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DIFFERENTIAL SCHATTEN $*$-ALGEBRAS. APPROXIMATION PROPERTY AND APPROXIMATE IDENTITIES

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Abstract. For symmetric operators S, we consider differential Schatten algebras $C_S^{p,q}$ of compact operators A from C^p with $SA - AS$ belonging to C^q. These algebras are analogues of the Sobolev $W^1_{p,q}$ spaces. We study their approximation property: whether every operator is approximated by finite rank operators, and the existence of approximate identities. For non-selfadjoint S, we show that $C_S^{p,q}$ have no bounded approximate identities and the product of any two operators is approximated by finite rank operators. For selfadjoint S, $C_S^{p,q}$ have approximate identities consisting of finite rank operators and hence, have the approximation property. These identities are bounded only if $p = \infty$. The existence of a bounded identity for $C_S^{\infty,1}$ is equivalent to 1-semidiagonality of S.

Keywords: *Schatten classes, differential algebras, derivations, approximate identities.*

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1. INTRODUCTION AND PRELIMINARIES

Extensive development of non-commutative differential geometry requires elaborating of the theory of differential Banach $*$-algebras, that is, dense $*$-subalgebras of C^*-algebras whose properties in many respects are analogous to the properties of algebras of differentiable functions. In this paper we study the class of the differential Schatten $*$-algebra $C_S^{p,q}$, $p, q \in [1, \infty]$, associated with symmetric operators S. These algebras consist of operators X from C^p with the derivative $\delta_S(X) = \mathrm{i}(SX - XS)$ belonging to C^q. In the same way as the Schatten ideals C^p are non-commutative analogous of L^p-spaces, the differential Schatten $*$-algebra $C_S^{p,q}$ are the analogous of the Sobolev $W^1_{p,q}$-spaces. We investigate the structure and various properties of these algebras; in particular, the approximation property and the existence of approximate identities.

Blackadar and Cuntz ([1]) and the authors ([9]) introduced and studied various classes of differential Banach $*$-algebras; the most interesting class consists of $\mathbb{D}$-subalgebras of C^*-algebras $(\mathfrak{A}, \|\cdot\|)$, that is, dense $*$-subalgebras A of $\mathfrak{A}$ which, in turn, are Banach $*$-algebras with respect to another norm $\|\cdot\|_1$ and the norms $\|\cdot\|$ and $\|\cdot\|_1$ on A satisfy the inequality:

$$\|xy\|_1 \leqslant D(\|x\|\,\|y\|_1 + \|x\|_1\,\|y\|), \qquad \text{for } x, y \in A, \tag{1.1}$$

for some $D > 0$. This class contains, for example, the algebras of differentiable functions, symmetrically normed ideals of $C(H)$ ([5]) and the domains $D(\delta)$ of closed unbounded $*$-derivations δ of C^*-algebras with the norm $\|\cdot\|_1$ defined by the formula $\|x\|_1 = \|x\| + \|\delta(x)\|$, for $x \in D(\delta)$. In all these cases $D = 1$; the case when $D > 1$ corresponds, roughly speaking, to the higher derivations.

In [11] and [12] the authors studied some $\mathbb{D}$-subalgebras of the algebra of all compact operators. This paper is a continuation of this study and investigates various properties of differential Schatten $*$-algebras.

Any closed symmetric operator S on a Hilbert space H implements closed $*$-derivations of various operator C^*-algebras on H; the largest one, δ_S, whose domain we denote by $\mathcal{A}_S$ is defined as follows:

$$\begin{aligned}\mathcal{A}_S = \{A \in B(H) : AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S),\\ (SA - AS)|D(S) \text{ extends to a bounded operator}\}\end{aligned} \tag{1.2}$$
and
$$\delta_S(A) = \mathrm{i}\,\mathrm{Closure}(SA - AS), \quad \text{for } A \in \mathcal{A}_S.$$

The algebra $\mathcal{A}_S$ is a unital Banach $*$-algebra with respect to the norm

$$\|A\|_S = \|A\| + \|\delta_S(A)\|$$

and contains the domains of all derivations implemented by S.

By $C(H)$ we denote the algebra of all compact operators on H. For $A \in C(H)$, let $\{s_i(A)\}_{i=1}^{\infty}$ be all eigenvalues of the positive compact operator $(A^*A)^{1/2}$. For any $0 < p < \infty$, set

$$|A|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p\right)^{1/p}.$$

The Schatten class $C^p = C^p(H)$ consists of all compact operators A for which $|A|_p < \infty$. In particular, C^1 consists of all trace class operators on H. Set

$$C^\infty = C(H), \qquad \text{with } |A|_\infty = \|A\|, \text{ for } A \in C(H),$$

and

$$C^{\mathrm{b}} = B(H), \quad \text{with } |A|_{\mathrm{b}} = \|A\|, \text{ for } A \in B(H),$$

and assume that $\infty < \mathrm{b}$. For $p \geqslant 1$, $(C^p, |\cdot|_p)$ is a Banach $*$-algebra.

Let S be a closed symmetric operator on H. For $p, q \in [1, \infty]$,

$$C_S^{p,q} = \{A \in C^p \cap \mathcal{A}_S : \delta_S(A) = \mathrm{i}\,\mathrm{Closure}(SA - AS) \in C^q\}$$

are dense $*$-subalgebras of $C(H)$ and are the domains of the largest closed $*$-derivations from C^p into C^q implemented by S. When endowed with the norms

$$|A|_{p,q} = |A|_p + |\delta_S(A)|_q, \quad \text{for } A \in C_S^{p,q},$$

they become Banach $*$-algebras which we call the differential Schatten $*$-algebras. By $\mathcal{F}_S^{p,q}$ we denote the closure with respect to $|\cdot|_{p,q}$ of the set of all finite rank operators in $C_S^{p,q}$.

The Schatten ideals C^p are non-commutative analogous of L^p-spaces. Similar to the classical Sobolev's construction, for any derivation δ from C^p into C^q, one can consider the algebra $W_{p,q}^\delta$ which consists of operators X from C^p with $\delta(X)$ belonging to C^q. In Section 2 we show that if $\mathcal{A}$ is a $\mathbb{D}$-subalgebra of an operator C^*-algebra $\mathfrak{A}$ on H and if $C(H) \subseteq \mathfrak{A}$ then any closed derivation δ from $\mathcal{A}$ into C^p, $1 \leqslant p$, is implemented by a symmetric operator $S : \delta = \delta_S | D(\delta)$. Hence any closed derivation δ from C^p into C^q, $1 \leqslant p, q$, is implemented by a symmetric operator S, so that $W_{p,q}^\delta = C_S^{p,q}$.

The algebras $C_S^{p,q}$ constitute a wide class of symmetric Banach $*$-algebras with a rich and interesting structure. The natural problems which arise for these algebras — the structure of their ideals and representations and the existence of an approximate identity — are closely linked with important problems of Operator Theory. In particular, the theory developed in this paper relies heavily on Voiculescu's theory of quasidiagonalization modulo operator ideals and on the analogue of Weyl-von Neumann theorem for Schatten ideals.

In Section 3 we show that the algebras $\mathcal{F}_S^{p,q}$ and $C_S^{p,q}$ are $\mathbb{D}$-subalgebras of $C(H)$ and find necessary and sufficient conditions for the algebras $C_S^{p,q}$ and $C_S^{p',q'}$ to coincide in the case when S is selfadjoint.

Sections 4 and 5 focus on the investigation of two problems concerning the algebras $C_S^{p,q}$. The first one is the approximation property: whether every operator in $C_S^{p,q}$ is approximated by finite rank operators. In other words, whether $C_S^{p,q} = \mathcal{F}_S^{p,q}$. The second one is the existence of a bounded or unbounded approximate identity. The existence of an approximate identity consisting of finite rank operators implies, clearly, a positive answer to the first problem.

As can be easily predicted, the algebras $C_S^{p,q}$ fall into two categories. The first category consists of the algebras corresponding to symmetric but non-selfadjoint operators S. In Proposition 5.8 we prove that these algebras have no bounded approximate identities. However, it is unknown whether they have unbounded approximate identities or the approximation property. Section 4 is devoted to establishing the fact that any product of two operators from $C_S^{p,q}$ is approximated by finite rank operators, that is, $(C_S^{p,q})^2 \subseteq \mathcal{F}_S^{p,q}$.

The second category — easier to work with — consists of the differential Schatten algebras $C_S^{p,q}$ corresponding to selfadjoint operators S. In Theorem 5.4 we show that in this case the algebras $C_S^{p,q}$ have approximate identities consisting of finite rank operators and, hence, have the approximation property. Moreover, the approximate identity is bounded only if $p = \infty$. For $q \neq 1$ and $p = \infty$, a bounded approximate identity always exists.

The case when $(p,q) = (\infty, 1)$ is more subtle. If S is a bounded selfadjoint operator of finite multiplicity, the algebra $C_S^{\infty,1}$ has a bounded approximate identity. If S has infinite multiplicity or is unbounded, we prove in Theorem 5.9

that the existence of a bounded approximate identity in $C_S^{\infty,1}$ is equivalent to 1-semidiagonality of S in terms of [15]. Voiculescu ([16]) found an excellent characterization of this condition: the integral of the spectral multiplicity of the absolutely continuous part of S converges (see Theorem 5.6).

In Section 6 we apply the results of Section 5 to describe the dual and the second dual spaces of the differential Schatten algebras. We show that, for $1 < p, q < \infty$ and any symmetric S, the algebras $\mathcal{F}_S^{p,q}$ and $C_S^{p,q}$ are reflexive. If S is bounded, $C_S^{\infty,\infty} = C(H)$ and $C_S^{\mathrm{b},\mathrm{b}} = \mathcal{A}_S = B(H)$, so that $C_S^{\mathrm{b},\mathrm{b}}$ is isometrically isomorphic to the second dual of $C_S^{\infty,\infty}$. In [12] this result was extended to unbounded selfadjoint operators S. Making use of the approximation property of the algebras $C_S^{p,\infty}$ and $C_S^{\infty,q}$ for selfadjoint S, we establish the full analogy with the bounded case: the algebras $C_S^{p,\mathrm{b}}$ (respectively $C_S^{\mathrm{b},q}$) are isometrically isomorphic to the second duals of the algebras $C_S^{p,\infty}$ (respectively $C_S^{\infty,q}$). These results will be used in the subsequent paper on positive functionals and representations of the algebras $C_S^{p,q}$ on the Pontryagin Π_κ-spaces.

2. DERIVATIONS OF ALGEBRAS C^p

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a normed $*$-algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach $\mathcal{A}$-bimodule with involution $*$ such that $\|B^*\|_{\mathcal{B}} = \|B\|_{\mathcal{B}}$, for $B \in \mathcal{B}$. A map δ from a dense $*$-subalgebra $D(\delta)$ of A (called the *domain of* δ) into $\mathcal{B}$ is a $*$-*derivation* if

$$\delta(AB) = A\delta(B) + \delta(A)B \quad \text{and} \quad \delta(A^*) = \delta(A)^*, \quad \text{for } A, B \in \mathcal{A}. \tag{2.1}$$

A derivation δ is *closable* if $A_n \to 0$ and $\delta(A_n) \to B$ implies $B = 0$ and *closed* if $A_n \to A$ and $\delta(A_n) \to B$ implies $A \in D(\delta)$ and $\delta(A) = B$.

It was proved in [2] that if $\mathcal{A}$ is a C^*-subalgebra of $B(H)$ containing the ideal $C(H)$ then any closable $*$-derivation δ of $\mathcal{A}$ into $B(H)$ is implemented by a densely defined symmetric operator S on H, that is,

$$AD(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|D(S) = \mathrm{i}(SA - AS)|D(S), \quad \text{for } A \in D(\delta). \tag{2.2}$$

Theorem 2.1 extends this result to $\mathbb{D}$-subalgebras of $B(H)$.

Let $\mathcal{A}$ be a $*$-subalgebra of $B(H)$ and a Banach $*$-algebra with respect to a norm $\|\cdot\|_{\mathcal{A}}$. By $\overline{\mathcal{A}}$ we denote the uniform closure of $\mathcal{A}$ in $B(H)$. Let $\mathcal{B}$ be a symmetric linear manifold in $B(H)$, that is, $B \in \mathcal{B}$ implies $B^* \in \mathcal{B}$ and assume that $\mathcal{B}$ has a norm $\|\cdot\|_{\mathcal{B}}$ such that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space,

$$\|B^*\|_{\mathcal{B}} = \|B\|_{\mathcal{B}} \quad \text{and} \quad \|B\| \leqslant \|B\|_{\mathcal{B}}, \quad \text{for } B \in \mathcal{B}. \tag{2.3}$$

The algebra $\mathcal{A}$ has two norms $\|\cdot\|$ and $\|\cdot\|_{\mathcal{A}}$. Suppose that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a $(\mathcal{A}, \|\cdot\|)$-bimodule, that is,

$$\|AB\|_{\mathcal{B}} \leqslant \|A\|\,\|B\|_{\mathcal{B}} \;\text{ and }\; \|BA\|_{\mathcal{B}} \leqslant \|B\|_{\mathcal{B}}\,\|A\|, \quad \text{if } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{2.4}$$

THEOREM 2.1. *Let δ be a closed $*$-derivation from $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ into $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and let formulae* (2.3) *and* (2.4) *hold. If $\mathcal{A}$ is a $\mathbb{D}$-subalgebra of the C^*-algebra $\overline{\mathcal{A}}$ and $C(H) \subseteq \overline{\mathcal{A}}$ then δ is a closable derivation from $(\overline{\mathcal{A}}, \|\cdot\|)$ into $B(H)$ and is implemented by a symmetric operator.*

Proof. Set $\|A\|_\delta = \|A\|_{\mathcal{A}} + \|\delta(A)\|_{\mathcal{B}}$, for $A \in D(\delta)$. Since δ is closed, $(D(\delta), \|\cdot\|_\delta)$ is a Banach space and

$$\|A^*\|_\delta = \|A^*\|_{\mathcal{A}} + \|\delta(A^*)\|_{\mathcal{B}} = \|A\|_{\mathcal{A}} + \|\delta(A)^*\|_{\mathcal{B}} = \|A\|_{\mathcal{A}} + \|\delta(A)\|_{\mathcal{B}} = \|A\|_\delta.$$

It follows from (2.1) and (2.4) that for $A, B \in D(\delta)$,

$$\begin{aligned} \|AB\|_\delta &= \|AB\|_{\mathcal{A}} + \|\delta(AB)\|_{\mathcal{B}} \leqslant \|A\|_{\mathcal{A}}\,\|B\|_{\mathcal{A}} + \|A\delta(B)\|_{\mathcal{B}} + \|\delta(A)B\|_{\mathcal{B}} \\ &\leqslant \|A\|_{\mathcal{A}}\,\|B\|_{\mathcal{A}} + \|A\|\,\|\delta(B)\|_{\mathcal{B}} + \|\delta(A)\|_{\mathcal{B}}\,\|B\|. \end{aligned} \tag{2.5}$$

Taking into account (2.3), we obtain that $\|AB\|_\delta \leqslant \|A\|_\delta\,\|B\|_\delta$, so $(D(\delta), \|\cdot\|_\delta)$ is a Banach $*$-algebra.

It is well known (see [3], Section 1.3.7) that $\|A\| \leqslant \|A\|_{\mathcal{A}}$, for $A \in \mathcal{A}$. Since $D(\delta)$ is dense in A with respect to $\|\cdot\|_{\mathcal{A}}$, it is also dense in $\overline{\mathcal{A}}$ with respect to the norm $\|\cdot\|$. The algebra $D(\delta)$ has three norms $\|\cdot\|, \|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_\delta$. Since $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a $\mathbb{D}$-subalgebra of $\overline{\mathcal{A}}$, there is $D > 0$ such that

$$\|AB\|_{\mathcal{A}} \leqslant D(\|A\|\,\|B\|_{\mathcal{A}} + \|A\|_{\mathcal{A}}\,\|B\|), \quad \text{for } A, B \in D(\delta).$$

It follows from (2.5) that

$$\|AB\|_\delta \leqslant \|A\|\,\|B\|_\delta + \|A\|_{\mathcal{A}}\,\|B\|_{\mathcal{A}} + \|A\|_\delta\,\|B\|$$

and we conclude that $D(\delta)$ is a differential algebra of order 2. (Differential algebras of order $p \in \mathbb{N}$ were introduced in [1] and studied in [1] and [9].)

Adding if necessary the identity $\mathbb{1}$ to $\mathcal{A}$ and setting $\delta(\mathbb{1}) = 0$, we may assume that $\mathbb{1} \in D(\delta)$. Since $C(H) \subseteq \overline{\mathcal{A}}$, it follows from Lemma 6 and Theorem 13 of [9] that $C(H) \cap D(\delta)$ is dense in $C(H)$.

Let $A^* = A \in C(H) \cap D(\delta)$, let $0 \neq \lambda \in \mathrm{Sp}(A)$ and H_λ be the finite-dimensional subspace of all eigenvectors of A corresponding to λ. Choose a neighbourhood U of λ such that $U \cap \mathrm{Sp}(A) = \lambda$ and let $f(t)$ be an infinitely differentiable function on $\mathbb{R}$ vanishing outside U and $f(\lambda) = 1$. Then $f(A)$ is the projection on H_λ and, by Theorem 12 of [9], it belongs to $D(\delta)$.

Since $C(H) \cap D(\delta)$ is dense in $C(H)$, H_λ is a cyclic set for $\mathcal{A}$: linear combinations of vectors Ax, $A \in \mathcal{A}$, $x \in H_\lambda$, are dense in H. Making use of Corollary 27.18 of [10], we obtain that there exists a densely defined symmetric operator S which implements δ.

To show that δ is closable with respect to the norm $\|\cdot\|$ on $\mathcal{A}$ and $\mathcal{B}$, we assume that operators A_n from $D(\delta)$ converge to 0 and $\delta(A_n)$ converge to B with respect to $\|\cdot\|$. Then for any $x \in D(S)$, it follows from (2.2) that

$$Bx = \lim \delta(A_n)x = \lim \mathrm{i}(SA_n - A_nS)x = \mathrm{i}\lim SA_nx - \mathrm{i}\lim A_nSx = \mathrm{i}\lim SA_nx.$$

Since $A_nx \to 0$ and S is closable, $Bx = 0$. Thus $B = 0$ and δ is closable in $\|\cdot\|$. ∎

For $p, q \in (0, \infty)$, let C^p and C^q be the Schatten classes of operators. It is well known (see [4] and [5]) that,

$$C^p \subseteq C^q \quad \text{and} \quad |A|_p \geqslant |A|_q \geqslant \|A\|, \qquad \text{if } p \leqslant q \text{ and } A \in C^p. \tag{2.6}$$

For any $A \in C^p$, the operator A^* belongs to C^p and

$$|A^*|_p = |A|_p. \tag{2.7}$$

Let $A \in C^p$, $B \in C^q$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $AB \in C^r$. If $r \geqslant 1$ then

$$|AB|_1 \leqslant |AB|_r \leqslant |A|_p \, |B|_q. \tag{2.8}$$

In particular, if $A \in C^p$ and $B \in C^{\mathrm{b}} = B(H)$ then AB, $BA \in C^p$ and

$$|AB|_p \leqslant |A|_p \, \|B\| \leqslant |A|_p \, |B|_p \quad \text{and} \quad |BA|_p \leqslant |A|_p \, \|B\| \leqslant |A|_p \, |B|_p. \tag{2.9}$$

For $p \geqslant 1$, $(C^p, |\cdot|_p)$ is a Banach $*$-algebra.

COROLLARY 2.2. *Let $\mathfrak{A}$ be a C^*-subalgebra of $B(H)$ containing $C(H)$ and let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a $\mathbb{D}$-subalgebra of $\mathfrak{A}$. Any closed $*$-derivation from $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ into $(C^p, |\cdot|_p)$, for $p \geqslant 1$, is a closable derivation from $(\mathfrak{A}, \|\cdot\|)$ into $B(H)$ and is implemented by a symmetric operator.*

Proof. We obtain from (2.6) and (2.9) that formulae (2.3) and (2.4), linking the norms $\|\cdot\|$ and $|\cdot|_p$, hold. Hence, the result follows from Theorem 2.1. ∎

The algebras C^p are dense in $C(H)$ and it follows from (2.9) and (1.1) that $(C^p, |\cdot|_p)$ are $\mathbb{D}$-subalgebras of $(C(H), \|\cdot\|)$.

COROLLARY 2.3. *Any closed $*$-derivation from $(C^p, |\cdot|_p)$ into $(C^q, |\cdot|_q)$, for $p, q \geqslant 1$, is a closable derivation from $(C(H), \|\cdot\|)$ into $B(H)$ and is implemented by a symmetric operator T such that either T is selfadjoint or the deficiency indices of T are $(0, \infty)$ or $(\infty, 0)$ or (∞, ∞).*

Proof. By Corollary 2.2, any closed $*$-derivation δ from C^p into C^q, $p, q \geqslant 1$, is a closable derivation from $C(H)$ into $B(H)$ and is implemented by a symmetric operator S, that is, (2.2) holds for all $A \in D(\delta)$. It follows from Theorem 3.11 (iii) (b) of [8] that there exists a symmetric extension T of S implementing δ with the deficiency indices which satisfy the conditions of the corollary. ∎

REMARK 2.4. The results of Corollaries 2.2 and 2.3 hold if the algebras C^p are replaced by any *symmetrically normable ideal* of $C(H)$ (see [5]).

3. DIFFERENTIAL ALGEBRAS $C_S^{p,q}$

Set $T = [1, \infty] \cup \{\mathrm{b}\}$ and let S be a closed densely defined symmetric operator on H. For $p, q \in T$, set

$$\begin{aligned} &C_S^{p,q} = \{A \in C^p \cap \mathcal{A}_S : \delta_S(A) = \mathrm{i}\,\mathrm{Closure}(SA - AS) \in C^q\} \quad \text{and} \\ &|A|_{p,q} = |A|_p + |\delta_S(A)|_q, \quad \text{for } A \in C_S^{p,q}. \end{aligned} \tag{3.1}$$

Since C^p, C^q and $\mathcal{A}_S$ (see (1.2)) are $*$-algebras and since δ_S is a closed $*$-derivation on $B(H)$ with domain $\mathcal{A}_S$, it follows from (2.6) that the restriction of δ_S to $C_S^{p,q}$ is a closed $*$-derivation from C^p into C^q. It also follows from (2.6) and (2.9) that formulae (2.3) and (2.4) hold for all norms $|\cdot|_p$. Hence, from this and from the discussion at the beginning of the proof of Theorem 2.1, we obtain the following result.

Proposition 3.1. *For any $p, q \in T$, $(C_S^{p,q}, |\cdot|_{p,q})$ is a Banach $*$-algebra and the domain of the largest $*$-derivation from C^p into C^q implemented by S.*

It is easy to see that $(C_S^{\mathrm{b,b}}, |\cdot|_{b,b}) = (\mathcal{A}_S, \|\cdot\|_S)$,

$$\begin{gathered} C_S^{p,q} \subseteq C_S^{r,t} \quad \text{and} \quad |A|_{p,q} \geqslant |A|_{r,t}, \quad \text{if } p \leqslant r,\ q \leqslant t \text{ and } A \in C_S^{p,q}; \\ |A|_{\infty,\infty} = |A|_{\infty,b} = |A|_{b,\infty} = |A|_{b,b} = \|A\|_S, \quad \text{for } A \in C_S^{\infty,\infty}. \end{gathered}$$

For $x, y \in H$, the rank one operator $x \otimes y$ on H is defined by the formula

$$(x \otimes y)z = (z, x)y. \tag{3.2}$$

It is easy to check that

$$\begin{aligned} &\|x \otimes y\| = \|x\|\,\|y\|, \quad (x \otimes y)^* = y \otimes x, \quad (x \otimes y)(u \otimes v) = (v, x)(u \otimes y), \\ &R(x \otimes y) = x \otimes Ry, \quad \text{and} \quad (x \otimes y)R \text{ extends to } (R^*x) \otimes y, \end{aligned} \tag{3.3}$$

if R is a densely defined operator, $y \in D(R)$ and $x \in D(R^*)$.

Let $\{e_j\}_{j=1}^\infty$ be a basis in H and $x, y \in H$. Then

$$\mathrm{Tr}(x \otimes y) = \sum_{j=1}^\infty ((x \otimes y)e_j, e_j) = \sum_{j=1}^\infty (e_j, x)(y, e_j) = (y, x). \tag{3.4}$$

The operator $A = x \otimes y$ belongs to C^p, for all $p > 0$. By (3.3), $A^*A = \|y\|^2(x \otimes x)$ and $(A^*A)^{1/2} = \frac{\|y\|}{\|x\|}(x \otimes x)$ has only one non-zero eigenvalue $\lambda = \|x\|\,\|y\|$. Hence

$$|x \otimes y|_p = \lambda = \|x\|\,\|y\| = \|x \otimes y\|. \tag{3.5}$$

Let S be a closed symmetric operator. It follows from Lemma 3.1 of [11] that any finite rank operator in $\mathcal{A}_S$ has the form

$$A = \sum_{i=1}^n x_i \otimes y_i \quad \text{where } x_i, y_i \in D(S). \tag{3.6}$$

By Φ_S we denote the set of all finite rank operators in $\mathcal{A}_S$. We obtain from (3.3) that, for any $A \in \Phi_S$, the operator $\delta_S(A)$ also has a finite rank. Hence

$$\Phi_S \subseteq C_S^{p,q} \subseteq C^p, \quad \text{for any } p, q \in T.$$

For $x, y \in H$ and $z, u \in D(S)$, the operator $A = x \otimes y - z \otimes u$ has rank less or equal to two. Hence $(A^*A)^{1/2}$ has not more than two non-zero eigenvalues and they are less than $\|(A^*A)^{1/2}\| = \|A\|$. Therefore $|A|_p \leqslant 2\|A\|$ and, by (3.3),

$$\|A\| = \|x\otimes y - z\otimes u\| \leqslant \|x\otimes y - x\otimes u\| + \|x\otimes u - z\otimes u\| = \|x\|\,\|y-u\| + \|x-z\|\,\|u\|.$$

Since $D(S)$ is dense in H, we have that Φ_S is dense in the set of all finite rank operators with respect to all norms $|\cdot|_p$, $0 < p \leqslant \infty$. It is well known that finite rank operators are dense in the algebra $C(H)$ and in all algebras C^p, $1 \leqslant p < \infty$ (see [4], Lemma XI.9.11). This yields the following result.

LEMMA 3.2. *For any $p \in T \setminus \{\mathrm{b}\}$ and $q \in T$, the algebras Φ_S and $C_S^{p,q}$ are dense in the algebra $(C^p, |\cdot|_p)$.*

We denote by $\mathcal{F}_S^{p,q}$ the closure of Φ_S with respect to the norm $|\cdot|_{p,q}$. Clearly, $\mathcal{F}_S^{p,q}$ are closed $*$-subalgebras of $C_S^{p,q}$,

$$\mathcal{F}_S^{p,q} \subseteq \mathcal{F}_S^{r,t}, \quad \text{if } p \leqslant r \text{ and } q \leqslant t,$$
$$\mathcal{F}_S^{\infty,q} = \mathcal{F}_S^{\mathrm{b},q}, \quad \mathcal{F}_S^{p,\infty} = \mathcal{F}_S^{p,\mathrm{b}} \quad \text{and} \quad \mathcal{F}_S^{\infty,\infty} = \mathcal{F}_S^{\infty,\mathrm{b}} = \mathcal{F}_S^{\mathrm{b},\infty} = \mathcal{F}_S^{\mathrm{b},\mathrm{b}}.$$

PROPOSITION 3.3. (i) *The algebras $\mathcal{F}_S^{p,q}$ and $C_S^{p,q}$, $p, q \in T$, are semisimple. The algebras $\mathcal{F}_S^{p,q}$ have no closed two-sided ideals and $\mathcal{F}_S^{p,q} \subseteq I$ for any closed two-sided non-trivial ideal I of $C_S^{p,q}$.*

(ii) *Let B be a bounded selfadjoint operator and $R = S + B$. If $p \leqslant q$ then $\mathcal{F}_S^{p,q} = \mathcal{F}_R^{p,q}$ and $C_S^{p,q} = C_R^{p,q}$.*

Proof. Let I be a closed two-sided ideal of $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$. Let $A \in I$ and $x \in D(S)$ be such that $A^*x \neq 0$. For any $y \in D(S)$, $x \otimes y \in \Phi_S \subseteq \mathcal{F}_S^{p,q}$. Hence, by (3.3), $(A^*x) \otimes y = (x \otimes y)A \in I$. Since $A \in \mathcal{A}_S$, we have that $A^*x \in D(S)$. Therefore, for any $z \in D(S)$, the operator $z \otimes A^*x$ belongs to Φ_S. Hence, by (3.3),

$$((A^*x) \otimes y)(z \otimes (A^*x)) = \|A^*x\|^2 (z \otimes y) \in I.$$

Thus $\Phi_S \subseteq I$, so $\mathcal{F}_S^{p,q} \subseteq I$.

Similarly, $\mathcal{F}_S^{p,q} \subseteq I$ if I is a closed two-sided ideal of $(C_S^{p,q}, |\cdot|_{p,q})$.

If R is the radical of $C_S^{p,q}$ or $\mathcal{F}_S^{p,q}$ and $R \neq 0$ then $\mathcal{F}_S^{p,q} \subseteq R$. For $x \in D(S)$ and $\|x\| = 1$, $x \otimes x \in \mathcal{F}_S^{p,q}$ and $(x \otimes x)^n = x \otimes x$. Hence $x \otimes x \notin R$. This contradiction shows that $C_S^{p,q}$ and $\mathcal{F}_S^{p,q}$ are semisimple. Part (i) is proved.

Since B is bounded, $D(R) = D(S)$. If $A \in \mathcal{A}_S$ then

$$\delta_R(A)|D(R) = \delta_S(A)|D(S) + \mathrm{i}(BA - AB)|D(S) \tag{3.7}$$

is a bounded operator. Hence $\mathcal{A}_S = \mathcal{A}_R$.

Let $A \in C_S^{p,q}$. Then $A \in C^p \cap \mathcal{A}_S = C^p \cap \mathcal{A}_R$ and $\delta_S(A) \in C^q$. Since, by (2.9), $BA - AB \in C^p \subseteq C^q$, it follows from (3.7) that $\delta_R(A) \in C^q$. Thus $A \in C_R^{p,q}$, so $C_S^{p,q} \subseteq C_R^{p,q}$. Similarly, $C_R^{p,q} \subseteq C_S^{p,q}$, so that $C_R^{p,q} = C_S^{p,q}$.

It follows from (2.9), (3.1) and (3.7) that the norms $|\cdot|_{p,q}$ generated by S and R are equivalent. Since $\mathcal{F}_R^{p,q}$ and $\mathcal{F}_S^{p,q}$ are the closures of Φ_S in these norms, they coincide. ∎

By $\mathfrak{A}_S^q$ we denote the uniform closures of the algebras $C_S^{\mathrm{b},q}$. Since $\Phi_S \subseteq C_S^{\mathrm{b},q}$ and since Φ_S is dense in $C(H)$, $\mathfrak{A}_S^q$ are C^*-algebras containing $C(H)$.

PROPOSITION 3.4. *The algebras* $(C_S^{p,q}, |\cdot|_{p,q})$, $p \neq \mathrm{b}$, *and* $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ *are* $\mathbb{D}$*-subalebras of* $C(H)$ *with constant* $D = 1$. *The algebras* $(C_S^{\mathrm{b},q}, |\cdot|_{\mathrm{b},q})$ *are* $\mathbb{D}$*-subalgebras of* $\mathfrak{A}_S^q$.

Proof. For any $A, B \in C_S^{p,q}$ it follows from (2.9) and (3.1) that

$$|AB|_{p,q} = |AB|_p + |\delta_S(AB)|_q \leqslant \|A\|\,|B|_p + |A\delta_S(B)|_q + |\delta_S(A)B|_q$$
$$\leqslant \|A\|\,|B|_p + \|A\|\,|\delta_S(B)|_q + |\delta_S(A)|_q\,\|B\| \leqslant \|A\|\,|B|_{p,q} + |A|_{p,q}\,\|B\|.$$

Thus $(C_S^{\mathrm{b},q}, |\cdot|_{\mathrm{b},q})$ are $\mathbb{D}$-subalgebras of $\mathfrak{A}_S^q$. Since $\Phi_S \subseteq \mathcal{F}_S^{p,q} \subseteq C_S^{p,q} \subseteq C(H)$, for $p \neq \mathrm{b}$, and since Φ_S is dense in $C(H)$, it follows that $(C_S^{p,q}, |\cdot|_{p,q})$, $p \neq \mathrm{b}$, and $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ are $\mathbb{D}$-subalgebras of $C(H)$. ∎

If S is bounded, the operator $\delta_S(A) = \mathrm{i}(SA - AS)$ belongs to C^p for any $A \in C^p$, so that $C_S^{p,q} = C_S^{p,p}$, for all $q \geqslant p$. If S is unbounded then all algebras $C_S^{p,q}$ are distinct. To establish this we need the following lemma.

LEMMA 3.5. *Let S be a selfadjoint operator on H.*
(i) *For any $p \in T$, there exists $A \in C^p \cap \mathcal{A}_S$ such that $\delta_S(A) \in C^1$ and $A \notin C^{p-\varepsilon}$ for any $\varepsilon > 0$.*
(ii) *If S is unbounded then, for any $p \in T$, there exists $B \in C^1 \cap \mathcal{A}_S$ such that $\delta_S(B) \in C^p$ and $\delta_S(B) \notin C^{p-\varepsilon}$ for any $\varepsilon > 0$.*

Proof. There is a decomposition $H = \bigoplus\limits_{n=-\infty}^{\infty} H(n)$ such that all $H(n)$ reduce S and $S|H(n) = \lambda_n \mathbb{1}_{H(n)} + T_n$, where $\|T_n\| \leqslant 1$ and $0 \in \mathrm{Sp}(T_n)$. Therefore, in every $H(n)$ we can choose x_n such that $\|x_n\| = 1$ and $\|T_n x_n\| \leqslant n^{-2}$.

Let $p < \infty$. Set $A = \sum\limits_{n=-\infty}^{\infty} \alpha_n (x_n \otimes x_n)$, where $\alpha_n = (|n| \ln^2 |n|)^{-1/p}$. Clearly, A belongs to C^p and does not belong to $C^{p-\varepsilon}$ for any $\varepsilon > 0$.

If $y = \sum\limits_{n=-\infty}^{\infty} y_n \in D(S)$, where $y_n \in H(n)$, then $\sum\limits_{n=-\infty}^{\infty} \|\lambda_n y_n + T_n y_n\|^2 < \infty$. Hence $\sum\limits_{n=-\infty}^{\infty} |\lambda_n|^2\, \|y_n\|^2 < \infty$. We have that $Ay = \sum\limits_{n=-\infty}^{\infty} \alpha_n (y_n, x_n) x_n$ and

$$\|SAy\|^2 = \sum_{n=-\infty}^{\infty} \alpha_n^2 |(y_n, x_n)|^2\, \|\lambda_n x_n + T_n x_n\|^2 \leqslant \sum_{n=-\infty}^{\infty} \alpha_n^2 \|y_n\|^2 (|\lambda_n|^2 + 1) < \infty,$$

so that $AD(S) \subseteq D(S)$. It follows from (3.3) and (3.5) that

$$|\delta_S(A)|_1 = |SA - AS|_1 = \sum_{n=-\infty}^{\infty} \alpha_n |x_n \otimes T_n x_n - T_n x_n \otimes x_n|_1$$
$$\leqslant \sum_{n=-\infty}^{\infty} \alpha_n\, 2\|x_n\|\,\|T_n x_n\| \leqslant 2 \sum_{n=-\infty}^{\infty} \alpha_n n^{-2} < \infty.$$

Therefore $A \in C^p \cap \mathcal{A}_S$ and $\delta_S(A) \in C^1$.

If $p = \infty$, set $\alpha_n = (\ln |n|)^{-1}$. Then $A \in C^\infty$ but it does not belong to any C^q, $q < \infty$. Similar to the case $p < \infty$, we obtain that $\delta_S(A) \in C^1$ and $A \in C^\infty \cap \mathcal{A}_S$.

If $p = \mathrm{b}$, set $A = \mathbb{1}_H$. Part (i) is proved.

Let S be unbounded. Then we can assume that $|\lambda_n| \geqslant n$, in the decomposition of S. Choose $\{n_k\}_{k=1}^{\infty}$ such that $\lambda_{n_{k+1}} - \lambda_{n_k} \geqslant 2^k$. Let $p < \infty$. Set

$$B = \sum_{k=1}^{\infty} \beta_k (x_{n_k} \otimes x_{n_{k+1}}), \quad \text{where } \beta_k = (\lambda_{n_{k+1}} - \lambda_{n_k})^{-1} (k \ln^2 k)^{-1/p}.$$

Then $s_k(B) = \beta_k \leqslant 2^{-k}$, so that $B \in C^1$. As above, it is easy to check that $BD(S) \subseteq D(S)$, so $B \in C^1 \cap \mathcal{A}_S$. We also have that

$$\begin{aligned}
\delta_S(B) &= \mathrm{i}(SB - BS) \\
&= \mathrm{i}\sum_{k=1}^{\infty} \beta_k (x_{n_k} \otimes (\lambda_{n_{k+1}} x_{n_{k+1}} + T_{n_{k+1}} x_{n_{k+1}})) - \mathrm{i}\sum_{k=1}^{\infty} \beta_k ((\lambda_{n_k} x_{n_k} + T_{n_k} x_{n_k}) \otimes x_{n_{k+1}}) \\
&= \mathrm{i}\sum_{k=1}^{\infty} \beta_k (\lambda_{n_{k+1}} - \lambda_{n_k}) x_{n_k} \otimes x_{n_{k+1}} + \mathrm{i}\sum_{k=1}^{\infty} \beta_k (x_{n_k} \otimes T_{n_{k+1}} x_{n_{k+1}} - T_{n_k} x_{n_k} \otimes x_{n_{k+1}}) \\
&= \mathrm{i}\sum_{k=1}^{\infty} (k \ln^2 k)^{-1/p} (x_{n_k} \otimes x_{n_{k+1}}) + \mathrm{i}\sum_{k=1}^{\infty} \beta_k (x_{n_k} \otimes T_{n_{k+1}} x_{n_{k+1}} - T_{n_k} x_{n_k} \otimes x_{n_{k+1}}).
\end{aligned}$$

The second term belongs to C^1. The first term belongs to C^p and does not belong to $C^{p-\varepsilon}$ for any $\varepsilon > 0$.

If $p = \infty$, set $\beta_k = (\lambda_{n_{k+1}} - \lambda_{n_k})^{-1} (\ln k)^{-1}$. Then $B \in C^1 \cap \mathcal{A}_S$ and $\delta_S(B)$ belongs to C^∞ but does not belong to any C^q, $q < \infty$. If $p = \mathrm{b}$, set $\beta_k = (\lambda_{n_{k+1}} - \lambda_{n_k})^{-1}$. Then $B \in C^1 \cap \mathcal{A}_S$ and $\delta_S(B)$ is a bounded but not compact operator. ∎

The next theorem describes the cases when $C_S^{p,q} = C_S^{h,t}$ for selfadjoint S.

THEOREM 3.6. *Let S be a selfadjoint operator on H.*

(i) *If S is unbounded and $(p,q) \neq (h,t)$ then $C_S^{p,q} \neq C_S^{h,t}$.*

(ii) *Let S be bounded. Then $C_S^{p,q} = C_S^{h,t}$ if and only if $p = h$ and there exist $\lambda \in \mathbb{R}$ and $r > 1$ such that $S - \lambda \mathbb{1}_H \in C^r$ and $\frac{1}{p} + \frac{1}{r} \geqslant \max(\frac{1}{q}, \frac{1}{t})$. In this case*

$$C_S^{p,q} = C_S^{h,t} = C^p.$$

Proof. Part (i) follows from Lemma 3.5.

Let S be bounded and $S = \lambda \mathbb{1}_H + T$, where $T \in C^r$, $r \in [1,\infty] \cup \{\mathrm{b}\}$. We obtain from (2.9) that, for any A in C^p, $\delta_S(A) = \mathrm{i}(SA - AS)$ belongs to C^s where $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$. Hence $C_S^{p,s} = C^p$. If $s \leqslant q$ and $s \leqslant t$ then $C_S^{p,q} = C_S^{p,t} = C_S^{p,s} = C^p$.

Conversely, let $C_S^{p,q} = C_S^{h,t}$. It follows from Lemma 3.5 (i) that $p = h$. Since S is bounded, we have that $C_S^{p,q} = C_S^{p,p} = C^p$, for all $q \geqslant p$. Therefore, we only have to consider the case when $q \leqslant p$ and $t \leqslant p$. Suppose that $t < q \leqslant p$.

It was shown in [7] that if $SA - AS \in C^t$, for all $A \in C^p$, then $S = \lambda \mathbb{1}_H + T$ where $\lambda \in \mathbb{R}$, $T \in C^r$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{t}$. Therefore, to finish the proof it suffices to show that $C_S^{p,q} = C_S^{p,t} = C^p$.

Since S is bounded, $C^q = C_S^{q,q} \subseteq C_S^{p,q}$. Set

$$u = \sup\{y : C^y \subseteq C_S^{p,q}\}.$$

Then $q \leqslant u \leqslant p$. We shall show now that $u = p$. Let d be such that $\frac{1}{q} + \frac{1}{d} = \frac{1}{t}$. Choose y such that $q \leqslant y \leqslant u$ and $\frac{1}{y} - \frac{1}{2d} \leqslant \frac{1}{u}$. Then $C^y \subseteq C_S^{p,q} = C_S^{p,t}$, so that $SA - AS \in C^t$ for all $A \in C^y$. It follows from the mentioned above result obtained in [7] that in this case $S = \lambda \mathbb{1}_H + T$ where $T \in C^r$ and $\frac{1}{y} + \frac{1}{r} = \frac{1}{t}$.

Let z be such that $\frac{1}{z} + \frac{1}{d} = \frac{1}{y}$. Then $z > y$ and it follows from (2.9) that for all $A \in C^z$, $SA - AS = TA - AT \in C^x$, where $\frac{1}{z} + \frac{1}{r} = \frac{1}{x}$. We have that

$$\frac{1}{x} = \frac{1}{z} + \frac{1}{r} = \frac{1}{y} - \frac{1}{d} + \frac{1}{r} = \frac{1}{t} - \frac{1}{d} = \frac{1}{q}$$

and

$$\frac{1}{z} = \frac{1}{y} - \frac{1}{d} < \frac{1}{y} - \frac{1}{2d} \leqslant \frac{1}{u}.$$

Hence $u < z$ and $x = q$, so that $SA - AS \in C^q$, for all $A \in C^z$. If $u < p$, we have a contradiction. Thus $u = p$ and $p < z$, so that $C^p = C_S^{p,q} = C_S^{p,t}$. ∎

Although, as we have seen above, the algebras $C_S^{p,q}$ and $C_S^{h,t}$ do not coincide if $p \neq h$, in some cases they may only differ by "diagonal parts".

By Z_S^p we denote the subalgebra of all operators in C^p commuting with S:

$$Z_S^p = C^p \cap S'.$$

Since the commutant S' is a weakly closed $*$-subalgebra of $B(H)$, Z_S^p is a closed $*$-subalgebra of C^p. If $q \leqslant p$ then $C_S^{q,q} + Z_S^p \subseteq C_S^{p,q}$. We consider below necessary and sufficient conditions on the selfadjoint operator S under which

$$C_S^{p,q} = C_S^{q,q} + Z_S^p, \quad \text{for all } q \leqslant p,$$

but first we need some results about Schur multipliers.

Let $\{e_i\}_{i=-\infty}^{\infty}$ be an orthogonal basis in a Hilbert space $\mathfrak{H}$. Every $T \in B(\mathfrak{H})$ has a matrix representation $T = (t_{ij})$ where $t_{ij} = (Te_j, e_i)$. A matrix $M = (m_{ij})$ is called a *Schur multiplier*, if $M \circ T = (m_{ij} t_{ij})$ is a matrix representation of a bounded operator for any $T \in B(\mathfrak{H})$. In this case the map $T \to M \circ T$ of $B(\mathfrak{H})$ into itself is bounded; it will also be denoted by M and its norm by $\|M\|$.

A matrix M is a *Schur C^p-multiplier*, $1 \leqslant p \leqslant \infty$, if $M \circ T \in C^p$ for any $T \in C^p$. The map $T \to M \circ T$ of C^p into itself is bounded; by $\|M\|_p$ we denote its norm. For example, since $T = (t_{ij}) \in C^2$ if and only if $\sum_{i,j} |t_{ij}|^2 < \infty$, it follows that $M = (m_{ij})$ is a Schur C^2-multiplier if and only if $\max_{i,j} |m_{ij}| < \infty$. In this case

$$\|M\|_2 = \max |m_{ij}|. \tag{3.8}$$

For any matrix M set

$$\|M\|_{l_\infty(l_2)} = \sup_j \left(\sum_i |m_{ij}|^2 \right)^{1/2}.$$

If $\|M\|_{l_\infty(l_2)} < \infty$ then (see [13]) M is a Schur multiplier on $B(\mathfrak{H})$ as well as a Schur C^p-multiplier for any $1 \leqslant p \leqslant \infty$ and

$$\|M\|_p \leqslant \|M\| \leqslant \|M\|_{l_\infty(l_2)}. \tag{3.9}$$

Let $H = \bigoplus_{i=-\infty}^{\infty} H_i$ be the orthogonal sum of Hilbert spaces H_i. Every operator T in $B(H)$ has a block-matrix representation (T_{ij}) where T_{ij} are bounded operators from H_j into H_i. For $M = (m_{ij})$ set

$$\widetilde{M} \times T = (m_{ij}T_{ij}), \quad \text{for } T \in B(H).$$

If $\widetilde{M} \times T \in B(H)$ for any $T \in B(H)$, we denote by $\|\widetilde{M}\|$ the norm of the map $T \to \widetilde{M} \times T$. If $\widetilde{M} \times T \in C^p$ for any $T \in C^p$, we denote by $\|\widetilde{M}\|_p$ the norm of the map $T \to \widetilde{M} \times T$.

PROPOSITION 3.7. *Let $H = \bigoplus_{i=-\infty}^{\infty} H_i$ and $M = m_{ij}$. If $\|M\|_{l_\infty(l_2)} < \infty$ then, for any $1 \leqslant p \leqslant \infty$, $\widetilde{M} \times C^p \subseteq C^p$ and $\|\widetilde{M}\|_p \leqslant \|\widetilde{M}\| = \|M\| \leqslant \|M\|_{l_\infty(l_2)}$.*

Proof. It was shown in [12] that if M is a Schur multiplier on $B(\mathfrak{H})$ then $\widetilde{M} \times B(H) \subseteq B(H)$ and $\|\widetilde{M}\| = \|M\|$. If $\|M\|_{l_\infty(l_2)} < \infty$, we obtain from (3.9) that M is a Schur multiplier on $B(\mathfrak{H})$, so that $\widetilde{M} \times B(H) \subseteq B(H)$.

It is well known (see, for example, [13]) that any Schur multiplier is also a Schur C^∞-multiplier and the norms coincide. Therefore $\widetilde{M} \times C(H) \subseteq C(H)$ and $\|\widetilde{M}\|_\infty = \|\widetilde{M}\|$. Making use of complex interpolation (see [6], Theorem 3.5.2), one can derive that $\widetilde{M} \times C^p \subseteq C^p$, for any $1 \leqslant p < \infty$, and $\|\widetilde{M}\|_p \leqslant \|\widetilde{M}\|$. ∎

We say that a selfadjoint operator S has *uniformly discrete spectrum* if

$$d = \inf\{|\lambda - \mu| : \lambda,\, \mu \in \mathrm{Sp}(S),\, \lambda \neq \mu\} > 0.$$

THEOREM 3.8. *If S has uniformly discrete spectrum then*

$$C_S^{p,q} = C_S^{q,q} + Z_S^p, \tag{3.10}$$

for all $q < p$. Conversely, if (3.10) *holds for some $q < p$ then S has uniformly discrete spectrum.*

Proof. Suppose that S has uniformly discrete spectrum. Then $\mathrm{Sp}(S) = \{s_n\}$ where s_n are eigenvalues of S. Let H_n be the corresponding eigenspaces. We can renumber them, if necessary, in such a way that $s_{n+1} - s_n \geqslant d$, so that

$$|s_n - s_k| \geqslant |n-k|d.$$

Let P_n be the projections on the subspaces H_n. For any $A \in C_S^{p,q}$, the operator $A_0 = \sum_n P_nAP_n$ is bounded and commutes with S, the operator $A' = A - A_0$ belongs to $\mathcal{A}_S$ and $B = \delta_S(A) = \delta_S(A')$. With respect to the decomposition $H = \bigoplus_n H_n$ the operators A and B have the block-matrix representation $A = (A_{nk})$, $B = (B_{nk})$. Since $B|D(S) = \mathrm{i}(SA - AS)|D(S)$, we have

$$B_{nn} = 0 \quad \text{and} \quad B_{nk} = \mathrm{i}(s_n - s_k)A_{nk}. \tag{3.11}$$

Consider the matrix $M = m_{nk}$ where $m_{nk} = -\mathrm{i}(s_n - s_k)^{-1}$ and $m_{nn} = 0$. Then

$$\|M\|_{l_\infty(l_2)} = \sup_k \left(\sum_n |m_{nk}|^2\right)^{1/2} \leqslant d^{-1}\left(\sum_{n\neq k} |n-k|^{-2}\right)^{1/2} < \infty.$$

We obtain from (3.11) that $A' = \widetilde{M} \times B = (m_{nk}B_{nk})$. It follows from Proposition 3.7 that $\widetilde{M} \times C^q \subseteq C^q$. Since $B \in C^q$, we have that $A' \in C^q$. Hence $A_0 = A - A'$ belongs to C^p and commutes with S, so that $A_0 \in Z_S^p$. Thus $C_S^{p,q} = C_S^{q,q} + Z_S^p$.

Let now (3.10) hold for some $q < p$. Suppose that $K = H \ominus \bigoplus_n H_n \neq \{0\}$. Then K reduces S and the restriction S_K of S to K is selfadjoint and has no eigenvalues. If a compact operator commutes with S_K, its adjoint also commutes with S_K, so there is a selfadjoint compact operator $A \neq 0$ commuting with S_K. Hence, every eigenspace of A is invariant under S_K and, since all of them are finite-dimensional, S_K has eigenvalues. This contradiction shows that there is no compact operator commuting with S_K, that is, $Z_{S_K}^p = \{0\}$ for $1 \leqslant p \leqslant \infty$.

By Lemma 3.5 (i), there exists an operator B on K such that $B \in C_{S_K}^{p,q}$ and $B \notin C^{p-\varepsilon}(K)$, for any $\varepsilon > 0$. Let Q be the projection on K. It commutes with S. We can consider B as an operator on H. Then $B \in C_S^{p,q}$ and $QB = BQ = B$.

Since (3.10) holds, $B = A + C$, where $A \in C_S^{q,q}$ and $C \in Z_S^p$. We have that $B = QBQ = QAQ + QCQ$. Since Q commutes with S, $QCQ \in Z_{S_K}^p$. By the above argument, $QCQ = 0$, so $B = QAQ$. Since $A \in C^q$, B also belongs to C^q. This contradiction shows that $K = \{0\}$, so $H = \bigoplus_n H_n$.

Assume now that $d = 0$. We can choose pairs (n_j, k_j), $2 \leqslant j < \infty$, such that $n_j > k_j$, all numbers $n_1, k_1, n_2, k_2, \ldots$ are distinct and $|s_{n_j} - s_{k_j}| \leqslant 2^{-j}$. Consider an operator $A = (A_{nk})$ such that $A_{nk} = 0$ if (n,k) does not coincide with any of the pairs (n_j, k_j) and $A_{n_jk_j}$ is a rank one operator with $\|A_{n_jk_j}\| = j^{-1/p}\ln^{-2/p}(j)$. Due to the choice of the pairs (n_j, k_j), the operator $R = A^*A = R_{nk}$ is block-diagonal such that the only non-zero elements are

$$R_{k_jk_j} = (A_{n_jk_j})^* A_{n_jk_j} \quad \text{and} \quad \|R_{k_jk_j}\| = \|A_{n_jk_j}\|^2 = j^{-2/p}\ln^{-4/p}(j).$$

It follows from (3.3) and (3.5) that all $R_{k_jk_j}$ are rank one positive operators with eigenvalues $\mu_j = \|R_{k_jk_j}\|$. Hence the eigenvalues of the operator $R^{1/2}$ are $\lambda_j = (\mu_j)^{1/2} = \|R_{k_jk_j}\|^{1/2}$. Therefore

$$|A|_p = \left(\sum_j \lambda_j^p\right)^{1/p} = \left(\sum_j [j^{-1/p}\ln^{-2/p}(j)]^p\right)^{1/p} = \left(\sum_j j^{-1}\ln^{-2}(j)\right)^{1/p} < \infty,$$

so that $A \in C^p \cap \mathcal{A}_S$ and $A \notin C^{p-\varepsilon}$, for any $\varepsilon > 0$.

Let $D = (D_{nk}) \in Z_S^p$. Then D is block-diagonal and $D +A \in C^p \cap \mathcal{A}_S$. One can check that, due to the choice of the pairs (n_j, k_j), among the eigenvalues of the operator $(D + A)^*(D + A)$ there are distinct eigenvalues λ_{m_j} such that

$$\lambda_{m_j} \geqslant \|(D_{k_jk_j})^* D_{k_jk_j} + (A_{n_jk_j})^* A_{k_jn_j}\| \geqslant \|(A_{n_jk_j})^* A_{k_jn_j}\| = j^{-2/p}\ln^{-4/p}(j).$$

Therefore, the eigenvalues $(\lambda_{m_j})^{1/2}$ of $[(D + A)^*(D + A)]^{1/2}$ are not smaller than $j^{-1/p}\ln^{-2/p}(j)$. From this it follows that $D + A \notin C^{p-\varepsilon}$ for any $\varepsilon > 0$.

We obtain from (3.11) that $B_{nk} = 0$ if (n,k) does not coincide with any of the pairs (n_j, k_j) and $B_{n_jk_j}$ is a rank one operator with

$$\|B_{n_jk_j}\| = |s_{n_j} - s_{k_j}|\, \|A_{n_jk_j}\| \leqslant 2^{-j} j^{-1/p}\ln^{-2/p}(j).$$

It is easy to see that $B \in C^1$ and, therefore, it belongs to any C^q, $1 \leqslant q \leqslant p$. From this we conclude that $A \in C_S^{p,q}$ and $A \notin C_S^{q,q} + Z_S^p$, so $C_S^{p,q} \neq C_S^{q,q} + Z_S^p$. Thus, we have shown that $C_S^{p,q} = C_S^{q,q} + Z_S^p$, for some $q < p$, implies that the spectrum of S is uniformly discrete. ∎

4. APPROXIMATION PROPERTY OF ALGEBRAS $C_S^{p,q}$

An operator algebra $\mathcal{A}$ containing finite rank operators is said to possess the *approximation property* if any operator in $\mathcal{A}$ is approximated by finite rank operators. For the algebras $C_S^{p,q}$ this means that $C_S^{p,q} = \mathcal{F}_S^{p,q}$. By Proposition 3.3, $C_S^{p,q}$ has approximation property if and only if it is simple.

The subalgebras $\mathcal{F}_S^{\infty,\infty}$, $C_S^{\infty,\infty}$ and $C_S^{\infty,\mathrm{b}}$ of $\mathcal{A}_S$ were studied in [11] and [12]. It was shown that if S is selfadjoint, the algebra $C_S^{\infty,\infty}$ has approximation property. In Corollary 5.5 we extend this result and prove that all algebras $C_S^{p,q}$, $p, q \in T \setminus \{\mathrm{b}\}$, have approximation property.

For non-selfadjoint S, we do not know whether the algebras $C_S^{p,q}$ have approximation property. However, making use of the results about the structure of $\mathbb{D}$-algebras established in [11], we show in this section that the closures of $(C_S^{p,q})^2$ with respect to $|\cdot|_{p,q}$ have this property, that is, they coincide with $\mathcal{F}_S^{p,q}$. Thus, the approximation problem is equivalent to the problem of the density of $(C_S^{p,q})^2$ in $C_S^{p,q}$. We start with the following lemma.

Lemma 4.1. *Let* $\frac{1}{p}+\frac{1}{x}=\frac{1}{r}$, $\frac{1}{p}+\frac{1}{y}=\frac{1}{s}$, $\frac{1}{q}+\frac{1}{x}=\frac{1}{t}$ *and* $a=\max(s,t)$. *Then*

$$\Phi_S \subseteq \mathcal{F}_S^{p,q} C_S^{x,y} \subseteq \mathcal{F}_S^{r,a}.$$

Proof. We obtain easily from (3.3) that $(\Phi_S)^2 = \Phi_S$. Since all algebras $\mathcal{F}_S^{p,q}$ and $C_S^{p,q}$, $p, q \in T$, contain Φ_S, we obtain that $\Phi_S \subseteq \mathcal{F}_S^{p,q} C_S^{x,y}$.

Let $X \in \mathcal{F}_S^{p,q}$ and $Y \in C_S^{x,y}$. Let $\{X_n\}_{n=1}^{\infty}$ be elements of Φ_S converging to X in $|\cdot|_{p,q}$, that is, $|X - X_n|_p + |\delta_S(X - X_n)|_q \to 0$, as $n \to \infty$. Since Y preserves $D(S)$, we have that $X_n Y \in \Phi_S$. Then, by (2.6) and (2.8),

$$\begin{aligned} |XY - X_nY|_{r,a} &= |(X - X_n)Y|_r + |\delta_S((X - X_n)Y)|_a \\ &\leqslant |X - X_n|_p\, |Y|_x + |\delta_S(X - X_n)Y|_a + |(X - X_n)\delta_S(Y)|_a \\ &\leqslant |X - X_n|_p\, |Y|_x + |\delta_S(X - X_n)Y|_t + |(X - X_n)\delta_S(Y)|_s \\ &\leqslant |X - X_n|_p\, |Y|_x + |\delta_S(X - X_n)|_q\, |Y|_x + |X - X_n|_p\, |\delta_S(Y)|_y \to 0 \end{aligned}$$

so that $XY \in \mathcal{F}_S^{r,a}$. ∎

Definition 4.2. Let i be an injective bounded linear map from a normed space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ into a normed space $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$. A sequence $\{x_n\}$ in $\mathfrak{X}$ is called $(\sim)$-*converging to* $x \in \mathfrak{X}$ *with respect to* i if

$$\|i(x) - i(x_n)\|_{\mathfrak{Y}} \to 0 \quad \text{and} \quad \sup_n \|x_n\|_{\mathfrak{X}} < \infty.$$

A subset M in $\mathfrak{X}$ is $(\sim)$-closed if it contains all $(\sim)$-limits of its elements.

Let φ be a closed linear map from a normed space $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ into a normed space $(\mathfrak{Z}, \|\cdot\|_{\mathfrak{Z}})$. The domain $\mathfrak{X}$ of φ is a Banach space with respect to the norm $\|x\|_{\mathfrak{X}} = \|x\|_{\mathfrak{Y}} + \|\varphi(x)\|_{\mathfrak{Z}}$. By i we denote the identity map of $\mathfrak{X}$ into $\mathfrak{Y}$.

Let $\mathfrak{Z}^*$ be the dual space of $\mathfrak{Z}$. By Ω_φ we denote the set of all $F \in \mathfrak{Z}^*$ such that the functional F_φ on $\mathfrak{X}: F_\varphi(x) = F(\varphi(x))$, is bounded with respect to $\|\cdot\|_{\mathfrak{Y}}$. The following lemma was proved in [11] for subspaces but the proof easily extends to closed convex subsets.

Lemma 4.3. *If Ω_φ is norm dense in $\mathfrak{Z}^*$ then any closed convex subset in $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is $(\sim)$-closed with respect to i.*

For any $p \in [1, \infty]$, let p' be the conjugate exponent of p

$$\frac{1}{p} + \frac{1}{p'} = 1, \text{ if } 1 < p \leqslant \infty, \quad \text{and} \quad p' = \mathrm{b} \text{ if } p = 1. \tag{4.1}$$

We denote by $j_{p,q}$ the identity maps of $C_S^{p,q}$ into C^p.

Lemma 4.4. *For $p \neq \mathrm{b}$ and $q > 1$, any closed convex subset in $(C_S^{p,q}, |\cdot|_{p,q})$ is $(\sim)$-closed with respect to $j_{p,q}$.*

Proof. Replacing $\mathfrak{Y}$ by C^p, $\mathfrak{Z}$ by C^q and φ by δ_S in Lemma 3.7, we obtain that in order to prove our lemma it suffices to show that the set Ω_δ is norm dense in the dual space $(C^q)^*$.

Since $1 < q$, $q' \in [1, \infty)$ and the algebra $(C^{q'}, |\cdot|_{q'})$ is isometrically isomorphic to $(C^q)^*$: any bounded linear functional on C^q has the form $F_T(A) = \mathrm{Tr}(AT)$ where $T \in C^{q'}$. It follows from (3.4) that, for any $T = x \otimes y$, $x, y \in D(S)$, the functional $(F_T)_\delta$:

$$\begin{aligned}(F_T)_\delta(A) = F_T(\delta_S(A)) &= \mathrm{Tr}(\delta_S(A)T) = \mathrm{i}\mathrm{Tr}(x \otimes ((SA - AS)y)) \\ &= \mathrm{i}(Ay, S^*x) - \mathrm{i}(ASy, x)\end{aligned}$$

extends to a bounded functional on C^p. Thus $F_T \in \Omega_\delta$. By Lemma 3.2, the set of all linear combinations of such operators T is norm dense in $C^{q'}$, so that Ω_δ is norm dense in $(C^q)^*$. ∎

We denote by $i_{p,q}$ the identity maps of $C_S^{p,q}$ into $C^\infty = C(H)$.

Lemma 4.5. *If $1 < p \leqslant \infty$ and $1 < q$, then any closed convex subset in $(C_S^{p,q}, |\cdot|_{p,q})$ is $(\sim)$-closed with respect to $i_{p,q}$.*

Proof. If $p = \infty$ then $i_{\infty,q} = j_{\infty,q}$, so the result follows from Lemma 3.8.

Let $1 < p < \infty$ and M be a closed convex subset in $(C_S^{p,q}, |\cdot|_{p,q})$. Let X_n from $M(\sim)$-converge to X with respect to $i_{p,q} : \|X - X_n\| \to 0$ and $\sup_n |X_n|_{p,q} \leqslant K$; we have to show that $X \in M$.

Clearly, $|X_n|_p \leqslant K$ for all n. Since C^p is isomorphic to the dual space of $C^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, the ball of C^p of radius K is compact in the weak $\sigma(C^p, C^{p'})$ topology on C^p. Hence the sequence $\{X_n\}_{n=1}^\infty$ has a cluster point Y.

Since $1 < p$, $C^{p'}$ is isomorphic to the dual space of C^p and it follows from Hahn-Banach Theorem that, for any m, Y belongs to the closed, in $|\cdot|_p$, convex set generated by $X_n, m \leqslant n$. Hence, there are positive $\alpha_m^{(m)}, \dots, \alpha_{\kappa(m)}^{(m)}$ such that

$$\alpha_m^{(m)} + \cdots + \alpha_{\kappa(m)}^{(m)} = 1 \quad \text{and} \quad |Y - Y_m|_p \leqslant \frac{1}{m},$$

where $Y_m = \alpha_m^{(m)} X_m + \cdots + \alpha_{\kappa(m)}^{(m)} X_{\kappa(m)}$. Then $Y_m \in M$, $|Y - Y_m|_p \to 0$, as $m \to \infty$, and

$$\sup_m |Y_m|_{p,q} = \sup_m \left(\alpha_m^{(m)} |X_m|_{p,q} + \cdots + \alpha_{\kappa(m)}^{(m)} |X_{\kappa(m)}|_{p,q}\right) \leqslant K.$$

Thus Y_m $(\sim)$-converge to Y with respect to $j_{p,q}$. By Lemma 4.4, M is $(\sim)$-closed with respect to $j_{p,q}$. Hence $Y \in M$.

We also have that

$$\begin{aligned}\|X - Y\| &\leqslant \|X - Y_m\| + \|Y_m - Y\| \\ &\leqslant \alpha_m^{(m)} \|X - X_m\| + \cdots + \alpha_{\kappa(m)}^{(m)} \|X - X_{\kappa(m)}\| + |Y_m - Y|_p \to 0\end{aligned}$$

as $m \to \infty$. Hence $X = Y$, so $X \in M$ and M is $(\sim)$-closed with respect to $i_{p,q}$. ∎

Lemma 4.6. *Let A, B belong to $C_S^{\infty,1}$ and let them be $(\sim)$-limits of operators from Φ_S with respect to $i_{\infty,1}$. Then $AB \in \mathcal{F}_S^{\infty,1}$.*

Proof. Let A_n from $\Phi_S(\sim)$-converge to A with respect to $i_{\infty,1}$:

$$\|A - A_n\| \to 0 \quad \text{and} \quad \sup_n |\delta_S(A_n)|_1 < \infty.$$

Then $\sup_n |\delta_S(A_n)|_2 \leqslant \sup_n |\delta_S(A_n)|_1 < \infty$. Since C^2 is a Hilbert space, the unit ball of C^2 is weakly compact. Hence the set $\{\delta_S(A_n)\}_{n=1}^{\infty}$ has a weak cluster point $Z \in C^2$. By Hahn-Banach Theorem, for any m, Z belongs to the closed in $|\cdot|_2$ convex set spanned by $\delta_S(A_n), m \leqslant n$. Hence there exist convex finite linear combinations A'_m of the operators $\{A_n\}_{n \geqslant m}$ such that $\delta_S(A'_m)$ converge to $\delta_S(A)$ in $|\cdot|_2$. Then

$$\|A - A'_m\| \to 0 \quad \text{and} \quad \|Z - \delta_S(A'_m)\| \leqslant |Z - \delta_S(A'_m)|_2 \to 0,$$

as $m \to \infty$, where $A'_m \in \Phi_S$. Since δ_S is closed, $Z = \delta_S(A)$. Therefore, replacing A_n by A'_m if necessary, we may assume that

$$\|A - A_n\| \to 0, \quad |\delta_S(A) - \delta_S(A_n)|_2 \to 0 \quad \text{and} \quad \sup_n |\delta_S(A_n)|_1 \leqslant \infty. \tag{4.2}$$

The unit ball of C^1 is $\sigma(C^1, C^\infty)$-compact. Since $\sup_n |\delta_S(A_n)|_1 < \infty$, the set $\{\delta_S(A_n)\}_{n=1}^{\infty}$ has a cluster point $Y \in C^1$ in the $\sigma(C^1, C^\infty)$ topology. Since $\delta_S(A) \in C^1$, we obtain from (2.8) and (4.2) that, for any $T \in C^2$,

$$\begin{aligned}\left|\mathrm{Tr}\big((Y - \delta_S(A))T\big)\right| &\leqslant \left|\mathrm{Tr}\big((Y - \delta_S(A_n))T\big)\right| + \left|\mathrm{Tr}\big((\delta_S(A_n) - \delta_S(A))T\big)\right| \\ &\leqslant \left|\mathrm{Tr}\big((Y - \delta_S(A_n))T\big)\right| + \left|(\delta_S(A_n) - \delta_S(A))T\right|_1 \\ &\leqslant \left|\mathrm{Tr}\big((Y - \delta_S(A_n))T\big)\right| + |\delta_S(A_n) - \delta_S(A)|_2 \, |T|_2 \to 0,\end{aligned}$$

as $n \to \infty$. Therefore, $\mathrm{Tr}((Y - \delta_S(A))T) = 0$. Since C^2 is dense in $(C^\infty, |\cdot|_\infty)$, $\mathrm{Tr}((Y - \delta_S(A))T) = 0$ for all $T \in C^\infty$. Hence $Y = \delta_S(A)$.

Let now $B \in C_S^{\infty,1}$ be the $(\sim)$-limit with respect to $i_{\infty,1}$ of a sequence of operators B_n from Φ_S. As above, $\delta_S(B)$ is a cluster point of $\{\delta_S(B_n)\}$ in the $\sigma(C^1, C^\infty)$ topology. We will show now that $\delta_S(AB)$ is a cluster point of the set $\{\delta_S(A_nB_n)\}_{n=1}^\infty$ in the $\sigma(C^1, B(H))$ topology on C^1. Indeed,

$$\delta_S(AB) - \delta_S(A_nB_n) = A\delta_S(B) + \delta_S(A)B - A_n\delta_S(B_n) - \delta_S(A_n)B_n = U_n + V_n,$$

where $U_n = A\delta_S(B) - A_n\delta_S(B_n)$ and $V_n = \delta_S(A)B - \delta_S(A_n)B_n$. By (2.9), for $R \in B(H)$,

$$\begin{aligned}|\mathrm{Tr}(U_nR)| &\leqslant |\mathrm{Tr}(A(\delta_S(B) - \delta_S(B_n))R)| + |\mathrm{Tr}((A - A_n)\delta_S(B_n)R)| \\ &\leqslant |\mathrm{Tr}((\delta_S(B) - \delta_S(B_n))RA)| + |(A - A_n)\delta_S(B_n)R|_1 \\ &\leqslant |\mathrm{Tr}((\delta_S(B) - \delta_S(B_n))RA)| + \|A - A_n\|\, |\delta_S(B_n)|_1\, \|R\|\end{aligned}$$

and, similarly,

$$|\mathrm{Tr}(V_nR)| \leqslant |\mathrm{Tr}((\delta_S(A) - \delta_S(A_n))BR)| + \|B - B_n\|\, |\delta_S(A_n)|_1\, \|R\|.$$

It follows from (4.2) that $\|A - A_n\| \to 0$ and $\|B - B_n\| \to 0$, as $n \to \infty$, and

$$\sup_n |\delta_S(A_n)|_1 < \infty \quad \text{and} \quad \sup_n |\delta_S(B_n)|_1 < \infty.$$

Since A and B are compact, RA and BR belong to $C(H)$. Since $\delta_S(A)$ and $\delta_S(B)$ are cluster points of $\{\delta_S(A_n)\}_{n=1}^\infty$ and $\{\delta_S(B_n)\}_{n=1}^\infty$, respectively, in the $\sigma(C^1, C^\infty)$ topology on C^1, we obtain that $\delta_S(AB)$ is a cluster point of $\{\delta_S(A_nB_n)\}_{n=1}^\infty$ in the $\sigma(C^1, B(H))$ topology on C^1.

By the Hahn-Banach Theorem, for any m, $\delta_S(AB)$ belongs to the closed, in $|\cdot|_1$, convex set spanned by $\delta_S(A_nB_n)$, $m \leqslant n$. Hence, there are positive numbers $\alpha_m^{(m)}, \dots, \alpha_{\kappa(m)}^{(m)}$ such that $\alpha_m^{(m)} + \dots + \alpha_{\kappa(m)}^{(m)} = 1$ and $|\delta_S(AB) - \delta_S(Z_m)|_1 \to 0$, as $m \to \infty$, where $Z_m = \alpha_m^{(m)} A_mB_m + \dots + \alpha_{\kappa(m)}^{(m)} A_{\kappa(m)}B_{\kappa(m)}$. Then $Z_m \in \Phi_S$ and

$$\begin{aligned}\|AB - Z_m\| &\leqslant \alpha_m^{(m)}\|AB - A_mB_m\| + \dots + \alpha_{\kappa(m)}^{(m)}\|AB - A_{\kappa(m)}B_{\kappa(m)}\| \\ &\leqslant \sup_{m\leqslant n} \|AB - A_nB_n\| \leqslant \sup_{m\leqslant n} \|A\|\, \|B - B_n\| + \sup_{m\leqslant n} \|A - A_n\|\, \|B_n\| \to 0\end{aligned}$$

as $m \to \infty$. Hence, Z_m converge to AB in $|\cdot|_{\infty,1}$, so that $AB \in \mathcal{F}_S^{\infty,1}$. ∎

By Ξ we denote the set of all functions in $C^\infty(\mathbb{R})$ that vanish in a neighbourhood of 0. To prove the main theorem of this section we need the following results obtained in Theorems 2.2, 2.5, 2.8 and Corollary 2.9 (iii) of [11].

Theorem 4.7. ([11]) *Let $(\mathcal{A}, \|\cdot\|_1)$ be a $\mathbb{D}$-subalgebra of a C^*-algebra $(\mathfrak{A}, \|\cdot\|)$.*

(i) *Let M be a subspace of $\mathcal{A}$ such that $\varphi(x) \in M$ for any $x = x^* \in \mathcal{A}$ and $\varphi \in \Xi$. If M is $(\sim)$-closed in $\mathcal{A}$ with respect to the injection of $\mathcal{A}$ into $\mathfrak{A}$, then $\mathcal{A}^2 \subseteq M$.*

(ii) *$\overline{\mathcal{A}^2} = \overline{\mathcal{A}^n}$ for $n \geqslant 2$.*

(iii) *Let $\mathcal{A}_+$ be the set of all positive elements in $\mathcal{A}$. For any $A \in \mathcal{A}_+$, there exist $\varphi_n \in \Xi$ such that $\varphi_n(A)$ $(\sim)$-converge to A with respect to the injection of $\mathcal{A}$ into $\mathfrak{A}$: $\|A - \varphi_n(A)\| \to 0$ and $\sup_n \|\varphi_n(A)\|_1 < \infty$.*

We are now ready to prove the main theorem of this section.

THEOREM 4.8. *For $p \neq \mathrm{b}$, the closure of $(C_S^{p,q})^2$ with respect to $|\cdot|_{p,q}$ coincides with $\mathcal{F}_S^{p,q}$.*

Proof. *Case* 1. *Let $1 < p$ and $1 < q$.* The algebra $\mathcal{F}^{p,q}$ is closed in $(C^{p,q}, |\cdot|_{p,q})$. Hence, by Lemma 4.5, it is $(\sim)$-closed in $(C^{p,q}, |\cdot|_{p,q})$ with respect to the map $i_{p,q}$ of $C^{p,q}$ in $C(H)$.

Let $A = A^* \in C_S^{p,q}$ and $\varphi \in \Xi$. Since, by Proposition 3.4, $(C_S^{p,q}, |\cdot|_{p,q})$ is a $\mathbb{D}$-subalgebra of $C(H)$ and $\varphi \in C^\infty(\mathbb{R})$, it follows from Proposition 6.4 of [1] (see also Theorem 12 of [9]) that $\varphi(A) \in C_S^{p,q}$. Since $p \neq \mathrm{b}$, A is compact. Hence $\varphi(A)$ is a finite rank operator and thus, it belongs to $\mathcal{F}_S^{p,q}$. Applying Lemma 4.1 and Theorem 4.7 (i), we obtain that $\Phi_S \subseteq (C_S^{p,q})^2 \subseteq \mathcal{F}_S^{p,q}$, which completes the proof.

Case 2. Let $p = 1$ and $1 < q$. Then $C_S^{1,q} \subseteq C_S^{2,q}$. It follows from Case 1 that $(C_S^{1,q})^2 \subseteq (C_S^{2,q})^2 \subseteq \mathcal{F}_S^{2,q}$. Hence, by Lemma 4.1, $(C_S^{1,q})^3 \subseteq \mathcal{F}_S^{2,q} C_S^{1,q} \subseteq \mathcal{F}_S^{1,q}$, so the closure of $(C_S^{1,q})^3$ in $|\cdot|_{1,q}$ coincides with $\mathcal{F}_S^{1,q}$. By Theorem 4.7 (ii),

$$\overline{(C_S^{1,q})^2} = \overline{(C_S^{1,q})^3} = \mathcal{F}_S^{1,q}.$$

Case 3. *Let $p < \infty$ and $q = 1$.* Choose $r > 1$ such that $\frac{1}{p} + \frac{1}{r} > 1$. Then $C^{p,1} \subseteq C^{p,r}$. It follows from Case 1 that $(C_S^{p,1})^2 \subseteq (C_S^{p,r})^2 \subseteq \mathcal{F}_S^{p,r}$. Hence, by Lemma 4.1, $(C_S^{p,1})^3 \subseteq \mathcal{F}_S^{p,r} C_S^{p,1} \subseteq \mathcal{F}_S^{p,1}$, so the closure of $(C_S^{p,1})^3$ in $|\cdot|_{p,1}$ coincides with $\mathcal{F}_S^{p,1}$. Using, as in Case 2, Theorem 4.7 (ii), we have that

$$\overline{(C_S^{p,1})^2} = \overline{(C_S^{p,1})^3} = \mathcal{F}_S^{p,1}.$$

Case 4. *Let $p = \infty$ and $q = 1$.* We have $C_S^{\infty,1} \subseteq C_S^{\infty,2}$. It follows from Case 1 that $(C_S^{\infty,1})^2 \subseteq (C_S^{\infty,2})^2 \subseteq \mathcal{F}_S^{\infty,2}$. Hence $(C_S^{\infty,1})^2 \subseteq \mathcal{F}_S^{\infty,2} \cap C_S^{\infty,1}$.

If A is a positive operator in $C_S^{\infty,1}$, it follows from Theorem 4.7 (iii) that there are φ_n in Ξ such that A is a $(\sim)$-limit of $\varphi_n(A)$ with respect to $i_{\infty,1}$, that means, $\|A - \varphi_n(A)\| \to 0$ and $\sup_n \|\varphi_n(A)\|_1 < \infty$. Since A is compact, the operators $\varphi_n(A)$ belong to Φ_S. Thus, A is a $(\sim)$-limit, with respect to $i_{\infty,1}$, of operators from Φ_S.

Let $A, B \in C_S^{\infty,1}$. Then $A^2, B^2, (A+B)^2$ and $(A+\mathrm{i}B)(A+\mathrm{i}B)^*$ are positive operators in $(C_S^{\infty,1})^2$ and

$$AB = \frac{1}{2}\big((A+B)^2 - A^2 - B^2 + \mathrm{i}(A+\mathrm{i}B)(A+\mathrm{i}B)^* - \mathrm{i}A^2 - \mathrm{i}B^2\big).$$

Hence, every operator in $(C_S^{\infty,1})^2$ is a linear combination of positive operators from $(C_S^{\infty,1})^2$ and, by the above argument, is a $(\sim)$-limit, with respect to $i_{\infty,1}$, of operators from Φ_S. Applying now Lemma 4.6, we obtain that $(C_S^{\infty,1})^4 \subseteq \mathcal{F}_S^{\infty,1}$. Therefore, it follows from Theorem 4.7 (ii) that $\overline{(C_S^{\infty,1})} = \overline{(C_S^{\infty,1})^4} = \mathcal{F}_S^{\infty,1}$. ∎

COROLLARY 4.9. *Let $r \leqslant p$, $t \leqslant q$, $p \neq \mathrm{b}$ and $q \neq 1$. Then the closures of $(C_S^{r,t})^2$ and $C_S^{r,t} C_S^{p,q}$ with respect to $|\cdot|_{p,q}$ coincide with $\mathcal{F}_S^{p,q}$.*

Proof. Since $\Phi_S \subseteq C_S^{r,t} \subseteq C_S^{p,q}$, it follows from Theorem 4.8 that

$$\Phi_S = (\Phi_S)^2 \subseteq (C_S^{r,t})^2 \subseteq C_S^{r,t} C_S^{p,q} \subseteq (C_S^{p,q})^2 \subseteq \mathcal{F}_S^{p,q}.$$

Taking the closure with respect to $|\cdot|_{p,q}$, we obtain the result. ∎

5. Approximate identities and the approximation property of the algebras $C_S^{p,q}$ for selfadjoint S

In this section we study mainly the case when S is a selfadjoint operator; this case is easier to deal with, since we can now employ the Spectral Theorem. The section is primarily devoted to the establishing of the existence of approximate identities in the algebras $C_S^{p,q}$, for selfadjoint S. Since these identities consist of finite rank operators, this provides an affirmative answer to the approximation problem. We also show that, for non-selfadjoint S, the algebras $C_S^{p,q}$ have no bounded approximate identities and that, for selfadjoint S, they have bounded approximate identities only if $p = \infty$.

First we observe that the algebras C^p, $1 \leqslant p \leqslant \infty$, have *two-sided approximate identities which are bounded if and only if* $p = \infty$. Indeed, by Theorem III.6.3 of [5], any sequence $\{E_n\}_{n=1}^\infty$ of operators from C^p strongly converging to $\mathbb{1}_H$ is a two-sided approximate identity. If $p = \infty$ and all E_n are projections, the identity is bounded. Let $p \neq \infty$ and suppose that there exists a bounded approximate identity $\{E_n\}_{n=1}^\infty$ in $C^p : \sup\limits_n\{|E_n|_p\} = C < \infty$. Let Q be an m-dimensional projection. Then $|Q|_p = m^{1/p}$ and, by (2.9),

$$|QE_n|_p \leqslant \|Q\|\,|E_n|_p = |E_n|_p \leqslant C, \quad \text{for all } n. \tag{5.1}$$

On the other hand, $\big||Q|_p - |QE_n|_p\big| \leqslant |Q - QE_n|_p \to 0$ as $n \to \infty$, so that $|QE_n|_p \to m^{1/p}$. Comparing this with (5.1), we obtain a contradiction which shows that the algebras C^p, $1 \leqslant p < \infty$, have no bounded approximate identity.

A closed subspace L of H reduces an operator S if $D(S) = D_L(S) \oplus D_{L^\perp}(S)$ where

$$D_L(S) = D(S) \cap L, \quad D_{L^\perp}(S) = D(S) \cap L^\perp,$$

and if $SD_L(S) \subseteq L$ and $SD_{L^\perp}(S) \subseteq L^\perp$. We start by establishing the existence of approximate identities in the algebras $C_S^{p,q}$ in the following simplest cases.

Proposition 5.1. *Let S be a selfadjoint operator on H.*

(i) *If $H = \bigoplus_{j=1}^{\infty} H(j)$, where all subspaces $H(j)$ reduce S and $S|H(j) = s_j \mathbb{1}_{H(j)}$, then, for $p, q \in T \setminus \{\mathrm{b}\}$, the algebra $C_S^{p,q}$ has a countable two-sided approximate identity which consists of finite-dimensional projections converging to $\mathbb{1}_H$ in the strong operator topology. If $p = \infty$, the approximate identity is bounded.*

(ii) *If $1 \leqslant p \leqslant q \leqslant \infty$, the algebra $C_S^{p,q}$ has a countable two-sided approximate identity which consists of finite-dimensional projections converging to $\mathbb{1}_H$ in the strong operator topology. If $p = \infty$, the approximate identity is bounded.*

Proof. In every $H(j)$ choose an increasing sequence of finite-dimensional projections Q_j^n converging to $\mathbb{1}_{H(j)}$ in the strong operator topology as $n \to \infty$. The finite-dimensional projections $Q_n = \bigoplus_{j=-n}^{n} Q_j^n$ belong to Φ_S, converge to $\mathbb{1}_H$ in the strong operator topology and commute with S. Therefore $\delta_S(Q_n) = 0$ and

$\delta_S(Q_nA) = Q_n\delta_S(A)$ for $A \in C_S^{p,q}$. Since $\{Q_n\}_{n=1}^{\infty}$ is a two-sided approximate identity in C^r for any $r \leqslant \infty$ (see Theorem III.6.3 of [5]), it follows that

$$\begin{aligned}|A - Q_nA|_{p,q} &= |A - Q_nA|_p + |\delta_S(A - Q_nA)|_q \\ &= |A - Q_nA|_p + |\delta_S(A) - Q_n\delta_S(A)|_q \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Similarly, $|A - Q_nA|_{p,q} \to 0$ so $\{Q_n\}$ is a two-sided approximate identity in $C_S^{p,q}$.

If $p = \infty$ then $|Q_n|_{\infty,q} = |Q_n|_\infty + |\delta_S(Q_n)|_q = \|Q_n\| = 1$, so that $\{Q_n\}_{n=1}^{\infty}$ is a bounded approximate identity in $C_S^{\infty,q}$. Part (i) is proved.

Let $E_S(\lambda)$ be the spectral measure of S. For every $n \in \mathbb{Z}$, set

$$P_S(n) = E_S(n+1) - E_S(n) \quad \text{and} \quad [S] = \sum_{-\infty}^{\infty} nP_S(n).$$

The operator $[S]$ is selfadjoint and $\mathrm{Sp}([S]) \subseteq \mathbb{Z}$. By (i), the algebra $C_{[S]}^{p,q}$ has a two-sided approximate identity $\{Q_n\}$ which consists of finite-dimensional projections strongly converging to $\mathbb{1}_H$. Since the operator $S - [S]$ is bounded and $p \leqslant q$, it follows from Proposition 3.3 (ii) that $C_S^{p,q} = C_{[S]}^{p,q}$ and the norms are equivalent. Hence $\{Q_n\}_{n=1}^{\infty}$ is a two-sided approximate identity in $C_S^{p,q}$. ∎

To prove the existence of an approximate identity in the general case, we need the following extension of the result due to Voiculescu ([17]) about the existence of quasicentral approximate units relative to C^q.

PROPOSITION 5.2. *Let H be a subspace of a Hilbert space $\mathfrak{H}$ and S be a bounded selfadjoint operator on H of finite multiplicity. Let $q \in (1, \infty]$ and let $X_1, \dots, X_k \in C(\mathfrak{H})$. There exist positive finite rank operators B_m on H strongly converging to $\mathbb{1}_H$ such that $\|B_m\| = 1$, $\sup |\delta_S(B_m)|_1 < \infty$ and*

$$|\delta_S(B_m)|_q + \sum_{i=1}^{k} |\delta_S(B_m)X_i|_1 < \frac{1}{m}.$$

Proof. Let S have multiplicity $N < \infty$. Then there exist $x_1, \dots, x_N \in H$ such that the linear span of S^kx_i, $1 \leqslant i \leqslant N$ and $0 \leqslant k < \infty$, is dense in H. Let H_n be the subspaces spanned by S^kx_i, $1 \leqslant i \leqslant N$ and $0 \leqslant k \leqslant n$. The projections P_n on H_n strongly converge to $\mathbb{1}_H$, $P_{n-1} \leqslant P_n$ and, for all $n \geqslant 1$, $P_nSP_{n-1} = SP_{n-1}$. Hence $P_{n-1}SP_n = P_{n-1}S$. Using this and setting $A_n = SP_n - P_nS$, we obtain that

$$A_nP_{n-1} = 0 \quad \text{and} \quad A_nP_{n+1} = A_n.$$

Taking into account that $A_n^* = -A_n$, we have $P_{n-1}A_n = 0$ and $P_{n+1}A_n = A_n$, so

$$A_n = (P_{n+1} - P_{n-1})A_n = A_n(P_{n+1} - P_{n-1}).$$

Thus A_n act on the r-dimensional subspaces $H_{n+1} \ominus H_{n-1}$, $r \leqslant 2N$, and, therefore, there exists $K \geqslant 0$ such that $|A_n|_1 \leqslant K\|A_n\| \leqslant 2K\|S\|$, for all n.

Let Q be the projection on H. By Theorem III.6.3 of [5], $\{P_n\}_{n=1}^{\infty}$ is a two-sided approximate identity in $C(H)$. For any $X \in C(\mathfrak{H})$, $QXQ \in C(H)$. Since $A_n = QA_nQ$, it follows from (2.9) that

$$\begin{aligned} &|\mathrm{Tr}(A_nX)| = |\mathrm{Tr}(QA_nQX)| = |\mathrm{Tr}(A_nQXQ)| \leqslant |A_nQXQ|_1 \\ (5.2)\quad &= |A_n(P_{n+1} - P_{n-1})QXQ|_1 \leqslant |A_n|_1 \|(P_{n+1} - P_{n-1})QXQ\| \\ &\leqslant 2K\|S\|\big(\|P_{n+1}QXQ - QXQ\| + \|QXQ - P_{n-1}QXQ\|\big) \to 0, \ \text{ as } n \to \infty. \end{aligned}$$

The dual space of $C^1(\mathfrak{H})$ is isomorphic to $B(\mathfrak{H})$ and the dual space of $C^q(\mathfrak{H})$ is isomorphic to $C^{q'}(\mathfrak{H})$ where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence, the dual space of the direct sum

$$C^1_{(k)} \dot{+} C^q = \underbrace{C^1 \dot{+} \cdots \dot{+} C^1}_{k} \dot{+} C^q$$

is isomorphic to the direct sum of k copies of $B(\mathfrak{H})$ and one copy of $C^{q'}(\mathfrak{H})$: any bounded linear functional on $C^1_{(k)} \dot{+} C^q$ has the form

$$(5.3)\quad F_{T_1,\dots,T_k,T}(R_1 \dot{+} \cdots \dot{+} R_k \dot{+} R) = \mathrm{Tr}(R_1T_1) + \cdots + \mathrm{Tr}(R_kT_k) + \mathrm{Tr}(RT),$$

for $R_1, \dots, R_k \in C^1(\mathfrak{H})$ and $R \in C^q(\mathfrak{H})$, where $T_1, \dots, T_k \in B(\mathfrak{H})$ and $T \in C^{q'}(\mathfrak{H})$.

If $X_1, \dots, X_k \in C(\mathfrak{H})$, all X_iT_i belong to $C(\mathfrak{H})$. Therefore, by (5.2) and (5.3),

$$\begin{aligned} &F_{T_1,\dots T_k,T}(A_nX_1 \dot{+} \cdots \dot{+} A_nX_k \dot{+} A_n) \\ &\qquad = \mathrm{Tr}(A_nX_1T_1) + \cdots + \mathrm{Tr}(A_nX_kT_k) + \mathrm{Tr}(A_nT) \to 0, \end{aligned}$$

as $n \to \infty$, hence $Y_n = A_nX_1 \dot{+} \cdots \dot{+} A_nX_k \dot{+} A_n$ weakly converge to 0 in $C^1_{(k)} \dot{+} C^q$. By the Hahn-Banach Theorem, 0 belongs to the closed convex set generated by $\{Y_n\}_{n \geqslant m}$, for any m. Hence, there are positive numbers $\alpha_m^{(m)}, \dots, \alpha_{\omega(m)}^{(m)}$ such that $\alpha_m^{(m)} + \cdots + \alpha_{\omega(m)}^{(m)} = 1$ and the norm of $\alpha_m^{(m)}Y_m + \cdots + \alpha_{\omega(m)}^{(m)}Y_{\omega(m)}$ in $C^1_{(k)} \dot{+} C^q$ is less than $\frac{1}{m}$. Therefore,

$$\big|\alpha_m^{(m)}A_m + \cdots + \alpha_{\omega(m)}^{(m)}A_{\omega(m)}\big|_q + \sum_{i=1}^{k} \big|\big(\alpha_m^{(m)}A_m + \cdots + \alpha_{\omega(m)}^{(m)}A_{\omega(m)}\big)X_i\big|_1 < \frac{1}{m}.$$

Set $B_m = \alpha_m^{(m)}P_m + \cdots + \alpha_{\omega(m)}^{(m)}P_{\omega(m)}$. Then

$$\delta_S(B_m) = \mathrm{i}(SB_m - B_mS) = \mathrm{i}\big(\alpha_m^{(m)}A_m + \cdots + \alpha_{\omega(m)}^{(m)}A_{\omega(m)}\big),$$

so that

$$|\delta_S(B_m)|_q + \sum_{i=1}^{k} |\delta_S(B_m)X_i|_1 < \frac{1}{m}.$$

Since the projections P_n increase, $B_m = P_m + \sum\limits_{i=m}^{\omega(m)-1} \beta_j^{(m)}(P_{i+1} - P_i)$ where $\beta_j^{(m)} = \sum\limits_{j=i+1}^{\omega(m)} \alpha_j^{(m)} \leqslant 1$. Hence

$$(5.4)\qquad P_m \leqslant B_m \leqslant P_{\omega(m)},$$

so the finite rank operators B_m strongly converge to $\mathbb{1}_H$ and $\|B_m\| = 1$. We also have that

$$\sup |\delta_S(B_m)|_1 \leqslant \sup \left(\alpha_m^{(m)}|A_m|_1 + \cdots + \alpha_{\omega(m)}^{(m)}|A_{\omega(m)}|_1\right) \leqslant 2K\|S\|. \qquad \blacksquare$$

Suppose that $H = \bigoplus_{i=1}^{n} H_i$ and $A = \bigoplus_{i=1}^{n} A_i$, where the operators A_i belong to the class $C^p(H_i)$ for some p. Then $A \in C^p(H)$ and

$$(|A|_p)^p = \sum_{i=1}^{n}(|A_i|_p)^p. \tag{5.5}$$

COROLLARY 5.3. *Let S be a selfadjoint operator on H. If $q \in (1, \infty]$ then there exist positive finite rank operators B_m strongly converging to $\mathbb{1}_H$ such that $\|B_m\| = 1$, $B_m D(S) \subseteq D(S)$ and $|\delta_S(B_m)|_q \to 0$.*

Proof. There is a decomposition $H = \bigoplus_{i=1}^{\infty} H(i)$ such that all $H(i)$ reduce S and $S_i = S|H(i)$ are bounded operators with finite multiplicity.

By Proposition 5.2, for any i, there are positive finite rank operators $\{B_m^i\}_{m=1}^{\infty}$ on $H(i)$, strongly converging to $\mathbb{1}_{H(i)}$, as $m \to \infty$, such that $\|B_m^i\| = 1$, $B_m^i \leqslant \mathbb{1}_{H(i)}$ and $|\delta_{S_i}(B_m^i)|_q \leqslant 2^{-(i+m)}$. The finite rank operators $B_m = \bigoplus_{i=1}^{m} B_m^i$ strongly converge to $\mathbb{1}_H$, $B_m D(S) \subseteq D(S)$, $\|B_m\| = 1$ and $0 \leqslant B_m \leqslant \mathbb{1}_H$. Since $\delta_S(B_m) = \bigoplus_{i=1}^{m} \delta_{S_i}(B_m^i)$, it follows from (5.5) that

$$|\delta_S(B_m)|_q \leqslant \left(\sum_{i=1}^{m}(|\delta_{S_i}(B_m^i)|_q)^q\right)^{1/q} \leqslant \left(\sum_{i=1}^{m} 2^{-q(i+m)}\right)^{1/q} \leqslant 2^{-m}. \qquad \blacksquare$$

Making use of Proposition 5.2 and Corollary 5.3, we will prove now that, if S is selfadjoint then all algebras $C_S^{p,q}$ have approximate identities.

THEOREM 5.4. *Let S be selfadjoint. For $1 \leqslant p, q \leqslant \infty$, the algebra $C_S^{p,q}$ has a two-sided approximate identity $\{B_\lambda\}_{\lambda\in\Lambda}$ which consists of positive, finite rank operators such that $\|B_\lambda\| = 1$. If $(p, q) \neq (\infty, 1)$, the approximative identity can be chosen countable.*

Proof. Step 1. Let $q \neq 1$ and let $\{B_m\}_{m=1}^{\infty}$ be the set of finite rank operators constructed in Corollary 5.3. Since they strongly converge to $\mathbb{1}_H$, it follows from Theorem III.6.3 of [5] that $\{B_m\}_{m=1}^{\infty}$ is a two-sided approximate identity in C^p and C^q. We obtain from Corollary 5.3 that, for any $X \in C_S^{p,q}$,

$$\begin{aligned} |X - B_m X|_{p,q} &= |X - B_m X|_p + |\delta_S(X - B_m X)|_q \\ &\leqslant |X - B_m X|_p + |\delta_S(X) - B_m \delta_S(X)|_q + |\delta_S(B_m)X|_q \to 0 \end{aligned} \tag{5.6}$$

as $m \to \infty$, since $|\delta_S(B_m)X|_q \leqslant |\delta_S(B_m)|_q\|X\|$. Similarly, $|X - XB_m|_{p,q} \to 0$. Hence $\{B_m\}_{m=1}^{\infty}$ is a two-sided approximate identity for $C_S^{p,q}$.

Step 2. Let $q = 1$. The case $p = 1$ was proved in Proposition 5.1. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $p' \in (1, \infty)$. By Corollary 5.3, there exist positive finite rank operators $\{B_m\}_{m=1}^{\infty}$ strongly converging to $\mathbb{1}_H$ such that $\|B_m\| = 1$, $B_m \leqslant \mathbb{1}_H$, $B_m D(S) \subseteq D(S)$ and $|\delta_S(B_m)|_{p'} \to 0$. For $X \in C_S^{p,1}$, we obtain from (2.8) that

$$|\delta_S(B_m)X|_1 \leqslant |\delta_S(B_m)|_{p'}\, |X|_p \to 0, \quad \text{as } m \to \infty.$$

Combining this with (5.6) yields that $\{B_m\}_{m=1}^{\infty}$ is a two-sided approximate identity for $C_S^{p,1}$.

Step 3. Let $(p, q) = (\infty, 1)$. There is a decomposition $H = \bigoplus_{i=1}^{\infty} H(i)$ such that all $H(i)$ reduce S and the operators $S|H(i)$ are bounded with finite multiplicity. The projections Q_k on $\mathfrak{H}_k = \bigoplus_{i=1}^{k} H(i)$ commute with S and $\mathcal{S}_k = Q_k S$ are bounded selfadjoint operators of finite multiplicity. Since Q_k strongly converge to $\mathbb{1}_H$, it follows from Theorem III.6.3 of [5] that $\{Q_k\}_{k=1}^{\infty}$ is a two-sided approximate identity in all C^p, $p \in [1, \infty]$.

Fix $\varepsilon > 0$ and let $X_1, \ldots, X_n \in C_S^{\infty,1}$. Choose k such that for all $i = 1, \ldots, n$,

$$|X_i - Q_k X_i Q_k|_\infty < \varepsilon, \quad |X_i - X_i Q_k|_\infty < \varepsilon,$$
$$|\delta_S(X_i) - Q_k \delta_S(X_i) Q_k|_1 < \varepsilon \quad \text{and} \quad |\delta_S(X_i) - \delta_S(X_i) Q_k|_1 < \varepsilon.$$

Since the operator $\mathcal{S}_k = Q_k S$ is bounded with finite multiplicity, it follows from Proposition 5.2 that there exist positive finite rank operators B_m on $\mathfrak{H}_k$ strongly converging to $\mathbb{1}_{\mathfrak{H}_k}$ such that $B_m \leqslant \mathbb{1}_{\mathfrak{H}_k}$, $\|B_m\| = 1$ and $|\delta_{\mathcal{S}_k}(B_m) X_i|_1 < \frac{1}{m}$, for $i = 1, \ldots, n$. We have

$$|X_i - B_m X_i|_{\infty,1} = |X_i - B_m X_i|_\infty + |\delta_S(X_i - B_m X_i)|_1.$$

Since $B_m = Q_k B_m = B_m Q_k$,

$$\begin{aligned}
&|X_i - B_m X_i|_\infty \\
&\quad = |X_i - Q_k X_i Q_k|_\infty + |Q_k X_i Q_k - B_m Q_k X_i Q_k|_\infty + |B_m Q_k X_i Q_k - B_m X_i|_\infty \\
&\quad \leqslant \varepsilon + |Q_k X_i Q_k - B_m Q_k X_i Q_k|_\infty + \|B_m\|\, |X_i Q_k - X_i|_\infty \\
&\quad \leqslant 2\varepsilon + |Q_k X_i Q_k - B_m Q_k X_i Q_k|_\infty.
\end{aligned}$$

Similarly,

$$|\delta_S(X_i) - B_m \delta_S(X_i)|_1 \leqslant 2\varepsilon + |Q_k \delta_S(X_i) Q_k - B_m Q_k \delta_S(X_i) Q_k|_1.$$

Since $\delta_S(B_m) = \delta_{\mathcal{S}_k}(B_m)$, we obtain, therefore, that

$$\begin{aligned}
|\delta_S(X_i - B_m X_i)|_1 &\leqslant |\delta_S(X_i) - B_m \delta_S(X_i)|_1 + |\delta_{\mathcal{S}_k}(B_m) X_i|_1 \\
&\leqslant 2\varepsilon + |Q_k \delta_S(X_i) Q_k - B_m Q_k \delta_S(X_i) Q_k|_1 + \frac{1}{m}.
\end{aligned}$$

Thus

$$\begin{aligned}
&|X_i - B_m X_i|_{\infty,1} \\
&\quad \leqslant 4\varepsilon + |Q_k X_i Q_k - B_m Q_k X_i Q_k|_\infty + |Q_k \delta_S(X_i) Q_k - B_m Q_k \delta_S(X_i) Q_k|_1 + \frac{1}{m}.
\end{aligned}$$

Since B_m strongly converge to $\mathbb{1}_{\mathfrak{H}_k}$, by Theorem III.6.3 of [5], $\{B_m\}_{m=1}^{\infty}$ is a two-sided approximate identity in $C^{\infty}(\mathfrak{H}_k)$ and $C^1(\mathfrak{H}_k)$. Since $Q_k X_i Q_k \in C^{\infty}(\mathfrak{H}_k)$, $Q_k \delta_S(X_i) Q_k \in C^1(\mathfrak{H}_k)$, we can find B_m such that $|X_i - B_m X_i|_{\infty,1} \leqslant 5\varepsilon$.

Let Λ be the set of all finite subsets of $C_S^{\infty,1}$. By the above argument, for any $\lambda \in \Lambda$, there exists a positive finite rank operator B_λ on H such that $B_\lambda \leqslant \mathbb{1}_H$, $\|B_\lambda\| = 1$ and $|X - B_\lambda X|_{\infty,1} \leqslant \frac{1}{n}$, for any $X \in \lambda$, where n is the number of elements in λ. Since $|X - XB_\lambda|_{\infty,1} = |X^* - B_\lambda X^*|_{\infty,1}$, we obtain that $\{B_\lambda\}_{\lambda\in\Lambda}$ is a two-sided approximate identity for $C_S^{\infty,1}$. ∎

COROLLARY 5.5. *If S is a selfadjoint operator on H then $C_S^{p,q} = \mathcal{F}_S^{p,q}$ for $1 \leqslant p, q \leqslant \infty$.*

Proof. By Theorem 5.4, the algebras $C_S^{p,q}$ have two-sided approximate identities $\{B_\lambda\}_{\lambda\in\Lambda}$ which consist of finite rank operators. For $A \in C_S^{p,q}$, $B_\lambda A$ are finite rank operators. Since $|A - B_\lambda A|_{p,q} \to 0$, we have that $C_S^{p,q} = \mathcal{F}_S^{p,q}$. ∎

Let $1 \leqslant p \leqslant \infty$. We call an operator S on H *p-semidiagonal* if there exists a sequence of positive finite rank operators $\{Q_n\}_{n=1}^{\infty}$ which preserve the domain $D(S)$, strongly converge to $\mathbb{1}_H$ and

$$\sup_n |SQ_n - Q_n S|_p < \infty. \tag{5.7}$$

Clearly, if S is p-semidiagonal, it is q-semidiagonal for $p \leqslant q$.

THEOREM 5.6. (i) *Any selfadjoint operator S is p-semidiagonal for $p > 1$. Moreover, there are $\{Q_n\}_{n=1}^{\infty}$ such that $\lim |SQ_n - Q_n S|_p = 0$.*

(ii) *A selfadjoint operator S is 1-semidiagonal if and only if*

$$\int_{\mathbb{R}} k_S(t)\,\mathrm{d}t < \infty, \tag{5.8}$$

where $k_S(t)$ is the spectral multiplicity of the absolutely continuous part of S. In particular, S is 1-semidiagonal if it is bounded and has finite multiplicity.

(iii) *Closed non-selfadjoint symmetric operators are not p-semidiagonal for any $1 \leqslant p \leqslant \infty$.*

Proof. Part (i) follows from Corollary 5.3.

If S is a bounded selfadjoint operator with finite multiplicity, it follows from Proposition 5.2 that S is 1-semidiagonal.

Let S be selfadjoint and 1-semidiagonal and let $\{Q_n\}_{n=1}^{\infty}$ be positive finite rank operators which preserve the domain $D(S)$, strongly converge to $\mathbb{1}_H$ and $\sup_n |SQ_n - Q_n S|_p = C < \infty$. Let S be bounded. It follows from Proposition 1.5 of [17] and Remark 2.3 of [17] that there exists a universal constant α such that

$$\int_{-\|S\|}^{\|S\|} k_S(t)\,\mathrm{d}t \leqslant \alpha C. \tag{5.9}$$

Let S be unbounded. For any $0 < \lambda < \infty$, let $P_\lambda = P_S[-\lambda, \lambda]$ be the spectral projection of S. Then $S_\lambda = P_\lambda S$ is a bounded selfadjoint operator on $H_\lambda = P_\lambda H$, finite rank operators $P_\lambda Q_n P_\lambda$ strongly converge to P_λ, as $n \to \infty$, and

$$|S_\lambda P_\lambda Q_n P_\lambda - P_\lambda Q_n P_\lambda S_\lambda|_1 = |P_\lambda(SQ_n - Q_n S)P_\lambda|_1 \leqslant \|P_\lambda\|\,|SQ_n - Q_n S|_1\,\|P_\lambda\| \leqslant C.$$

Hence S_λ is 1-semidiagonal and, by (5.9),

$$\int_{-\lambda}^{\lambda} k_S(t)\,\mathrm{d}t = \int_{-\lambda}^{\lambda} k_{S_\lambda}(t)\,\mathrm{d}t \leqslant \alpha C.$$

Therefore

$$\int_{-\infty}^{\infty} k_S(t)\,\mathrm{d}t = \alpha C < \infty.$$

Conversely, let (5.8) hold. Set

$$\alpha_m = \int_{m}^{m+1} k_S(t)\,\mathrm{d}t, \quad P_m = P_S[m, m+1], \quad S_m = P_m S \quad \text{and} \quad H_m = P_m H.$$

Since $\alpha_m < \infty$, it follows from Remark 2.3 of [17] and the definition on page 5 in [17] that there exists an increasing sequence $\{R_n(m)\}_{n=1}^{\infty}$ of positive finite rank contractions on H_m strongly converging to P_m, as $n \to \infty$, such that

$$\lim_n |S_m R_n(m) - R_n(m) S_m|_1 = \frac{\alpha_m}{\pi}, \quad \text{for all } m.$$

For every m, we choose a subsequence $A_r(m) = R_{n_r}(m)$ such that

$$|S_m A_r(m) - A_r(m) S_m|_1 \leqslant \frac{\alpha_m}{\pi} + \frac{1}{2^{r(|m|+1)}}.$$

Set $Q_n = \bigoplus_{m=-n}^{n} A_n(m)$. Since $\|Q_n\| = \sup_m \|A_n(m)\| \leqslant 1$, Q_n are positive finite rank contractions preserving $D(S)$. For $x \in H_m$, $Q_n x = A_n(m)x \to x$, as $n \to \infty$. Since linear combinations of elements from all H_m, $-\infty < m < \infty$, are dense in H and since the sequence $\{Q_n\}$ is bounded, it follows that Q_n strongly converge to $\mathbb{1}_H$. Moreover,

$$\begin{aligned}|SQ_n - Q_n S|_1 &= \Big|\sum_{m=-n}^{n} \oplus (S_m A_n(m) - A_n(m) S_m)\Big|_1 \leqslant \sum_{m=-n}^{n} \Big(\frac{\alpha_m}{\pi} + \frac{1}{2^{n(|m|+1)}}\Big) \\ &\leqslant \frac{1}{\pi}\int_{-n}^{n} k_S(t)\,\mathrm{d}t + 2^{2-n} \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} k_S(t)\,\mathrm{d}t + 4.\end{aligned}$$

Hence S is 1-semidiagonal. Part (ii) is proved.

Part (iii) follows from the lemma below.

Lemma 5.7. *Let S be a closed non-selfadjoint symmetric operator on H and $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{A}_S$ (see (1.2)) weakly converging to $\mathbb{1}_H$. If $A_n D(S^*) \subseteq D(S)$, for all n, then $\|SA_n - A_nS\| \to \infty$, as $n \to \infty$. In particular, if A_n are finite rank operators in $\mathcal{A}_S$ weakly converging to $\mathbb{1}_H$ then $\|SA_n - A_nS\| \to \infty$.*

Proof. By (1.2), the operators $SA_n - A_nS$ extend to bounded operators R_n. For $x \in D(S)$ and $y \in D(S^*)$,

$$(Sx, A_n^*y) = (A_nSx, y) = (SA_nx, y) - (R_nx, y) = (x, A_n^*S^*y) - (x, R_n^*y),$$

so that $A_n^*y \in D(S^*)$ and $R_n^*|D(S^*) = A_n^*S^* - S^*A_n^*$. Hence $R_n^*|D(S) = A_n^*S - SA_n^*$.

Making use of this, we obtain that

$$(Sx, A_ny) = (A_n^*Sx, y) = (SA_n^*x, y) + (R_n^*x, y) = (x, A_nS^*y) + (x, R_ny),$$

so that $A_ny \in D(S^*)$ and $R_n|D(S^*) = S^*A_n - A_nS^*$.

Since A_n weakly converge to $\mathbb{1}_H$, we have that, for $x, y \in D(S)$,

$$(R_nx, y) = (SA_nx, y) - (A_nSx, y) = (A_nx, Sy) - (A_nSx, y) \to (x, Sy) - (Sx, y) = 0,$$

as $n \to \infty$. If $\sup_n \|R_n\| = \sup_n \|SA_n - A_nS\| < \infty$, it follows from the above formula that $(R_nu, z) \to 0$, as $n \to \infty$, for all $u, z \in H$, so R_n weakly converge to 0.

Let $x \in D(S^*)$. Since $A_nD(S^*) \subseteq D(S)$, we have that, for any $z \in H$,

$$(SA_nx, z) = (S^*A_nx, z) = (A_nS^*x, z) + (R_nx, z) \to (S^*x, z),$$

as $n \to \infty$. Hence $A_nx \oplus SA_nx$ weakly converges to $x \oplus S^*x$ in $H \oplus H$. Since S is closed, the subspace $L = \{u \oplus Su : u \in D(S)\}$ is closed in $H \oplus H$ and, hence, weakly closed. Since all A_nx belong to $D(S)$, we have that $A_nx \oplus SA_nx \in L$. Therefore $x \oplus S^*x \in L$, so that $x \in D(S)$ and $Sx = S^*x$. Thus, S is selfadjoint and this contradicts the assumption of the lemma.

If, in particular, all A_n are finite rank operators in $\mathcal{A}_S$ then $A_nD(S) \subseteq D(S)$ implies that $A_nH \subseteq D(S)$. Hence $\|SA_n - A_nS\| \to \infty$. ∎

The problem of the existence of bounded approximate identities in the algebras $C_S^{p,q}$ and $\mathcal{F}_S^{p,q}$ can be now solved in full generality.

Proposition 5.8. *If S is a non-selfadjoint operator, the algebras $C_S^{p,q}$ and $\mathcal{F}_S^{p,q}$, $1 \leqslant p, q \leqslant \infty$, have no bounded approximate identities.*

Proof. Suppose that $\{E_n\}$ is a bounded approximate identity in $\mathcal{F}_S^{p,q}$. Then there exist $Q_n \in \Phi_S$ such that $|E_n - Q_n|_{p,q} \leqslant \frac{1}{n}$. Clearly, $\{Q_n\}$ is a bounded approximate identity in $\mathcal{F}_S^{p,q}$, so $\|\delta_S(Q_n)\| \leqslant |Q_n|_{p,q} \leqslant C < \infty$.

On the other hand, by (3.5), for any $x, y \in D(S)$,

$$\|x\|\,\|Q_ny - y\| = |x \otimes (Q_ny - y)|_p = |Q_n(x \otimes y) - x \otimes y|_p \leqslant |Q_n(x \otimes y) - x \otimes y|_{p,q} \to 0,$$

as $n \to \infty$. Hence $\|Q_ny - y\| \to 0$, as $n \to \infty$. Since $\|Q_n\| \leqslant |Q_n|_{p,q} < \infty$, it follows easily that $\|Q_nz - z\| \to 0$, as $n \to \infty$, for any $z \in H$, so Q_n strongly converge to $\mathbb{1}_H$. Therefore, by Lemma 5.7, $\|\delta_S(Q_n)\| = \|SQ_n - Q_nS\| \to \infty$, as $n \to \infty$. This contradiction proves the proposition for the algebras $\mathcal{F}_S^{p,q}$.

Assume now that $\{E_n\}$ is a bounded approximate identity in $C_S^{p,q}$. By Theorem 4.8, all E_n^2 belong to $\mathcal{F}_S^{p,q}$ and, for any $A \in \mathcal{F}_S^{p,q}$,

$$|A - E_n^2A|_{p,q} \leqslant |A - E_nA|_{p,q} + |E_nA - E_n^2A|_{p,q} \leqslant |A - E_nA|_{p,q} + |E_n|_{p,q}\,|A - E_nA|_{p,q}.$$

Hence, $\{E_n^2\}$ is a bounded approximate identity in $\mathcal{F}_S^{p,q}$ and this contradicts the discussion at the beginning of the proof. ∎

THEOREM 5.9. *Let S be a selfadjoint operator on H and let $k_S(t)$ be the spectral multiplicity of the absolutely continuous part of S.*

(i) *If $p \neq \infty$, then the algebra $C_S^{p,q}$ has no bounded approximate identities.*

(ii) *If $q \neq 1$, then the algebra $C_S^{\infty,q}$ has a bounded approximate identity.*

(iii) *The algebra $C_S^{\infty,1}$ has a bounded approximate identity if and only if*

$$\int_{-\infty}^{\infty} k_S(t)\,\mathrm{d}t < \infty. \tag{5.10}$$

Proof. Let $p < \infty$ and $\{E_n\}_{n=1}^{\infty}$ be a bounded approximate identity in $C_S^{p,q}$. Then $|E_n|_p \leqslant |E_n|_{p,q} \leqslant K$, for some $K > 1$, and, for any $A \in C_S^{p,q}$,

$$|A - E_nA|_p \leqslant |A - E_nA|_{p,q} \to 0, \quad \text{as } n \to \infty.$$

Let $B \in C^p$ and $\varepsilon > 0$. Since the algebra $C_S^{p,q}$ is dense in C^p, choose $A \in C_S^{p,q}$ and n_ε such that $|B - A|_p \leqslant \frac{\varepsilon}{3K}$ and $|A - E_nA|_p \leqslant \frac{\varepsilon}{3}$ for $n \geqslant n_\varepsilon$. By (2.9),

$$|B - E_nB|_p \leqslant |B - A|_p + |A - E_nA|_p + |E_nA - E_nB|_p \leqslant \frac{2\varepsilon}{3} + |E_n|_p\,|A - B|_p \leqslant \varepsilon,$$

so $\{E_n\}_{n=1}^{\infty}$ is a bounded approximate identity in C^p. This contradiction shows that $C_S^{p,q}$ has no bounded approximate identity if $p < \infty$. Part (i) is proved.

If $q \neq 1$, it follows from Corollary 5.3 and Theorem 5.4 that there is an approximate identity $\{B_n\}_{n=1}^{\infty}$ in $C_S^{\infty,q}$ which consists of finite rank operators such that $\|B_n\| = 1$ and $|\delta_S(B_n)|_q \to 0$, as $n \to \infty$. The idenity is bounded, since

$$|B_n|_{\infty,q} = \|B_n\| + |\delta_S(B_n)|_q = 1 + |\delta_S(B_n)|_q \to 1 \quad \text{as } n \to \infty.$$

Assume that $\{E_n\}$ is a bounded approximate identity in $C_S^{\infty,1}$. As in Proposition 5.8, we obtain that there exists a bounded approximate identity $\{Q_n\}$ in $C_S^{\infty,1}$ which consists of finite rank operators strongly converging to $\mathbb{1}_H$. Since $|SQ_n - Q_nS|_1 = |\delta_S(Q_n)|_1 \leqslant |Q_n|_{\infty,1} \leqslant C$, the operator S is 1-semidiagonal. It follows from Theorem 5.6 (ii) that (5.10) holds.

Conversely, let (5.10) hold. Set $P_m = P_S[m, m+1]$, $S_m = P_mS$ and $H_m = P_mH$. It follows from the proof of Theorem 5.6 (ii) that, for every m, there exists an increasing sequence $\{R_n(m)\}_{n=1}^{\infty}$ of positive finite rank contractions on H_m strongly convering to P_m, as $n \to \infty$, such that the operators $Q_n = \bigoplus_{m=-n}^{n} R_n(m)$ are positive finite rank contractions, preserving $D(S)$ and strongly converging to $\mathbb{1}_H$, and, for all n,

$$|\delta_S(Q_n)|_1 = |SQ_n - Q_nS|_1 \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} k_s(t)\,\mathrm{d}t + 4.$$

Hence $\sup_n |Q_n|_{\infty,1} \leqslant \sup_n \|Q_n\| + \sup_n |\delta_S(Q_n)|_1 < \infty$.

If $y \in H_k$ then $S_kR_n(k)y \to S_ky = Sy$, as $n \to \infty$, since S_k is bounded. Hence

$$\delta_S(Q_n)y = \mathrm{i}(SQ_ny - Q_nSy) = \mathrm{i}(S_kR_n(k)y - Q_nSy) \to \mathrm{i}(Sy - Sy) = 0,$$

as $n \to \infty$. Since linear combinations of elements from H_k, $-\infty < k < \infty$, are dense in H and since $\sup_n \|\delta_S(Q_n)\| \leqslant \sup_n |\delta_S(Q_n)|_1 < \infty$, the operators $\delta_S(Q_n)$ strongly converge to 0.

The sequence $\{Q_n\}_{n=1}^{\infty}$ is bounded in $|\cdot|_{\infty,1}$ and, for $A \in C_S^{\infty,1}$,

$$\begin{aligned}|A - Q_nA|_{\infty,1} &= \|A - Q_nA\| + |\delta_S(A - Q_nA)|_1 \\ &\leqslant \|A - Q_nA\| + |\delta_S(A) - Q_n\delta_S(A)|_1 + |\delta_S(Q_n)A|_1.\end{aligned}$$

Since $\{Q_n\}_{n=1}^{\infty}$ strongly converges to $\mathbb{1}_H$, it is an approximate identity in $C(H)$ and in C^1. Hence, to prove that $\{Q_n\}_{n=1}^{\infty}$ is a bounded approximate identity in $C_S^{\infty,1}$, it suffices to show that $|\delta_S(Q_n)A|_1 \to 0$, as $n \to \infty$.

By Theorem 5.5, $C_S^{\infty,1} = \mathcal{F}_S^{\infty,1}$, so Φ_S is dense in $C_S^{\infty,1}$. Since $\sup_n |\delta_S(Q_n)|_1 < \infty$, it follows from (2.9) that, in order to prove that $|\delta_S(Q_n)A|_1 \to 0$ for $A \in C_S^{\infty,1}$, it only suffices to show this for all $A \in \Phi_S$. Since (see (3.6)) Φ_S is the linear span of rank one operators $x \otimes y$, for $x, y \in D(S)$, it is only sufficient to show that $|\delta_S(Q_n)(x \otimes y)|_1 \to 0$, as $n \to \infty$, for $x, y \in D(S)$. Making use of (3.5) and taking into account that $\delta_S(Q_n)$ strongly converge to 0, we obtain that

$$|\delta_S(Q_n)(x \otimes y)|_1 = |x \otimes \delta_S(Q_n)y|_1 = \|x\| \, \|\delta_S(Q_n)y\| \to 0, \quad \text{as } n \to \infty. \quad \blacksquare$$

6. DUAL AND SECOND DUAL SPACES OF THE ALGEBRAS $C_S^{p,q}$

Let p' be the conjugate exponent of p, $1 \leqslant p \leqslant \infty$ (see (4.1)). The dual space of the algebra C^p is isometrically isomorphic to $C^{p'}$: for any $T \in C^{p'}$,

$$F_T(A) = \mathrm{Tr}(AT), \quad A \in C^p,$$

is a bounded linear functional on C^p and $\|F_T\| = |T|_{p'}$ (see [5]). Therefore, the algebras C^p, $1 < p < \infty$, are reflexive and the second dual of the algebra $C^\infty = C(H)$ is isometrically isomorphic to the algebra $C^{\mathrm{b}} = B(H)$.

In [12] it was shown that if S is a unbounded selfadjoint operator then the second dual of the algebra $C_S^{\infty,\infty}$ is isometrically isomorphic to the algebra $C_S^{\mathrm{b,b}} = \mathcal{A}_S$. In this section we show that, for any symmetric S, the algebras $C_S^{p,q}$, $1 < p, q < \infty$, are reflexive, and that for selfadjoint S, the second duals of the algebras $C_S^{p,\infty}$ and $C_S^{\infty,p}$, $1 < p < \infty$, are isometrically isomorphic to the algebras $C_S^{p',\mathrm{b}}$ and $C_S^{\mathrm{b},p'}$ respectively.

Let X, Y be Banach spaces. Their direct sum $X \oplus Y$ will be considered with two equivalent norms:

$$\|x \oplus y\| = \|x\|_X + \|y\|_Y \quad \text{and} \quad \|x \oplus y\|^{\sim} = \max(\|x\|_X, \|y\|_Y), \quad \text{for } x \in X \text{ and } y \in Y.$$

For clarity, we will write in the second case $X \widetilde{\oplus} Y$ instead of $X \oplus Y$.

If X^* and Y^* are their dual spaces then

$$(X \oplus Y)^* = X^* \widetilde{\oplus} Y^* \quad \text{and} \quad (X \widetilde{\oplus} Y)^* = X^* \oplus Y^*.$$

Clearly, if X and Y are reflexive, $X \oplus Y$ is also reflexive.

Let Z be a linear subspace of X. The *annihilator*

$$Z^\perp = \{F \in X^* : F(z) = 0, \text{ for all } z \in Z\}$$

of Z in X^* is a closed subspace of X^* and from the general theory of Banach spaces (see [4], II.4.18 and [14], III, Problem 30) we have the following lemma.

Lemma 6.1. *The dual space Z^* of a closed subspace Z of X is isometrically isomorphic to the quotient space $X^*/Z^\perp$ and the second dual Z^{**} of Z is isometrically isomorphic to $Z^{\perp\perp}$ where*

$$Z^{\perp\perp} = \{\theta \in X^{**} : \theta(F) = 0, \text{ for all } F \in Z^\perp\}.$$

Let $1 \leqslant p, q \leqslant \infty$ and p' and q' be their conjugate exponents (see (4.1)). From the above discussion it follows that the space $C^{p'} \widetilde{\oplus} C^{q'}$ can be identified with the dual space of the Banach space $C^p \oplus C^q$ by the formula:

$$F_{R\oplus T}(A \oplus B) = \mathrm{Tr}(AR) + \mathrm{Tr}(BT), \quad \text{for } A \oplus B \in C^p \oplus C^q, \tag{6.1}$$

and $\|F_{R\oplus T}\| = \|R \oplus T\|^{\sim}_{p',q'}$. This yields

Lemma 6.2. (i) *If $1 < p, q < \infty$, then the space $C^p \oplus C^q$ is reflexive.*

(ii) *If $1 < q < \infty$, then the second dual space of $C^\infty \oplus C^q$ is isometrically isomorphic to $C^{\mathrm{b}} \oplus C^q$.*

(iii) *The second dual space of $C^\infty \oplus C^\infty$ is isometrically isomorphic to $C^{\mathrm{b}} \oplus C^{\mathrm{b}}$.*

Let $p, q \in T$ and S be a symmetric operator on H. The linear manifolds

$$\widehat{\mathcal{F}}_S^{p,q} = \{A \oplus \delta_S(A) : A \in \mathcal{F}_S^{p,q}\} \quad \text{and} \quad \widehat{C}_S^{p,q} = \{A \oplus \delta_S(A) : A \in C_S^{p,q}\}$$

in $C^p \oplus C^q$ are, clearly, isometrically isomorphic to the algebras $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ and $(C_S^{p,q}, |\cdot|_{p,q})$, respectively. Hence they are closed subspaces of $C^p \oplus C^q$.

Proposition 6.3. *For $1 < p, q < \infty$, the algebras $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ and $(C_S^{p,q}, |\cdot|_{p,q})$ are reflexive.*

Proof. It is well known that any closed subspace of a reflexive space is also reflexive. Since $\widehat{\mathcal{F}}_S^{p,q}$ and $\widehat{C}_S^{p,q}$ are closed subspaces of the reflexive space $C^p \oplus C^q$, we obtain that $\widehat{\mathcal{F}}_S^{p,q}$ and $\widehat{C}_S^{p,q}$ are reflexive. Since $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ and $(C_S^{p,q}, |\cdot|_{p,q})$ are, respectively, isometrically isomorphic to $(\widehat{\mathcal{F}}_S^{p,q} \|\cdot\|_{p,q})$ and $(\widehat{C}_S^{p,q} \|\cdot\|_{p,q})$, they are reflexive. ∎

For $p, q \in T$, set

$$\mathfrak{T}_S^{p,q} = \{T \in C^p : TD(S) \subseteq D(S^*),\ T^*D(S) \subseteq D(S^*) \text{ and } \mathrm{i}(S^*T - TS)|D(S) \\ \text{extends to a bounded operator } T_S \text{ from the class } C^q\}.$$

It follows that $C_S^{p,q} \subseteq \mathfrak{T}_S^{p,q}$. If S is selfadjoint, $T_S = \delta_S(T)$ for any $T \in \mathfrak{T}_S^{p,q}$, so $\mathfrak{T}_S^{p,q} = C_S^{p,q}$. Clearly, $\mathfrak{T}_S^{p,q}$ is a linear subspace in C^p.

For $T \in \mathfrak{T}_S^{p,q}$ and $z, u \in D(S)$,

$$((T_S)^*z, u) = (z, T_S u) = (z, \mathrm{i}(S^*T - TS)u) = (\mathrm{i}(S^*T^* - T^*S)z, u),$$

so that

$$(T_S)^*|D(S) = \mathrm{i}(S^*T^* - T^*S)|D(S) = (T^*)_S|D(S). \tag{6.2}$$

From this and from (2.7) it follows that $(T^*)_S = (T_S)^* \in C^q$. Hence $T^* \in \mathfrak{T}_S^{p,q}$.

By $\widetilde{\mathfrak{T}}_S^{p,q}$ we denote the following linear manifold in $C^q \widetilde{\oplus} C^p$:

$$\widetilde{\mathfrak{T}}_S^{p,q} = \{T_S \oplus T : T \in \mathfrak{T}_S^{p,q}\}.$$

Lemma 6.4. *For any $p, q \in T$, $\widetilde{\mathfrak{T}}_S^{p,q}$ is a closed subspace in $C^q \widetilde{\oplus} C^p$.*

Proof. Let $T_n \in \mathfrak{T}_S^{p,q}$ and suppose that $T_n \to T$ in C^p and $(T_n)_S \to R$ in C^q. By (2.6), T_n converge to T and $(T_n)_S$ converge to R with respect to the norm $\|\cdot\|$ in $B(H)$. Let $x, y \in D(S)$. Since $S^*T_n|D(S) = T_nS|D(S) - \mathrm{i}(T_n)_S|_{D(S)}$, we have that

$$(Tx, Sy) = \lim(T_nx, Sy) = \lim(S^*T_nx, y) = \lim(T_nSx, y) - \mathrm{i}\lim((T_n)_Sx, y)$$
$$= (TSx, y) - \mathrm{i}(Rx, y).$$

Therefore $Tx \in D(S^*)$ and $S^*Tx = TSx - \mathrm{i}Rx$. Thus, $TD(S) \subseteq D(S^*)$ and $R = T_S$.

It follows from (6.2) that $(T_n)^* \in \mathfrak{T}_S^{p,q}$ and $((T_n)_S)^* = ((T_n)^*)_S$. Hence, we obtain from (2.7) that $(T_n)^* \to T^*$ in C^p and $((T_n)^*)_S \to R^*$ in C^q. Repeating the above argument, we obtain that $T^*D(S) \subseteq D(S^*)$, so that $T \in \mathfrak{T}_S^{p,q}$. Thus, $\widetilde{\mathfrak{T}}_S^{p,q}$ is a closed subspace in $C^q \widetilde{\oplus} C^p$. ∎

Let $x, y \in D(S^*)$ and $T = x \otimes y$. By (3.2) and (3.3), for $z \in H$,

$$\begin{aligned} &Tz = (z,x)y \in D(S^*), \quad T^*z = (y \otimes x)z = (z,y)x \in D(S^*), \quad \text{and} \\ &T_S = \mathrm{i}[S^*T - TS] = \mathrm{i}[x \otimes S^*y - (S^*x) \otimes y], \end{aligned} \tag{6.3}$$

so $T \in \mathfrak{T}_S^{p,q}$ for any $p > 0$ and $q > 0$.

Let $1 \leqslant p, q \leqslant \infty$ and let p' and q' be their conjugate exponents. Since $\widehat{C}_S^{p,q} \subseteq C^p \oplus C^q$, the annihilator $(\widehat{C}_S^{p,q})^\perp$ is a closed subspace of $C^{p'} \widetilde{\oplus} C^{q'}$.

PROPOSITION 6.5. *For* $1 \leqslant p, q \leqslant \infty$, $(\widehat{\mathcal{F}}_S^{p,q})^\perp = \widetilde{\mathfrak{T}}_S^{q',p'}$.

Proof. Let $T_S \oplus T \in \widetilde{\mathfrak{T}}_S^{q',p'}$ and $A = x \otimes y$, for $x, y \in D(S)$. By (3.3) and (6.2),

$$\begin{aligned} \delta_S(A) &= \mathrm{i}[S(x \otimes y) - (x \otimes y)S] = \mathrm{i}[x \otimes Sy - (Sx) \otimes y], \\ \delta_S(A)T &= \mathrm{i}(x \otimes Sy)T - \mathrm{i}((Sx) \otimes y)T = \mathrm{i}(T^*x) \otimes Sy - \mathrm{i}(T^*Sx) \otimes y, \\ AT_S &= (x \otimes y)T_S = ((T_S)^*x) \otimes y = \mathrm{i}((T^*S - S^*T^*)x) \otimes y. \end{aligned} \tag{6.4}$$

It follows from (3.4), (6.1) and (6.4) that

$$F_{T_S \oplus T}(A \oplus \delta_S(A)) = \mathrm{Tr}(AT_S) + \mathrm{Tr}(\delta_S(A)T)$$
$$= \mathrm{i}(y, (T^*S - S^*T^*)x) + \mathrm{i}(Sy, T^*x) - \mathrm{i}(y, T^*Sx) = 0.$$

Hence $F_{T_S \oplus T}(A \oplus A_S) = 0$ for any $A \in \Phi_S$.

Since Φ_S is dense in $\mathcal{F}_S^{p,q}$ and $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ is isometrically isomorphic to $(\widehat{\mathcal{F}}_S^{p,q}, \|\cdot\|_{p,q})$, the operators $A \oplus \delta_S(A)$, where $A \in \Phi_S$, are dense in $\widehat{\mathcal{F}}_S^{p,q}$. Since $T_S \oplus T \in C^{p'} \widetilde{\oplus} C^{q'}$, $F_{T_S \oplus T}$ is a continuous functional on $C^p \oplus C^q$. Therefore, $F_{T_S \oplus T}(A \oplus \delta_S(A)) = 0$, for $A \in \mathcal{F}_S^{p,q}$. Thus $F_{T_S \oplus T} \in (\widehat{\mathcal{F}}_S^{p,q})^\perp$, so $\widetilde{\mathfrak{T}}_S^{q',p'} \subseteq (\widehat{\mathcal{F}}_S^{p,q})^\perp$.

Conversely, let $R \oplus T \in (\widehat{\mathcal{F}}_S^{p,q})^\perp \subseteq C^{p'} \widetilde{\oplus} C^{q'}$ and $A = x \otimes y \in \Phi_S$, where $x, y \in D(S)$. From (3.3), (3.4), (6.1) and (6.4) it follows that

$$\begin{aligned} 0 = F_{R \oplus T}(A \oplus \delta_S(A)) &= \mathrm{Tr}(AR) + \mathrm{Tr}(\delta_S(A)T) \\ &= \mathrm{Tr}((R^*x) \otimes y) + \mathrm{Tr}[\mathrm{i}(T^*x) \otimes Sy - \mathrm{i}(T^*Sx) \otimes y] \\ &= (y, R^*x) + \mathrm{i}(Sy, T^*x) - \mathrm{i}(y, T^*Sx). \end{aligned}$$

Hence

$$(Sy, T^*x) = (y, (T^*S - \mathrm{i}R^*)x), \quad \text{for } x, y \in D(S).$$

Therefore $T^*x \in D(S^*)$ and $S^*T^*x = (T^*S - \mathrm{i}R^*)x$. Thus $T^*D(S) \subseteq D(S^*)$ and

$$(Sx, Ty) = (T^*Sx, y) = (S^*T^*x, y) + \mathrm{i}(R^*x, y) = (x, TSy) - (x, \mathrm{i}Ry).$$

From this it follows that $Ty \in D(S^*)$ and $S^*Ty = TSy - \mathrm{i}Ry$. Hence

$$TD(S) \subseteq D(S^*) \quad \text{ and } \quad R|D(S) = \mathrm{i}[S^*T - TS]\,|D(S).$$

Hence $T \in \mathfrak{T}_S^{q',p'}$ and $R = T_S$. Thus $(\widehat{\mathcal{F}}_S^{p,q})^\perp \subseteq \widetilde{\mathfrak{T}}_S^{q',p'}$, so $(\widehat{\mathcal{F}}_S^{p,q})^\perp = \widetilde{\mathfrak{T}}_S^{q',p'}$. ∎

Since the Banach spaces $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ and $(\widehat{\mathcal{F}}_S^{p,q}, \|\cdot\|_{p,q})$ are isometrically isomorphic, Lemma 6.1 and Proposition 6.5 yield

Corollary 6.6. *Let $1 \leqslant p, q \leqslant \infty$ and p' and q' be their conjugate exponents. The dual space of the Banach $*$-algebra $(\mathcal{F}_S^{p,q}, |\cdot|_{p,q})$ is isometrically isomorphic to the quotient space $(C^{p'} \widetilde{\oplus} C^{q'})/\widetilde{\mathfrak{T}}_S^{q',p'}$.*

By φ we denote the isomorphism of $C^p \oplus C^q$ on $C^q \oplus C^p$:

$$\varphi(A \oplus B) = B \oplus A, \quad A \in C^p \text{ and } B \in C^q.$$

If S is selfadjoint then $\mathfrak{T}_S^{p,q} = C_S^{p,q}$, so $\widetilde{\mathfrak{T}}_S^{p,q} = \varphi(\widehat{C}_S^{p,q})$. Combining this with Proposition 6.5 and Corollaries 5.5 and 6.6, we obtain the following result.

Corollary 6.7. *Let S be a selfadjoint operator and $1 \leqslant p, q \leqslant \infty$. Then $(\widehat{C}_S^{p,q})^\perp = \varphi(\widehat{C}_S^{q',p'})$ and the dual space of the algebra $(C_S^{p,q}, |\cdot|_{p,q})$ is isometrically isomorphic to the quotient space $(C^{p'} \widetilde{\oplus} C^{q'})/\varphi(\widehat{C}_S^{q',p'})$.*

In Proposition 6.3 it was shown that, for $1 < p, q < \infty$, the algebras $C_S^{p,q}$ are reflexive for any symmetric operator S. Below we consider the case when S is selfadjoint and either $p = \infty$ or $q = \infty$.

Theorem 6.8. *Let S be a selfadjoint operator on H.*

(i) *If $1 < p < \infty$ then $(\widehat{C}_S^{p,\infty})^{\perp\perp} = \widehat{C}_S^{p,\mathrm{b}}$, so that the second dual space of the algebra $C_S^{p,\infty}$ is isometrically isomorphic to $C_S^{p,\mathrm{b}}$.*

(ii) ([12]) *$(\widehat{C}_S^{\infty,\infty})^{\perp\perp} = \widehat{C}_S^{\mathrm{b},\mathrm{b}}$, so that the second dual space of the algebra $C_S^{\infty,\infty}$ is isometrically isomorphic to $C_S^{\mathrm{b},\mathrm{b}} = \mathcal{A}_S$.*

(iii) *Let $1 < q < \infty$ then $(\widehat{C}_S^{\infty,q})^{\perp\perp} = \widehat{C}_S^{\mathrm{b},q}$, so that the second dual space of the algebra $C_S^{\infty,q}$ is isometrically isomorphic to $C_S^{\mathrm{b},q}$.*

Proof. First observe that $\varphi(\widehat{C}_S^{p,q})^\perp = \varphi((\widehat{C}_S^{p,q})^\perp)$. Since $\infty' = 1$, it follows from Corollary 6.7 that $(\widehat{C}_S^{p,\infty})^\perp = \varphi(\widehat{C}_S^{1,p'})$. If $1 < p < \infty$ then $1 < p' < \infty$. Since $1' = \mathrm{b}$, we obtain from Corollary 6.7 that $(\widehat{C}_S^{1,p'})^\perp = \varphi(\widehat{C}_S^{p,\mathrm{b}})$. Hence

$$(\widehat{C}_S^{p,\infty})^{\perp\perp} = \varphi(\widehat{C}_S^{1,p'})^\perp = \varphi((\widehat{C}_S^{1,p'})^\perp) = \varphi(\varphi(\widehat{C}_S^{p,\mathrm{b}})) = \widehat{C}_S^{p,\mathrm{b}}$$

and it follows from Lemma 6.1 that the second dual space of $(\widehat{C}_S^{p,\infty}, \|\cdot\|_{p,\infty})$ is isometrically isomorphic to $(\widehat{C}_S^{p,\mathrm{b}}, \|\cdot\|_{p,\mathrm{b}})$. Taking into account that $C_S^{p,\infty}$ is isometrically isomorphic to $(\widehat{C}_S^{p,\infty}, \|\cdot\|_{p,\infty})$ and that $C_S^{p,\mathrm{b}}$ is isometrically isomorphic to $(\widehat{C}_S^{p,\mathrm{b}}, \|\cdot\|_{p,\mathrm{b}})$, we obtain the proof of part (i). In the same way we prove parts (ii) and (iii). ∎

REMARK. In Example 3.4 of [12] a non-selfadjoint operator S was considered such that the second dual space of $C_S^{\infty,\infty}$ is a *proper* subspace of $C_S^{\mathrm{b,b}}$. ∎

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