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INVARIANT SUBSPACES FOR COMMUTING CONTRACTIONS

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Abstract. It is shown that each finite commuting system of contractions which possesses a unitary dilation and for which the Harte spectrum is dominating in the open unit polydisc possesses non-trivial joint invariant subspaces. Since by a well-known result of Ando each commuting pair of contractions admits a unitary dilation, we obtain in particular that each commuting pair of contractions with dominating Harte spectrum in the bidisc possesses non-trivial joint invariant subspaces.

Keywords: *Invariant subspaces, commuting contractions, Harte spectrum, dual operator algebras, Henkin measures.*

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Let $T \in L(H)$ be a contraction on a complex Hilbert space. A result of Brown, Chevreau, and Pearcy ([10]) says that T has a non-trivial invariant subspace if the spectrum of T is dominating in the open unit disc. In the present paper we prove the same result for finite commuting systems of contractions that possess a unitary dilation and for which the Harte spectrum is dominating in the open unit polydisc in $\mathbb{C}^n$. By a well-known result of Ando ([3]) each commuting pair of contractions possesses a unitary dilation. Hence, in this case, the richness of the Harte spectrum is sufficient to guarantee the existence of non-trivial joint invariant subspaces.

The condition that T possesses a unitary dilation ensures that T satisfies von Neumann's inequality over the unit polydisc, or equivalently, that T admits a contractive functional calculus over the polydisc algebra $A(\mathbb{D}^n)$. A classical result of Sz.-Nagy and Foiaş saying that a Hilbert-space contraction which is neither of type $C_{0\cdot}$ nor of type $C_{\cdot 0}$, and which is not a scalar multiple of the identity operator, always has a non-trivial hyperinvariant subspace, allows us to reduce the assertion to the case where T is absolutely continuous, that is, the $A(\mathbb{D}^n)$-functional calculus of T extends to a w*-continuous algebra homomorphism $\Phi : H^\infty(\mathbb{D}^n) \to L(H)$. Finally, results of Apostol from [4] on the existence of invariant subspaces for suitable polynomially bounded n-tuples of operators can be used to reduce the assertion to the case that the representation Φ is of type $C_{\cdot 0}$.

To prove the existence of invariant subspaces for single contractions one may assume that the spectrum of the given contraction T coincides with its right essential spectrum. Then a standard application of the Scott Brown technique shows that each operator that admits an H^∞-functional calculus of type $C_{\cdot 0}$ over an open set in $\mathbb{C}$ in which the right essential spectrum is dominating possesses an extremely rich invariant subspace lattice. The main remaining difficulty in proving the existence of joint invariant subspaces for commuting contractions with rich Harte spectrum is that the last reduction concerning the type of the spectrum of T is not available in the multidimensional case.

Let $T \in L(H)^n$ be a commuting system of Hilbert-space contractions and let $S \in L(K)^n$ be a dilation of T on a larger Hilbert space K. In Section 1 we extend results of Bekken ([5]), Briem, Davie, Øksendal ([8]), and Kosiek ([21]) on Henkin measures over the n-dimensional torus $\mathbb{T}^n$. We use the properties of Henkin measures and a decomposition theorem of Mlak ([23]) for representations of function algebras to show that, if S is a minimal dilation of T, then each w*-continuous H^∞-functional calculus of T over the unit polydisc admits a dilation to a w*-continuous H^∞-functional calculus of S.

In Section 2 we construct, for each commuting system $T \in L(H)^n$ with a unitary dilation U, an extension to a commuting n-tuple C of co-isometries on a larger Hilbert space such that C admits a Wold-type decomposition

$$C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$$

into a commuting tuple of unitary operators $R \in L(\mathcal{R})^n$ and a system $S^* \in L(\mathcal{S})^n$ satisfying a suitable weak $C_{0\cdot}$-condition. The results of Section 1 imply that, if T is absolutely continuous, then S^* and R are absolutely continuous, and that in this case the H^∞-functional calculus of T extends to a w*-continuous H^∞-functional calculus of C. We extend some one-dimensional methods from [12] to prove that, if T satisfies a suitable weak $C_{\cdot 0}$-condition and if its H^∞-functional calculus allows the almost factorization of elements in the predual of $H^\infty(\mathbb{D}^n)$, then T satisfies an actual factorization property strong enough to guarantee that the dual algebra $\mathfrak{A}_T$ generated by T has property $(\mathbb{A}_{1,\chi_0})$.

In Section 3 we show that under the above conditions the dual algebra $\mathfrak{A}_T$ satisfies a strengthened version of property $(\mathbb{A}_{1,\chi_0})$. In the one-variable case, and in the case of spherical contractions, this factorization condition can be used to prove the reflexivity of the dual algebra $\mathfrak{A}_T$. In the polydisc case corresponding reflexivity results can at least be proved in the case where the tuple T is in addition subnormal. These results will be presented elsewhere.

In Section 4 we prove the main invariant-subspace result explained at the beginning of this introduction. Furthermore, we show that each commuting system $T \in L(H)^n$ of contractions with a unitary dilation such that T is of type $C_{0\cdot}$ or of type $C_{\cdot 0}$, and such that the essential Harte spectrum is dominating in the unit polydisc $\mathbb{D}^n$, satisfies the factorization property $(\mathbb{A}_{1,\chi_0})$.

In the one-variable case each absolutely continuous contraction $T \in L(H)$ with isometric H^∞-functional calculus allows the almost factorization of elements in the predual of $H^\infty(\mathbb{D})$. This observation of Bercovici lies at the heart of the results of Bercovici ([6]) and Chevreau ([11]) saying that each contraction of class $\mathbb{A}$ satisfies the factorization property $(\mathbb{A}_1)$, and also of the result of Brown and Chevreau ([9]) showing that contractions of class $\mathbb{A}$ are reflexive. If the corresponding almost factorization property is true for isometric w*-continuous representations of $H^\infty(\mathbb{D}^n)$, then the methods of this paper immediately yield the existence of joint invariant subspaces for commuting systems of contractions with a unitary dilation and isometric w*-continuous H^∞-functional calculus over the unit polydisc $\mathbb{D}^n$. But at present it seems that additional ideas are necessary to decide this more general question.

0. PRELIMINARIES

Let H be a complex Hilbert space, and let $T = (T_1, \dots, T_n) \in L(H)^n$ be a tuple of commuting continuous linear operators on H. We denote by $\sigma(T)$ the Taylor spectrum of T, i.e., the set of all points $\lambda \in \mathbb{C}^n$ for which the induced Koszul complex $K^\bullet(\lambda - T, H)$ is not exact (see [17]). The Harte spectrum $\sigma^{\mathcal{H}}(T)$ of T is the subset of the Taylor spectrum given by all points $\lambda \in \mathbb{C}^n$ for which the map

$$H \to H^n, \quad x \mapsto \big((\lambda_i - T_i)x\big)_{i=1}^n$$

is not bounded from below or the map

$$H^n \to H, \quad (x_i)_{i=1}^n \mapsto \sum_{i=1}^n (\lambda_i - T_i)x_i$$

is not onto. The left essential spectrum $\sigma_{\mathrm{le}}(T)$ of T consists of those points $\lambda \in \mathbb{C}^n$ for which the first of the above two maps has non-closed range or infinite-dimensional kernel. The right essential spectrum $\sigma_{\mathrm{re}}(T)$ of T is the set of all points $\lambda \in \mathbb{C}^n$ for which the range of the second map has infinite codimension. The essential Harte spectrum of T is the union $\sigma_{\mathrm{e}}^{\mathcal{H}}(T) = \sigma_{\mathrm{le}}(T) \cup \sigma_{\mathrm{re}}(T)$.

Let $\lambda \in \mathbb{C}^n$. It is well known (see [17]) that $\lambda \in \sigma_{\mathrm{le}}(T)$ if and only if there is an orthonormal sequence (x_k) in H with $\lim_{k\to\infty} (\lambda_i - T_i)x_k = 0$ for $i = 1, \dots, n$, and that $\lambda \in \sigma_{\mathrm{re}}(T)$ if and only if $\overline{\lambda} \in \sigma_{\mathrm{le}}(T^*)$, where $T^* = (T_1^*, \dots, T_n^*)$ is the adjoint tuple.

For T as above and $k \in \mathbb{N}^n$ ($k \in \mathbb{N}$, respectively), we write $T^k = T_1^{k_1} \cdot \dots \cdot T_n^{k_n}$ $\big(T^k = (T_1 \cdot \dots \cdot T_n)^k$, respectively$\big)$. Analogous notations are used for powers of n-tuples of complex numbers. A unitary dilation of T is by definition a tuple $U \in L(K)^n$ of commuting unitary operators on a larger Hilbert space $K \supset H$ with

$$T^k = PU^k|H, \quad k \in \mathbb{N}^n,$$

where P is the orthogonal projection from K onto the closed subspace H. A unitary dilation $U \in L(K)^n$ of T is minimal if the only reducing subspace for U containing H is the space K itself.

Let $H^\infty(\mathbb{D}^n)$ be the Banach algebra of all bounded analytic functions on the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$ equipped with the norm $\|f\| = \sup\{|f(z)|;\, z \in \mathbb{D}^n\}$. A subset σ of $\mathbb{C}^n$ is dominating in $\mathbb{D}^n$ if $\|f\| = \sup\{|f(z)|;\, z \in \mathbb{D}^n \cap \sigma\}$ for all $f \in H^\infty(\mathbb{D}^n)$. The space $H^\infty(\mathbb{D}^n)$ is a w*-closed subspace of $L^\infty(\mathbb{D}^n)$ relative to the duality $\langle L^1(\mathbb{D}^n), L^\infty(\mathbb{D}^n)\rangle$ (formed with respect to the $2n$-dimensional Lebesgue measure). Thus $H^\infty(\mathbb{D}^n)$ is isometrically isomorphic to the norm-dual of the Banach space $Q = L^1(\mathbb{D}^n)/{}^\perp H^\infty(\mathbb{D}^n)$. A sequence (f_k) in $H^\infty(\mathbb{D}^n)$ is a w*-zero sequence if and only if it is norm-bounded and pointwise convergent to zero. For $\lambda \in \mathbb{D}^n$, the w*-continuous linear functional

$$\mathcal{E}_\lambda : H^\infty(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto f(\lambda)$$

is regarded as an element in Q. The Banach algebra $L(H)$ of all continuous linear operators on H is regarded as the norm-dual of the Banach space $\mathcal{C}^1(H)$ of all trace-class operators via the duality

$$\mathcal{C}^1(H) \times L(H) \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{Tr}(AB).$$

For a commuting system $T \in L(H)^n$, the smallest w*-closed unital subalgebra $\mathfrak{A}_T$ of $L(H)$ that contains $T_1, \dots, T_n$ is identified with the norm-dual of the Banach space $Q_T = \mathcal{C}^1(H)/{}^\perp\mathfrak{A}_T$. Both $H^\infty(\mathbb{D}^n)$ and $\mathfrak{A}_T$ are examples of dual algebras, that is, of Banach algebras A which are isometrically isomorphic to the norm-dual of a suitable Banach space A_* such that the multiplication in A is separately w*-continuous (we refer to [14] for this terminology).

Suppose that $T \in L(H)^n$ possesses a unitary dilation. Then T satisfies von Neumann's inequality over $\mathbb{D}^n$, that is,

$$\|p(T)\| \leqslant \|p\|$$

for all polynomials $p \in \mathbb{C}[z]$ in n variables. Since the polynomials are dense in the polydisc algebra $A(\mathbb{D}^n) = \{f \in C(\overline{\mathbb{D}}^n);\ f|\mathbb{D}^n \text{ is analytic}\}$ equipped with the supremum norm on $\mathbb{D}^n$, von Neumann's inequality is equivalent to the existence of a contractive algebra homomorphism

$$\Phi : A(\mathbb{D}^n) \to L(H)$$

which extends the polynomial functional calculus of T. We call T absolutely continuous if Φ extends to a w*-continuous algebra homomorphism $\widehat{\Phi} : H^\infty(\mathbb{D}^n) \to L(H)$. A continuous algebra homomorphism $\Phi : H^\infty(\mathbb{D}^n) \to L(H)$ is said to be of type $C_{0\cdot}$ ($C_{\cdot 0}$, respectively) if $(\Phi(f_k))$ $((\Phi(f_k)^*)$, respectively) tends to zero strongly for each w*-zero sequence (f_k) in $H^\infty(\mathbb{D}^n)$.

Let $\mathbb{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n;\ |z_i| = 1 \text{ for } i = 1, \dots, n\}$ be the n-torus in $\mathbb{C}^n$, and let $M(\mathbb{T}^n)$ be the Banach space of all complex regular Borel measures on $\mathbb{T}^n$. We write $M^+(\mathbb{T}^n)$ ($M_1^+(\mathbb{T}^n)$, respectively) for the subsets of $M(\mathbb{T}^n)$ consisting of all positive (respectively, probability) measures on $\mathbb{T}^n$. We denote by $B(\mathbb{T}^n)$ the σ-algebra of all Borel measurable subsets of $\mathbb{T}^n$. A sequence (f_k) in $A(\mathbb{D}^n)$ is called a Montel sequence if $(f_k|\mathbb{D}^n)$ is a w*-zero sequence in $H^\infty(\mathbb{D}^n)$. A measure $\mu \in M(\mathbb{T}^n)$ is a weak Henkin measure if

$$\int_{\mathbb{T}^n} f_k \, \mathrm{d}\mu \xrightarrow{k} 0$$

for each Montel sequence (f_k) in $A(\mathbb{D}^n)$. A measure $\mu \in M(\mathbb{T}^n)$ is a Henkin measure if

$$\int_{\mathbb{T}^n} f_k g \, \mathrm{d}\mu \xrightarrow{k} 0$$

for each Montel sequence (f_k) in $A(\mathbb{D}^n)$ and each function $g \in L^1(|\mu|)$. We write $\mathrm{HM}(\mathbb{T}^n)$ for the set of all Henkin measures on $\mathbb{T}^n$, and we denote by $M_0(\mathbb{T}^n)$ the set of all measures $\rho \in M_1^+(\mathbb{T}^n)$ which represent the point evaluation at zero on $A(\mathbb{D}^n)$ in the sense that

$$f(0) = \int_{\mathbb{T}^n} f \,\mathrm{d}\rho, \quad f \in A(\mathbb{D}^n).$$

The set $A(\mathbb{D}^n)^\perp$ consists of all measures $\mu \in M(\mathbb{T}^n)$ with $\int_{\mathbb{T}^n} f \,\mathrm{d}\mu = 0$ for all $f \in A(\mathbb{D}^n)$.

A closed linear subspace $M \subset M(\mathbb{T}^n)$ is called a band of measures if each measure $\nu \in M(\mathbb{T}^n)$ for which there is a measure $\mu \in M$ with $\nu \ll \mu$ (i.e. $\nu \ll |\mu|$) belongs to M. The set $\mathrm{HM}(\mathbb{T}^n)$ of all Henkin measures on $\mathbb{T}^n$ is a band. If $M \subset M(\mathbb{T}^n)$ is arbitrary, then we write $\mathcal{B}(M)$ for the smallest band of measures in $M(\mathbb{T}^n)$ that contains M.

Let $\mu \in M(\mathbb{T}^n)$ be a positive measure. We write $P^2(\mu)$ for the norm-closure of the set of all polynomials in $L^2(\mu)$, and we denote by $P^\infty(\mu)$ the w*-closure of $\mathbb{C}[z]$ in $L^\infty(\mu)$ with respect to the duality $\langle L^1(\mu), L^\infty(\mu)\rangle$. The space $P^\infty(\mu)$ becomes a dual algebra with predual $Q(\mu) = L^1(\mu)/{}^\perp P^\infty(\mu)$. If μ is a Henkin measure, then there is a uniquely determined w*-continuous algebra homomorphism $r = r_\mu : H^\infty(\mathbb{D}^n) \to P^\infty(\mu)$ with $r(p) = p$ for each polynomial p.

Let $T \in L(H)^n$ be a commuting system. For $x, y \in H$, we denote by $[x \otimes y] \in Q_T$ the equivalence class of the rank-one operator $H \to H$, $\xi \mapsto \langle \xi, y\rangle x$. Let p, q be any cardinal numbers with $1 \leqslant p, q \leqslant \chi_0$. The dual algebra $\mathfrak{A}_T$ is said to possess property $(\mathbb{A}_{p,q})$ if, for each matrix (L_{ij}) of functionals $L_{ij} \in Q_T$ $(0 \leqslant i < p$, $0 \leqslant j < q)$, there are vectors $(x_i)_{0\leqslant i<p}$ and $(y_j)_{0\leqslant j<q}$ in H with

$$L_{ij} = [x_i \otimes y_j], \quad 0 \leqslant i < p,\, 0 \leqslant j < q.$$

If $p = q$, then we write $(\mathbb{A}_p)$ instead of $(\mathbb{A}_{p,p})$.

Suppose that there is a w*-continuous algebra homomorphism $\Phi : H^\infty(\mathbb{D}^n) \to L(H)$ that extends the polynomial functional calculus of T. For $x, y \in H$,

$$x \otimes y : H^\infty(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto \langle \Phi(f)x, y\rangle$$

defines an element in Q with $\Phi_*([x \otimes y]) = x \otimes y$, where $\Phi_* : Q_T \to Q$ is the predual of $\Phi : H^\infty(\mathbb{D}^n) \to \mathfrak{A}_T$. There is an obvious way to define the properties $(\mathbb{A}_{p,q})$ $(1 \leqslant p, q \leqslant \chi_0)$ for the representation Φ. Since Φ_* is injective, it follows that property $(\mathbb{A}_{p,q})$ for Φ implies property $(\mathbb{A}_{p,q})$ for the dual algebra $\mathfrak{A}_T$.

1. HENKIN MEASURES AND UNITARY DILATIONS

In this section we study the properties of Henkin measures in the unit polydisc in $\mathbb{C}^n$. We apply our observations to construct dilations of H^∞-functional calculi. In the first part we extend and clarify corresponding two-dimensional results of Briem, Davie, Øksendal ([8]), Bekken ([5]), and Kosiek ([22]). The n-dimensional case was also considered by Kosiek in [21].

For completeness sake and for the convenience of the reader, we give full proofs for all the results that are used in the sequel.

As before we denote by $\mathrm{HM}(\mathbb{T}^n)$ the set of all Henkin measures on $\mathbb{T}^n$ and by $M_0(\mathbb{T}^n)$ the set of all probability measures ρ on $\mathbb{T}^n$ representing the point evaluation at zero on $A(\mathbb{D}^n)$. We write $\mathcal{B}\big(M_0(\mathbb{T}^n)\big)$ for the band of measures on $\mathbb{T}^n$ generated by $M_0(\mathbb{T}^n)$. A Borel set $N \subset \mathbb{T}^n$ is a null set if $\rho(N) = 0$ for all measures $\rho \in M_0(\mathbb{T}^n)$. A subset $E \subset \mathbb{T}^n$ is called an (H)-set (Henkin set) if there is an integer $q \in \{1, \ldots, n-1\}$ and a null set $N \subset \mathbb{T}^q$ such that, for some indices $1 \leqslant j_1 < \cdots < j_q \leqslant n$, the set E can be written as

$$E = \{(z_1, \ldots, z_n) \in \mathbb{T}^n;\ (z_{j_1}, \ldots, z_{j_q}) \in N\}.$$

Our first aim is to show that

$$\mathrm{HM}(\mathbb{T}^n) = \mathcal{B}\big(M_0(\mathbb{T}^n)\big).$$

We prove first that the set on the right is contained in the set on the left.

Proposition 1.1. *Each measure in $\mathcal{B}\big(M_0(\mathbb{T}^n)\big)$ is a Henkin measure on $\mathbb{T}^n$.*

Proof. We prove this result by induction on the dimension n.
For $n = 1$, it is well known that $M_0(\mathbb{T}) = \{m\}$ and that

$$\mathrm{HM}(\mathbb{T}) = \mathcal{B}\big(M_0(\mathbb{T})\big) = \{\mu \in M(\mathbb{T});\ \mu \ll m\},$$

where m is the normalized linear Lebesgue measure on $\mathbb{T}$.

Let us fix a natural number $n \geqslant 2$, and let us suppose that the assertion has been proved in dimension $q = 1, \ldots, n-1$. Using the induction hypothesis we prove the following result on Henkin measures in dimension n.

Lemma 1.2. *Let $\mu \in M(\mathbb{T}^n)$ be a weak Henkin measure such that $|\mu|(E) = 0$ for all* (H)*-sets $E \subset \mathbb{T}^n$. Then μ is a Henkin measure.*

Proof. Let μ be a weak Henkin measure as in the lemma. Let (f_k) be a Montel sequence. We have to show that

$$\int_{\mathbb{T}^n} f_k g \,\mathrm{d}\mu \xrightarrow{k} 0, \quad g \in L^1(|\mu|).$$

Since (f_k) is bounded in $L^\infty(|\mu|)$, it suffices to check this condition for all functions g in a suitable total subset of $L^1(|\mu|)$. Using the fact that $C(\mathbb{T})\widehat{\otimes}_\varepsilon \cdots \widehat{\otimes}_\varepsilon C(\mathbb{T}) \subset L^1(|\mu|)$ is dense and that the rational functions with poles off $\mathbb{T}$ are uniformly dense in $C(\mathbb{T})$, we reduce the assertion to the case where g is of the form $g = u/(z_1 - w_1) \cdot \cdots \cdot (z_n - w_n)$, where $u \in A(\mathbb{D}^n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{D}^n$. Replacing f_k by $f_k u$ we may suppose that $u = 1$.

Let us fix a point $w \in \mathbb{D}^n$. Note that, for any function $h \in A(\mathbb{D}^n)$, the function $H \in A(\mathbb{D}^n)$ defined by

$$H(z_1, \ldots, z_n) = \sum_{p=0}^{n} (-1)^p \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} h(z_1, \ldots, w_{j_1}, \ldots, w_{j_p}, \ldots, z_n)$$

vanishes on the set $\{z \in \overline{\mathbb{D}}^n;\, z_i = w_i \text{ for some } i = 1, \ldots, n\}$. Therefore there is a function $H^w \in A(\mathbb{D}^n)$ with

$$(z_1 - w_1) \cdot \cdots \cdot (z_n - w_n) H^w(z) = H(z), \quad z \in \overline{\mathbb{D}}^n.$$

It is elementary to check that the sequence (F_k^w) arising in this way from the sequence (f_k) is a Montel sequence again.

Since μ is a weak Henkin measure, the numbers

$$\int_{\mathbb{T}^n} \frac{f_k(z)}{(z_1 - w_1) \cdots (z_n - w_n)} \, \mathrm{d}\mu(z) + \sum_{p=1}^{n} (-1)^p \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \int_{\mathbb{T}^n} \frac{f_k(z_1, \ldots, w_{j_1}, \ldots, w_{j_p}, \ldots, z_n)}{(z_1 - w_1) \cdots (z_n - w_n)} \, \mathrm{d}\mu(z)$$

converge to zero as k tends to infinity. To conclude the proof of the lemma, it therefore suffices to show that, for each index tuple $1 \leqslant j_1 < \cdots < j_p \leqslant n$ $(p = 1, \ldots, n-1)$, the corresponding summand in the above sum converges to zero as k tends to infinity. Let us fix such an index tuple. Set $q = n - p$. Denote by $\nu \in M(\mathbb{T}^q)$ the unique measure with

$$\int_{\mathbb{T}^q} g \, \mathrm{d}\nu = \int_{\mathbb{T}^n} \Big(g(z_{i_1}, \ldots, z_{i_q}) / \prod_{k=1}^{p} (z_{j_k} - w_{j_k}) \Big) \, \mathrm{d}\mu(z_1, \ldots, z_n)$$

for all functions $g \in C(\mathbb{T}^q)$. Here $1 \leqslant i_1 < \cdots < i_q \leqslant n$ is the unique ordered index tuple with $\{j_1, \ldots, j_p\} \cup \{i_1, \ldots, i_q\} = \{1, \ldots, n\}$. It suffices to show that ν is a Henkin measure.

By the Glicksberg-König-Seever decomposition theorem (Theorem 9.4.4 in [25]) we have $\nu = \nu_{\mathrm{a}} + \nu_{\mathrm{s}}$, where $\nu_{\mathrm{a}} \in \mathcal{B}\big(M_0(\mathbb{T}^q)\big)$ and ν_{s} is concentrated on a null set $N \subset \mathbb{T}^q$. By induction hypothesis it suffices to show that $|\nu|(N) = 0$. Let $\varepsilon > 0$ be arbitrary. Denote by $\pi : \mathbb{T}^n \to \mathbb{T}^q$ the canonical projection. Since by hypothesis $|\mu|\big(\{z \in \mathbb{T}^n;\, (z_{i_1}, \ldots, z_{i_q}) \in N\}\big) = 0$, a regularity argument shows that there is an open set $U \subset \mathbb{T}^q$ containing N with $|\mu|\big(\pi^{-1}(U)\big) < \varepsilon$. But then we obtain, for each function $g \in C_{\mathrm{c}}(U)$ with $\|g\|_{\infty, U} \leqslant 1$,

$$\Big| \int_U g \, \mathrm{d}\nu \Big| \leqslant \varepsilon / \prod_{k=1}^{p} (1 - |w_{j_k}|).$$

Therefore $|\nu|(N) = \inf\{|\nu|(V);\, V \supset N \text{ open}\} = 0$, and the proof of the lemma is complete. ∎

Proof of Proposition 1.1. To conclude the inductive proof of Proposition 1.1 let us fix a measure $\rho \in M_0(\mathbb{T}^n)$. Since the Henkin measures form a band, it suffices to show that $\rho \in \mathrm{HM}(\mathbb{T}^n)$. Obviously, ρ is a weak Henkin measure. To check that ρ satisfies the hypothesis of the preceding lemma, it suffices to observe that each (H)-set $E \subset \mathbb{T}^n$ is a null set. To see this note that, for each measure $\delta \in M_0(\mathbb{T}^n)$ and each projection $\pi : \mathbb{T}^n \to \mathbb{T}^q$, $z \mapsto (z_{j_1}, \dots, z_{j_q})$ where $1 \leqslant j_1 < \dots < j_q \leqslant n$, the image measure $\delta^\pi \in M(\mathbb{T}^q)$ induced by δ belongs to $M_0(\mathbb{T}^q)$. ∎

To prove that, conversely, each Henkin measure belongs to the band generated by the measures in $M_0(\mathbb{T}^n)$, we use the abstract Glicksberg-König-Seever decomposition theorem (Theorem 9.4.4 in [25]) and a Forelli-type lemma.

Lemma 1.3. *Let $\mu \in M(\mathbb{T}^n)$ be a weak Henkin measure. Then there is a unique decomposition*

$$\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}},$$

where $\mu_{\mathrm{a}} \in M(\mathbb{T}^n)$ is absolutely continuous with respect to some measure ρ in $M_0(\mathbb{T}^n)$ and $\mu_{\mathrm{s}} \in M(\mathbb{T}^n)$ is concentrated on a null set of type F_σ. The measure μ_{s} in this decomposition belongs to $A(\mathbb{D}^n)^\perp$.

Proof. By the Glicksberg-König-Seever theorem (Theorem 9.4.4 in [25]) there is a unique decomposition $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$ such that $\mu_{\mathrm{a}} \in M(\mathbb{T}^n)$ is absolutely continuous with respect to some measure $\rho \in M_0(\mathbb{T}^n)$ and μ_{s} is concentrated on a null set E of type F_σ. We only have to show that $\mu_{\mathrm{s}} \in A(\mathbb{D}^n)^\perp$.

As an application of Forelli's lemma (Lemma 9.5.5 in [25]) we obtain a sequence (g_k) in $A(\mathbb{D}^n)$ with $\|g_k\| \leqslant 1$ for all k such that $\lim_{k\to\infty} g_k(x) = 0$ for every $x \in E$ and $\lim_{k\to\infty} g_k(x) = 1$ ρ-almost everywhere for every measure $\rho \in M_0(\mathbb{T}^n)$. The last property implies that $\lim_{k\to\infty} g_k(0) = 1$. A normal family argument together with the maximum modulus principle shows that $(f_k) = (1 - g_k)$ is a Montel sequence in $A(\mathbb{D}^n)$. Since μ_{s} is concentrated on E, and since μ_{s} is a weak Henkin measure by Proposition 1.1, it follows that

$$\int_{\mathbb{T}^n} f \,\mathrm{d}\mu_{\mathrm{s}} = \lim_{k\to\infty} \int_{\mathbb{T}^n} f f_k \,\mathrm{d}\mu_{\mathrm{s}} = 0, \quad f \in A(\mathbb{D}^n). \quad ∎$$

As an elementary application we obtain the converse of Proposition 1.1.

Theorem 1.4. *For each natural number $n \geqslant 1$, we have the equality*

$$\mathrm{HM}(\mathbb{T}^n) = \mathcal{B}\big(M_0(\mathbb{T}^n)\big).$$

Proof. Since the Henkin measures form a band, all that is left to prove is that each positive Henkin measure belongs to $\mathcal{B}\big(M_0(\mathbb{T}^n)\big)$. But if μ is such a measure, then both parts in the decomposition $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$ explained in Lemma 1.3 are positive measures, and hence

$$\|\mu_{\mathrm{s}}\| = \mu_{\mathrm{s}}(\mathbb{T}^n) = \int_{\mathbb{T}^n} 1 \,\mathrm{d}\mu_{\mathrm{s}} = 0.$$

Thus we obtain that $\mu = \mu_{\mathrm{a}} \in \mathcal{B}\big(M_0(\mathbb{T}^n)\big)$. ∎

On the Euclidean unit ball B in $\mathbb{C}^n$ weak Henkin measures and Henkin measures are the same. On the polydisc we obtain the same result for positive measures.

COROLLARY 1.5. *Let $\mu \in M(\mathbb{T}^n)$ be a positive measure. Suppose that μ is a weak Henkin measure. Then μ is a Henkin measure.*

Proof. Again the measure μ_{s} in the decomposition of μ given by Lemma 1.3 is the zero measure. Hence the assertion follows from Theorem 1.4. ∎

In the following we apply the above measure theoretic results to study Hilbert-space representations of the polydisc algebra $A(\mathbb{D}^n)$. The methods that we use go back to Mlak ([23]) and subsequent papers of Kosiek ([21], [22]). We start with an elementary observation that can be proved in exactly the same way as in the case of the unit ball (see Lemma 1.1 in [15]).

LEMMA 1.6. *Let $\Phi : A(\mathbb{D}^n) \to L(H)$ be a norm-continuous algebra homomorphism. Then Φ can be extended to a* w**-continuous algebra homomorphism $\widehat{\Phi} : H^\infty(\mathbb{D}^n) \to L(H)$ if and only if $\mathrm{w}^*\text{-}\lim\limits_{k\to\infty} \Phi(f_k) = 0$ for each Montel sequence (f_k) in $A(\mathbb{D}^n)$.*

Let $\Phi : A(\mathbb{D}^n) \to L(H)$ be a norm-continuous algebra homomorphism. A collection of measures $\mu(x, y) \in M(\mathbb{T}^n)$ $(x, y \in H)$ is a family of representing measures for Φ if

$$\langle \Phi(f)x, y\rangle = \int\limits_{\mathbb{T}^n} f \,\mathrm{d}\mu(x, y), \quad f \in A(\mathbb{D}^n),\ x, y \in H.$$

By the theorem of Hahn-Banach there is always a family of representing measures $\mu(x, y)$ $(x, y \in H)$ with $\|\mu(x, y)\| \leqslant \|\Phi\|\,\|x\|\,\|y\|$ for all $x, y \in H$.

Let us call a measure $\mu \in M(\mathbb{T}^n)$ absolutely continuous if it is absolutely continuous with respect to some measure $\rho \in M_0(\mathbb{T}^n)$. A measure $\mu \in M(\mathbb{T}^n)$ is said to be singular if it is concentrated on a null set $N \subset \mathbb{T}^n$ of type F_σ. Following Mlak ([23]) we call Φ absolutely continuous (singular) if Φ possesses a family of representing measures $\mu(x, y)$ $(x, y \in H)$ that are absolutely continuous (singular).

The following result is a version of a decomposition theorem of Mlak ([23]) adapted to the case of the unit polydisc. The proof is based on the Glicksberg-König-Seever decomposition theorem (Theorem 9.4.4 in [25]) and the Glicksberg-König-Seever generalization of the classical F. and M. Riesz theorem (Theorem 9.5.6 in [25]).

THEOREM 1.7. *Let $\Phi : A(\mathbb{D}^n) \to L(H)$ be a unital norm-continuous algebra homomorphism. Then there are unique norm-continuous algebra homomorphisms $\Phi_{\mathrm{a}}, \Phi_{\mathrm{s}} : A(\mathbb{D}^n) \to L(H)$ such that:*

(i) $\Phi = \Phi_{\mathrm{a}} + \Phi_{\mathrm{s}}$*;*

(ii) Φ_{a} *is absolutely continuous and* Φ_{s} *is singular.*

In this case, $\|\Phi_{\mathrm{a}}\| \leqslant \|\Phi\|$, $\|\Phi_{\mathrm{s}}\| \leqslant \|\Phi\|$, *and*

(iii) $\Phi_{\mathrm{a}}(f)\Phi_{\mathrm{s}}(g) = 0 = \Phi_{\mathrm{s}}(g)\Phi_{\mathrm{a}}(f)$ *for all* $f, g \in A(\mathbb{D}^n)$.

Again the proof of the ball case given in detail in [15] (Theorem 1.5) carries over to the case of the polydisc. For further use, we only sketch the main idea. Let

$\mu(x,y)$ $(x,y \in H)$ be a family of representing measures for Φ such that $\|\mu(x,y)\| \leqslant \|\Phi\|\,\|x\|\,\|y\|$ $(x,y \in H)$. For each pair of vectors $x,y \in H$, let

$$\mu(x,y) = \mu_{\mathrm{a}}(x,y) + \mu_{\mathrm{s}}(x,y)$$

be the Glicksberg-König-Seever decomposition of $\mu(x,y)$ into an absolutely continuous measure $\mu_{\mathrm{a}}(x,y)$ and a singular measure $\mu_{\mathrm{s}}(x,y)$. Then $\Phi_{\mathrm{a}}, \Phi_{\mathrm{s}} : A(\mathbb{D}^n) \to L(H)$ are the unique norm-continuous algebra homomorphisms defined by

$$\langle \Phi_{\mathrm{a}}(f)x, y\rangle = \int_{\mathbb{T}^n} f \,\mathrm{d}\mu_{\mathrm{a}}(x,y), \quad \langle \Phi_{\mathrm{s}}(f)x, y\rangle = \int_{\mathbb{T}^n} f \,\mathrm{d}\mu_{\mathrm{s}}(x,y)$$

for $f \in A(\mathbb{D}^n)$ and $x,y \in H$. Thus Φ_{a} and Φ_{s} possess families of representing measures $\big(\mu_{\mathrm{a}}(x,y)\big)_{x,y\in H}$ and $\big(\mu_{\mathrm{s}}(x,y)\big)_{x,y\in H}$, respectively. For details we refer the reader to [15].

As an application we obtain a useful characterization of those representations of $A(\mathbb{D}^n)$ that can be extended to a w*-continuous representation of $H^\infty(\mathbb{D}^n)$.

COROLLARY 1.8. *Let $\Phi : A(\mathbb{D}^n) \to L(H)$ be a norm-continuous unital algebra homomorphism. Then the following are equivalent:*

(i) Φ *extends to a* w**-continuous algebra homomorphism* $\widehat{\Phi} : H^\infty(\mathbb{D}^n) \to L(H)$*;*

(ii) Φ *possesses a family of representing measures* $\mu(x,y)$ $(x,y \in H)$ *that are weak Henkin measures;*

(iii) Φ *is absolutely continuous.*

Proof. The equivalence of (i) and (ii) follows directly from Lemma 1.6. By Lemma 1.3 condition (ii) implies condition (iii), while the converse of this implication is obvious. ∎

The next result gives sufficient conditions for the existence of dilations of H^∞-functional calculi. A particular case of this result is contained in [22].

Let $T \in L(H)^n$ and $S \in L(K)^n$ be commuting systems of contractions on Hilbert spaces H and K such that S is a dilation of T, that is, $H \subset K$ and

$$T^k = PS^k|H, \quad k \in \mathbb{N}^n,$$

where P is the orthogonal projection from K onto H. Let S, and hence also T, satisfy von Neumann's inequality over $\mathbb{D}^n$. We denote by $\Phi : A(\mathbb{D}^n) \to L(H)$ and $\Psi : A(\mathbb{D}^n) \to L(K)$ the unique contractive $A(\mathbb{D}^n)$-functional calculi of T and S. Then $\Phi(f) = P\Psi(f)|H$ for all $f \in A(\mathbb{D}^n)$, or equivalently,

$$\langle \Phi(f)x, y\rangle = \langle \Psi(f)x, y\rangle, \quad f \in A(\mathbb{D}^n),\ x,y \in H.$$

Let $\Phi = \Phi_{\mathrm{a}} + \Phi_{\mathrm{s}}$ and $\Psi = \Psi_{\mathrm{a}} + \Psi_{\mathrm{s}}$ be the decompositions of Φ and Ψ into their absolutely continuous and singular parts according to Theorem 1.7, and let $H = H_{\mathrm{a}} + H_{\mathrm{s}}$ and $K = K_{\mathrm{a}} + K_{\mathrm{s}}$ be the corresponding orthogonal decompositions of the underlying spaces.

We call a dilation S of T as above minimal if K is the only reducing subspace for S that contains H.

COROLLARY 1.9. *Under the above assumptions we have:*
(i) $H_{\rm a} \subset K_{\rm a}$, $H_{\rm s} \subset K_{\rm s}$, *and* $\Psi_{\rm a}|K_{\rm a}$, $\Psi_{\rm s}|K_{\rm s}$ *are dilations of* $\Phi_{\rm a}|H_{\rm a}$ *and* $\Phi_{\rm s}|H_{\rm s}$*;*
(ii) *if* Φ *is absolutely continuous and if* S *is a minimal dilation of* T*, then* Ψ *is absolutely continuous.*

Proof. The construction of the representations $\Phi_{\rm a}, \Phi_{\rm s}$, and $\Psi_{\rm a}, \Psi_{\rm s}$ described in the sketch of the proof of Theorem 1.7 shows that

$$\langle \Phi_{\rm a}(f)x, y\rangle = \langle \Psi_{\rm a}(f)x, y\rangle \quad \text{ and } \quad \langle \Phi_{\rm s}(f)x, y\rangle = \langle \Psi_{\rm s}(f)x, y\rangle$$

for $f \in A(\mathbb{D}^n)$ and $x, y \in H$. In particular, we have

$$\begin{aligned} \|x\|^2 &= \langle \Phi_{\rm a}(1)x, x\rangle = \langle \Psi_{\rm a}(1)x, x\rangle = \|\Psi_{\rm a}(1)x\|^2, \\ \|y\|^2 &= \langle \Phi_{\rm s}(1)y, y\rangle = \langle \Psi_{\rm s}(1)y, y\rangle = \|\Psi_{\rm s}(1)y\|^2 \end{aligned}$$

for $x \in H_{\rm a}$ and $y \in H_{\rm s}$. Hence $H_{\rm a} \subset K_{\rm a}$, $H_{\rm s} \subset K_{\rm s}$, and $\Psi_{\rm a}|K_{\rm a}$, $\Psi_{\rm s}|K_{\rm s}$ are dilations of $\Phi_{\rm a}|H_{\rm a}$ and $\Phi_{\rm s}|H_{\rm s}$, respectively.

If Φ is absolutely continuous, then $\Phi = \Phi_{\rm a}$ and therefore $H = H_{\rm a} \subset K_{\rm a}$. Since $K_{\rm a}$ is reducing for S, the assertion of part (ii) follows. ∎

2. CO-ISOMETRIC EXTENSIONS

Let $T = (T_1, \ldots, T_n) \in L(H)^n$ be a commuting tuple of contractions on a Hilbert space H. We call T unitary if all components T_i of T are unitary operators, and we say T is completely non-unitary if there is no non-zero reducing subspace M for T such that $T|M$ is unitary.

LEMMA 2.1. *Let* $T = (T_1, \ldots, T_n) \in L(H)^n$ *be a tuple of commuting contractions on a Hilbert space* H*. Then* T *is completely non-unitary if and only if the product* $T_1 \cdot \cdots \cdot T_n$ *is a completely non-unitary contraction.*

Proof. Let T be completely non-unitary. It suffices to show that the unitary part

$$H_{\rm u} = \{x \in H;\ \|T^k x\| = \|x\| = \|T^{*k}x\| \text{ for all } k \in \mathbb{N}\}$$

of the product $T_1 \cdot \cdots \cdot T_n$ is the zero space (see [26]). Since $H_{\rm u}$ is contained in the unitary part of each component T_i $(i = 1, \ldots, n)$, it suffices to show that the space $H_{\rm u}$ is reducing for the tuple T. To check this we only need to observe that

$$T_i T_i^* x = x = T_i^* T_i x, \quad x \in H_{\rm u},\ i = 1, \ldots, n.$$

Thus we have shown that the product $T_1 \cdot \cdots \cdot T_n$ is a completely non-unitary contraction. The reverse implication obviously holds. ∎

If $S = (S_1, \ldots, S_n) \in L(H)^n$ is a subnormal tuple of commuting contractions and if S is completely non-unitary, then as an application of Lemma 2.1 we obtain that

$$\|S^{*k}x\| \xrightarrow{k} 0, \quad x \in H.$$

This observation follows as a particular application of the next result.

Lemma 2.2. *Let $T \in L(H)^n$ be a commuting tuple of contractions. Then there is a unique orthogonal decomposition $H = \mathcal{S} \oplus \mathcal{R}$ with reducing spaces $\mathcal{S}$ and $\mathcal{R}$ for T such that $T|\mathcal{S}$ is completely non-unitary and $T|\mathcal{R}$ is unitary. If T is subnormal, then*

$$\mathcal{S} = \Big\{x \in H;\ \inf_{k\in\mathbb{N}^n} \|T^{*k}x\| = 0\Big\}.$$

Proof. The space

$$\mathcal{R} = \bigvee(M;\ M \text{ is a reducing subspace for } T \text{ such that } T|M \text{ is unitary})$$

reduces T and $T|\mathcal{R}$ is unitary while T restricted to $\mathcal{S} = H \ominus \mathcal{R}$ is completely non-unitary. Obviously such a decomposition is unique.

Suppose that T is subnormal. By Lemma 2.1 the subnormal contraction $(T_1 \cdot \dots \cdot T_n)|\mathcal{S}$ is completely non-unitary. Hence (see, e.g., Corollary 2.4 in [15])

$$\inf_{k\in\mathbb{N}^n} \|T^{*k}x\| = \inf_{k\in\mathbb{N}} \|T^{*k}x\| = \lim_{k\to\infty} \|T^{*k}x\| = 0$$

for all $x \in \mathcal{S}$. Conversely, if $x \in H$ is any vector for which the above infimum is zero, then writing $x = y + z$ with $y \in \mathcal{S}$ and $z \in \mathcal{R}$ one obtains that

$$0 = \lim_{k\to\infty} \|T^{*k}x\|^2 = \lim_{k\to\infty} (\|T^{*k}y\|^2 + \|T^{*k}z\|^2) = \|z\|^2.$$

This observation completes the proof. ∎

Let $T \in L(H)^n$ be a commuting tuple of contractions such that T has a unitary dilation. Fix a unitary dilation $U \in L(K)^n$ of the adjoint tuple T^* on a Hilbert space $K \supset H$. Let

$$K_+ = \bigvee(U^k H;\ k \in \mathbb{N}^n) \in \mathrm{Lat}(U)$$

be the smallest invariant subspace for U that contains H. Then the restriction $V = U|K_+ \in L(K_+)^n$ is a commuting tuple of isometries. As in the one-variable case, the tuple V^* yields a co-isometric extension of T.

Lemma 2.3. *The tuple V^* leaves the space H invariant and $T = V^*|H$. The space K_+ decomposes into an orthogonal sum $K_+ = \mathcal{S} \oplus \mathcal{R}$ such that $\mathcal{S}$ and $\mathcal{R}$ reduce V, $V|\mathcal{S}$ is completely non-unitary, and $V|\mathcal{R}$ is unitary. This decomposition is unique and*

$$\mathcal{S} = \{x \in K_+;\ \inf_{k\in\mathbb{N}^n} \|V^{*k}x\| = 0\}.$$

Proof. Since V is subnormal, we only have to prove the first part of the assertion. Obviously the space

$$K_+ \ominus H = \bigvee\big((U^k - T^{*k})H;\ k \in \mathbb{N}^n\big)$$

is invariant under V, and hence $H \in \mathrm{Lat}(V^*)$. Since

$$\langle V_i^* h, k\rangle = \langle h, U_i k\rangle = \langle h, T_i^* k\rangle = \langle T_i h, k\rangle, \quad h, k \in H,\ i = 1, \dots, n,$$

it follows that $V^*|H = T$. ∎

Suppose in addition that $U \in L(K)^n$ is a minimal unitary dilation of T^*. Using the notations from Lemma 2.3 we define

$$S = V|\mathcal{S}, \quad R = (V|\mathcal{R})^*, \quad C = V^*.$$

Then T possesses the co-isometric extension $C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$, where R is unitary and $S^* \in L(\mathcal{S})^n$ satisfies the weak $C_{0\cdot}$-condition

$$\inf\{\|S^{*k}x\|;\, k \in \mathbb{N}^n\} = 0, \quad x \in \mathcal{S}.$$

Let us denote by $A : K_+ \to K_+$ and $Q : K_+ \to K_+$ the orthogonal projections from K_+ onto $\mathcal{R}$ and $\mathcal{S}$, respectively.

LEMMA 2.4. *Suppose that $\mathcal{R} \neq \{0\}$. Then there is a measure $\mu \in M_1^+(\mathbb{T}^n)$ such that $\mathcal{R}$ contains a reducing subspace $\mathcal{R}_0$ for R with:*

(i) *$R_0 = R|\mathcal{R}_0$ is unitarily equivalent to the tuple M_z given by the multiplication with the coordinate functions on $L^2(\mu)$;*

(ii) *the subspace $\mathcal{R}_0^+$ of $\mathcal{R}_0$ corresponding to $P^2(\mu)$ under the unitary equivalence in part* (i) *satisfies $\mathcal{R}_0^+ \subset \overline{AH}$.*

If T is absolutely continuous, then μ is necessarily a Henkin measure.

Proof. Since $R_i(Ah) = AT_ih$ ($h \in H$ and $i = 1, \ldots, n$), it follows that $\overline{AH} \in \mathrm{Lat}(R)$. We claim that R is the minimal normal extension of $R|\overline{AH}$. Note first that, since $U|\mathcal{R}$ is unitary, the space $\mathcal{R}$ reduces U. Let M be a reducing subspace for R that contains $\overline{AH}$. Then $M \oplus (K \ominus \mathcal{R})$ is a reducing subspace for U containing H. The minimality of U (as a unitary dilation of T^*) implies that $K = M \oplus (K \ominus \mathcal{R})$, and hence that $M = \mathcal{R}$.

Since $\mathcal{R} \neq \{0\}$, we obtain in particular that $\overline{AH} \neq \{0\}$. By the multidimensional version of Proposition V.17.14 from [14] (which can be proved in exactly the same way as the one-dimensional case) there is a separating vector e for $W^*(R)$ in $\overline{AH}$. Denote by $\mu \in M_1^+(\mathbb{T}^n)$ the scalar spectral measure for R given by the separating vector e, that is,

$$\mu(A) = \langle E\big(A \cap \sigma(R)\big)e, e\rangle, \quad A \in B(\mathbb{T}^n),$$

where E is the projection-valued spectral measure of R.

The space $\mathcal{R}_0 = \bigvee(R^kR^{*m}e;\, k, m \in \mathbb{N}^n)$ is reducing for R, and the (unique) unitary operator $W : L^2(\mu) \to \mathcal{R}_0$ with

$$W\, p(z, \overline{z}) = p(R, R^*)e$$

for all polynomials p in $(2n)$ variables intertwines $M_z \in L\big(L^2(\mu)\big)^n$ and $R_0 = R|\mathcal{R}_0$ componentwise. Since e was chosen in $\overline{AH}$, it follows that the space $\mathcal{R}_0^+ = W\, P^2(\mu)$ is contained in $\overline{AH}$.

Suppose that T is absolutely continuous. Then by Corollary 1.9 the tuple R is absolutely continuous, and hence the same is true for the multiplication tuple M_z on $L^2(\mu)$. Let (f_k) be a Montel sequence in $A(B)$, and let $g \in L^1(\mu)$. Then $g = u\overline{v}$ with suitable functions $u, v \in L^2(\mu)$ and by Lemma 1.6

$$\int_{\mathbb{T}^n} f_k g \,\mathrm{d}\mu = \langle f_k u, v\rangle_{L^2(\mu)} \xrightarrow{k} 0.$$

Thus we have shown that μ is a Henkin measure. ∎

Let us fix a probability measure $\mu \in M_1^+(\mathbb{T}^n)$ as in the preceding proof. For any pair of vectors $x, y \in \mathcal{R}$, we write $x \cdot y \in L^1(\mu)$ for the Radon-Nikodym derivative of the measure

$$\mu_{x,y} : B(\mathbb{T}^n) \to \mathbb{C}, \quad \mu_{x,y}(A) = \langle E\big(A \cap \sigma(R)\big)x, y\rangle$$

with respect to μ. We denote by $x \odot y = [x \cdot y]$ the induced equivalence class in $Q(\mu) = L^1(\mu)/{}^{\perp}P^{\infty}(\mu)$. We write

$$\Psi_R : L^{\infty}(\mu) \to W^*(R)$$

for the isomorphism of von Neumann algebras associated with the unitary tuple R.

Let $W\colon\ L^2(\mu) \to \mathcal{R}_0$ be the unitary operator chosen in the proof of Lemma 2.4. For $x \in \mathcal{R}_0$, we write $\{x\}$ for the function in $L^2(\mu)$ corresponding to x under the operator W. Let $x, y \in \mathcal{R}_0$. Since

$$\int_A \{x\}\overline{\{y\}}\,\mathrm{d}\mu = \langle \chi_A\{x\}, \{y\}\rangle_{L^2(\mu)} = \langle W(\chi_A\{x\}), W\{y\}\rangle_{\mathcal{R}_0} = \mu_{x,y}(A)$$

for all Borel sets $A \subset \mathbb{T}^n$, it follows that $x \cdot y = \{x\}\overline{\{y\}}$ in $L^1(\mu)$.

Suppose that the tuple T is absolutely continuous, that is, T has a w*-continuous H^{∞}-functional calculus $\Phi : H^{\infty}(\mathbb{D}^n) \to L(H)$. Then the same is true for T^*, for the minimal unitary dilation U of T^* (Corollary 1.9), and for the parts S^* and R of the co-isometric extension C of T obtained above. Furthermore,

$$\Phi(f) = \Phi_C(f)|H = [\Phi_{S^*}(f) \oplus \Phi_R(f)]|H, \quad f \in H^{\infty}(\mathbb{D}^n),$$

where Φ_C, Φ_{S^*}, and Φ_R denote the w*-continuous H^{∞}-functional calculi of C, S^*, and R, respectively. For $x, y \in K_+$, we regard the w*-continuous linear functional

$$x \otimes y : H^{\infty}(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto \langle \Phi_C(f)x, y\rangle$$

as an element in the predual $Q = L^1(\mathbb{D}^n)/{}^{\perp}H^{\infty}(\mathbb{D}^n)$ of $H^{\infty}(\mathbb{D}^n)$.

DEFINITION 2.5. Let $\rho > 0$ and let $0 \leqslant \theta < \gamma$ be real numbers.

(i) The tuple T has the *ρ-almost factorization property* if, for each $L \in Q$ and each $\varepsilon > 0$, there are vectors $x, y \in H$ with $\|x\|, \|y\| \leqslant \rho\|L\|^{1/2}$ and

$$\|L - x \otimes y\| < \varepsilon.$$

If this condition holds with $\rho = 1$, then T is said to possess the *almost factorization property*.

(ii) We write $E_{\theta}^{\mathrm{r}}(T)$ for the set of all $L \in Q$ such that there are sequences $(x_k)_{k\geqslant 1}$ and $(y_k)_{k\geqslant 1}$ in the closed unit ball of H and K_+, respectively, with

$$\overline{\lim_{k\to\infty}}\|L - x_k \otimes y_k\| \leqslant \theta, \quad (x_k \otimes z) \xrightarrow{k} 0 \text{ for all } z \in H, \quad (z \otimes y_k) \xrightarrow{k} 0 \text{ for all } z \in \mathcal{S}.$$

(iii) We say that the tuple T has *property $E_{\theta,\gamma}^{\mathrm{r}}$* if

$$\{L \in Q;\ \|L\| \leqslant \gamma\} \subset \overline{\Gamma}\big(E_{\theta}^{\mathrm{r}}(T)\big).$$

For $\lambda \in \mathbb{D}$, consider the conformal map of the unit disc defined by

$$\varphi_{\lambda} : \mathbb{D} \to \mathbb{D}, \quad z \mapsto \frac{z - \lambda}{1 - \overline{\lambda}z}.$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, the biholomorphic map

$$\varphi_\lambda : \mathbb{D}^n \to \mathbb{D}^n, (z_1, \dots, z_n) \mapsto \big(\varphi_{\lambda_1}(z_1), \dots, \varphi_{\lambda_n}(z_n)\big)$$

extends to a holomorphic $\mathbb{C}^n$-valued map on a neighbourhood of $\overline{\mathbb{D}}^n$ which induces a homeomorphism $\overline{\mathbb{D}}^n \to \overline{\mathbb{D}}^n$. For any commuting tuple $A \in L(H)^n$ of contractions on a Hilbert space, the tuple $A_\lambda = \varphi_\lambda(A) = \big(\varphi_{\lambda_1}(A_1), \dots, \varphi_{\lambda_n}(A_n)\big)$ is again a commuting tuple of contractions. We write A_λ^* for the tuple $(A_\lambda)^* = \big(\varphi_{\lambda_1}(A_1)^*, \dots, \varphi_{\lambda_n}(A_n)^*\big)$.

PROPOSITION 2.6. *Suppose that T has the ρ-almost factorization property and that*

$$\inf_{k\in\mathbb{N}} \|T_\lambda^{*k} x\| = 0,$$

for each $x \in H$ and $\lambda \in \mathbb{D}^n$. Then T satisfies property $E^{\mathrm{r}}_{0,\gamma}$ with $\gamma = 1/\rho^2$.

Proof. We show that $\gamma \mathcal{E}_\lambda \in E_0^{\mathrm{r}}(T)$ for all $\lambda \in \mathbb{D}^n$.

Let $\lambda = 0$. The functionals

$$L_k : H^\infty(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto (\partial_1^{2k} \cdots \partial_n^{2k} f)(0)/\big((2k)!\big)^n, \quad k \in \mathbb{N}$$

are w*-continuous with $\|L_k\| \leqslant 1$ (Theorem 2.2.7 in [20]). By hypothesis there are sequences $(u_k)_{k\geqslant 1}$, $(v_k)_{k\geqslant 1}$ in H with $(L_k - u_k \otimes v_k) \xrightarrow{k} 0$ and $\|u_k\|, \|v_k\| \leqslant \rho$, $k \geqslant 1$. Define $x_k = T^k u_k \in H$ and $y_k = (S^k \oplus R^{*k}) v_k \in K_+$ for $k \geqslant 1$. Then $\|x_k\|$, $\|y_k\| \leqslant \rho$ and, for each $z \in H$,

$$\langle \Phi(f) x_k, z\rangle = \langle \Phi(f) u_k, T^{*k} z\rangle \xrightarrow{k} 0$$

uniformly for f in the closed unit ball of $H^\infty(\mathbb{D}^n)$. For each $z \in \mathcal{S}$,

$$\langle \Phi_{S^*}(f) z, y_k\rangle = \langle \Phi_{S^*}(f) S^{*k} z, v_k\rangle \xrightarrow{k} 0$$

uniformly for f in the closed unit ball of $H^\infty(\mathbb{D}^n)$. Thus $(x_k \otimes z) \xrightarrow{k} 0$ for all $z \in H$ and $(z \otimes y_k) \xrightarrow{k} 0$ for all $z \in \mathcal{S}$. For each $k \geqslant 1$, choose a function h_k in the closed unit ball of $H^\infty(\mathbb{D}^n)$ with $\|L_0 - x_k \otimes y_k\| = \langle L_0 - x_k \otimes y_k, h_k\rangle$. Then

$$\begin{aligned}\|L_0 - x_k \otimes y_k\| &\leqslant |\langle L_0, h_k\rangle - \langle L_k, z^{2k} h_k\rangle| + |\langle L_k - u_k \otimes v_k, z^{2k} h_k\rangle| \\ &\leqslant \|L_k - u_k \otimes v_k\| \xrightarrow{k} 0.\end{aligned}$$

Let $\lambda \in \mathbb{D}^n$. Write $\gamma_* : Q \to Q$ for the predual of the dual algebra isomorphism

$$\gamma = \gamma_\lambda : H^\infty(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n), \quad f \mapsto f \circ \varphi_\lambda.$$

The composition $\Phi_\lambda = \Phi \circ \gamma_\lambda$ is a w*-continuous H^∞-functional calculus of the tuple T_λ. Define $\psi_\lambda = \varphi_{\overline{\lambda}}$, that is, $\psi_\lambda(z) = \overline{\varphi_\lambda(\overline{z})}$. Note that T_λ is the restriction of $C_\lambda = (S^*)_\lambda \oplus R_\lambda$ onto H. The tuple R_λ is unitary and $(S^*)_\lambda$ is the adjoint of the completely non-unitary subnormal tuple $\psi_\lambda(S) = \psi_\lambda(U)|\mathcal{S}$. By Lemma 2.2

$$\inf_{k\in\mathbb{N}^n} \|(S^*)_\lambda^k x\| = 0, \quad x \in \mathcal{S}.$$

The functional calculus Φ_λ of T_λ extends to the w*-continuous H^∞-functional calculus of C_λ, that is,

$$\Phi_\lambda(f) = \Phi_{C_\lambda}(f)|H, \quad f \in H^\infty(\mathbb{D}^n).$$

For $x, y \in K_+$, we define

$$x \otimes_\lambda y : H^\infty(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto \langle \Phi_{C_\lambda}(f)x, y\rangle.$$

Using the fact that $\gamma_* : Q \to Q$ is an isometric isomorphism with $\gamma_*(x \otimes y) = x \otimes_\lambda y$ for $x, y \in K_+$, one easily obtains that T_λ satisfies the ρ-almost factorization property.

Now the first part of the proof applied to T_λ allows us to choose sequences $(x_k)_{k\geqslant 1}$ in H and $(y_k)_{k\geqslant 1}$ in K_+ with $\|x_k\|, \|y_k\| \leqslant \rho$ for all k and

$$\mathcal{E}_0 = \lim_{k\to\infty} x_k \otimes_\lambda y_k, \quad (x_k \otimes_\lambda z) \xrightarrow{k} 0 \text{ for all } z \in H, \quad (z \otimes_\lambda y_k) \xrightarrow{k} 0 \text{ for all } z \in \mathcal{S}.$$

But then $x_k \otimes z = \gamma_*^{-1}(x_k \otimes_\lambda z) \xrightarrow{k} 0$ for all $z \in H$, $z \otimes y_k = \gamma_*^{-1}(z \otimes_\lambda y_k) \xrightarrow{k} 0$ for all $z \in \mathcal{S}$, and

$$\|\mathcal{E}_\lambda - x_k \otimes y_k\| = \|\gamma_*^{-1}(\mathcal{E}_0 - x_k \otimes_\lambda y_k)\| \xrightarrow{k} 0.$$

Thus the proof is complete. ∎

To prove factorization results for the unitary part R of C we need the following measure-theoretic result proved in [18], Lemma 1.3.

LEMMA 2.7. *Let $\nu \in M_1^+(X)$ be a probability measure on a compact set $X \subset \mathbb{C}^n$. For each $L \in Q(\nu)$ and each $\varepsilon > 0$, there are functions $f, g \in P^2(\nu)$ with $\|f\|, \|g\| \leqslant \|L\|^{1/2}$ and*

$$\|L - [f\overline{g}]\| < \varepsilon.$$

Secondly, we recall a well-known result on the boundary values of functions in $A(\mathbb{D}^n)$.

LEMMA 2.8. *Let $\kappa : \mathbb{T}^n \to \mathbb{R}$ be a Borel measurable function such that $c \leqslant \kappa \leqslant d$, where $c, d > 0$ are given real numbers. Then, for any finite positive Borel measure ν on $\mathbb{T}^n$ and any real numbers $\varepsilon, \delta > 0$, there is a function $f \in A(\mathbb{D}^n)$ with $|f| \leqslant d$ on $\mathbb{T}^n$ and*

$$\nu\big(\{z \in \mathbb{T}^n;\ |\kappa(z) - |f(z)|| \geqslant \delta\}\big) < \varepsilon.$$

Proof. Set $\eta = \delta/2$. By Lusin's theorem (p. 227 in [13]) there is a continuous function $r : \mathbb{T}^n \to \mathbb{R}$ with $c \leqslant r \leqslant d$ and

$$\nu\big(\{z \in \mathbb{T}^n;\ r(z) \neq \kappa(z)\}\big) < \varepsilon.$$

Since each positive continuous function on $\mathbb{T}^n$ can uniformly be approximated by moduli of polynomials ([19]), there is a polynomial p such that $\big||p| - r\big| < \eta$ on $\mathbb{T}^n$. Then $f = p/(1 + \eta/d)$ is a polynomial with $|f| \leqslant d$ and $\big||p| - |f|\big| \leqslant \eta$ on $\mathbb{T}^n$. Clearly, f has all needed properties. ∎

In the remainder of Section 2 we suppose that the unitary part R of the co-isometric extension $C = S^* \oplus R$ of T is non-trivial. We apply the preceding two results to the scalar spectral measure $\mu \in M_1^+(\mathbb{T}^n)$ of the unitary tuple R chosen before. With the notation from Lemma 2.4, we write $A_0 \in L(K_+)$ and $A_1 \in L(K_+)$ for the orthogonal projections from K_+ onto $\mathcal{R}_0$ and $\mathcal{R} \ominus \mathcal{R}_0$, respectively.

PROPOSITION 2.9. *Let $L \in Q(\mu)$ and let $0 < \rho < 1$ be a real number. For given vectors $a \in H$, $b \in \mathcal{R}$, and any real number $\varepsilon > 0$, there are vectors $x \in H$, $c \in \mathcal{R}$, and a Borel set $Z \subset \mathbb{T}^n$ with $\mu(Z) < \varepsilon$, $c - b \in \mathcal{R}_0$, and*

$$\begin{aligned}
&\|L - A(a+x) \odot c\| < \varepsilon,\\
&\|x\| \leqslant 2\|L - Aa \odot b\|^{1/2}, \quad \|Qx\| < \varepsilon, \quad \|A_1 x\| < \varepsilon,\\
&\|c\| \leqslant \frac{1}{\rho}\big(\|b\| + \|L - Aa \odot b\|^{1/2}\big),\\
&|\{A_0(a+x)\}| \geqslant \rho|\{A_0 a\}| \qquad \text{on } \mathbb{T}^n \setminus Z.
\end{aligned}$$

Proof. Fix a positive real number $\delta < \rho$. Set $d = \|L - Aa \odot b\|^{1/2}$. By Lemma 2.7 there are vectors $\alpha, \beta \in \mathcal{R}_0^+$ with $\|\alpha\|, \|\beta\| \leqslant d$ and

$$\|L - Aa \odot b - \alpha \odot \beta\| < \delta.$$

By Lemma 2.4 there is a vector $x' \in H$ with $\|\alpha - Ax'\| < \delta$. Note that

$$R^k\alpha \cdot R^k\beta = |z^k|^2\{\alpha\}\overline{\{\beta\}} = \alpha \cdot \beta, \quad k \in \mathbb{N}.$$

Choose $k \in \mathbb{N}$ such that $\|QT^k x'\| = \|S^{*k}Qx'\| < \delta/2$. Then the vectors defined by

$$u = R^k\alpha \in \mathcal{R}_0^+, \quad v = R^k\beta \in \mathcal{R}_0^+, \quad \widetilde{x} = T^k x' \in H$$

satisfy the estimates $\|u\|, \|v\| \leqslant d$, $\|Q\widetilde{x}\| < \delta/2$, $\|u - A\widetilde{x}\| < \delta$, and

$$\|L - Aa \odot b - u \odot v\| < \delta.$$

Choose a constant $\eta > 0$ such that

$$\int_Z (|\{A_0 a\}\{A_0 b\}| + |\{A_0\widetilde{x}\}\{v\}|)\, \mathrm{d}\mu < \delta$$

for each Borel set $Z \subset \mathbb{T}^n$ with $\mu(Z) < \eta$. Define a measurable function $\kappa : \mathbb{T}^n \to \mathbb{R}$ by

$$\kappa(z) = \begin{cases} 1 + \rho & |\{A_0 a\}(z)| \leqslant |\{A_0\widetilde{x}\}(z)|;\\ 1 - \rho & \text{otherwise.}\end{cases}$$

Here as in the following we write $\{A_0 a\}, \{A_0\widetilde{x}\}, \ldots,$ when we really mean some fixed representatives of the corresponding equivalence classes in $L^2(\mu)$. Lemma 2.8 allows us to choose a function $f \in A(\mathbb{D}^n)$ with $|f| \leqslant 1 + \rho$ on $\mathbb{T}^n$ and such that

$$Z = \{z \in \mathbb{T}^n;\ \big|\kappa(z) - |f(z)|\big| \geqslant \delta\}$$

is a Borel set of measure $\mu(Z) < \eta$.

Define $x = \Phi(f)\widetilde{x} \in H$. Then $\|Qx\| < \delta$, $\|x\| \leqslant (1+\rho)\|\widetilde{x}\|$, and

$$\begin{aligned}
&\{A_0 x\} = \{\Phi_R(f) A_0\widetilde{x}\} = f\{A_0\widetilde{x}\},\\
&\|A_1 x\| = \|\Phi_R(f) A_1\widetilde{x}\| \leqslant (1+\rho)\|A_1(A\widetilde{x} - u)\| < 2\delta.
\end{aligned}$$

For $z \in \mathbb{T}^n \setminus Z$ with $|\{A_0 a\}(z)| \leqslant |\{A_0\widetilde{x}\}(z)|$, we have

$$|\{A_0(a+x)\}(z)| \geqslant (|f(z)|-1)|\{A_0\widetilde{x}\}(z)| \geqslant (\rho-\delta)|\{A_0\widetilde{x}\}(z)|.$$

For the remaining points z in $\mathbb{T}^n \setminus Z$, we have

$$|\{A_0(a+x)\}(z)| \geqslant \big(1-|f(z)|\big)|\{A_0 a\}(z)| \geqslant (\rho-\delta)|\{A_0 a\}(z)|.$$

Combining these two estimates we obtain on $\mathbb{T}^n \setminus Z$ the inequality

$$|\{A_0(a+x)\}| \geqslant (\rho-\delta)|\max(|\{A_0\widetilde{x}\}|, |\{A_0 a\}|).$$

Set $h_0 = (A_0 a)\cdot(A_0 b) + (A_0\widetilde{x})\cdot v \in L^1(\mu)$. Define $h : \mathbb{T}^n \to \mathbb{C}$ by setting

$$\overline{h} = h_0/\{A_0(a+x)\}$$

on $\mathbb{T}^n \setminus Z$ (with $t/0 = 0$ for all $t \in \mathbb{C}$) and $h|Z = 0$. Then $h \in L^2(\mu)$, and we can choose a vector $c_0 \in \mathcal{R}_0$ with $h = \{c_0\}$ and $\|c_0\| \leqslant (\|A_0 b\| + \|v\|)/(\rho-\delta)$. Our choices guarantee that

$$\|A_0(a+x)\cdot c_0 - h_0\|_{L^1(\mu)} \leqslant \delta.$$

Define $c = c_0 + A_1 b \in \mathcal{R}$. Since $r\cdot s = 0$ for $r \in \mathcal{R}_0$ and $s \in \mathcal{R}\ominus\mathcal{R}_0$, we obtain that

$$\begin{aligned}\|L - A(a+x)\odot c\| &\leqslant \|L - Aa\odot b - A_0\widetilde{x}\odot v\| \\ &\quad + \|Aa\odot b + A_0\widetilde{x}\odot v - A_0(a+x)\odot c_0 - A_1(a+x)\odot A_1 b\|.\end{aligned}$$

We abbreviate the first term on the right by r and the second by r'. Then

$$\begin{aligned} r &\leqslant \|L - Aa\odot b - u\odot v\| + \|(u - A_0\widetilde{x})\odot v\| \leqslant \delta(1+d), \\ r' &\leqslant \|[h_0] - A_0(a+x)\odot c_0\| + \|(A_1 x)\odot(A_1 b)\| \leqslant \delta(1+2\|A_1 b\|).\end{aligned}$$

We have $c - b = c_0 - b + A_1 b = c_0 - A_0 b \in \mathcal{R}_0$ and

$$\|x\| \leqslant (1+\rho)\|\widetilde{x}\| = (1+\rho)\|u - (u - A\widetilde{x}) + Q\widetilde{x}\| \leqslant (1+\rho)(d+2\delta).$$

Note that the norm of c can be estimated by

$$\|c\|^2 = \|c_0\|^2 + \|A_1 b\|^2 \leqslant (\|b\|+d)^2/(\rho-\delta)^2.$$

This observation completes the proof. ∎

In the next step we obtain almost factorizations for finite systems in $Q(\mu)$.

Proposition 2.10. *Let $N \geqslant 1$ be an integer, let $L_1, \dots, L_N \in Q(\mu)$, and let $\varepsilon > 0$, $\mu_1, \dots, \mu_N > 0$ be given real numbers. Suppose that $a \in H$, $b_1, \dots, b_N \in \mathcal{R}$ are vectors such that*

$$\|L_k - Aa\odot b_k\| < \mu_k, \quad k = 1, \dots, N.$$

Then there are vectors $x \in H$ and $y_1, \dots, y_N \in \mathcal{R}$ with $y_k - b_k \in \mathcal{R}_0$ and

$$\begin{aligned} &\|L_k - A(a+x)\odot y_k\| < \varepsilon, \\ &\|Qx\| < \varepsilon, \quad \|A_1 x\| < \varepsilon, \quad \|x\| < 2\sum_{i=1}^N \mu_i^{1/2}, \\ &\|y_k\| < \|b_k\| + \mu_k^{1/2}\end{aligned}$$

for $k = 1, \ldots, N$.

Proof. Fix positive real numbers $\eta < \varepsilon$ and $\rho < 1$. By Proposition 2.9 there are vectors $x_1 \in H$, $c_1 \in \mathcal{R}$, and a Borel set $Z_1 \subset \mathbb{T}^n$ with $\mu(Z_1) < \eta$ and

$$\begin{aligned}
&\|L_1 - A(a + x_1) \odot c_1\| < \varepsilon,\\
&\|x_1\| < 2\mu_1^{1/2}, \quad \|Qx_1\| < \varepsilon, \quad c_1 - b_1 \in \mathcal{R}_0,\\
&\|c_1\| < \|b_1\| + \mu_1^{1/2}, \quad \|A_1 x_1\| < \eta,\\
&|\{A_0(a + x_1)\}| \geqslant \rho|\{A_0 a\}| \quad \text{on } \mathbb{T}^n \setminus Z_1.
\end{aligned}$$

Define $f_1 \in L^\infty(\mathbb{T}^n, \mu)$ by setting

$$\overline{f_1} = \{A_0 a\}/\{A_0(a + x_1)\}$$

on $\mathbb{T}^n \setminus Z_1$ (with $t/0 = 0$ for all $t \in \mathbb{C}$) and $f_1|Z_1 = 0$. Set

$$b_k(1) = A_1 b_k + \Psi_R(f_1)A_0 b_k \in \mathcal{R}, \quad k = 1, \ldots, N.$$

If η is chosen small enough, then one obtains the estimates

$$\begin{aligned}
&\|L_k - A(a + x_1) \odot b_k(1)\|\\
&\quad \leqslant \|L_k - A_1 a \odot A_1 b_k - A_0(a + x_1) \odot \Psi_R(f_1)A_0 b_k\| + \eta\|b_k\|\\
&\quad \leqslant \|L_k - Aa \odot b_k\| + \int\limits_{Z_1} |\{A_0 a\}\{A_0 b_k\}| \mathrm{d}\mu + \eta\|b_k\| < \mu_k
\end{aligned}$$

for $k = 1, \ldots, N$.

In the next step we repeat the above constructions, but this time with L_1, a, b_1 replaced by $L_2, a + x_1, b_2(1)$, and $b_1, \ldots, b_N$ replaced by $b_1(1), \ldots, b_N(1)$. Inductively we obtain vectors $x_1, \ldots, x_N \in H$, $c_1, \ldots, c_N \in \mathcal{R}$, and Borel sets $Z_1, \ldots, Z_N \subset \mathbb{T}^n$ with

$$\begin{aligned}
&\Big\|L_k - A\Big(a + \sum_{i=1}^{k} x_i\Big) \odot c_k\Big\| < \varepsilon,\\
&\|x_k\| < 2\mu_k^{1/2}, \quad \|Qx_k\| < \varepsilon, \quad c_k - b_k(k-1) \in \mathcal{R}_0,\\
&\|c_k\| < \|b_k(k-1)\| + \mu_k^{1/2}, \quad \|A_1 x_k\| < \varepsilon,\\
&\Big|\Big\{A_0\Big(a + \sum_{i=1}^{k} x_i\Big)\Big\}\Big| \geqslant \rho\Big|\Big\{A_0\Big(a + \sum_{i=1}^{k-1} x_i\Big)\Big\}\Big| \quad \text{on } \mathbb{T}^n \setminus Z_k
\end{aligned}$$

for $k = 1, \ldots, N$. Here

$$b_k(k-1) = A_1 b_k + \Psi_R(f_{k-1} \cdot \cdots \cdot f_1)A_0 b_k \quad (= b_1 \text{ for } k = 1),$$

where $f_1, \ldots, f_N \in L^\infty(\mathbb{T}^n, \mu)$ are defined by

$$\overline{f_k} = \Big\{A_0\Big(a + \sum_{i=1}^{k-1} x_i\Big)\Big\}/\Big\{A_0\Big(a + \sum_{i=1}^{k} x_i\Big)\Big\}$$

on $\mathbb{T}^n \setminus Z_k$ and $f_k = 0$ on Z_k. The sets $Z_1, \dots, Z_N$ can be chosen in such a way that in addition

$$\int\limits_{Z_k} \Big| \Big\{ A_0\Big(a + \sum_{i=1}^{j} x_i\Big)\Big\} \, \{A_0 c_j\} \Big| \, \mathrm{d}\mu < \frac{\varepsilon}{N}$$

for $k = 2, \dots, N$ and $j = 1, \dots, k-1$.

Let us define $x = \sum\limits_{i=1}^{N} x_i \in H$, $y_N = c_N \in \mathcal{R}$, and

$$y_k = A_1 c_k + \Psi_R(f_{k+1} \cdot \dots \cdot f_N) A_0 c_k \in \mathcal{R}, \quad 1 \leqslant k < N.$$

Then we obtain the estimates

$$\|y_k\| \leqslant (1/\rho)^{N-1}\big(\|b_k\| + \mu_k^{1/2}\big)$$

for $k = 1, \dots, N$. For $k = 1, \dots, N-1$, it follows that

$$\begin{aligned}
&\|L_k - A(a+x) \odot y_k\| \\
&= \Big\| L_k - A_1\Big(a + \sum_{i=1}^{k} x_i\Big) \odot A_1 c_k - A_1\Big(\sum_{i=k+1}^{N} x_i\Big) \odot A_1 c_k \\
&\qquad\qquad - A_0(a+x) \odot \Psi_R(f_{k+1} \cdot \dots \cdot f_N) A_0 c_k \Big\| \\
&\leqslant \Big\| L_k - A_1\Big(a + \sum_{i=1}^{k} x_i\Big) \odot A_1 c_k - A_0\Big(a + \sum_{i=1}^{k} x_i\Big) \odot A_0 c_k \Big\| \\
&\qquad\qquad + \sum_{i=k+1}^{N} \|A_1 x_i\| \, \|A_1 c_k\| + \sum_{j=k+1}^{N} \int\limits_{Z_j} \Big| \Big\{ A_0\Big(a + \sum_{i=1}^{k} x_i\Big)\Big\}\{A_0 c_k\} \Big| \, \mathrm{d}\mu \\
&< \varepsilon(2 + N\|b_k\|),
\end{aligned}$$

while for $k = N$, we obtain the inequality

$$\|L_N - A(a+x) \odot y_N\| < \varepsilon.$$

Furthermore, our choices guarantee that $y_k - b_k \in \mathcal{R}_0$ for $k = 1, \dots, N$.

By starting the proof with ε replaced by a sufficiently small number $\varepsilon' > 0$, with μ_i replaced by suitable numbers $\mu_i' < \mu_i$, and by choosing ρ appropriately, one easily obtains vectors $x \in H$ and $y_1, \dots, y_N \in \mathcal{R}$ satisfying precisely the estimates stated in Proposition 2.10. ∎

3. FACTORIZATIONS OF TYPE $(\mathbb{A}_{1,\aleph_0})$

In this section we prove that the dual algebra generated by a commuting tuple $T \in L(H)^n$ with a unitary dilation such that T is absolutely continuous and possesses property $E^{\mathrm{r}}_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma \leqslant 1$ has the factorization property $(\mathbb{A}_{1,\chi_0})$. This observation will be used later to deduce concrete invariant-subspace results.

Let $T \in L(H)^n$ be a commuting system of contractions with a unitary dilation. As explained in Section 2 we use a minimal unitary dilation of T^* to construct a co-isometric extension

$$C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$$

of T. Let us suppose that T is absolutely continuous, that is, T possesses a w*-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{D}^n) \to L(H)$. By the results of Section 1 the systems C, S^*, and R are absolutely continuous. The H^∞-functional calculi of C, S^*, and R satisfy

$$\Phi(f) = \Phi_C(f)|H = [\Phi_{S^*}(f) \oplus \Phi_R(f)]|H, \quad f \in H^\infty(\mathbb{D}^n).$$

For $x, y \in \mathcal{S} \oplus \mathcal{R}$, we regard the functional

$$x \otimes y : H^\infty(\mathbb{D}^n) \to \mathbb{C}, \quad f \mapsto \langle \Phi_C(f)x, y\rangle$$

as an element in the predual $Q = L^1(\mathbb{D}^n)/{}^\perp H^\infty(\mathbb{D}^n)$ of $H^\infty(\mathbb{D}^n)$.

In the case when $\mathcal{R} \neq \{0\}$, we fix a Henkin probability measure $\mu \in M_1^+(\mathbb{T}^n)$ as explained in Lemma 2.4, and we denote by $r_* : Q(\mu) \to Q$ the predual of the canonical w*-continuous algebra homomophism (cf. the preliminaries)

$$r : H^\infty(\mathbb{D}^n) \to P^\infty(\mu)$$

associated with μ. For $x, y \in \mathcal{R}$, let $x \odot y \in Q(\mu)$ be defined as in the considerations following Lemma 2.4. An elementary exercise shows that $r_*(x \odot y) = x \otimes y$ for all $x, y \in \mathcal{R}$. If $\mathcal{R} = \{0\}$, then we set $Q(\mu) = \{0\} = P^\infty(\mu)$.

DEFINITION 3.1. Let $0 \leqslant \theta < \gamma \leqslant 1$ be real numbers. We say that the tuple T possesses *property* $F^{\mathrm{r}}_{\theta,\gamma}$ if there is a co-isometric extension C of T as above such that the set

$$\overline{\Gamma}\big(E^{\mathrm{r}}_\theta(T) \cup r_*\{L \in Q(\mu);\ \|L\| \leqslant 1\}\big)$$

contains the closed ball $\{L \in Q;\ \|L\| \leqslant \gamma\}$.

Our aim is to show that property $F^{\mathrm{r}}_{\theta,\gamma}$ implies the factorization property $(\mathbb{A}_{1,\chi_0})$ for the dual algebra generated by T.

PROPOSITION 3.2. *Suppose that T has property $F^{\mathrm{r}}_{\theta,\gamma}$ for some $0 < \theta < \gamma \leqslant 1$. Let $L_1, \dots, L_N \in Q$ and $\mu_1, \dots, \mu_N > 0$ be given. For any vectors $a \in H$, $w_1, \dots, w_N \in \mathcal{S}$ and $b_1, \dots, b_N \in \mathcal{R}$ with*

$$\|L_j - a \otimes (w_j + b_j)\| < \mu_j, \quad j = 1, \dots, N,$$

there are $a' \in H$, $w'_1, \dots, w'_N \in \mathcal{S}$, and $b'_1, \dots, b'_N \in \mathcal{R}$ such that, for $j = 1, \dots, N$,

$$\|L_j - a' \otimes (w'_j + b'_j)\| < (\theta/\gamma)\mu_j,$$

$$\|a' - a\| < (3/\gamma^{1/2}) \sum_{i=1}^{N} \mu_i^{1/2},$$

$$\|w'_j - w_j\| < (\mu_j/\gamma)^{1/2}, \quad \|b'_j\| < \|b_j\| + (\mu_j/\gamma)^{1/2}.$$

Proof. For $j = 1, \dots, N$, define

$$L'_j = L_j - a \otimes (w_j + b_j) \in Q, \quad d_j = \max\Big(\|L'_j\|, \frac{\mu_j}{2}\Big).$$

Choose $\varepsilon > 0$ such that $(\theta/\gamma)d_j + 5\varepsilon < (\theta/\gamma)\mu_j$ for all j. Define $s_j = (\theta/\gamma)d_j + \varepsilon$. Arguing exactly as in the proof of Proposition 3.4 from [12] one can choose integers $0 = k_0 < k_1 < \cdots < k_N$, elements $K_i \in E^{\mathrm{r}}_\theta(T)$, $\alpha_i \in \mathbb{C}$ $(i = 1, \dots, k_N)$, functions $f_j \in L^1(\mu)$ $(j = 1, \dots, N)$ with

$$\|L'_j - r_*([f_j]) - \sum_{i \in I_j} \alpha_i K_i\| < \frac{\varepsilon}{2}$$

and

$$\|f_j\|_{L^1(\mu)} + \sum_{i \in I_j} |\alpha_i| < \frac{d_j}{\gamma}$$

for $j = 1, \dots, N$ where $I_j = \{k_{j-1} + 1, \dots, k_j\}$.

Since $K_i \in E^{\mathrm{r}}_\theta(T)$, we can choose sequences $(x^i_k)_{k \geqslant 1}$ and $(y^i_k)_{k \geqslant 1}$ $(i = 1, \dots, k_N)$ in the closed unit ball of H and K_+, respectively, with $(x^i_k \otimes z) \xrightarrow{k} 0$, for $z \in H$, $(z \otimes y^i_k) \xrightarrow{k} 0$, for $z \in \mathcal{S}$ and such that, for each tuple $\nu = (n_1, \dots, n_{k_N}) \in \mathbb{N}^{k_N}$,

$$\Big\|L'_j - r_*([f_j]) - \sum_{i \in I_j} \alpha_i x^i_{n_i} \otimes y^i_{n_i}\Big\| < s_j, \quad j = 1, \dots, N.$$

Following closely [12] we define, for ν as above and $j = 1, \dots, N$,

$$A^j(\nu) = a \otimes b_j + r_*([f_j]) + \sum_{i \in I_j} \alpha_i \, A x^i_{n_i} \otimes y^i_{n_i},$$

$$Q^j(\nu) = a \otimes w_j + \sum_{i \in I_j} \alpha_i Q x^i_{n_i} \otimes y^i_{n_i},$$

and we observe that $\|L_j - Q^j(\nu) - A^j(\nu)\| < s_j$.

Choose complex numbers β_i $(i = 1, \dots, k_N)$ with $\beta_i^2 = \alpha_i$ and define

$$u_\nu = \sum_{i=1}^{k_N} \beta_i x^i_{n_i}, \quad v_{\nu,j} = \sum_{i \in I_j} \overline{\beta_i} Q y^i_{n_i}$$

for $j = 1, \dots, N$ and ν arbitrary.

Let $\eta > 0$ be arbitrary. Inductively one can choose, for each index $i = 1, \dots, k_N$, a natural number $n_i \geqslant 1$ such that

$$\begin{aligned} &\|Qx^i_{n_i} \otimes y^\ell_{n_\ell}\| < \eta, \quad \|Qa \otimes y^i_{n_i}\| < \eta, \quad |\langle x^i_{n_i}, x^\ell_{n_\ell}\rangle| < \eta, \\ &\|x^i_{n_i} \otimes w_j\| < \eta, \qquad \|x^i_{n_i} \otimes b_j\| < \eta, \quad |\langle Qy^i_{n_i}, y^\ell_{n_\ell}\rangle| < \eta \end{aligned}$$

for $i, \ell = 1, \dots, k_N$ with $i \neq \ell$ and each $j = 1, \dots, N$. If η is small enough and if $\nu = (n_1, \dots, n_{k_N})$ is chosen as above, then $\|u_\nu\|^2 < \sum_{i=1}^{N} \mu_i/\gamma$, $\|v_{\nu,j}\|^2 < \mu_j/\gamma$, $\|u_\nu \otimes b_j\| < \varepsilon$, and

$$\|Q^{(j)}(\nu) - (a + u_\nu) \otimes (w_j + v_{\nu,j})\| < \varepsilon, \quad j = 1, \dots, N.$$

We fix $\nu = (n_1, \dots, n_{k_N})$ such that the above estimates hold and define

$$\begin{aligned} &a_1 = a + u_\nu \in H, \quad w'_j = w_j + v_{\nu,j} \in \mathcal{S}, \\ &x^i = x^i_{n_i} \in H, \quad y^i = y^i_{n_i} \in K_+, \\ &h_j = f_j + \sum_{i \in I_j} \alpha_i A x^i \cdot A y^i \in L^1(\mu) \end{aligned}$$

for $j = 1, \dots, N$ and $i = 1, \dots, k_N$.

Since $\|h_j\| < d_j/\gamma$, Proposition 2.10 allows us to choose $x \in H$, $b'_1, \dots, b'_N \in \mathcal{R}$ with

$$\begin{aligned} &\|Aa_1 \odot b_j + [h_j] - A(a_1 + x) \odot b'_j\| < \varepsilon, \\ &\|Qx\| < \varepsilon/(\|w'_j\| + 1), \quad \|x\| < 2\sum_{i=1}^{N} (d_i/\gamma)^{1/2}, \\ &\|b'_j\| < \|b_j\| + (d_j/\gamma)^{1/2} \end{aligned}$$

for $j = 1, \dots, N$. By comparing the definitions of $A^j(\nu)$ and h_j, we obtain that

$$\|(a_1 + x) \otimes b'_j - A^j(\nu) + (u_\nu \otimes b_j)\| < \varepsilon.$$

Gathering all estimates prepared up to now, we obtain that

$$\begin{aligned} &\|L_j - (a_1 + x) \otimes (w'_j + b'_j)\| \\ &\quad = \|L_j - (a + u_\nu) \otimes (w_j + v_{\nu,j}) - (a_1 + x) \otimes b'_j - x \otimes w'_j\| \\ &\quad < \|L_j - (a + u_\nu) \otimes (w_j + v_{\nu,j}) - (a_1 + x) \otimes b'_j + u_\nu \otimes b_j\| + 2\varepsilon \\ &\quad < \|L_j - Q^j(\nu) - A^j(\nu)\| + 4\varepsilon < s_j + 4\varepsilon < (\theta/\gamma)\mu_j. \end{aligned}$$

To complete the proof it suffices to define $a' = a_1 + x = a + u_\nu + x$ and to observe that with this definition all claimed estimates hold. ∎

In the case when $\mathcal{R} = \{0\}$ the whole proof remains valid with $\mu = 0$ if one forgets everywhere the terms coming from $\mathcal{R}$ or $L^1(\mu)$. In this case Proposition 2.10 is not needed for the proof of Proposition 3.2.

Before we prove property $(\mathbb{A}_{1,\chi_0})$, we show how to replace the almost factorization obtained in Proposition 3.2 by an actual factorization.

COROLLARY 3.3. *Suppose that T has property $F^{\mathrm{r}}_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma \leqslant 1$. Let $L_1, \dots, L_N \in Q$ and let $\mu_1, \dots, \mu_N > 0$ be given real numbers. Then, for any vectors $a \in H$, $w_1, \dots, w_N \in \mathcal{S}$, and $b_1, \dots, b_N \in \mathcal{R}$ with*

$$\|L_j - a \otimes (w_j + b_j)\| < \mu_j, \quad j = 1, \dots, N,$$

there are $a' \in H$, $w'_1, \dots, w'_N \in \mathcal{S}$, and $b'_1, \dots, b'_N \in \mathcal{R}$ such that

$$\begin{aligned} &L_j = a' \otimes (w'_j + b'_j), \\ &\|a' - a\| < 3\alpha \sum_{i=1}^{N} \mu_i^{1/2}, \quad \|w'_j - w_j\| < \alpha \mu_j^{1/2}, \\ &\|b'_j\| < \|b_j\| + \alpha \mu_j^{1/2} \end{aligned}$$

for $j = 1, \dots, N$, where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$.

Proof. Without loss of generality we may suppose that $\theta > 0$. Otherwise, one can replace μ_j and $\theta = 0$ by suitable numbers $\mu'_j < \mu_j$ and $\theta' > 0$.

An inductive application of Proposition 3.2 allows us to choose sequences $(a_k)_{k \geqslant 1}$ in H, $(w_{kj})_{k \geqslant 1}$ in $\mathcal{S}$, and $(b_{kj})_{k \geqslant 1}$ in $\mathcal{R}$ for $j = 1, \dots, N$ such that (with $a_0 = a$, $w_{0j} = w_j$, and $b_{0j} = b_j$)

$$\begin{aligned} &\|L_j - a_k \otimes (w_{kj} + b_{kj})\| < (\theta/\gamma)^k \mu_j, \\ &\|a_k - a_{k-1}\| < (3/\gamma^{1/2}) \Big(\sum_{i=1}^{N} \mu_i^{1/2} \Big) (\theta/\gamma)^{(k-1)/2}, \\ &\|w_{kj} - w_{k-1,j}\| < (\mu_j/\gamma)^{1/2} (\theta/\gamma)^{(k-1)/2}, \\ &\|b_{kj}\| < \|b_{k-1,j}\| + (\mu_j/\gamma)^{1/2} (\theta/\gamma)^{(k-1)/2} \end{aligned}$$

for $j = 1, \dots, N$ and $k \geqslant 1$.

Obviously the sequences $(a_k)_{k \geqslant 1}$ and $(w_{kj})_{k \geqslant 1}$ ($j = 1, \dots, N$) are Cauchy sequences and their limits a' and w'_j satisfy the right estimates. Since

$$\|b_{kj}\| < \|b_j\| + (\mu_j/\gamma)^{1/2} \sum_{i=0}^{k-1} (\theta/\gamma)^{i/2}$$

for $k \geqslant 1$ and $j = 1, \dots, N$, we may suppose, after dropping to suitable subsequences, that the weak limits

$$b'_j = \operatorname*{w-lim}_{k \to \infty} b_{kj}, \quad j = 1, \dots, N$$

exist. Then $\|b'_j\| \leqslant \|b_j\| + \alpha \mu_j^{1/2}$ and $L_j = a' \otimes (w'_j + b'_j)$ for $j = 1, \dots, N$. ∎

Our next result shows that the dual algebra generated by a commuting tuple T of contractions satisfying property $F^{\mathrm{r}}_{\theta,\gamma}$ for some real numbers $0 \leqslant \theta < \gamma \leqslant 1$ has the factorization property $(\mathbb{A}_{1,\chi_0})$.

THEOREM 3.4. *Suppose that T satisfies the property $F^{\mathrm{r}}_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma \leqslant 1$. Let $(L_k)_{k\geqslant 1}$ be a sequence in Q and let $\delta_k > \|L_k\|$ be real numbers with $\delta = \sum\limits_{k=1}^{\infty} \delta_k^{1/2} < \infty$. Then, for any given vector $a \in H$, there are vectors $x \in H$, $y_k \in \mathcal{S}$, and $z_k \in \mathcal{R}$ such that $\|x - a\| < 3\alpha\delta$ and such that*

$$L_k = x \otimes (y_k + z_k), \quad \|y_k\| < \alpha\delta_k^{1/2}, \quad \|z_k\| < \alpha\delta_k^{1/2}$$

for all $k \geqslant 1$. Here, as before, $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$.

Proof. Choose real numbers μ_k with $\|L_k\| < \mu_k < \delta_k$ for $k \geqslant 1$. Define a sequence of positive real numbers c_k $(k \geqslant 1)$ by

$$c_k^{1/2} = \min\Big(\frac{1}{2^k}, \delta_k^{1/2} - \mu_k^{1/2}\Big)$$

and set $\varepsilon_{kj} = c_k c_j$, $k > j \geqslant 1$. Then the real numbers ε_{kj} satisfy the conditions

$$\sum_{k=j+1}^{\infty} \varepsilon_{kj}^{1/2} < \delta_j^{1/2} - \mu_j^{1/2} \qquad \text{for } j \geqslant 1,$$
$$\sum_{j=1}^{k-1} \varepsilon_{kj}^{1/2} < \delta_k^{1/2} - \mu_k^{1/2} \qquad \text{for } k \geqslant 1.$$

An iterative application of Corollary 3.3 allows us to choose vectors $a_k \in H$, $w_{kj} \in \mathcal{S}$, and $b_{kj} \in \mathcal{R}$ $(k \geqslant j \geqslant 1)$ with

$$L_j = a_k \otimes (w_{kj} + b_{kj}), \quad k \geqslant j \geqslant 1,$$

and such that (with $a_0 = a$)

$$\begin{aligned}
&\|a_k - a_{k-1}\| < 3\alpha\Big(\mu_k^{1/2} + \sum_{j=1}^{k-1} \varepsilon_{kj}^{1/2}\Big), && k \geqslant 1,\\
&\|w_{kj} - w_{k-1,j}\| < \alpha\,\varepsilon_{kj}^{1/2}, && k > j \geqslant 1,\\
&\|w_{jj}\| < \alpha\,\mu_j^{1/2}, && j \geqslant 1,\\
&\|b_{kj}\| < \|b_{k-1,j}\| + \alpha\,\varepsilon_{kj}^{1/2}, && k > j \geqslant 1,\\
&\|b_{jj}\| < \alpha\,\mu_j^{1/2}, && j \geqslant 1.
\end{aligned}$$

Note that the sequences $(a_k)_{k\geqslant 1}$ and $(w_{kj})_{k\geqslant j}$ $(j \geqslant 1)$ converge and that their limits $x \in H$ and $y_j \in \mathcal{S}$ satisfy

$$\|x - a\| < 3\alpha \sum_{k=1}^{\infty} \delta_k^{1/2} = 3\alpha\delta, \quad \|y_j\| < \alpha\delta_j^{1/2}, \quad j \geqslant 1.$$

Since

$$\|b_{kj}\| < \alpha\Big(\mu_j^{1/2} + \sum_{i=j+1}^{k} \varepsilon_{ij}^{1/2}\Big) < \alpha\delta_j^{1/2}$$

for $k \geqslant j \geqslant 1$, we can choose, for each $j \geqslant 1$, a weakly convergent subsequence of the sequence $(b_{kj})_{k\geqslant j}$. The limits $z_j \in \mathcal{R}$ of these subsequences satisfy $\|z_j\| < \alpha\delta_j^{1/2}$ and $L_j = x \otimes (y_j + z_j)$. Thus all conditions in Theorem 3.4 are satisfied. ∎

The following is the version of Theorem 3.4 that will be used in the applications.

Corollary 3.5. *Suppose that T satisfies the property $F^{\mathrm{r}}_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma \leqslant 1$. For each $\varepsilon > 0$, there is a constant $C = C(\varepsilon, \theta, \gamma) > 0$ such that, for each sequence $(L_k)_{k\geqslant 1}$ in Q and each vector $a \in H$, there are elements $x, y_k \in H$ $(k \geqslant 1)$ with $\|x - a\| < \varepsilon$ and*

$$L_k = x \otimes y_k, \quad \|y_k\| \leqslant Ck^2\|L_k\|, \quad k \geqslant 1.$$

Proof. Define $d_k = \|L_k\|$ if $L_k \neq 0$ and $d_k = 1$ otherwise. Set

$$\beta = 3\alpha \sum_{j=1}^{\infty} \frac{1}{j^2}$$

(where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$) and

$$\delta_k = \varepsilon^2/\beta^2 k^4, \quad M_k = \delta_k(L_k/2d_k), \quad k \geqslant 1.$$

By Theorem 3.4 we can choose $x \in H$, $w_k \in \mathcal{S}$, and $b_k \in \mathcal{R}$ with $\|x - a\| < \varepsilon$ and

$$M_k = x \otimes (w_k + b_k), \quad \|w_k\| < \alpha\delta_k^{1/2}, \quad \|b_k\| < \alpha\delta_k^{1/2}$$

for all $k \geqslant 1$. Then $L_k = x \otimes y_k$ with $y_k = (2d_k/\delta_k)\,(w_k + b_k)$. Because

$$\|y_k\| \leqslant \left(\frac{1}{\varepsilon}\right)\beta^2 k^2 d_k, \quad k \geqslant 1,$$

one can choose

$$C(\varepsilon, \theta, \gamma) = C/\big(\varepsilon(\gamma^{1/2} - \theta^{1/2})^2\big)$$

with a suitable universal constant $C > 0$. ∎

4. INVARIANT SUBSPACES

Let $T \in L(H)^n$ be a commuting tuple of contractions on a Hilbert space H such that T possesses a unitary dilation. In this section we show that, if the Harte spectrum $\sigma^{\mathcal{H}}(T)$ of T is dominating in $\mathbb{D}^n$, then T possesses non-trivial joint invariant subspaces. In the one-dimensional case this result specializes to a well-known theorem of Brown, Chevreau, and Pearcy saying that each contraction on a Hilbert space with dominating spectrum in the open unit disc has non-trivial invariant subspaces. For spherical contractions the corresponding result was obtained in [15].

The existence of a unitary dilation implies that T satisfies von Neumann's inequality over the unit polydisc, or equivalently, that T possesses a contractive $A(\mathbb{D}^n)$-functional calculus. Since by a classical result of Sz.-Nagy and Foiaş (Theorem II.5.4 in [26]) each single contraction that is neither of type $C_{0\cdot}$ nor of type $C_{\cdot 0}$ either possesses a non-trivial hyperinvariant subspace or is a scalar multiple of the identity operator, we may suppose that each component T_i $(i = 1, \ldots, n)$ of T is of type $C_{0\cdot}$ or $C_{\cdot 0}$. But then results of Apostol (Theorem 1.7 and Proposition 1.8 in [4]) imply that T is absolutely continuous, that is, T possesses a w*-continuous contractive H^{∞}-functional calculus.

In [16] (Theorem 2.11) it is shown that the dual algebra generated by a commuting tuple of contractions with dominating Harte spectrum and w*-continuous contractive H^∞-functional calculus such that, for some indices $i, j \in \{1, \dots, n\}$, T_i^* and T_j are of type $C_{0\cdot}$, has property $(\mathbb{A}_{\chi_0})$. Thus to prove the existence of joint invariant subspaces for our tuple $T \in L(H)^n$ we may suppose that all components of T are of type $C_{\cdot 0}$. Again a result of Apostol [4] (Proposition 1.8) shows that in this case the H^∞-functional calculus Φ of T is of class $C_{\cdot 0}$. Finally, it is useful to observe that, if there is a point $\lambda \in \sigma^{\mathcal{H}}(T) \setminus \sigma_{\mathrm{e}}^{\mathcal{H}}(T)$, then $\bigcap\limits_{i=1}^{n} \operatorname{Ker}(\lambda_i - T_i)$ or $\sum\limits_{i=1}^{n} (\lambda_i - T_i)H$ is a non-trivial joint invariant subspace for T.

Our invariant-subspace construction is based on the results of the preceding sections and on the following lemma.

Lemma 4.1. *Let $T \in L(H)^n$ be a commuting tuple with a contractive* w*-*continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{D}^n) \to L(H)$. Suppose that Φ is of type $C_{\cdot 0}$ and that $\sigma_{\mathrm{e}}^{\mathcal{H}}(T)$ is dominating in $\mathbb{D}^n$. Then T satisfies the almost factorization property.*

Proof. It suffices to show that, for each functional $L \in Q$ with $\|L\| \leqslant 1$ and each $\varepsilon > 0$, there are vectors $x, y \in H$ with $\max(\|x\|, \|y\|) \leqslant 1$ and $\|L - x \otimes y\| < \varepsilon$. Since $\sigma_{\mathrm{e}}^{\mathcal{H}}(T)$ is dominating in $\mathbb{D}^n$, we only need to consider functionals L of the form

$$L = \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda^j},$$

where $\lambda^1, \dots, \lambda^r \in \sigma_{\mathrm{e}}^{\mathcal{H}}(T)$ and $c_1, \dots, c_r$ are complex numbers with $\sum\limits_{j=1}^{r} |c_j| \leqslant 1$. By changing the order, if necessary, we may of course assume that $\lambda^1, \dots, \lambda^s \in \sigma_{\mathrm{le}}(T)$ and $\lambda^{s+1}, \dots, \lambda^r \in \sigma_{\mathrm{re}}(T)$ for some natural number $s \in \{0, \dots, r\}$.

Let $\varepsilon > 0$ be arbitrary. We choose pairwise orthogonal unit vectors x_k^j ($j = 1, \dots, r$ and $k \in \mathbb{N}$) such that (cf. Lemma 6.5.2 in [17])

$$\max_{1 \leqslant i \leqslant n} \|(\lambda_i^j - T_i) x_k^j\| \xrightarrow{k} 0, \quad j = 1, \dots, s,$$

and

$$\max_{1 \leqslant i \leqslant n} \|(\overline{\lambda}_i^j - T_i^*) x_k^j\| \xrightarrow{k} 0, \quad j = s+1, \dots, r.$$

By the open mapping principle there is a constant $C > 0$ such that, for each function $f \in H^\infty(\mathbb{D}^n)$ with $\|f\| \leqslant 1$ and each $j = 1, \dots, r$, there are functions $f_1, \dots, f_n \in H^\infty(\mathbb{D}^n)$ (depending on j) with $\|f_i\| \leqslant C$ and

$$f(z) - f(\lambda^j) = \sum_{i=1}^{n} (z_i - \lambda_i^j) f_i(z), \quad z \in \mathbb{D}^n.$$

Let $f \in H^\infty(\mathbb{D}^n)$ be a function with $\|f\| \leqslant 1$. For $\lambda \in \{\lambda^1, \ldots, \lambda^r\}$, $f_1, \ldots, f_n$ as above, and each unit vector $x \in H$, we obtain

$$\begin{aligned}|\langle x \otimes x - \mathcal{E}_\lambda, f\rangle| &\leqslant \sum_{i=1}^n |\langle \Phi(f_i)(\lambda_i - T_i)x, x\rangle| \\ &\leqslant C \sum_{i=1}^n \min\big(\|(\lambda_i - T_i)x\|, \|(\overline{\lambda}_i - T_i^*)x\|\big).\end{aligned}$$

Similarly, one obtains for $\lambda, f_1, \ldots, f_n$ as above, and any unit vectors $x, y \in H$,

$$\begin{aligned}|\langle x \otimes y, f\rangle| &\leqslant |\langle x, y\rangle| + \sum_{i=1}^n |\langle \Phi(f_i)(\lambda_i - T_i)x, y\rangle| \\ &\leqslant |\langle x, y\rangle| + C \sum_{i=1}^n \min\big(\|(\lambda_i - T_i)x\|, \|(\overline{\lambda}_i - T_i^*)y\|\big).\end{aligned}$$

After cancelling finitely many terms in each of the sequences $(x_k^j)_{k\in\mathbb{N}}$ $(j = 1, \ldots, r)$, we may therefore suppose that

$$\|\mathcal{E}_{\lambda^j} - x_k^j \otimes x_k^j\| < \frac{\varepsilon}{2}$$

for all $j = 1, \ldots, r$ and $k \in \mathbb{N}$, and that

$$\|x_k^p \otimes x_l^q\| < \frac{\varepsilon}{2r^2}, \quad k, l \in \mathbb{N},$$

for $p, q \in \{1, \ldots, r\}$ with $p \neq q$ and such that $p \leqslant s$ or $q > s$.

Define $x^j = x_0^j$ for $j = 1, \ldots, s$. Since Φ is of type $C_{\cdot 0}$, there is a natural number $k \in \mathbb{N}$ such that with $x^j = x_k^j$ $(j = s+1, \ldots, r)$ the inequalities

$$\|x^p \otimes x^q\| < \frac{\varepsilon}{2r^2}, \quad p = s+1, \ldots, r,\ q = 1, \ldots, s$$

hold. We fix complex numbers $d_1, \ldots, d_r$ with $d_j^2 = c_j$ for all j, and we define

$$x = \sum_{j=1}^r d_j x^j \quad \text{and} \quad y = \sum_{j=1}^r \overline{d}_j x^j.$$

Then $\|x\|, \|y\| \leqslant 1$ and

$$\|L - x \otimes y\| \leqslant \sum_{j=1}^r |c_j|\, \|\mathcal{E}_{\lambda^j} - x^j \otimes x^j\| + \sum_{\substack{p,q=1 \\ p\neq q}}^r \|x^p \otimes x^q\| < \varepsilon. \quad \blacksquare$$

Now we have gathered all pieces that we need to prove our main invariant-subspace result for commuting contractions.

THEOREM 4.2. *Let $T \in L(H)^n$ be a commuting tuple of contractions that possesses a unitary dilation. Suppose that the Harte spectrum $\sigma^{\mathcal{H}}(T)$ of T is dominating in $\mathbb{D}^n$. Then* $\mathrm{Lat}(T)$ *is non-trivial.*

Proof. As explained at the beginning of this section, we are allowed to assume that $\sigma^{\mathcal{H}}(T) = \sigma_{\mathrm{e}}^{\mathcal{H}}(T)$ and that T possesses a contractive H^∞-functional calculus

$$\Phi : H^\infty(\mathbb{D}^n) \to L(H)$$

of type $C_{\cdot 0}$. By Lemma 4.1 the tuple T possesses the almost factorization property. Since, for each $\lambda \in \mathbb{D}^n$, the tuple $T_\lambda = \varphi_\lambda(T)$ (see Section 2 for the notation) possesses the $H^\infty(\mathbb{D}^n)$-functional calculus

$$\Phi_\lambda : H^\infty(\mathbb{D}^n) \to L(H), \quad f \mapsto \Phi(f \circ \varphi_\lambda),$$

which is again of type $C_{\cdot 0}$, it follows from Proposition 2.6 that T satisfies property $E_{0,1}^{\mathrm{r}}$. By Corollary 3.5 the dual algebra generated by T satisfies property $(\mathbb{A}_{1,\chi_0})$. In particular T possesses non-trivial joint invariant subspaces. ∎

By a well-known result of Ando ([3]) each commuting pair of Hilbert-space contractions possesses a unitary dilation. Therefore we obtain the following consequence of Theorem 4.2 in dimension $n = 2$.

COROLLARY 4.3. *Each commuting pair $T = (T_1, T_2) \in L(H)^2$ of contractions with dominating Harte spectrum in the bidisc $\mathbb{D}^2$ possesses non-trivial joint invariant subspaces.*

The proof of Theorem 4.2 yields the following factorization result.

COROLLARY 4.4. *Let $T \in L(H)^n$ be a commuting tuple of contractions with a unitary dilation. Suppose that all components of T are of type $C_{\cdot 0}$ or that all components of T are of type $C_{0\cdot}$. If $\sigma_{\mathrm{e}}^{\mathcal{H}}(T)$ is dominating in $\mathbb{D}^n$, then T satisfies condition $E_{0,1}^{\mathrm{r}}$. In particular, the dual algebra $\mathfrak{A}_T$ has property $(\mathbb{A}_{1,\chi_0})$.*

The condition that the Harte spectrum of T is dominating in $\mathbb{D}^n$ has only been used to ensure that the tuple T satisfies the almost factorization property. Without this hypothesis one obtains the following variant of Theorem 4.2.

THEOREM 4.5. *Let $T \in L(H)^n$ be a commuting tuple of contractions such that T possesses a unitary dilation. Suppose that either T is not absolutely continuous or that T is absolutely continuous and satisfies the ρ-almost factorization property for some real number $\rho > 0$. Then* $\mathrm{Lat}(T)$ *is non-trivial.*

Proof. Because of the theorem of Sz.-Nagy and Foiaş stated at the beginning of this section we are allowed to assume that each component T_i $(i = 1, \ldots, n)$ of T is of type $C_{0\cdot}$ or of type $C_{\cdot 0}$. Then Apostol's results from [4] show that T is absolutely continuous.

Suppose that T satisfies the ρ-almost factorization property for some real number $\rho > 0$. Set $\gamma = 1/\rho^2$. It follows from Lemma 2.7 in [16] that, if T_i^* and T_j are of type $C_{\cdot 0}$ for some indices $i, j \in \{1, \ldots, n\}$, then the H^∞-functional calculus of T satisfies condition $(\Delta_{0,\gamma})$ (see [16] for this notion). Since in this case the dual algebra generated by T has property $(\mathbb{A}_{\chi_0})$ (Proposition 1.6 in [16]), we are allowed to assume that each component of T, and hence also the H^∞-functional calculus of T, is of type $C_{\cdot 0}$. Then Proposition 2.6 shows that T satisfies condition $E_{0,\gamma}^{\mathrm{r}}$. By Corollary 3.5 the dual algebra generated by T has property $(\mathbb{A}_{1,\chi_0})$. Hence in any case the existence of joint invariant subspaces is shown. ∎

Each absolutely continuous contraction with isometric H^∞-functional calculus satisfies the almost factorization property ([6]). It would be interesting to know whether the corresponding result holds true in the multivariable setting. If the answer to this question is positive, then Theorem 4.2 and Corollary 4.3 hold true with the Taylor spectrum instead of the Harte spectrum. Moreover, in this case, each commuting tuple T of contractions with a unitary dilation such that T is absolutely continuous with isometric H^∞-functional calculus over $\mathbb{D}^n$ would possess a non-trivial joint invariant subspace.

The methods of this paper can be used to show that each completely non-unitary subnormal tuple with an isometric $H^\infty(\mathbb{D}^n)$-functional calculus is reflexive. This result will be the subject of a sequel to this paper.

After completing this paper B. Chevreau showed with different methods that the dual algebra generated by an absolutely continuous commuting tuple of contractions with a unitary dilation and dominating essential Harte spectrum possesses the factorization property $(\mathbb{A}_{\chi_0})$.

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