [20] and [21], or [18])), it is known that for any $y \in \mathcal{H}$ one has

$$(T_i^{*n} z, y)_{\mathcal{H}} = c_{-n}(\phi_{T_i^*}([z \otimes y]_{T_i^*}) \to 0,$$

where c_j is the j-th Fourier coefficient. Therefore $T_i^{*n} z \to 0$ weakly. But then

$$\|T_i^{*^n} z\|^2 = (T_i^n T_i^{*^n} z, z) = \langle T_i^n, [T_i^{*^n} z \otimes z]_{T_i} \rangle \leqslant \|[T_i^{*^n} z \otimes z]\|_{T_i} \leqslant \|[T_i^{*^n} z \otimes z]\|_{\mathcal{A}}.$$

Then it is clear that the first limit in (2.7) need not hold, and we have the result by contraposition. The proof for $T_i \in C_{0\cdot}$ (i.e., $T_i^* \in C_{\cdot 0}$) is similar.) Note that it is not known whether $T_i \in C_{00}$ for all i, $1 \leqslant i \leqslant N$, implies Φ is of C_{00} type, because there is a mistake in the proof of [17], Lemma 5.1 (for example, M depends on ε in [17], p. 412 as has been pointed out by Frédéric Jaeck).

The next lemma completes the proof of the theorem.

LEMMA 2.8. *Suppose Φ has property $(\mathbb{A}_1)$, is bounded below, and is of C_{00} type. Then $\mathcal{A}$ has property $X_{0,1/\|\Phi\|}$, and thus property $(\mathbb{A}_{\aleph_0})$.*

Proof. Since $\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, by Lemma 2.5 we have

$$(1/\|\Phi\|) \cdot B_{0,1} \subset \overline{\mathrm{aco}}\{[C_\lambda] : \lambda \in \mathbb{D}^N\}. \tag{2.8}$$

We will prove that $\{[C_\lambda] : \lambda \in \mathbb{D}^N\} \subset \mathcal{X}_0(\mathcal{A})$. Since $T_i \in C_{00}$, according to Lemma 2.6, it is sufficient to show that for any $\lambda \in \mathbb{D}^N$ there exists a sequence $\{f_j\}_{j=1}^\infty \subset \mathcal{H}$, such that $f_j \to 0$ weakly and

$$[f_j \otimes f_j] = [C_\lambda], \quad j = 1, 2, \ldots.$$

We consider first the case $\lambda = 0$; fix $m \in \mathbb{N}$ for the moment, fix ε such that $1 > \varepsilon > 0$, and let J_m be the operator on $\mathbb{C}^m$ whose matrix (with respect to some orthonormal basis) is

$$J_m = \begin{bmatrix} 0 & \varepsilon & & & 0 & \\ & 0 & \varepsilon & & & \\ & & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & \varepsilon \\ & & & & & 0 \end{bmatrix}. \tag{2.9}$$

We construct a representation Ψ as follows: define Ψ on $H^\infty(\mathbb{D}^N)$, first on the p_i, by

$$\Psi(p_i^n) = J_m^n, \quad i = 1, \dots, N,\ n = 0, 1, 2, \dots. \tag{2.10}$$

Extend the definition of Ψ linearly and multiplicatively. As before (see (1.5)), for any polynomial $p(z_1, \dots, z_N) \in H^\infty(\mathbb{D}^N)$, we have

$$\|\Psi(p)\| = \|p(J_m, \dots, J_m)\| \leqslant \|p\|_\infty. \tag{2.11}$$

We may extend Ψ to $H^\infty(\mathbb{D}^N)$ using limits of polynomials as usual, and Ψ is "well-defined" and a contractive representation. Hence by [14], Propositions 4.1 and 4.6, Ψ is weak*-continuous.

Note also that Ψ is $(1, 1)$-cyclic. Since Φ has property $(\mathbb{A}_1)$, by Proposition 1.11, there exists a Φ semi-invariant subspace $\mathcal{M} \ominus \mathcal{N}$ with $\dim(\mathcal{M} \ominus \mathcal{N}) = m$ and a similarity $S : \mathcal{M} \ominus \mathcal{N} \to \mathbb{C}^m$ so that $S^{-1}\Psi(u)S = \Phi_{\mathcal{M}\ominus\mathcal{N}}(u)$ for all $u \in H^\infty(\mathbb{D}^N)$. Let

$$\mathcal{L}_j = \bigcap_{i=1}^{N} \operatorname{Ker}(\Phi_{\mathcal{M}\ominus\mathcal{N}}(z_i^j)), \quad j = 1, \dots, m.$$

It is easy to show, using S and Ψ, that the subspaces $\mathcal{L}_1, \mathcal{L}_2 \ominus \mathcal{L}_1, \dots, \mathcal{L}_m \ominus \mathcal{L}_{m-1}$ are each of dimension one. Further, if the finite sequence $\{e_j^m\}_{j=1}^m$ is chosen by choosing a unit vector from each of these latter subspaces, one has

$$[C_0]_{\mathcal{A}} = [e_j^m \otimes e_j^m]_{\mathcal{A}}, \quad j = 1, \dots, m. \tag{2.12}$$

Observe that we may perform this construction for each m. Now applying the usual method in the theory of dual algebras (see, for example, the proof of [3], Theorem 6.6), we can extract from the finite families $\{e_j^{(m)}\}_{j=1}^m$, $m = 1, 2, \dots$, a sequence $\{f_j\}$ so that $f_j \to 0$ weakly and $[f_j \otimes f_j] = [C_0]$ for all $j = 1, \dots$.

To extend the result to arbitrary $\lambda \in \mathbb{D}^N$, we use another standard device, which is the introduction of the Möbius transform $\varphi_\mu : \mathbb{D} \to \mathbb{D}$ defined by

$$\varphi_\mu(z) = (z - \mu)/(1 - \overline{\mu} z).$$

Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$ be fixed, and consider the map $F : \mathbb{D}^N \to \mathbb{D}^N$ defined by

$$F(z_1, \dots, z_N) = (\varphi_{\lambda_1}(z_1), \dots, \varphi_{\lambda_N}(z_N)), \quad z = (z_1, \dots, z_N) \in \mathbb{D}^N.$$

Consider the representation Φ_λ given by

$$\Phi_\lambda(u) = \Phi(u \circ F), \quad u \in H^\infty(\mathbb{D}^N).$$

Note that Φ and Φ_λ have the same range, and thus Φ_λ has property $(\mathbb{A}_1)$. A computation shows that the sequence of vectors $\{f_j\}_{j=1}^\infty$ that we may construct via the argument above, weakly convergent to zero and satisfying

$$[C_0]_{\Phi_\lambda(H^\infty(\mathbb{D}^N))} = [f_j \otimes f_j]_{\Phi_\lambda(H^\infty(\mathbb{D}^N))}, \quad j = 1, \ldots,$$

satisfies also

$$[C_\lambda]_{\Phi(H^\infty(\mathbb{D}^N))} = [f_j \otimes f_j]_{\Phi(H^\infty(\mathbb{D}^N))}, \quad j = 1, \ldots.$$

This is the requisite sequence of vectors, and since λ was arbitrary, we have the lemma and thus the proof of the theorem. ∎

We write $w(T_1, \ldots, T_N)$ for the weak closed algebra generated by T_i, $i = 1, \ldots, N$. The following may be compared to [17], Proposition 5.3.

COROLLARY 2.9. *Suppose Φ is an isometric representation of C_{00} type. If Φ has property $(\mathbb{A}_1)$, then $w(T_1, \ldots, T_N)$ is reflexive, where $T_i = \Phi(p_i)$, $i = 1, \ldots, N$.*

Proof. Since $\Phi(H^\infty(\mathbb{D}^N))$ has property $X_{0,1}$, it is closed in the weak operator topology and is reflexive using [3], Theorem 9.22. ∎

From Propositions 1.4 and 1.6 we obtain immediately the following corollary.

COROLLARY 2.10. *Suppose Φ is a $(1,1)$-elementary representation of C_{00} type. Then $w(T_1, \ldots, T_N)$ is reflexive, where $T_i = \Phi(p_i)$, $i = 1, \ldots, N$.*

3. PAIRS OF COMMUTING CONTRACTIONS

We turn in this section to consideration of one of the important examples of a representation from $H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ with $N \neq 1$, namely that arising from the functional calculus for a pair of commuting and absolutely continuous contractions. (Recall that a contraction is absolutely continuous if its unitary part is absent or has spectral measure absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$.) The choice of $N = 2$, that is, a *pair* of commuting contractions is occasioned, of course, by the fact that such a pair has a (minimal) joint isometric or unitary dilation and joint coisometric extension, not necessarily available for n-tuples of commuting contractions (see [24]). From now on T will denote a pair (T_1, T_2) of commuting contractions yielding an absolutely continuous representation. (This is not known to be the same as each of the T_i absolutely continuous; see, for example, [16], Lemma 2.2.) The effort to extend the Scott Brown technique from a single contraction to pairs is only partially complete: our goal is to make a modest contribution to the still open question of whether $\mathcal{A}_T$ having property $(\mathbb{A}_{\aleph_0})$ implies $\mathcal{A}_T$ has some property $X_{\theta,\gamma}$.

We shall use the standard machinery of the functional calculus for a pair of commuting contractions yielding an absolutely continuous representation and matters of dilations and extensions without further comment; [22] is a source for our approach (see also [25] and [6]). Various authors have studied pairs (even

n-tuples) of commuting contractions under various hypotheses (often including spectral conditions) which yield some property $X_{\theta,\gamma}$ with $0 \leqslant \theta < \gamma \leqslant 1$, and therefore both property $(\mathbb{A}_{\aleph_0})$ for, and reflexivity, of the relevant algebra (see, for example, [17], [16], [15], [11], and [1]). In [23] the author considers explicitly the question of when, for a pair of commuting contractions, property $(\mathbb{A}_{\aleph_0})$ implies property $X_{\theta,\gamma}$, and announces the following result [23], Main Theorem:

Theorem 3.1. *If $T = (T_1, T_2)$ is a pair of commuting contractions yielding an isometric absolutely continuous representation, $T_1 \in C_{0\cdot}$, and $T_2 \in C_{\cdot 0}$, and $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$, then $\mathcal{A}_T$ has property $X_{0,1}$.*

(We remark that we are unable to obtain Lemmas 4.4 and 4.5 of that paper in the generality stated; a citation is made to Lemma 3.3 of [16], but no substitute is provided for the assumption of that lemma that a certain sequence $\{x_n\}$ satisfies a condition of the form $\|(T_2-\lambda)x_n\| \to 0$. However, the theorem may be recovered by a more direct use of [16], Lemma 3.3.)

Let us assemble some definitions for our related result. If $T = (T_1, T_2)$ is a pair of commuting contractions they are said to doubly commute in case $T_1T_2^* = T_2^*T_1$. If (T_1, T_2) acts on $\mathcal{H}$ there is a joint coisometric extension (B_1^*, B_2^*) of T, minimal in a standard sense. Note that two minimal joint coisometric extensions of T need not be isomorphic. We call a pair $T = (T_1, T_2)$ *diagonally extendable* if there is a minimal joint coisometric extension (B_1^*, B_2^*) acting on $\mathcal{K}$ containing $\mathcal{H}$ such that, for j either 1 or 2, if $\mathcal{K}$ is decomposed as $\mathcal{K} = \mathcal{S}_j \oplus \mathcal{R}_j$, with $S_j^* \triangleq B_j^*|\mathcal{S}_j$ a backward shift and $R_j \triangleq B_j^*|\mathcal{R}_j$ a unitary operator, then each of $\mathcal{S}_j$ and $\mathcal{R}_j$ is reducing for B_k^*, $k \neq j$. We will call such a pair (B_1^*, B_2^*) a (D.E.)-m.j.c.e. of T. We will use from [22], Theorem 2.5 and [25], Lemma 1 the following proposition.

Proposition 3.2. *With the notation as above, the pair (T_1, T_2) is diagonally extendable under any of the following conditions:*
(i) *R_1 has no part of uniform infinite multiplicity;*
(ii) *R_2 has no part of uniform infinite multiplicity;*
(iii) *T_1 and T_2 doubly commute.*

It is useful to generalize a certain "hereditary" condition on minimal isometric dilations studied in [13]. Let $T = (T_1, T_2)$ acting on $\mathcal{H}$ be a pair of commuting contractions with $U = (U_1, U_2)$ a minimal joint isometric dilation acting on $\mathcal{K}$. Note that contrary to the one variable case, minimal joint unitary dilations of a pair of contractions are not always isometrically similar. For each i, $i = 1, 2$, we may decompose the space $\mathcal{K}$ as $\mathcal{K} = \mathcal{K}_{\rm s}^i \oplus \mathcal{K}_{\rm r}^i$, where $U_i|\mathcal{K}_{\rm s}^i$ is a (forward) unilateral shift of some multiplicity, and $U_i|\mathcal{K}_{\rm r}^i$ is a unitary operator (of course, some of the spaces may happen to be (0)). If $\mathcal{M} \subseteq \mathcal{H}$ is a subspace invariant for T, we may consider the pair of contractions $(\widetilde{T_1}, \widetilde{T_2})$ given by $\widetilde{T_i} = T_i|\mathcal{M}$, $i = 1, 2$. Since $U = (U_1, U_2)$ is a joint isometric dilation of $(\widetilde{T_1}, \widetilde{T_2})$, there is some subspace $\widetilde{\mathcal{K}}$ of $\mathcal{K}$, invariant for U, so that $(U_1, U_2)|\widetilde{\mathcal{K}}$ is a minimal joint isometric dilation of $(\widetilde{T_1}, \widetilde{T_2})$.

Denote by $\widetilde{U} = (\widetilde{U_1}, \widetilde{U_2})$ the dilation obtained in this fashion, and call it the canonical m.j.i.d. of $(\widetilde{T_1}, \widetilde{T_2})$ induced by U. Of course, $\widetilde{\mathcal{K}}$ may itself be decomposed into shift and unitary spaces, so let $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_s^i \oplus \widetilde{\mathcal{K}}_r^i$, where $\widetilde{U_i}|\widetilde{\mathcal{K}}_s^i$ is a (forward) unilateral shift of some multiplicity, and $\widetilde{U_i}|\widetilde{\mathcal{K}}_r^i$ is a unitary operator.

We use this notation in the definition that follows. Observe also that the definition requires a modification from the single variable case, again since minimal dilations of a pair (T_1, T_2) need not be isometrically similar.

DEFINITION 3.3. Let $T = (T_1, T_2)$ acting on $\mathcal{H}$ be a pair of commuting contractions with $U = (U_1, U_2)$ acting on $\mathcal{K}$ a minimal joint isometric dilation. We say that the pair (T_1, T_2) satisfies *property* (H) if there exists (U_1, U_2) acting on $\mathcal{K}$ a minimal joint isometric dilation such that, for each (non-zero) subspace $\mathcal{M}$, $\widetilde{\mathcal{K}}_s^i \subseteq \mathcal{K}_s^i$, $i = 1, 2$; call such a U an (H)-m.j.i.d. of T.

Remark that it is easy to show that $\widetilde{\mathcal{K}}_r^i \subseteq \mathcal{K}_r^i$, $i = 1, 2$. Note also that there is a weakened version of this property in which the containment is assumed only for one value of i; we leave to the reader consideration of which of the results to follow extend under this weaker hypothesis. Finally, this property could equally well have been defined in terms of the minimal joint coisometric extension of (T_1^*, T_2^*).

We record the following result and omit the proof.

PROPOSITION 3.4. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions with $U = (U_1, U_2)$ a minimal joint isometric dilation acting on $\mathcal{K}$. If U is an* (H)-m.j.i.d. *of T, and $\mathcal{M}$ is any non-zero subspace invariant for T, then the pair $(T_1|\mathcal{M}, T_2|\mathcal{M})$ has property* (H)*, and in fact the canonical* m.j.i.d. *of $(T_1|\mathcal{M}, T_2|\mathcal{M})$ is an* (H)-m.j.i.d. *of $(T_1|\mathcal{M}, T_2|\mathcal{M})$. Also, if neither U_1 nor U_2 contains a bilateral shift, then (T_1, T_2) has property* (H) *and* any *minimal joint isometric dilation of T is an* (H)-m.j.i.d. *of T.*

In the next lemma we consider simultaneously property (H) and that of being diagonally extendable.

LEMMA 3.5. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions with property* (H) *yielding an absolutely continuous representation, and that U is an* (H)-m.j.i.d. *of T such that U^* is a* (D.E.)-m.j.c.e. *of the pair (T_1^*, T_2^*). Suppose $\mathcal{M}$ is any subspace invariant for T. Then the pair $((T_1|\mathcal{M})^*, (T_2|\mathcal{M})^*)$ is diagonally extendable. In fact, the canonical* m.j.i.d. *for $(T_1|\mathcal{M}, T_2|\mathcal{M})$ is an* (H)-m.j.i.d. *for the pair and its adjoint is a* (D.E.)-m.j.c.e. *for the pair $((T_1|\mathcal{M})^*, (T_2|\mathcal{M})^*)$.*

Proof. Denote the pair $((T_1|\mathcal{M})^*, (T_2|\mathcal{M})^*)$ by $(\widetilde{T_1}^*, \widetilde{T_2}^*)$. With this notation, we must be concerned with a minimal joint coisometric extension of $(\widetilde{T_1}^*, \widetilde{T_2}^*)$. Any such extension is the adjoint of a minimal joint isometric dilation of $(\widetilde{T_1}, \widetilde{T_2})$. Let (U_1, U_2) acting on $\mathcal{K}$ be as in the hypothesis. Since (T_1, T_2) is a joint dilation of $(\widetilde{T_1}, \widetilde{T_2})$, we may produce the canonical m.j.i.d $(\widetilde{U_1}, \widetilde{U_2})$ of $(\widetilde{T_1}, \widetilde{T_2})$ as a restriction of (U_1, U_2) to some jointly invariant subspace $\widetilde{\mathcal{K}}$.

Suppose we write $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_s^1 \oplus \widetilde{\mathcal{K}}_r^1$, where $\widetilde{U_1}|\widetilde{\mathcal{K}}_s^1$ is a (forward) unilateral shift of some multiplicity, and $\widetilde{U_1}|\widetilde{\mathcal{K}}_r^1$ is a unitary operator. We claim first that $\widetilde{U_2}$ is (also) diagonal with respect to this decomposition. Since U is an (H)-m.j.i.d. for T, we have $\widetilde{\mathcal{K}}_s^1 \subseteq \mathcal{K}_s^1$, and always $\widetilde{\mathcal{K}}_r^1 \subseteq \mathcal{K}_r^1$. Writing $\widetilde{U_1} = \widetilde{S_1} \oplus \widetilde{R_1}$, and writing the space $\mathcal{K}$ with respect to the decomposition

$$\mathcal{K} = \widetilde{\mathcal{K}}_s^1 \oplus (\mathcal{K}_s^1 \ominus \widetilde{\mathcal{K}}_s^1) \oplus \widetilde{\mathcal{K}}_r^1 \oplus (\mathcal{K}_r^1 \ominus \widetilde{\mathcal{K}}_r^1),$$

we have

$$U_1 = \begin{bmatrix} \widetilde{S_1} & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & \widetilde{R_1} & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Observe that U_2 is diagonal with respect to the decomposition $\mathcal{K} = \mathcal{K}_{\rm s}^1 \oplus \mathcal{K}_{\rm r}^1$ (this follows easily from U^* a (D.E.)-m.j.c.e. for (T_1^*, T_2^*)). If use this to write U_2 with respect to the decomposition above, and note $\widetilde{U_2}$ is the restriction of U_2 to the invariant subspace $\widetilde{\mathcal{K}}_{\rm s}^1 \oplus \widetilde{\mathcal{K}}_{\rm r}^1$, we see that $\widetilde{U_2}$ is indeed diagonal with respect to this decomposition.

Of course $(\widetilde{U_1}^*, \widetilde{U_2}^*)$ is a minimal joint coisometric extension of $(\widetilde{T_1}^*, \widetilde{T_2}^*)$, and upon using the diagonality deduced above, and taking adjoints, we have indeed that $\widetilde{U_2}^*$ is diagonal when $\widetilde{\mathcal{K}}$ is written as a direct sum of the (backwards) shift space of $\widetilde{U_1}^*$ and the unitary space of $\widetilde{U_1}^*$. This is half of what is required to show that $(\widetilde{T_1}^*, \widetilde{T_2}^*) = ((T_1|\mathcal{M})^*, (T_2|\mathcal{M})^*)$ is diagonally extendable, and the proof with indices reversed is entirely similar. The final observation is obtained by noting that the canonical m.j.i.d. for $(\widetilde{T_1}, \widetilde{T_2})$ has been the one used throughout the proof. ∎

The next lemma provides, in circumstances based on a dilation, some of the "vanishing conditions" required to achieve property $X_{\theta,\gamma}$.

Lemma 3.6. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions with property* (H) *yielding an absolutely continuous representation, and that U is an* (H)-m.j.i.d. *of T such that U^* is a* (D.E.)*-m.j.c.e of the pair (T_1^*, T_2^*). Assume further that there is some infinite dimensional subspace $\mathcal{M}$, jointly semi-invariant for (T_1, T_2), and scalars λ_1 and λ_2 in the disk $\mathbb{D}$ so that $T_{1\mathcal{M}} = \lambda_1 I_{\mathcal{M}}$ and $T_{2\mathcal{M}} = \lambda_2 I_{\mathcal{M}}$. Suppose $\{e_n\}$ is an orthonormal basis for $\mathcal{M}$. Then*

$$\|[e_n \otimes z]\|_{\mathcal{A}_T} \to 0, \quad z \in \mathcal{H}.$$

Proof. We consider only the case $\lambda_1 = \lambda_2 = 0$; the general case may be obtained by a modification of the argument, or by a standard device with Möbius transforms. We note for future use that the sequence $\{e_n\}$ tends weakly to zero. Since $\mathcal{M}$ is semi-invariant, it is some difference $\mathcal{N} \ominus \mathcal{L}$, where $\mathcal{N}$ and $\mathcal{L}$ are subspaces invariant for T with $\mathcal{L} \subseteq \mathcal{N}$. From the action of $\mathcal{A}_T$ on $[e_n \otimes z]_{\mathcal{A}_T}$, it clearly suffices to assume that $z \in \mathcal{N}$. By the previous proposition, the pair $(T_{1\mathcal{N}}, T_{2\mathcal{N}})$ satisfies exactly the same assumptions as (T_1, T_2). Thus we may assume without loss of generality that $\mathcal{N} = \mathcal{H}$, and that the vectors $\{e_n\}$ are in the kernels of T_1^* and T_2^* (that is, the dilations of $0 \cdot I_{\mathcal{M}}$ are to the orthogonal complement of an invariant subspace, and not merely a general semi-invariant subspace).

Let (V_1^*, V_2^*) be a joint coisometric extension of (T_1^*, T_2^*) meeting the requirement for diagonal extendibility (for example, U^*, but at this stage any (D.E.)-m.j.c.e. suffices). Observe that

$$\begin{aligned} \|[e_n \otimes z]\|_{\mathcal{A}_T} = \|[z \otimes e_n]\|_{\mathcal{A}_{T^*}} &= \sup_{h \in H^\infty(\mathbb{D}^2),\, \|h\| \leqslant 1} |\langle h(T_1^*, T_2^*) z, e_n\rangle| \\ &= \sup_{h \in H^\infty(\mathbb{D}^2),\, \|h\| \leqslant 1} |\langle h(V_1^*, V_2^*) z, e_n\rangle|. \end{aligned}$$

Let $V_1^* = S_1^* \oplus R_1^*$ act on $\mathcal{K} = \mathcal{K}_{\rm s}^1 \oplus \mathcal{K}_{\rm r}^1$ expressed as the shift and unitary spaces as usual, and, using diagonal extendibility, suppose also that $V_2^* = A \oplus B$ with respect to this decomposition. Observe that since $e_n \in \operatorname{Ker}(T_1^*)$, we have $e_n \in \operatorname{Ker}(V_1^*)$, so $e_n \in \mathcal{K}_{\rm s}^1$. These spaces are reducing for V_1^* (and V_2^*) so we may assume $z \in \mathcal{K}_{\rm s}^1$ as well. So

$$\|[e_n \otimes z]\|_{\mathcal{A}_T} = \sup_{h \in H^\infty(\mathbb{D}^2),\, \|h\| \leqslant 1} |\langle h(S_1^*, A)z, e_n\rangle| = \|[z \otimes e_n]\|_{\mathcal{A}_{(S_1^*, A)}}.$$

Note finally that since $e_n \in \operatorname{Ker}(T_2^*)$, $e_n \in \operatorname{Ker}(V_2^*)$, so $e_n \in \operatorname{Ker}(A)$. We may now use Lemma 3.3 of [16] to deduce $\|[e_n \otimes z]\|_{\mathcal{A}_T} \to 0$. ∎

By combining the lemma with its dual, we obtain the following.

Proposition 3.7. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions acting on $\mathcal{H}$ yielding an isometric absolutely continuous representation.*

Suppose (T_1, T_2) has an (H)-m.j.i.d. *whose adjoint is a* (D.E.)-m.j.c.e. *for T^*, and that (T_1^*, T_2^*) satisfies the same hypothesis. Assume further that there is some infinite dimensional subspace $\mathcal{M}$, jointly semi-invariant for (T_1, T_2), and scalars λ_1 and λ_2 in the disk $\mathbb{D}$ so that $T_{1\mathcal{M}} = \lambda_1 I_{\mathcal{M}}$ and $T_{2\mathcal{M}} = \lambda_2 I_{\mathcal{M}}$. Suppose $\{e_n\}$ is an orthonormal basis for $\mathcal{M}$. Then*

$$\|[e_n \otimes z]\|_{\mathcal{A}_T} + \|[z \otimes e_n]\|_{\mathcal{A}_T} \to 0, \quad z \in \mathcal{H}.$$

We finally have the theorem.

Theorem 3.8. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions acting on $\mathcal{H}$ yielding an isometric absolutely continuous representation. Suppose (T_1, T_2) has an* (H)-m.j.i.d. *whose adjoint is a* (D.E.)-m.j.c.e. *for T^*, and that (T_1^*, T_2^*) satisfies the same hypothesis. Suppose finally that $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$. Then $\mathcal{A}_T$ has property $X_{0,1}$.*

Proof. (⇐) See [3], Theorem 3.7.

(⇒) It suffices, by standard arguments, to show that $\{[C_\lambda] : \lambda \in \mathbb{D}^2\} \subset \mathcal{X}_0(\mathcal{A}_T)$. It is also well known from the dilation theory of pairs (T_1, T_2) with property $(\mathbb{A}_{\aleph_0})$ that there is, for each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, a subspace $\mathcal{M}$ semi-invariant for T of the type considered in the previous proposition. It is well known that for each e a unit vector in $\mathcal{M}$, we have $[C_\lambda] = [e \otimes e]$. It is then easy to use the previous proposition and the definition of $\mathcal{X}_0(\mathcal{A}_T)$ to finish. ∎

We close with the following corollary which is easily obtained from Theorem 3.8 and Propositions 3.2 and 3.4.

Corollary 3.9. *Suppose $T = (T_1, T_2)$ is a pair of commuting contractions acting on $\mathcal{H}$ yielding an absolutely continuous representation, and $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$. Suppose that none of the unitaries arising from the minimal isometric dilation and minimal coisometric extension of T contains a bilateral shift. Suppose finally that either at least one of the unitaries arising from the minimal isometric dilation of T, and one from the minimal coisometric extension of T, has no part of uniform multiplicity, or T_1 and T_2 doubly commute. Then $\mathcal{A}_T$ has property $X_{0,1}$.*

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