J. OPERATOR THEORY
45(2001), 251–264

K-THEORY OF CERTAIN C^*-ALGEBRAS ASSOCIATED WITH FREE PRODUCTS OF CYCLIC GROUPS

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Communicated by Norberto Salinas

Abstract. Let $\Gamma_\Lambda \leqslant \Gamma$ be free products of countably many cyclic groups and let $C(X_\Lambda) \times \Gamma$ denote the crossed product related to an action of Γ on a compact space X_Λ constructed from the homogeneous space Γ/Γ_Λ and the boundary $\partial\Gamma$. Assuming that either Γ is free or Γ_Λ is finite we determine the K-groups of the crossed products. Among the algebras considered there are both extensions of some Cuntz-Krieger algebras by the compacts and some purely infinite simple C^*-algebras (nuclear or not).

Keywords: *C^*-algebras, K-theory.*

MSC (2000): Primary 46L80; Secondary 46L35.

0. INTRODUCTION

This article is a continuation of our investigation of a class of C^*-algebras generated by the reduced group C^*-algebra $C^*\Gamma$ and a set of projections $\mathcal{P}_\Lambda$, denoted by $C^*_{\rm r}(\Gamma, \mathcal{P}_\Lambda)$, where Γ is a free product of countably many but at least two finite or infinite cyclic groups (see Section 1 for the construction). This class of C^*-algebras was considered initially by the second author in [24], [25] and later by both authors in [20], [21].

The C^*-algebras of the form $C^*_{\rm r}(\Gamma, \mathcal{P}_\Lambda)$ are either purely infinite, simple C^*-algebras (nuclear or not) or the extensions of certain Cuntz-Krieger algebras by the compacts (see 1.2 for references). In [21] we proved that $C^*_{\rm r}(\Gamma, \mathcal{P}_\Lambda)$ is $*$-isomorphic to the reduced crossed product $C(X_\Lambda) \times \Gamma$ described as follows. Let Γ_Λ be a suitable subgroup of Γ (see Section 1). The homogeneous space Γ/Γ_Λ has a natural compactification X_Λ, obtained by adding some (equivalence classes of) infinite words in the generators of Γ, i.e. elements of the boundary $\partial\Gamma$. The left action of Γ on Γ/Γ_Λ extends to an action on X_Λ ([21]). Then one forms the corresponding reduced crossed product $C(X_\Lambda) \times \Gamma$. A similar construction from a different (graph-theoretical) point of view has been very recently studied by Kumjian and Pask in [12].

This paper is devoted to calculation of K groups of these C^*-algebras when either Γ_Λ is finite or Γ is free. In the former case the algebra $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ can be realized as an inductive limit of extensions of suitable Cuntz-Krieger algebras by the compacts. The calculation of the K groups utilizes the six-term exact sequence of K-theory. These algebras can be also described as Cuntz-Krieger algebras related to infinite matrices in the sense of Exel and Laca ([8]). Typically the rows of the corresponding 0-1 matrices are infinite and, thus, their K groups cannot be calculated by the method of Pask and Raeburn ([13]). In the latter case the K groups are calculated with the help of the Pimsner-Voiculescu exact sequence. If Γ_Λ is free non-abelian then these algebras are non-nuclear. We believe that good understanding of these algebras can be helpful in future attempts of classification of non-nuclear, purely infinite, simple C^*-algebras.

1. PRELIMINARIES

1.0. Let $\Gamma = \mathbb{F}_n * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_r}$ be a free product, where $0 \leqslant n \leqslant \infty$, $0 \leqslant r \leqslant \infty$ with $n + r \geqslant 2$ and $m_i \geqslant 2$ (also excluding the case $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$). We assume that the free group $\mathbb{F}_n$ is generated by $a_1, \ldots, a_n$ and $\mathbb{Z}_{m_i}$ is generated by b_i (we fix these generators throughout). We say that $a_1, \ldots, a_n$ are free generators and $b_1, \ldots, b_r$ are torsion generators. We denote $\mathcal{G} = \{a_i, b_j \mid i, j\}$, the collection of all generators. For $\emptyset \neq \Lambda \subseteq \mathcal{G}$ we denote by Γ_Λ the subgroup of Γ generated by $\mathcal{G} \setminus \Lambda$.

Let $\{\xi_h \mid h \in \Gamma\}$ be the standard orthonormal basis of the Hilbert space $\ell^2(\Gamma)$, where $\xi_h : \Gamma \to \mathbb{C}$ is such that $\xi_h(s) = \delta_{h,s}$. Let $L : \Gamma \to \mathcal{L}(\ell^2(\Gamma))$ be the left regular representation, i.e., $L_h \xi_s = \xi_{hs}$. For any $s \in \Gamma$ we denote by P_s the projection from $\ell^2(\Gamma)$ onto the closed subspace spanned by all reduced words which begin with s. We set $\mathcal{P}_\Lambda = \{P_s \mid s \in \Lambda\}$ and denote by $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ the C^*-algebra generated by $\{L_s \mid s \in \Gamma\}$ and $\mathcal{P}_\Lambda$ (cf. [20], [21]).

For $s \in \Gamma$ we denote by Q_s the projection from $\ell^2(\Gamma)$ onto the one-dimensional subspace $\langle \xi_s \rangle$. For $i = 1, \ldots, r$ we denote $R_i = \sum\limits_{j=0}^{m_i-1} Q_{b_i^j}$. If Γ is finitely generated then all these projections are in $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}})$.

Let $\mathcal{A}_\Lambda$ be the smallest unital C^*-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ containing $\{P_g \mid g \in \Lambda\}$ and invariant under the action $\mathrm{Ad}\, L$. By [21], Proposition 2.1, $\mathcal{A}_\Lambda$ is abelian. X_Λ, the spectrum of $\mathcal{A}_\Lambda$, is a totally disconnected compact Hausdorff space. For a detailed description of X_Λ (which is constructed from the homogeneous space Γ/Γ_Λ and the boundary $\partial\Gamma$) we refer the reader to [21]. It is clear from the construction that $\mathcal{A}_\Lambda$ is an AF-algebra.

We now recall some results of [21].

1.1. $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ is isomorphic to the reduced crossed product $\mathcal{A}_\Lambda \underset{\mathrm{Ad}\, L, r}{\times} \Gamma$ ([21], Theorem 2.8).

1.2. If Γ is finitely generated and Γ_Λ is finite, then $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ is isomorphic to an extension of a simple Cuntz-Krieger algebra by the compacts. Otherwise, $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ is purely infinite and simple ([21], Theorem 3.1, [24], [25]).

1.3. $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ is nuclear if and only if Γ_Λ is amenable ([21], Theorem 4.5).

1.4. For $k \in \mathbb{N}$ we denote by Γ_k the subgroup of Γ generated by the first k elements of $\mathcal{G}$. We set $\Lambda_k = \Lambda \cap \Gamma_k$ and define $C^*_{\mathrm{r},k}(\Gamma, \mathcal{P}_\Lambda)$ as the C^*-subalgebra of $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ generated by $\{L_h \mid h \in \Gamma_k\}$ and $\{P_g \mid g \in \Lambda_k\}$. By [21], Proposition 4.4, there exists a C^*-algebra isomorphism $C^*_{\mathrm{r},k}(\Gamma, \mathcal{P}_\Lambda) \cong C^*_{\mathrm{r}}(\Gamma_k, \mathcal{P}_{\Lambda_k})$ and, hence, if Γ is infinitely generated, then $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ is isomorphic to an inductive limit $\varinjlim C^*_{\mathrm{r}}(\Gamma_k, \mathcal{P}_{\Lambda_k})$. Denoting by $\varphi_k : C^*_{\mathrm{r}}(\Gamma_k, \mathcal{P}_{\Lambda_k}) \to C^*_{\mathrm{r}}(\Gamma_{k+1}, \mathcal{P}_{\Lambda_{k+1}})$ the corresponding imbeddings, we have $\varphi_k(L_s) = L_s$ for $s \in \Gamma_k$ and $\varphi_k(P_t) = P_t$ for $t \in \Lambda_k$.

1.5. We now give a more detailed description of the algebra $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ when Γ is finitely generated and Γ_Λ is finite, i.e. $\Gamma_\Lambda = \langle e \rangle$ or $\Gamma_\Lambda = \langle b_r \rangle \cong \mathbb{Z}_{m_r}$. We denote $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$ by $\mathcal{T}_0$ in the former case and by $\mathcal{T}_1$ in the latter. As noted in 1.2 above, there exist short exact sequences for $\nu = 0, 1$

$$0 \longrightarrow \mathcal{J}_\nu \xrightarrow{\;i\;} \mathcal{T}_\nu \xrightarrow{\;\pi\;} \mathcal{O}_{A_\nu} \longrightarrow 0,$$

with $\mathcal{J}_\nu$ isomorphic to the compacts $\mathcal{K}$, and $\mathcal{O}_{A_\nu}$ a simple Cuntz-Krieger algebra corresponding to a suitable 0-1 matrix A_ν. In fact, $\mathcal{J}_0$ coincides with the algebra of all compact operators on $\ell^2(\Gamma)$. This is not the case with $\mathcal{J}_1$, as R_r is its minimal projection.

$\mathcal{O}_{A_0}$ is generated by $\{S_i = \pi(V_i)\}$, the images under $\pi : \mathcal{T}_0 \to \mathcal{O}_{A_0}$ of the following partial isometries (notice the following construction is quite different from [19] even restricted to the special case $\mathbb{Z}_2 * \mathbb{Z}_{n+1}$):

$$\begin{aligned} &V_{2i-1} = L_{a_i}(I - P_{a_i^{-1}}), \quad V_{2i} = L_{a_i^{-1}}(I - P_{a_i}), \quad i = 1, \dots, n, \\ &V_{2n+\sum_{l=1}^{i-1}(m_l-1)+j} = L_{b_i^j}(I - P_{b_i}), \quad i = 1, \dots, r,\ j = 1, \dots, m_i - 1. \end{aligned}$$

Notice that the partial isometries above are associated with the following decomposition of the identity:

$$I = \sum_{i=1}^{n}(P_{a_i} + P_{a_i^{-1}}) + \sum_{l=1}^{r}\left(\sum_{j=1}^{m_l-2}(P_{b_l^j} - P_{b_l^{j+1}}) + P_{b_l^{m_l-1}}\right) + Q_e.$$

$\mathcal{O}_{A_1}$ is generated by the images under $\pi : \mathcal{T}_1 \to \mathcal{O}_{A_1}$ of the same partial isometries as above, with the last $m_r - 1$ partial isometries V_i replaced by

$$V'_{2n+\sum_{l=1}^{r-1}(m_l-1)+j} = L_{b_r^j}(I - (P_{b_r} + Q_e)), \quad j = 1, \dots, m_r - 1.$$

These partial isometries are associated with the following decomposition of the identity:

$$I=\sum_{i=1}^{n}(P_{a_i}+P_{a_i^{-1}})+\sum_{l=1}^{r-1}\left(\sum_{j=1}^{m_l-2}(P_{b_l^j}-P_{b_l^{j+1}})+P_{b_l^{m_l-1}}\right)$$
$$+\sum_{j=1}^{m_r-2}(P_{b_r^j}-P_{b_r^{j+1}}-Q_{b_r^j})+(P_{b_r^{m_r-1}}-Q_{b_r^{m_r-1}})+R_r.$$

Also notice, as a subtle matter, that neither $P_{b_r^j}$ nor $Q_{b_r^j}$ for $j=0,\dots,m_r-1$ are in $\mathcal{T}_1$, but the projections $P_{b_r}+Q_e$, $P_{b_r^j}-P_{b_r^{j+1}}-Q_{b_r^j}$, $j=1,\dots,m_r-2$, $P_{b_r^{m_r-1}}-Q_{b_r^{m_r-1}}$, and R_r are in $\mathcal{T}_1$, because $P_{a_i},P_{a_i^{-1}},P_{b_1},\dots,P_{b_{r-1}}\in\mathcal{T}_1$, and the following equalities hold:

$$P_{b_r}+Q_e=I-\sum_{j=0}^{m_r-1}L_{b_r^j}\left(\sum_{i=1}^{n}(P_{a_i}+P_{a_i^{-1}})+\sum_{l=1}^{r-1}P_{b_l}\right)L_{b_r}^*,$$

$$P_{b_r^j}-P_{b_r^{j+1}}-Q_{b_r^j}=L_{b_r^j}\left(\sum_{i=1}^{n}(P_{a_i}+P_{a_i^{-1}})+\sum_{l=1}^{r-1}P_{b_l}\right)L_{b_r^j}^*,\quad j=1,\dots,m_r-2,$$

$$P_{b_r^{m_r-1}}-Q_{b_r^{m_r-1}}=L_{b_r^{m_r-1}}\left(\sum_{i=1}^{n}(P_{a_i}+P_{a_i^{-1}})+\sum_{l=1}^{r-1}P_{b_l}\right)L_{b_r^{m_r-1}}^*.$$

Thus, some caution is needed on this subtle point in dealing with $\mathcal{T}_1$.

Let $k=2n+\sum_{i=1}^{r}(m_i-1)$. Then $A_0=A_1$ is a symmetric matrix in $M_k(\{0,1\})$ of the form

$$A=\begin{bmatrix} B & C\\ C^{\mathrm{t}} & D\end{bmatrix}.$$

Here B is a $2n\times 2n$ matrix with n diagonal blocks equal to the 2×2 identity matrix, and all other entries 1. C is a $2n\times(k-2n)$ matrix with all entries 1. D is a $(k-2n)\times(k-2n)$ matrix with r diagonal blocks D_i, $i=1,\dots,r$, each equal to the zero matrix of the corresponding size $(m_i-1)\times(m_i-1)$, and all other entries 1. Thus the matrix A is irreducible and hence the C^*-algebra $\mathcal{O}_A$ is simple and purely infinite by [6], Theorem 2.14.

1.6. In what follows we will often use matrix forms of elements of $\mathcal{T}_\nu$. Namely, if $I=F_1+\dots+F_l$, with F_i's projections, and $X\in\mathcal{T}_\nu$, then we identify X with an $l\times l$ matrix $[F_iXF_j]_{i,j=1}^l$. We refer to this as to the matrix form of X with respect to the decomposition of the identity $I=F_1+\dots+F_l$. In particular, we consider the matrix form of L_{a_i} with respect to the decomposition $I=P_{a_1}+P_{a_1^{-1}}+P_{a_2}+\dots+P_{a_n^{-1}}+F$ (where $F=I-\sum_{j=1}^{n}(P_{a_j}+P_{a_j^{-1}})$). We observe that the only non-zero entries of this $(2n+1)\times(2n+1)$ matrix lie in the $(2i-1)^{\text{th}}$ row and $(2i)^{\text{th}}$ column. Moreover, the $(2i-1,2i)$ entry is zero. Indeed, $P_{a_i}L_{a_i}P_{a_i^{-1}}=0$, and $L_{a_i}(I-P_{a_i^{-1}})L_{a_i}^*=P_{a_i}$ implies $(I-P_{a_i})L_{a_i}(I-P_{a_i^{-1}})=0$.

2. K_1-GROUPS — Γ_Λ FINITE

This section is entirely devoted to proving Theorem 2.0 below, which describes the K_1-groups of algebras $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}})$ and $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}\setminus\{b_1\}})$. We denote by $\mathbb{Z}^n$ the direct sum of n copies of $\mathbb{Z}$. This includes the case $\mathbb{Z}^0 = \langle 0 \rangle$ and $\mathbb{Z}^\infty$ — the direct sum of countably many copies of $\mathbb{Z}$.

2.0. THEOREM. *Let E be either $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}})$ or $C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}\setminus\{b_1\}})$. Then:*

(i) $\mathrm{K}_1(E) \cong \mathbb{Z}^n$, *where n denotes the number of free generators of Γ.*

(ii) $\mathrm{K}_1(E)$ *is generated by $\{[L_{a_i}] \mid i = 1, \ldots, n\}$, i.e. the free generators.*

(iii) *If Γ is finitely generated, then the Fredholm index of any Fredholm operator in E is zero.*

The essential part of the theorem is to establish it when Γ is finitely generated, in which case $E = \mathcal{T}_0$ or $E = \mathcal{T}_1$, and there exists a short exact sequence as in 1.5. The standard six-term exact sequence of K-theory then yields (cf. [1]) for $\nu = 0, 1$

$$0 \longrightarrow \mathrm{K}_1(\mathcal{T}_\nu) \longrightarrow \mathrm{K}_1(\mathcal{O}_A) \xrightarrow{\delta} \mathbb{Z} \longrightarrow \mathrm{K}_0(\mathcal{T}_\nu) \longrightarrow \mathrm{K}_0(\mathcal{O}_A) \longrightarrow 0.$$

The main step in the proof is to show that the index map δ is trivial. If $\nu = 0$ then δ is just the usual Fredholm index, as in this case the ideal $\mathcal{J}_0$ coincides with the algebra of all compact operators on $\ell^2(\Gamma)$.

At first we determine $\mathrm{K}_1(\mathcal{O}_A)$, which by [4] is isomorphic to $\ker(I - A)$ on $\mathbb{Z}^k$ (note that $A = A^{\mathrm{t}}$).

2.1. LEMMA. *Let Γ be finitely generated and $v_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)^{\mathrm{t}}$ with 1 in the $(2i-1)^{\text{th}}$ place, for $i = 1, \ldots, n$, where n is the number of free generators of Γ. Then $\ker(I - A) \cong \mathbb{Z}^n$ with a basis $\{v_i \mid i = 1, \ldots n\}$.*

Proof. We use the description of the matrix A given in 1.5. Clearly, $\{v_i \mid i = 1, \ldots, n\}$ are independent elements of $\ker(I - A)$. So it suffices to show that $\mathrm{rank}(I - A) = n$. For this we eliminate from $I - A$ rows and columns $1, 3, \ldots, 2n-1$ and show that the resulting matrix M (of size $(k-n) \times (k-n)$) is invertible.

We have $M = T - Q$, where Q is a matrix whose all entries are 1 and T is block-diagonal. The diagonal blocks of T are $T_0, T_1, \ldots, T_r$, where T_0 is the $n \times n$ identity matrix, and T_i, $i = 1, \ldots, r$, is an $(m_i - 1) \times (m_i - 1)$ matrix with 2 on the diagonal and 1 elsewhere. Clearly, $T \geqslant I$ is invertible, and Q is of rank one with the range spanned by $\zeta = (1, \ldots, 1)^{\mathrm{t}}$.

If $\xi \in \ker(T - Q)$ then $\xi = T^{-1}Q\xi$, i.e. ξ is a multiple of $T^{-1}\zeta$. That is $\ker(T - Q) \subseteq \langle T^{-1}\zeta \rangle$. Since $T^{-1}\zeta = (1, \ldots, 1, m_1^{-1}, \ldots, m_1^{-1}, \ldots, m_r^{-1}, \ldots, m_r^{-1})^{\mathrm{t}}$ (where 1 is repeated n times and each m_i^{-1} is repeated $m_i - 1$ times, respectively), $T^{-1}\zeta$ is in the kernel of $T - Q$ if and only if

$$n + \frac{m_1 - 1}{m_1} + \cdots + \frac{m_r - 1}{m_r} = 1.$$

In view of our assumptions ($n + r \geqslant 2$ and if $n = 0$, $r = 2$, then at least one $m_i \neq 2$) this is impossible. ∎

2.2. LEMMA. *Assume that* Γ *is finitely generated. Then:*

(i) *The index map* δ *is trivial.*

(ii) *Every Fredholm operator in* $\mathcal{T}_\nu$, $\nu = 0, 1$, *has Fredholm index* 0.

(iii) $\mathrm{K}_1(\mathcal{T}_\nu) \cong \mathbb{Z}^n$, $\nu = 0, 1$, *where* n *is the number of free generators of* Γ. *Furthermore,* $\{[L_{a_i}] \mid i = 1, \ldots, n\}$ *is a basis of the* K_1 *group.*

Proof. To show (i) it suffices to find a collection of generators of $\mathrm{K}_1(\mathcal{O}_A)$ each of which has a lift to a unitary element in $\mathcal{T}_\nu$, $\nu = 0, 1$. Let $\{v_i \mid i = 1, \ldots, n\}$ be the basis of $\ker(I - A)$ in Lemma 2.1. Then to each $i = 1, \ldots, n$ there corresponds a unitary $U_i = T_i + W_i$ in $M_2(\mathcal{O}_A)$, where

$$T_i = \begin{bmatrix} S_{2i-1} & 0 \\ 0 & S_{2i}^* \end{bmatrix} \quad \text{and} \quad W_i = \begin{bmatrix} \pi(P_{a_i^{-1}}) & \pi(I - (P_{a_i} + P_{a_i^{-1}})) \\ 0 & \pi(P_{a_i}) \end{bmatrix}.$$

W_i is a partial isometry in $M_2(\Sigma)$ such that $W_i^* W_i = I - T_i^* T_i$ and $W_i W_i^* = I - T_i T_i^*$. Here Σ denotes the vector space spanned by the range projections of the partial isometries generating $\mathcal{O}_A$. It follows from Lemma 2.1 and [5], Proposition 3.1 (cf. also [16]) that $[U_i]$, $i = 1, \ldots, n$, generate $\mathrm{K}_1(\mathcal{O}_A)$. We define

$$X_i = \begin{bmatrix} V_{2i-1} + P_{a_i^{-1}} & I - (P_{a_i} + P_{a_i^{-1}}) \\ 0 & V_{2i}^* + P_{a_i} \end{bmatrix}.$$

Clearly, X_i is a unitary lift of U_i to $M_2(\mathcal{T}_1) \subset M_2(\mathcal{T}_0)$. Thus, part (i) of the lemma is proved. We now show that $[X_i] = [L_{a_i}]$ in $\mathrm{K}_1(\mathcal{T}_\nu)$, $\nu = 0, 1$, $i = 1, \ldots, n$. To this end we consider matrix forms of elements of $\mathcal{T}_1$, as explained in 1.6. According to 1.6, L_{a_i} has the form

$$\begin{bmatrix} 0 & \ldots & 0 & d_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & d_{2i-2} & 0 & \ldots & 0 \\ c_1 & \ldots & c_{2i-1} & 0 & c_{2i+1} & \ldots & c_{2n+1} \\ 0 & \ldots & 0 & d_{2i} & 0 & \ldots 0 & \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & \ldots & 0 & d_{2n+1} & 0 & \ldots & 0 \end{bmatrix},$$

with non-zero entries c_l in the $(2i-1)^{\text{th}}$ row and d_l in the $(2i)^{\text{th}}$ column. Similarly,

X_i can be written in the matrix form

$$
\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & P_{a_1} & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & & & & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & & P_{a_{i-1}^{-1}} & & & & \\
c_1 & \cdots & c_{2i-1} & 0 & c_{2i+1} & \cdots & c_{2n+1} & & & & 0 & & & \\
0 & \cdots & 0 & P_{a_i^{-1}} & 0 & \cdots & 0 & & & & & 0 & & \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & & & & \vdots & & P_{a_{i+1}} & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 & & & \ddots \\
 & & & & & & & & & & d_1 & & & \\
 & & & & & & & & & & \vdots & & & \\
 & & & & & & & & & & d_{2i-2} & & & \\
 & & & & & & & & & P_{a_i} & 0 & & & \\
 & & & & & & & & & & d_{2i} & & & \\
 & & & & & & & & & & \vdots & & & \\
 & & & & & & & & & & d_{2n+1} & & &
\end{bmatrix}.
$$

We define a unitary $Y_i \in M_2(\mathcal{T}_1)$ as the following elementary matrix

$$
\begin{bmatrix}
0 & & & & & & & P_{a_1} & & & & & & \\
 & \ddots & & & & & & & \ddots & & & & & \\
 & & 0 & & & & & & & P_{a_{i-1}} & & & & \\
 & & & P_{a_i^{-1}} & & & & & & & 0 & & & \\
 & & & & 0 & & & & & & & P_{a_i} & & \\
 & & & & & \ddots & & & & & & & \ddots & \\
 & & & & & & 0 & & & & & & & F \\
P_{a_1} & & & & & & & 0 & & & & & & \\
 & \ddots & & & & & & & \ddots & & & & & \\
 & & P_{a_{i-1}} & & & & & & & 0 & & & & \\
 & & & 0 & & & & & & & P_{a_i^{-1}} & & & \\
 & & & & P_{a_i} & & & & & & & 0 & & \\
 & & & & & \ddots & & & & & & & \ddots & \\
 & & & & & & F & & & & & & & 0
\end{bmatrix}
$$

and a unitary $Z_i \in M_2(\mathcal{T}_1)$ as another elementary matrix

$$\begin{bmatrix}
P_{a_1} &&&&&&&&&&&&& \\
& \ddots &&&&&&&&&&&& \\
&& P_{a_{i-1}^{-1}} &&&&&&&&&&& \\
&&& 0 &&&&&&& P_{a_i^{-1}} &&& \\
&&&& P_{a_i} &&&&&&&&& \\
&&&&& \ddots &&&&&&&& \\
&&&&&& F &&&&&&& \\
&&&&&&& P_{a_1} &&&&&& \\
&&&&&&&& \ddots &&&&& \\
&&&&&&&&& P_{a_{i-1}} &&&& \\
&&& P_{a_i^{-1}} &&&&&&& 0 &&& \\
&&&&&&&&&&& P_{a_i} && \\
&&&&&&&&&&&& \ddots & \\
&&&&&&&&&&&&& F
\end{bmatrix}.$$

We have $I \oplus L_{a_i} = Y_i X_i Z_i$; to see this equality one needs to think of the effects on the matrix X_i under the elementary row operations caused by Z_i and the elementary column operations caused by Y_i. Since two elementary matrices Y_i and Z_i are homotopic to the identity, it follows that X_i is homotopic to $I \oplus L_{a_i}$, as desired. Thus, part (iii) of the lemma is proved. To see that part (ii) holds consider T, a Fredholm operator in $\mathcal{T}_\nu$. Then $\pi(T)$ is invertible in $\mathcal{O}_A$. Since L_{a_i}'s are generators of both $\mathrm{K}_1(\mathcal{T}_0)$ and $\mathrm{K}_1(\mathcal{T}_1)$ as just shown, the path component $[\pi(T)]$ of $\pi(T)$ can be written as a product of the form

$$[\pi(L_{a_1})]^{k_1}[\pi(L_{a_2})]^{k_2}\cdots[\pi(L_{a_n})]^{k_n}$$

for some $k_i \in \mathbb{Z}$ (see [3], [5], and [6]). But $L_{a_1}^{k_1}\cdots L_{a_n}^{k_n}$ is a unitary whose Fredholm index is zero. Therefore, the Fredholm index of T is also zero, for the index is a homotopy invariant. ∎

2.3. *Proof of Theorem* 2.0. If Γ is finitely generated, Theorem 2.0 follows from the six-term exact sequence and Lemmas 2.1 and 2.2. If Γ is infinitely generated the theorem follows from continuity of the K-functors and the inductive limit described in 1.4. ∎

3. K_0-GROUPS — Γ_Λ FINITE

As an immediate consequence of Lemma 2.2 (and the six-term exact sequence) we obtain the following.

3.0. PROPOSITION. *If Γ is finitely generated and $\Gamma_\Lambda = \langle b_1 \rangle$ is finite, then there exists a short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathrm{K}_0(\mathcal{T}_\nu) \xrightarrow{\pi_*} \mathrm{K}_0(\mathcal{O}_A) \longrightarrow 0$$

for $\nu = 0, 1$, where $i_*(1) = [Q_e]$ for $\nu = 0$ and $i_*(1) = [R_1]$ for $\nu = 1$.

It appears impractical to try to determine K_0-groups resulting from all possible pairs (Γ, Λ) as above, due to their enormous variety and combinatorial difficulties involved. In this section we describe K_0-groups related to a relatively simple case when Γ has only one torsion generator, i.e. $r = 1$, which nevertheless provides a number of interesting examples. The inductive limit process described in 1.4 allows us to include infinitely generated groups Γ as well. To simplify notations in what follows we denote $m = m_1$, $b = b_1$, and $R = R_1$. We use the convention that $\mathbb{Z}^0 = \mathbb{Z}_1 = \langle 0 \rangle$, and $P_b = 0$ if $m = 1$.

3.1. THEOREM. *Let $\Gamma = \mathbb{F}_n * \mathbb{Z}_m$, where $1 \leqslant n \leqslant \infty$, $1 \leqslant m < \infty$, and either $n \geqslant 2$ or $m \geqslant 2$. Then*

$$\mathrm{K}_0(C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}})) \cong \mathbb{Z}^{n+1} \cong \mathrm{K}_0(C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}\setminus\{b\}})).$$

$[I - P_b]$, $[P_{a_i}]$, $i = 1, \ldots, n$, are generators of $\mathrm{K}_0(C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}}))$, while $[I]$, $[P_{a_i}]$, $i = 1, \ldots, n$, are generators of $\mathrm{K}_0(C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_{\mathcal{G}\setminus\{b\}}))$.

The proof requires two lemmas.

3.2. LEMMA. *With the same assumptions as in Theorem 3.1, but n finite, we have*

$$\mathrm{K}_0(\mathcal{O}_A) \cong \mathbb{Z}^n \oplus \mathbb{Z}_{nm-1},$$

with the free part generated by $[\pi(P_{a_i})]$, $i = 1, \ldots, n$, and the torsion part generated by $[\pi(I)]$.

Proof. As in 1.5, A is a symmetric $k \times k$ $(k = 2n + m - 1)$ matrix of the form

$$A = \begin{bmatrix} B & C \\ C^{\mathrm{t}} & 0 \end{bmatrix}.$$

Notice that $D = 0$ in the present case. For $i = 1, \ldots, n$ we define $w_i \in \mathbb{Z}^k$ as

$$w_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{t}},$$

with 1 in the $(2i-1)^{\mathrm{th}}$ place. We also set

$$w_{n+1} = (1, \ldots, 1)^{\mathrm{t}} \in \mathbb{Z}^k.$$

By virtue of [4] we have $\mathrm{K}_0(\mathcal{O}_A) \cong \mathbb{Z}^k/(I - A)\mathbb{Z}^k$. Since in our case $[\pi(P_{a_i})]$ corresponds to w_i and $[\pi(I)]$ corresponds to w_{n+1}, it suffices to prove the following:

(i) $\mathbb{Z}^k$ is spanned (over $\mathbb{Z}$) by $\{w_1, \ldots, w_{n+1}\}$ and $(I - A)\mathbb{Z}^k$.

(ii) If $x_i \in \mathbb{Z}$, $i = 1, \dots, n+1$, and $\sum\limits_{i=1}^{n+1} x_i w_i \in (I-A)\mathbb{Z}^k$, then $x_1 = \cdots = x_n = 0$ and $x_{n+1} \in (nm-1)\mathbb{Z}$.

(iii) $(nm-1)w_{n+1}$ belongs to $(I-A)\mathbb{Z}^k$.

At first, we observe that $(I-A)\mathbb{Z}^k$ is spanned by $\{u_i \in \mathbb{Z}^k \mid i = 1, \dots, n+m-1\}$, where

$$u_i = (1, \dots, 1, 0, 0, 1, \dots, 1)^{\mathrm{t}}, \quad i = 1, \dots, n,$$

with 0's in the $2i-1$ and $(2i)^{\mathrm{th}}$ places, and

$$u_i = (1, \dots, 1, 0, \dots, 0, -1, 0, \dots, 0)^{\mathrm{t}}, \quad i = n+1, \dots, n+m-1,$$

with 1 in the first $2n$ places and -1 in the $(n+i)^{\mathrm{th}}$ place.

Proof of (i). Clearly, the vectors $\{w_i,\ u_i - w_{n+1} \mid i = 1, \dots, n\}$ span (over $\mathbb{Z}$) the space $\{(t_1, \dots, t_{2n}, 0, \dots, 0)^{\mathrm{t}} \mid t_j \in \mathbb{Z}\}$. Thus, together with $\{u_i \mid i = n+1, \dots, n+m-1\}$ they span all of $\mathbb{Z}^k$.

Proof of (ii). Let $x_i, y_j \in \mathbb{Z}$, $i = 1, \dots, n+1$, $j = 1, \dots, n+m-1$, and suppose that $\sum\limits_{i=1}^{n+1} x_i w_i = \sum\limits_{j=1}^{n+m-1} y_j u_j$. This is equivalent to the following system of equations:

$$\begin{cases} \text{(a)} \quad x_i + x_{n+1} = \sum\limits_{j=1}^{n+m-1} y_j - y_i, & i = 1, \dots, n \\ \text{(b)} \quad x_{n+1} = \sum\limits_{j=1}^{n+m-1} y_j - y_i, & i = 1, \dots, n \\ \text{(c)} \quad x_{n+1} = \sum\limits_{j=1}^{n} y_j - y_{n+i}, & i = 1, \dots, m-1. \end{cases}$$

(a) and (b) imply $x_i = 0$ for $i = 1, \dots, n$, and (c) implies $y_{n+1} = \cdots = y_{n+m-1}$ $(= y)$. Then (a) and (b) imply $y_1 = \cdots = y_n$ $(= z)$. Thus, the whole system is equivalent to

$$\begin{cases} x_{n+1} = (n-1)z + (m-1)y \\ x_{n+1} = nz - y. \end{cases}$$

This implies $my = z$ and, hence, $x_{n+1} = (nm-1)y$, as desired.

Proof of (iii). With the notation from the proof of part (ii) we have

$$(nm-1)w_{n+1} = m \sum_{j=1}^{n} u_i + \sum_{j=n+1}^{n+m-1} u_j$$

and hence $(nm-1)w_{n+1} \in (I-A)\mathbb{Z}^k$. ∎

3.3. LEMMA. *With the same assumptions as in Lemma* 3.2 *we have:*
(i) $\mathrm{K}_0(\mathcal{T}_0) \cong \mathbb{Z}^{n+1}$ *with generators* $[I - P_b]$, $[P_{a_i}]$, $i = 1, \dots, n$;
(ii) $\mathrm{K}_0(\mathcal{T}_1) \cong \mathbb{Z}^{n+1}$ *with generators* $[I]$, $[P_{a_i}]$, $i = 1, \dots, n$.

Proof. Part (i). It follows from Proposition 3.0 and Lemma 3.2 that $\mathrm{K}_0(\mathcal{T}_0)$ is generated by $[Q_e]$, $[I]$, and $[P_{a_i}]$, $i=1,\dots,n$. Since $P_{a_i} = L_{a_i}(I - P_{a_i^{-1}})L^*_{a_i}$, we have $[P_{a_i}] + [P_{a_i^{-1}}] = [I]$. Thus,

$$I = \sum_{i=0}^{m-1} L_{b^i}\Big(Q_e + \sum_{j=1}^{n}(P_{a_j} + P_{a_j^{-1}})\Big)L^*_{b^i}$$

implies that $[I] = m([Q_e] + n[I])$, i.e. $m[Q_e] = (1-nm)[I]$. Since $I - P_b = Q_e + \sum_{i=1}^{n}(P_{a_i} + P_{a_i^{-1}})$, we have $[I - P_b] = [Q_e] + n[I]$. It follows that $[Q_e] = (1-nm)[I-P_b]$ and $[I] = m[I-P_b]$. Consequently, $\mathrm{K}_0(\mathcal{T}_0)$ is generated by $[I-P_b]$ and $[P_{a_i}]$, $i=1,\dots,n$. It remains to show that these generators are independent. Notice that $[Q_e] = (1-nm)[I-P_b]$ and Proposition 3.0 imply that $[I-P_b]$ has infinite order. Let $x_i \in \mathbb{Z}$, $i = 1,\dots,n+1$, and $\sum_{i=1}^{n} x_i[P_{a_i}] + x_{n+1}[I-P_b] = [0]$. Then $\sum_{i=1}^{n} x_i[\pi(P_{a_i})] + x_{n+1}[\pi(I-P_b)] = [0]$ in $\mathrm{K}_0(\mathcal{O}_A)$. As $[\pi(I-P_b)] = n[\pi(I)]$, Lemma 3.2 implies that $x_1 = \cdots = x_n = 0$ and, hence, $x_{n+1} = 0$ as well.

Part (ii). Proposition 3.0 and Lemma 3.2 imply that $\mathrm{K}_0(\mathcal{T}_1)$ is generated by $[R]$, $[I]$, and $[P_{a_i}]$, $i=1,\dots,n$. Since

$$I = R + \sum_{i=0}^{m-1} L_{b^i}\Big(\sum_{j=1}^{n}(P_{a_i} + P_{a_i^{-1}})\Big)L^*_{b^i},$$

we have $[I] = [R] + nm[I]$, i.e. $[R] = (1-nm)[I]$. The rest of the proof is as above. ∎

3.4. *Proof of Theorem* 3.1. It follows from 1.4 and Lemma 3.3. ∎

4. K-groups — Γ free

As in [21], we denote by Γ^Λ the set of all reduced words in Γ ending with either g or g^{-1} for some $g \in \Lambda$.

4.0. Lemma. $\mathrm{K}_0(\mathcal{A}_\Lambda)$ *is generated by* $[I]$ *and* $\{[P_s] \mid s \in \Gamma^\Lambda\}$.

Proof. Since $\mathcal{A}_\Lambda$ is an AF-algebra, $\mathrm{K}_0(\mathcal{A}_\Lambda)$ is generated by the projections in $\mathcal{A}_\Lambda$. But we observed in [21], Section 2 that any such projection can be approximated in norm by a combination of $[I]$ and $\{[P_s] \mid s \in \Gamma^\Lambda\}$. ∎

We now let $\Gamma = \mathbb{F}_n$, $2 \leqslant n \leqslant \infty$, with free generators $a_1, a_2, \ldots$. Let Λ be a non-empty subset of $\{a_1, a_2, \ldots\}$ and let Γ_Λ be the corresponding subgroup of Γ. We have $\Gamma_\Lambda \cong \mathbb{F}_m$ for some m, depending on n and Λ. We denote $\mathcal{F}_{n,m} = C^*_{\mathrm{r}}(\Gamma, \mathcal{P}_\Lambda)$. Let $|\Lambda|$ be the cardinality of Λ. We agree that $\infty + 1 = \infty$. We denote by $\mathbb{Z}^\infty$ the free abelian group on countably many generators.

4.1. THEOREM. *We have* $\mathrm{K}_0(\mathcal{F}_{n,m}) \cong \mathbb{Z}^{|\Lambda|+1}$, *with generators* $[I]$, $[P_{a_i}]$, $a_i \in \Lambda$, *and* $\mathrm{K}_1(\mathcal{F}_{n,m}) \cong \mathbb{Z}^n$, *with generators* $[L_{a_i}]$, $i = 1, \ldots, n$.

Proof. It suffices to prove the theorem for Γ finitely generated, as the other case follows from the inductive limit in 1.4. So we assume that $n < \infty$ and choose Γ_Λ generated by $\{a_1, \ldots, a_m\}$. At first, we look at the case $m = 0$, that is $\Lambda = \{a_1, \ldots, a_n\}$. In this case we denote $\mathcal{A}_\Lambda$ simply by $\mathcal{A}$. The K-groups of $\mathcal{F}_{n,0}$ were already determined in Theorems 2.0 and 3.1 as $\mathbb{Z}^{n+1} \cong \mathrm{K}_0(\mathcal{F}_{n,0}) = \langle [I], [P_{a_i}], i = 1, \ldots, n \rangle$ and $\mathbb{Z}^n \cong \mathrm{K}_1(\mathcal{F}_{n,0}) = \langle [P_{a_i}], i = 1, \ldots, n \rangle$. Nevertheless, remembering that $\mathcal{F}_{n,0} \cong \mathcal{A} \underset{\alpha}{\times} \mathbb{F}_n$ ($\alpha = \mathrm{Ad}\, L$) as in 1.1, we can apply the Pimsner-Voiculescu sequence ([15], Theorem 3.5) to this crossed product. Since $\mathcal{A}$ is an AF-algebra (1.0), we have $\mathrm{K}_1(\mathcal{A}) = 0$ and, hence, there is an exact sequence

$$0 \longrightarrow \mathrm{K}_1(\mathcal{F}_{n,0}) \xrightarrow{\delta} \mathrm{K}_0(\mathcal{A})^n \xrightarrow{\beta} \mathrm{K}_0(\mathcal{A}) \xrightarrow{(\mathrm{id} \otimes I)_*} \mathrm{K}_0(\mathcal{F}_{n,0}) \longrightarrow 0,$$

where $\beta(\gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} (\gamma_i - (\alpha_{a_i}^{-1})_*(\gamma_i))$. Therefore, $\ker(\beta) \cong \mathrm{K}_1(\mathcal{F}_{n,0}) \cong \mathbb{Z}^n$ is generated by $d_i = (0, \ldots, 0, [I], 0, \ldots, 0)$, $i = 1, \ldots, n$, (d_i has $[I]$ in the i^{th} place) and $\mathrm{coker}(\beta) \cong \mathrm{K}_0(\mathcal{F}_{n,0}) \cong \mathbb{Z}^{n+1}$ is generated by $[I]$, $[P_{a_i}]$, $i = 1, \ldots, n$.

After this preparation we now consider the case when $2 \leqslant n < \infty$ and $1 \leqslant m < n$. We have $\mathcal{F}_{n,m} \cong \mathcal{A}_\Lambda \underset{\alpha}{\times} \mathbb{F}_n$, with the same action α as above, and $\mathcal{A}_\Lambda$ an α-invariant subalgebra of $\mathcal{A}$. Again we consider the Pimsner-Voiculescu exact sequence

$$0 \longrightarrow \mathrm{K}_1(\mathcal{F}_{n,m}) \xrightarrow{\delta} \mathrm{K}_0(\mathcal{A}_\Lambda)^n \xrightarrow{\beta} \mathrm{K}_0(\mathcal{A}_\Lambda) \xrightarrow{(\mathrm{id} \otimes I)_*} \mathrm{K}_0(\mathcal{F}_{n,m}) \longrightarrow 0.$$

Since $i : \mathcal{A}_\Lambda \to \mathcal{A}$ is an inclusion of abelian AF-algebras, the induced map $i_* : \mathrm{K}_0(\mathcal{A}_\Lambda) \to \mathrm{K}_0(\mathcal{A})$ is injective. Thus, we can identify $\mathrm{K}_0(\mathcal{A}_\Lambda)$ with an appropriate subgroup of $\mathrm{K}_0(\mathcal{A})$. Considering the natural inclusion $i : \mathcal{F}_{n,m} \to \mathcal{F}_{n,0}$ we get a commuting diagram with rows and columns exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_1(\mathcal{F}_{n,0}) \arrow[r, "\delta"] & \mathrm{K}_0(\mathcal{A})^n \arrow[r, "\beta"] & \mathrm{K}_0(\mathcal{A}) \arrow[r, "(\mathrm{id}\otimes I)_*"] & \mathrm{K}_0(\mathcal{F}_{n,0}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{K}_1(\mathcal{F}_{n,m}) \arrow[r, "\delta"] \arrow[u, "i_*"'] & \mathrm{K}_0(\mathcal{A}_\Lambda)^n \arrow[r, "\beta"] \arrow[u, "i_*"'] & \mathrm{K}_0(\mathcal{A}_\Lambda) \arrow[r, "(\mathrm{id}\otimes I)_*"] \arrow[u, "i_*"'] & \mathrm{K}_0(\mathcal{F}_{n,m}) \arrow[r] \arrow[u, "i_*"'] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

An easy diagram chase shows that the map $i_* : \mathrm{K}_1(\mathcal{F}_{n,m}) \to \mathrm{K}_1(\mathcal{F}_{n,0})$ is injective. As $\mathrm{K}_1(\mathcal{F}_{n,m})$ contains $[L_{a_i}]$, $i = 1, \ldots, n$, the generators of $\mathrm{K}_1(\mathcal{F}_{n,0})$, we have $\mathrm{K}_1(\mathcal{F}_{n,m}) \cong \mathrm{K}_1(\mathcal{F}_{n,0}) \cong \mathbb{Z}^n$.

Lemma 4.0 and the above diagram imply that $\mathrm{K}_0(\mathcal{F}_{n,m})$ is generated by $[I]$, $[P_s]$, with $s \in \mathbb{F}_n$ which in reduced form end with either a_i or a_i^{-1} for $i \in \{m+1, \ldots, n\}$. However, if $s, t \in \mathbb{F}_n$, g a word of length 1, and $s = tg$ as a reduced word, then $P_s = L_t P_g L_t^*$ and, hence, $[P_s] = [P_g]$ in $\mathrm{K}_0(\mathcal{F}_{n,m})$. Thus, $\mathrm{K}_0(\mathcal{F}_{n,m})$ is generated by $[I]$, $[P_{a_i}]$, $i = m+1, \ldots, n$. Therefore, in order to complete the proof of the theorem it suffices to show that the map $i_* : \mathrm{K}_0(\mathcal{F}_{n,m}) \to$

$K_0(\mathcal{F}_{n,0})$ is injective. To this end we consider an element $c_0[I]+\sum_{i=m+1}^{n} c_i[P_{a_i}]$ of $K_0(\mathcal{F}_{n,m})$, where $c_0, c_{m+1}, \dots, c_n \in \mathbb{Z}$. Commutativity of the third square in the above diagram shows that $c_0[I]+\sum_{i=m+1}^{n} c_i[P_{a_i}]=0$ in $K_0(\mathcal{F}_{n,m})$ implies $c_0[I]+\sum_{i=m+1}^{n} c_i[P_{a_i}]=0$ in $K_0(\mathcal{F}_{n,0})$. Since $[I]$ and $[P_{a_i}]$, $i=m+1,\dots,n$, belong to the set of free generators of $K_0(\mathcal{F}_{n,0})$ (as we have already shown), it follows that $c_0[I]+\sum_{i=m+1}^{n} c_i[P_{a_i}]=0$ implies $c_0=c_{m+1}=\cdots=c_n$ and, hence, the map $i_*: K_0(\mathcal{F}_{n,m}) \to K_0(\mathcal{F}_{n,0})$ is injective, as required. ∎

Acknowledgements. The first mentioned author would like to thank all members of the Mathematics Department at the University of Cincinnati for their warm hospitality during his visit there, in 1997.

The first author was partially supported by the RMC grant 45/290/603, and the second author was partially supported by NSF grant DMS-9626592.

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Received August 23, 1998; revised November 3, 2000.