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BERGMAN PROJECTION AND BERGMAN SPACES

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Abstract. In this paper we study mapping properties of the Bergman projection P, i.e. which function spaces or classes are preserved by P. It is shown that the Bergman projection is of weak type $(1,1)$ and bounded on the Orlicz space $L^{\varphi}(\mathbb{D},\mathrm{d}A)$ iff $L^{\varphi}(\mathbb{D},\mathrm{d}A)$ is reflexive. So the dual space of the Bergman space L_a^{φ} is L_a^{ψ} if $L^{\varphi}(\mathbb{D},\mathrm{d}A)$ is reflexive, where φ and ψ are a pair of complementary Young functions. In addition, we also get that the Kolmogorov type inequality and the Zygmund type inequality hold for the Bergman projection.

Keywords: *Bergman projection, Bergman spaces, and Orlicz spaces.*

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1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathrm{d}A(z)$ the normalized area measure on $\mathbb{D}$. For $1 \leqslant p < \infty$, the Bergman space L_{a}^p is the set of analytic functions on $\mathbb{D}$ which are in $L^p(\mathbb{D},\mathrm{d}A)$. It can be easily verified by the mean value formula and the Holder inequality that L_{a}^p is a closed subspace of $L^p(\mathbb{D},\mathrm{d}A)$. This implies, in particular, that there exists an orthogonal projection P from $L^2(\mathbb{D},\mathrm{d}A)$ onto L_{a}^2, which is called the *Bergman projection* and can be represented as an integral operator

$$Pf(z) = \int_{\mathbb{D}} f(w)K(w,z)\,\mathrm{d}A(w) \tag{1.1}$$

for f in $L^2(\mathbb{D},\mathrm{d}A)$ where $K(z,w)$ is the Bergman reproducing kernel.

Bergman spaces have a long history. Its theory goes back to the book [3] in the early fifties by S. Bergman, where the first systematic treatment of L_{a}^2 was given, and since then there have been a lot of papers devoted to this area. One of important problems in the theory of Bergman spaces is to study mapping properties of P, i.e. which function spaces or classes are preserved by P. The boundedness of P on $L^2(\mathbb{D},\mathrm{d}A)$ follows immediately from the definition of P. But the boundedness of P on $L^p(\mathbb{D}, \mathrm{d}A)$ is not obvious at all. To our knowledge, Zakarjuta and Judovic ([22]) are the first to prove that the Bergman projection is bounded on $L^p(\mathbb{D}, \mathrm{d}A)$ for $1 < p < \infty$, using the theory of singular integral operators. Another proof based on the Schur's test, which has become standard in more recent literatures, was given by Forelli and Rudin ([6]) and was applied to the case of the strongly pseudo-convex domains by Phong and Stein ([14]). For a detailed presentation of the L^p-theory of Bergman spaces and further historical references, we refer to Axler′s survey paper ([1]) and Zhu′s book ([23]).

The aim of this paper is to deal with Bergman spaces with general Orlicz norms. The interest in such a study lies not only in the fact that we shall have a more general theory but more importantly in that it leads to certain finer results and shields light on the L^p-situation. To illustrate the main ideas and motivation of this paper, we begin by recalling some facts from the theory of Hardy spaces. We know that, as early as the twenties, one realized that the L^p-spaces themselves are insufficient to describe the mapping properties of operators such as the conjugate function operator, which is the periodic analog of the Hilbert transform. It is in connection with the conjugate function operator that A.N. Kolmogorov ([9]) in 1925 established the so-called weak type estimates and then A. Zgymund ([24]) and E.T. Titchmarsh ([18] and [19]) in 1928 independently introduced the spaces $L\log^+ L$ and $L^{\exp}$ which are actually two concrete Orlicz spaces. It is clear that, in order to reflect these classical results in the theory of Bergman spaces, a more general framework than one provided by the L^p-spaces will be required. In this paper we shall establish the weak type (1,1) estimate, the Kolmogorov type estimate, the Zygmund type estimate and the L^φ-boundedness for the Bergman projection in the general framework.

The main results of this paper are following theorems.

THEOREM 1.1. *The Bergman projection is of weak type* $(1,1)$*, i.e. there exists a constant* C *independent of* f *such that*

$$(Pf)_*(t) \leqslant \frac{C\|f\|_1}{t}$$

for all $f \in L^1(\mathbb{D}, \mathrm{d}A)$ *and* $t > 0$.

THEOREM 1.2. (Kolmogorov inequality) *If* $f \in L^1(\mathbb{D}, \mathrm{d}A)$*, then* $Pf \in L^p(\mathbb{D}, \mathrm{d}A)$ *for all* $0 < p < 1$*. More precisely, there is a constant* C_p *depending only,* p *such that*

$$\|Pf\|_p \leqslant C_p\|f\|_1.$$

THEOREM 1.3. (Zygmund inequality) *The Bergman projection* P *is a bounded operator from the Zygmund space* $L\log^+ L$ *into* L^1_{a}*, i.e. there is a constant* C *such that*

$$\|Pf\|_1 \leqslant C\|f\|_{L\log^+ L}$$

for all f *in* $L\log^+ L$.

THEOREM 1.4. *The Bergman projection is a bounded operator from $L^\infty(\mathbb{D}, \mathrm{d}A)$ to the Zygmund space $L^{\exp}(\mathbb{D}, \mathrm{d}A)$, i.e. there is a constant C such that*

$$\|Pf\|_{\exp} \leqslant C\|f\|_\infty$$

for all f in $L^\infty(\mathbb{D}, \mathrm{d}A)$.

The last two theorems are the counterparts for the Bergman projection of the well-known results in the theory of the Hardy spaces. Since the continuous embeddings

$$L^\infty(\mathbb{D}, \mathrm{d}A) \hookrightarrow L^{\exp}(\mathbb{D}, \mathrm{d}A) \hookrightarrow L^p(\mathbb{D}, \mathrm{d}A) \hookrightarrow L\log^+ L(\mathbb{D}, \mathrm{d}A) \hookrightarrow L^1(\mathbb{D}, \mathrm{d}A)$$

hold for $1 < p < \infty$, they can be also viewed as the substitutes for the $L^1(\mathbb{D}, \mathrm{d}A)$ and $L^\infty(\mathbb{D}, \mathrm{d}A)$ boundedness of the Bergman projection respectively.

THEOREM 1.5. *Suppose that φ and ψ are a pair of complementary Young functions. Then the Bergman projection is bounded iff both φ and ψ satisfy the Δ_2-condition iff the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is reflexive.*

Theorem 1.5 contains, in particular, the L^p-boundedness of the Bergman projection as its consequence. Through such a general result, one will be able to gain a better insight into the reason why P is bounded on $L^p(\mathbb{D}, \mathrm{d}A)$ for $1 < p < \infty$, but unbounded for $p = 1$, or $p = \infty$.

THEOREM 1.6. *For a reflexive Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$, the dual of the Bergman space L^φ_{a} can be identified with L^ψ_{a} where ψ is the complementary Young function of φ. More precisely, for every bounded linear functional l on L^φ_{a}, there exists some unique $g \in L^\psi_{\mathrm{a}}$ such that*

$$l(f) = \int_{\mathbb{D}} f\overline{g}\, \mathrm{d}A(z).$$

Furthermore, the norm $\|l\|$ of l is equivalent to $\|g\|_\psi$.

If the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is not reflexive, the duality problem is much more complicated. However, as with L^1_{a}, it is possible to give a satisfactory description of the dual and predual of L^φ_{a} by Besov spaces. Since this is lenghty, we shall publish those results in the forthcoming paper.

This paper is organized as follows. Section 2 contains preliminaries on Orlicz spaces and Bergman spaces. In Section 3 we show that the Bergman projection is of weak type (1, 1). In Section 4 we prove a Marcinkiewicz type interpolation theorem for Orlicz spaces. Using the interpolation theorem we prove "if part" of Theorem 1.5 in Section 6. Theorem 1.6 is also proved in the section. The "only if" part of Theorem 1.5 is proved in Section 5. In the last section we present the proofs of the Kolmogorov inequality and the Zygmund inequalities.

2. ORLICZ SPACES AND BERGMAN SPACES

The Orlicz spaces were first considered by Birnbaum and Orlicz ([4]) and Orlicz ([12] and [13]). In this section we present certain definitions, notation and related facts, which will be used later on. Most of them may be found in [10], [21] and [2].

DEFINITION 2.1. Let $p : [0,\infty] \to [0,\infty]$ be a non-decreasing and left-continuous function with $p(0) = 0$. Suppose p is non-trivial, i.e., it is neither identically zero nor identically infinite on $(0,\infty)$, Then the function, defined by

$$\varphi(t) = \int_0^1 p(s)\,\mathrm{d}s$$

for $t \geqslant 0$, is said to be a *Young function*.

Let $q(s) = \inf\{t : p(t) \geqslant s\}$ for $0 \leqslant s \leqslant \infty$, which is called the *left-continuous inverse of* p and has the same properties as p. Thus to the Young function φ we can associate its *complementary Young function* given by

$$\psi(t) = \int_0^t q(s)\,\mathrm{d}s$$

for $t \geqslant 0$.

The introduction of Young functions has been inspired by the obvious role played by the functions t^p in the definition of L^p spaces. We note that the complementary Young function of ψ is φ. It is also obvious that a Young function must be convex on the interval where it is finite. Furthermore, if φ and ψ are a pair of complementary Young functions, then one has the Young inequality

$$st \leqslant \varphi(t) + \psi(s)$$

for $s, t \geqslant 0$ with equality if and only if either $t = q(s)$ or $s = p(t)$ holds.

DEFINITION 2.2. A Young function φ is said to satisfy the Δ_2-*condition* (shortly, $\varphi \in \Delta_2$) if there exist constants $t_0 > 0$ and $c \geqslant 1$ such that

$$\varphi(2t) < c\varphi(t)$$

for $t \geqslant t_0$.

A Young function is said to satisfy the Δ_2^*-*condition* if its complementary Young function satisfies the Δ_2-condition.

We point out that the Δ_2-condition assures that the Young function $\varphi(t)$ does not increase too fast as t increases, more precisely, that $\varphi = \mathrm{O}(t^a)$ for some $1 \leqslant a < \infty$. But the converse is not true. In fact, as shown in [10], the Young function

$$\psi(t) = \int_0^t q(s)\,\mathrm{d}s$$

where

$$q(s) = \begin{cases} s & \text{if } s \in [0,1), \\ (k-1)! & \text{if } s \in [(k-1)!, k!),\ k = 2, 3, \ldots \end{cases}$$

increases no more rapidly than $t^2/2$, but does not satisfy the Δ_2-condition.

If φ is a Young function, following Lindberg ([11]), we can associate to it two numbers $\overline{\alpha}_\varphi$ and $\underline{\alpha}_\varphi$ given by

$$\overline{\alpha}_\varphi = \limsup_{t\to\infty} \frac{tp(t)}{\varphi(t)}, \qquad \underline{\alpha}_\varphi = \liminf_{t\to\infty} \frac{tq(t)}{\psi(t)}$$

which are called the *upper index* and the *lower index* of φ, respectively.

With these two indices, we can rephase Theorem 4.1 and Lemma 4.1 in [10] in the following form:

Lemma 2.3. *Let φ and ψ be a pair of complementary Young functions. Then φ satisfies the Δ_2-condition iff $\overline{\alpha}_\varphi < \infty$ iff $\underline{\alpha}_\psi > 1$.*

Definition 2.4. Let φ be a Young function. We define the *Orlicz class* $\mathcal{L}^\varphi(\mathbb{D}, \mathrm{d}A)$ as the set of all complex measurable functions f on $\mathbb{D}$ for which the integral

$$\rho(f,\varphi) = \int\limits_{\mathbb{D}} \varphi(|f(z)|)\,\mathrm{d}A(z)$$

is finite, and the *Orlicz space* $L^\varphi(\mathbb{D}, \mathrm{d}A)$ as the linear hull of $\mathcal{L}^\varphi(\mathbb{D}, \mathrm{d}A)$ with the Luxemberg norm

$$\|f\|_{(\varphi)} = \inf\{c > 0 : \rho(f/c, \varphi) \leqslant 1\}.$$

In the sequel we shall also use another norm on $L^\varphi(\mathbb{D}, \mathrm{d}A)$ known as the *Orlicz norm*, given by

$$\|f\|_\varphi = \sup_{\rho(g,\psi)\leqslant 1} \Big| \int\limits_{\mathbb{D}} f(z)\overline{g}(z)\,\mathrm{d}A(z)\Big|.$$

As it is well-known, the Luxemberg and Orlicz norms are equivalent, more precisely

$$\|f\|_{(\varphi)} \leqslant \|f\|_\varphi \leqslant 2\|f\|_{(\varphi)}.$$

Let φ and ψ be a pair of complementary Young functions. We shall call the space $L^\psi(\mathbb{D}, \mathrm{d}A)$ the *associate space* of $L^\varphi(\mathbb{D}, \mathrm{d}A)$. In general, the dual space of the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is not identical with its associate space $L^\psi(\mathbb{D}, \mathrm{d}A)$, but we have the following lemma.

Lemma 2.5. *Let φ and ψ be a pair of complementary Young functions. Suppose that φ satisfies the Δ_2-condition. Then the dual space of $L^\varphi(\mathbb{D}, \mathrm{d}A)$ can be "identified" with $L^\psi(\mathbb{D}, \mathrm{d}A)$. More precisely, a linear functional l on $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is uniquely represented by a function g in $L^\psi(\mathbb{D}, \mathrm{d}A)$ in the following way:*

$$l(f) = \int\limits_{\mathbb{D}} f(z)\overline{g(z)}\,\mathrm{d}A(z)$$

for all f in $L^\varphi(\mathbb{D}, \mathrm{d}A)$ and

$$\|l\| \leqslant \|g\|_\psi \leqslant C\|l\|$$

for some constant C which is independent of l.

The following well-known fact is important for our further discussion, which gives an elegant characterization of reflexivity of an Orlicz space in terms of the upper and lower indices of its Young function.

LEMMA 2.6. *Let φ be a Young function. Then the following are equivalent:*
(i) $L^\varphi(\mathbb{D}, \mathrm{d}A)$ *is reflexive;*
(ii) φ *satisfies both the* Δ_2- *and* Δ_2^*-*conditions;*
(iii) $1 < \underline{\alpha}_\varphi < \overline{\alpha}_\varphi < \infty$.

We say that two Young functions φ_1 and φ_2 are equivalent if there are constants k_1, k_2 and t_0 such that

$$\varphi_1(k_1 t) \leqslant \varphi_2(t) \leqslant \varphi_1(k_2 t)$$

for $t \geqslant t_0$.

Now let us look at several examples of Orlicz spaces. (i) The Lebesgue spaces L^p are the most commonly used Orlicz spaces. If $1 < p < \infty$, then we have $p(t) = pt^{p-1}$, $\varphi(t) = t^p$ and $\psi(t) = t^q$ with $1/p + 1/q = 1$. It is easily verified that the Young functions φ and ψ satisfy the Δ_2-condition if $1 < p < \infty$, hence in this case L^p is reflexive. If $p = 1$, then $p(t) = 1$ for $t > 0$ with $p(0) = 0$, $\varphi(t) = t$ and

$$\psi(t) = \begin{cases} 0 & 0 \leqslant t \leqslant 1, \\ \infty & 1 < t. \end{cases}$$

So $\varphi(t)$ in this case satisfies Δ_2-condition but $\psi(t)$ does not.

(ii) Another interesting examples of Orlicz spaces are the Zygmund spaces $L^p \log L$ with Young functions $\varphi = t^p \log^+ t$. It is easy to check that both φ and its complementary Young function ψ satisfy the Δ_2-condition when $p > 1$.

(iii) Let

$$p(t) = \begin{cases} 0 & 0 \leqslant t \leqslant 1, \\ \mathrm{e}^{t-1} & 1 < t. \end{cases}$$

Then we have the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ with the Young function

$$\varphi(t) = \begin{cases} t & 0 \leqslant t \leqslant 1, \\ \mathrm{e}^{t-1} & 1 < t. \end{cases}$$

A simple computation shows that the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is the associate space of the Zygmund space $L \log^+ L$ with the Young function $t \log^+ t$. Moreover, the Young function $\varphi(t)$ is equivalent to the Young function e^t. So the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is the Zygmund space $L^{\exp}(\mathbb{D}, \mathrm{d}A)$.

DEFINITION 2.7. Let φ be a Young function. The *Bergman space* L_{a}^φ is the subset of the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ consisting of functions analytic on $\mathbb{D}$.

As with L_{a}^p, it is easy to check that L_{a}^φ is a Banach space with the Orlicz norm $\|f\|_\varphi$. Since the evaluation at any fixed point z in $\mathbb{D}$ is a bounded functional on L_{a}^2, there is a function $K(z, w)$ in L_{a}^2 such that

$$f(z) = \langle f, K(z, \cdot) \rangle$$

for all f in $\mathrm{L}_{\mathrm{a}}^2$. $K(z, w)$ is called the *Bergman reproducing kernel.* In fact, for any orthogonal basis $\{e_n(w)\}$, $K(z, w)$ can be represented as

$$K(z, w) = \sum_1^\infty \overline{e_n(z)} e_n(w)$$

where the sum converges pointwise to $K(z, w)$. In the case of the unit disk

$$K(z, w) = \frac{1}{(1 - \overline{z} w)^2}. \tag{2.1}$$

$K(z,w)$ plays a very important role in the theory of the Bergman spaces, differential geometry and differential equations, since we can use it to define the Bergman metric on $\mathbb{D}$ as follows. For z in $\mathbb{D}$ and u,v in $\mathbb{C}$, the Bergman metric $H_z(u,v)$ is defined by

$$H_z(u,v) = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \log K(z,z)u\overline{v}.$$

Then $\mathbb{D}$ is a complete Hermitian symmetric space of noncompact type with the Bergman metric which gives the usual topology on $\mathbb{D}$. By definition, the Bergman distance $\beta(z,w)$ is given by

$$\beta(z,w) = \inf_{\gamma} \int_0^1 \sqrt{H_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,\mathrm{d}t$$

where the inf is taken over all geodesics in $\mathbb{D}$ which connect z and w. It is clear that the argument above can be carried over any domains in $\mathbb{C}^n$.

On the other hand, it is deep and well-known in Lie theory that any symmetric domains can be realized as a unit ball of $\mathbb{C}^n$ with some norm for some n. So it is natural to ask whether the theory of Bergman spaces and the Bergman projection on the unit disk still holds on any bounded symmetric domains with rank greater than one. However, as we know, the L^p-boundedness of the Bergman projection on the bounded symmetric domains is still an open question for any $1 < p < \infty$ except for $p = 2$.

Throughout this paper, C will stand for different constants from place to place for our convenience.

3. THE WEAK TYPE (1,1)

As mentioned before, the Bergman projection is unbounded on $L^1(\mathbb{D}, \mathrm{d}A)$ and $L^\infty(\mathbb{D}, \mathrm{d}A)$. To remedy this situation, one is led to look for certain substitute results. The substitute result for $L^\infty(\mathbb{D}, \mathrm{d}A)$ is well-known. It asserts that the Bergman projection is a bounded operator from $L^\infty(\mathbb{D}, \mathrm{d}A)$ onto the Bloch space. In this section, as a useful substitute for $L^1(\mathbb{D}, \mathrm{d}A)$, we shall show that the Bergman projection is of weak type $(1,1)$. Although its counterpart for the Hilbert transform have been known in the theory of the Hardy spaces for a long time, to our great surprise, we could not find this result in the literature. We present it on the one hand for it is of interest in its own right, and on the other hand also because it is indispensable to our further consideration.

First, we recall that an operator T defined in $L^p(\mathbb{D}, \mathrm{d}A)$ with values in the class of measurable functions of the measure space $(\mathbb{D}, \mathrm{d}A)$ is said to be of weak type (p,q) if there is a constant $A_{p,q}$ depending only on p and q such that

$$(Tf)_*(t) \leqslant \Big(A_{p,q}\frac{\|f\|_p}{t}\Big)^q$$

for $t > 0$ and all $f \in L^p(\mathbb{D}, \mathrm{d}A)$, where $(Tf)_*(t)$ stands for the normalized Lebesgue measure of the set

$$\{z \in \mathbb{D} : |(Tf)(z)| > t\}$$

and called the distribution function of Tf.

Proof of Theorem 1.1. It is obvious that the inequality holds for $t \leqslant \|f\|_1$, since then, by the definition of the distribution function,

$$(Pf)_*(t) = |\{z \in \mathbb{D} : |Pf(z)| > t\}| \leqslant |\mathbb{D}| = 1 \leqslant \frac{\|f\|_1}{t}.$$

Now suppose that $t > \|f\|_1$. To verify the desired inequality, let n be a fixed integer such that $\pi/2n < 1/2$ and divide $\mathbb{D}$ in a mesh consisting of the disk $\mathbb{D}_0 = \{z \in \mathbb{D} : |z| \leqslant 1 - 2\pi/n\}$ and n curvilinear squares

$$\mathbb{D}_j = \Big\{z = r\mathrm{e}^{\mathrm{i}\theta} : 1 - \frac{2\pi}{n} < r < 1,\ \frac{2(j-1)\pi}{n} < \theta < \frac{2j\pi}{n}\Big\}$$

for $j = 1, 2, \ldots, n$. Obviously, f can be written as $f = \sum\limits_{j=0}^{n} f_j$, where f_j is supported in $\mathbb{D}_j$. Since P is linear and

$$(Pf)_*(t) \leqslant \sum_{j=0}^{n} (Pf_j)_*\Big(\frac{t}{n+1}\Big),$$

it suffices to prove $(Pf_j)_*(t) \leqslant C\frac{\|f_j\|_1}{t}$ for all j.

Since the Bergman kernel $K(z, w)$ is bounded uniformly on $\mathbb{D} \times \mathbb{D}_0$, it is easy to see that there is a constant C independent of f such that $(Pf_0)_*(t) \leqslant C\frac{\|f\|_1}{t}$ for all f in $L^1(\mathbb{D}, \mathrm{d}A)$ and $t > 0$.

We next turn to the proof of inequality for $(Pf_j)_*(t)$ with $j \neq 0$. The proof rests on the so-called Calderon-Zygmund decomposition. We sketch it as follows.

FIRST STEP. We use the stopping time argument to decompose each curvlinear squares $\mathbb{D}_j$ as $\mathbb{D}_j = F_j \cup \Omega_j$ with $F_j \cap \Omega_j = 0$ so that

$$\begin{aligned} &\text{(i) } f_j(z) < t \qquad \text{a.e. } z \in F;\\ &\text{(ii) } \Omega_j = \bigcup Q_k^j;\\ &\text{(iii) } t < |Q_k^j|^{-1} \int\limits_{Q_k^j} |f_j(z)|\,\mathrm{d}A(z) < Ct; \end{aligned} \tag{3.1}$$

for k and j where Q_k^j denote the curvilinear squares with the same form as $\mathbb{D}_j$ and having pairwise disjoint interiors. The argument is the same as the standard one, except that in the proof of (iii) we need to use our assumption $t > \|f\|_1$, which has $|\mathbb{D}_j|^{-1} \int\limits_{\mathbb{D}_j} |f(w)|\,\mathrm{d}A(w) < Ct$ as a consequence, and the observation that if $Q_{k,m+1}^j$ is a dyadic subsquare obtained by bisecting each sides of the square $Q_{k,m}^j$ in the m^{th} step of the subdivision process, then there exists a constant c independent of m such that $\frac{1}{c}|Q_{k,m}^j| < |Q_{k,m+1}^j| < |Q_{k,m}^j|$. For more details concerning the stopping time argument, see [16] and [7].

Using the above decomposition of $\mathbb{D}_j$, we define the Calderon-Zygmund decomposition of f_j corresponding to a given $t > \|f\|_1$, as follows

$$g_j(z) = \begin{cases} f_j(z) & \text{if } z \in \mathbb{D}/\Omega_j,\\ \frac{1}{|Q_k^j|} \int\limits_{Q_k^j} f(w)\,\mathrm{d}A(w) & \text{if } z \in Q_k^j, \end{cases} \tag{3.2}$$

and

$$b_j(z) = f_j(z) - g_j(z). \tag{3.3}$$

Then g_j is in $L^2(\mathbb{D}, \mathrm{d}A)$, because the above construction gives

$$\begin{aligned}\|g_j\|_2^2 &= \int\limits_{F_j} |g_j(z)|^2 \,\mathrm{d}A(z) + \int\limits_{\Omega_j} |g_j(z)|^2 \,\mathrm{d}A(z)\\ &\leqslant \int\limits_{F_j} t|f_j(z)| \,\mathrm{d}A(z) + \int\limits_{\Omega_j} \Big|\sum_{k=1}^{\infty} |Q_k^j|^{-1} \int\limits_{Q_k^j} f_j(w)\,\mathrm{d}A(w)\Big|^2 \,\mathrm{d}A(z)\\ &\leqslant t\|f_1\|_1 + c^2t^2|\Omega_j| \leqslant t\|f_1\|_1 + c^2t^2 \sum_{k=1}^{\infty} \frac{1}{t}\Big|\int\limits_{Q_k^j} f_j(w)\,\mathrm{d}A(w)\Big| \leqslant (c^2+1))t\|f\|_1.\end{aligned}$$

In addition, P is bounded on $L^2(\mathbb{D}, \mathrm{d}A)$ by the definition. So we have

$$(Pg_j)_*(t) \leqslant \frac{1}{t^2}\frac{1}{t^2}\int\limits_{\mathbb{D}} |(Pg_j)(z)|^2 \,\mathrm{d}A(z) \leqslant \frac{C}{t^2}\int\limits_{\mathbb{D}} |g_j(z)|^2 \,\mathrm{d}A(z) \leqslant C\frac{\|f_j\|_1}{t}.$$

Second step. We now come to the estimate for $(Pb_j)_*(t)$. Let $b_{k,j}(z) = b(z)\chi_{Q_k^j}(z)$ where $\chi_{Q_k^j}(z)$ denotes the characteristic function of the set Q_k^j; then

$$b_j(z) = \sum_{k=1}^{\infty} b_{k,j}(z) \quad \text{and} \quad Pb_j(z) = \sum_{k=1}^{\infty} Pb_{k,j}(z).$$

For each k, let $\mathbb{S}_k$ be the circumscribed disk of Q_k^j, $\mathbb{S}_k^1$ the disk with the same center w_k as $\mathbb{S}_k$ and radius two times as that of $\mathbb{S}_k$. Let $\Omega' = \bigcup \mathbb{S}_k^1$. It is clear that there is a fixed constant C such that

$$|\Omega'| \leqslant C|\Omega| \leqslant C\frac{\|f\|_1}{t}. \tag{3.4}$$

Since

$$\int\limits_{Q_k^j} b_j(w)\,\mathrm{d}A(w) = 0,$$

we have

$$\begin{aligned}|(Pb_{k,j})(z)| &= \Big|\int\limits_{Q_k^j} (K(z,w) - K(z,w_k))b_{k,j}(w)\,\mathrm{d}A(w)\Big|\\ &\leqslant \int\limits_{Q_k^j} |K(z,w) - K(z,w_k)|\,|b_{k,j}(w)|\,\mathrm{d}A(w).\end{aligned}$$

Therefore

$$\int_{\mathbb{D}\setminus\Omega'} |Pb_j(z)|\,\mathrm{d}A(z) \leqslant \sum_{k=1}^{\infty} \int_{\mathbb{D}\setminus\mathbb{S}_k^1} |Pb_{k,j}(z)|\,\mathrm{d}A(z)$$

$$\leqslant \sum_{k=1}^{\infty} \int_{w\in Q_k^j} |b_{k,j}(w)| \int_{z\in\mathbb{D}\setminus\mathbb{S}_k^1} |K(z,w)-K(z,w_k)|\,\mathrm{d}A(z)\,\mathrm{d}A(w).$$

It follows from (2.1) and a simple computation that for $w \in Q_k^j$

$$\int_{z\in\mathbb{D}\setminus\mathbb{S}_k^1} |K(z,w)-K(z,w_k)|\,\mathrm{d}A(z) \leqslant C \int_{\substack{|z-w_k|\geqslant 2|w-w_k| \\ |z|<1}} \frac{|w-w_k|}{|z-w_k|^3}\,\mathrm{d}A(z)$$

$$\leqslant C \int_{|z|>2} \frac{1}{|z|^3}\,\mathrm{d}A(z) = C < \infty.$$

So we have

$$\int_{\mathbb{D}\setminus\Omega'} |Pb_j(z)|\,\mathrm{d}A(z) \leqslant \sum_{k=1}^{\infty} \int_{Q_k^j} A|b_{k,j}(z)|\,\mathrm{d}A(w).$$

On the other hand, by (3.1), (3.2) and (3.3) we have $|b_j(z)| \leqslant |f_j(z)| + |g_j(z)| \leqslant |f(z)| + Ct$. Thus

$$\int_{\mathbb{D}\setminus\Omega'} |Pb_j(z)|\,\mathrm{d}A(z) \leqslant A(\|f\|_1 + Ct|\Omega|) \leqslant A(C+1)\|f\|_1. \tag{3.5}$$

Moreover

$$(Pb)_*(t) \leqslant \left|(\mathbb{D}\setminus\Omega')\bigcap\{|Pb_j(z)|>t\}\right| + |\Omega'|. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we obtain that $(Pb)_*(t) \leqslant C\frac{\|f\|_1}{t}$.

FINAL STEP. Combining the estimates of $(Pg_j)_*(t)$ and $(Pb_j)_*(t)$ we have

$$(Pf_j)_*(t) < (Pg_j)_*\Big(\frac{t}{2}\Big) + (Pb_j)_*\Big(\frac{t}{2}\Big) \leqslant C\frac{\|f\|_1}{t}$$

for $j = 1, 2, \ldots, n$, thus completing the proof of the theorem. ∎

4. INTERPOLATION ON ORLICZ SPACES

In this section, following [5] and [17], we shall show a Marcinkiewicz type interpolation theorem for Orlicz spaces, which is good enough for our purpose and looks more natural and convenient than some interpolation theorems for Orlicz spaces given in [8], [15] and [20].

Definition 4.1. Let φ be a Young function and L^φ the Orlicz space. A linear operator T defined on L^φ is said to be of *mean strong type* (φ, φ) if

$$\int_{\mathbb{D}} \varphi(|Tf|)\,\mathrm{d}A(z) \leqslant C \int_{\mathbb{D}} \varphi(|f|)\,\mathrm{d}A(z) \tag{4.1}$$

for $f \in \mathcal{L}^\varphi(\mathbb{D}, \mathrm{d}A)$, and T is said to be *mean weak type* (φ, φ) if

$$|\{z : |Tf(z)| > t\}| \leqslant \frac{C \int_{\mathbb{D}} \varphi(|f|)\,\mathrm{d}A(z)}{\varphi(t)} \tag{4.2}$$

for $f \in \mathcal{L}^\varphi(\mathbb{D}, \mathrm{d}A)$ and $t > 0$, where C is independent of f.

It is easy to see that an operator T is of mean weak type (φ, φ) if it is of mean strong type (φ, φ). Moreover, note also that the mean strong type (t^p, t^p) and the usual strong type (p, p) coincide.

Lemma 4.2. *Let φ be a Young function. If T is of mean strong type (φ, φ), then T is a bounded operator on $L^\varphi(\mathbb{D}, \mathrm{d}A)$.*

The proof of the lemma is routine and so it is omitted.

Theorem 4.3. *Let φ_0, φ_1, and φ_2 be Young functions. Suppose that their upper and lower indices satisfy the following condition*

$$1 \leqslant \underline{\alpha}_{\varphi_0} \leqslant \overline{\alpha}_{\varphi_0} < \underline{\alpha}_{\varphi_2} \leqslant \overline{\alpha}_{\varphi_2} < \underline{\alpha}_{\varphi_1} \leqslant \overline{\alpha}_{\varphi_1} < \infty. \tag{4.3}$$

If T is of mean weak types (φ_0, φ_0) and (φ_1, φ_1), then it is of mean strong type (φ_2, φ_2). In particular, T is a bounded operator on the Orlicz space $L^{\varphi_2}(\mathbb{D}, \mathrm{d}A)$.

Proof. By Lemma 2.3 we see that the condition (4.3) implies that φ_i satisfy the Δ_2-condition. Since the measure of $\mathbb{D}$ is 1, we can find a Young function $\varphi'(t)$ equivalent to $\varphi(t)$ such that $\underline{\alpha}_{\varphi'} \leqslant \frac{p'(t)t}{\varphi'(t)} \leqslant \overline{\alpha}_{\varphi'}$ for $t > 0$ if $\varphi(t)$ satisfies Δ_2-condition. So, without loss of generality, we may assume that

$$\underline{\alpha}_{\varphi_i} \leqslant \frac{p_i(t)t}{\varphi_i(t)} \leqslant \overline{\alpha}_{\varphi_i} \tag{4.4}$$

for $t > 0$ and $i = 0, 1, 2$. Now we want to prove

$$\int_{\mathbb{D}} \varphi_2(|Tf|)\,\mathrm{d}A \leqslant C \int_{\mathbb{D}} \varphi_2(|f|)\,\mathrm{d}A \tag{4.5}$$

for $f \in \mathcal{L}^{\varphi_2}(\mathbb{D}, \mathrm{d}A) = L^{\varphi_2}(\mathbb{D}, \mathrm{d}A)$. To this end, for any $f \in L^{\varphi_2}(\mathbb{D}, \mathrm{d}A)$ and given $t > 0$, we break up f as $f_1 + f_2$ in the following way

$$f_1(z) = \begin{cases} f(z) & \text{if } |f(z)| < t, \\ t\frac{|f(z)|}{f(z)} & \text{otherwise;} \end{cases} \tag{4.6}$$

and $f_0(z) = f(z) - f_1(z)$. Since

$$|Tf(z)| \leqslant |Tf_0(z)| + |Tf_1(z)|, \tag{4.7}$$

we have

$$\{z \in \mathbb{D} : |Tf(z)| > s\} \subset \left\{z \in \mathbb{D} : |Tf_0(z)| > \frac{s}{2}\right\} \cup \left\{z \in \mathbb{D} : |Tf_1(z)| > \frac{s}{2}\right\}$$

or

$$(Tf)_*(s) \leqslant (Tf_0)_*\Big(\frac{s}{2}\Big) + (Tf_1)_*\Big(\frac{s}{2}\Big). \tag{4.8}$$

It is readily seen that

$$(Tf_1)_*(s) = \begin{cases} (Tf)_*(s) & \text{if } s \leqslant t, \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad (Tf_0)_*(s) = (Tf)_*(s+t).$$

Clearly, $f_1 \in L^{\varphi_1}(\mathbb{D}, \mathrm{d}A)$. On the other hand, by the condition (4.3) and (4.4), we have

$$\frac{sp_0(s)}{\varphi_0(s)} \leqslant \overline{\alpha}_{\varphi 0} \leqslant \underline{\alpha}_{\varphi_2} < \frac{sp_2(s)}{\varphi_2(s)} < \overline{\alpha}_{\varphi_2} < \underline{\alpha}_{\varphi_1} \leqslant \frac{sp_1(s)}{\varphi_1(s)}, \tag{4.9}$$

so there are constants C_1 and C_2 such that

$$C_1\varphi_0(s) \leqslant \varphi_2(s) \leqslant C_2\varphi_1(s). \tag{4.10}$$

From (4.9) and the fact that f is in $L^{\varphi_2}(\mathbb{D}, \mathrm{d}A)$ it follows at once that f_0 is in $L^{\varphi_0}(\mathbb{D}, \mathrm{d}A)$. Now, by the assumption of the theorem, T is of mean weak types (φ_0, φ_0) and (φ_1, φ_1), so there are constants B_0 and B_1 such that

$$(Tf_i)_*(s) \leqslant \frac{B_i \int_{\mathbb{D}} \varphi_i(|f_i|)\,\mathrm{d}A(z)}{\varphi_i(s)}$$

for $i = 0,\ 1$. This together with (4.8) gives

$$(Tf)_*(2t) \leqslant \sum_{i=0}^{1} \frac{B_i \int_{\mathbb{D}} \varphi_i(|f_i|)\,\mathrm{d}A(z)}{\varphi_i(s)}. \tag{4.11}$$

Since for a Young function φ and a measurable function g on $\mathbb{D}$ the relation

$$\rho(g, \varphi) = \int_{\mathbb{D}} \varphi(|g|)\,\mathrm{d}A(z) = \int_0^\infty g_*(s)p(s)\,\mathrm{d}s$$

holds, we can rewrite (4.11) in the following form

$$(Tf)_*(2t) \leqslant \frac{B_0 \int_0^\infty p_0(u)f_*(u+t)\,\mathrm{d}u}{\varphi_0} + \frac{B_1 \int_0^t p_1(u)f_*(u)\,\mathrm{d}u}{\varphi_1(t)}. \tag{4.12}$$

Multiplying both sides of above inequality by $p(t)$ and integrating with respect to t over $(0,\infty)$, one gets

$$\int_0^\infty p(t)(Tf)_*(2t)\,\mathrm{d}t \leqslant B_0\int_0^\infty \frac{p(t)}{\varphi_0(t)}\Big(\int_0^\infty p_0(u)f_*(u+t)\,\mathrm{d}u\Big)\,\mathrm{d}t + B_1\int_0^\infty \frac{p(t)}{\varphi_1(t)}\Big(\int_0^\infty p_1(u)f_*(u)\,\mathrm{d}u\Big)\,\mathrm{d}t = B_0I_0 + B_1I_1.$$

Let us first estimate the integral I_1. It can be easily seen from (4.9) that $\lim\limits_{t\to 0}\frac{\varphi(t)}{\varphi_1(t)} = 0$. Changing the order of integration in I_1 implies

$$I_1 = -\int_0^\infty f_*(u)\frac{p_1(u)\varphi_2(u)}{\varphi_1(u)}\,\mathrm{d}u + \int_0^\infty f_*(u)p_1(u)\Big(\int_u^\infty \frac{\varphi(t)p_1(t)}{\varphi_1^2(t)}\,\mathrm{d}t\Big)\,\mathrm{d}u.$$

By (4.9)

$$\frac{\varphi_2(t)p_1(t)}{\varphi_1^2(t)} \geqslant \underline{\alpha}_{\varphi_1}\frac{\varphi_2(t)}{\varphi_1(t)t} \geqslant \frac{\underline{\alpha}_{\varphi 1}p_2(t)}{\overline{\alpha}_{\varphi_2}\varphi_1(t)}.$$

Therefore we obtain

$$\begin{aligned} I_1 &\geqslant -\int_0^\infty f_*(u)\frac{p_1(u)\varphi_2(u)}{\varphi_1(u)}\,\mathrm{d}u + \frac{\underline{\alpha}_{\varphi_1}}{\overline{\alpha}_{\varphi_2}}\int_0^\infty f_*(u)p_1(u)\int_u^\infty \frac{p(t)}{\varphi_1(t)}\,\mathrm{d}t\,\mathrm{d}u \\ &= -\int_0^\infty f_*(u)\frac{p_1(u)\varphi_2(u)}{\varphi_1(u)}\,\mathrm{d}u + \frac{\underline{\alpha}_{\varphi 1}}{\overline{\alpha}_{\varphi_2}}I_1 \end{aligned}$$

or

$$\begin{aligned} I_1 &\leqslant \frac{\overline{\alpha}_{\varphi_2}}{\underline{\alpha}_{\varphi_1}-\overline{\alpha}_{\varphi_2}}\int_0^\infty f_*(u)\frac{p_1(u)\varphi_2(u)}{\varphi_1(u)}\,\mathrm{d}u \leqslant \frac{\overline{\alpha}_{\varphi_2}\overline{\alpha}_{\varphi_1}}{\underline{\alpha}_{\varphi_1}(\underline{\alpha}_{\varphi_1}-\overline{\alpha}_{\varphi_2})}\int_0^\infty f_*(u)p_2(u)\,\mathrm{d}u \\ &\leqslant \frac{\overline{\alpha}_{\varphi_2}\overline{\alpha}_{\varphi_1}}{\underline{\alpha}_{\varphi_1}(\underline{\alpha}_{\varphi_1}-\overline{\alpha}_{\varphi_2})}\int \varphi(|f|)\,\mathrm{d}A(z). \end{aligned}$$

We next come to estimate the integral I_0. Making the change of variables $u = s - t$ and then changing the order of integration, we get

$$\begin{aligned} I_0 &= \int_0^\infty \frac{p_2(t)}{\varphi_0(t)}\int_t^\infty f_*(s)p_0(s-t)\,\mathrm{d}s = \int_0^\infty f_*(s)\int_0^s \frac{p_2(t)}{\varphi_0(t)}p_0(s-t)\,\mathrm{d}t\,\mathrm{d}s \\ &\leqslant \int_0^\infty f_*(s)p_0(s)\int_0^s \frac{p_2(t)}{\varphi_0(t)}\,\mathrm{d}t. \end{aligned}$$

Write I_2 for the integral on the right hand side of the above inequality. Now since $\lim\limits_{t\to 0}\frac{\varphi(t)}{\varphi_0(t)}=0$, we obtain

$$I_2=\int_0^\infty f_*(s)\frac{p_0(s)\varphi_2(s)}{\varphi_0(s)}\,\mathrm{d}s+\int_0^\infty f_*(s)p_0(s)\int_0^s\frac{\varphi(t)p_0(t)}{\varphi_0^2(t)}\,\mathrm{d}t\,\mathrm{d}s.$$

Hence

$$\begin{aligned}I_2&\leqslant\int_0^\infty f_*(s)\frac{p_0(s)\varphi_2(s)}{\varphi_0(s)}\,\mathrm{d}s+\frac{\overline{\alpha}_{\varphi_0}}{\underline{\alpha}_{\varphi_2}}\int_0^\infty f_*(s)p_0(s)\int_0^s\frac{p_2(t)}{\varphi_0(t)}\,\mathrm{d}t\,\mathrm{d}s\\&=\frac{\overline{\alpha}_{\varphi_0}}{\underline{\alpha}_{\varphi_2}}\Big(\int\varphi(|f|)\,\mathrm{d}A(z)+I_2\Big)\end{aligned}$$

or

$$I_2\leqslant\frac{\overline{\alpha}_{\varphi_0}}{\underline{\alpha}_{\varphi_2}-\overline{\alpha}_{\varphi_0}}\int\varphi(|f|)\,\mathrm{d}A(z).$$

This implies

$$I_0\leqslant\frac{\overline{\alpha}_{\varphi 0}}{\underline{\alpha}_{\varphi_2}-\overline{\alpha}_{\varphi_0}}\int\varphi(|f|)\,\mathrm{d}A(z).$$

Combining estimates of I_0 and I_1, we finally obtain

$$\int_0^\infty p_2(t)(Tf)_*(2t)\,\mathrm{d}t\leqslant C\int_{\mathbb{D}}\varphi_2(|f|)\,\mathrm{d}A(z)$$

for some positive constant C. However we have

$$\int_0^\infty p_2(t)(Tf)_*(2t)\,\mathrm{d}t\geqslant\frac{1}{2^{\overline{\alpha}_{\varphi_2}}}\int_{\mathbb{D}}\varphi_2(|Tf|)\,\mathrm{d}A(z).$$

Hence

$$\int_{\mathbb{D}}\varphi_2(|Tf|)\,\mathrm{d}A(z)\leqslant C\int_{\mathbb{D}}\varphi_2(|f|)\,\mathrm{d}A(z).$$

as desired. The rest of the theorem is immediate consequence of Lemma 4.2. ∎

5. A NECESSARY CONDITION FOR THE L^φ-BOUNDEDNESS

In this section a necessary condition for the Bergman projection to be bounded on the Orlicz spaces is established. We will show that this condition is also sufficient in next section.

The following lemma shows that the Bergman projection ever maps the characteristic functions of curvilinear rectangles near the boundary of the disk into unbounded analytic functions on the disk.

LEMMA 5.1. *For $0 < r_0 < 1$ and $0 < \beta < 2\pi$, let $\chi_{r_0,\beta}(z)$ be the characteristic function of the set*

$$\{re^{i\theta} \in \mathbb{D} : \ r_0 < r < 1 \text{ and } |\theta| < \beta\}.$$

Then

$$(P\chi_{r_0,\beta})(z) = \frac{1}{i\pi}\Big(\log\frac{1-ze^{-i\beta}}{1-ze^{i\beta}} - r_0^2\log\frac{1-zr_0e^{-i\beta}}{1-zr_0e^{i\beta}} + \sum_{n=1}^{\infty}\frac{-1}{n(n+2)}z^n(e^{-in\beta}-e^{in\beta})(1-r_0^{n+2})\Big).$$

Proof. Since the Bergman kernel $K(z,w)$ has the power series expansion

$$K(z,w) = \sum_{n=1}^{\infty}(n+1)\overline{z}^n w^n,$$

it follows from the integral formula (1.1) of Bergman projection that

$$\begin{aligned} P\chi_{r_0,\beta}(z) &= \frac{1}{\pi}\int_{r_0}^{1}\int_{-\beta}^{\beta}\sum_{n=0}^{\infty}(n+1)z^n r^n e^{-in\theta} r\,dr\,d\theta \\ &= \frac{1}{i\pi}\sum\frac{n+1}{n(n+2)}z^n(e^{-in\beta}-e^{in\beta})(1-r_0^{n+2}). \end{aligned}$$

On the other hand, $\log(1-w) = \sum\limits_{n=1}^{\infty}\frac{w^n}{n}$. Thus we get

$$\begin{aligned} P\chi_{r_0,\beta}(z) &= \frac{1}{i\pi}\Big(\log\frac{1-ze^{-i\beta}}{1-ze^{i\beta}} - r_0^2\log\frac{1-zr_0e^{-i\beta}}{1-zr_0e^{i\beta}}\Big) \\ &\quad + \frac{1}{i\pi}\sum_{n=1}^{\infty}\frac{-1}{n(n+2)}z^n(e^{-in\beta}-e^{in\beta})(1-r_0^{n+2}). \end{aligned}$$ ∎

LEMMA 5.2. *For small enough $\beta > 0$, if $r_0 = 1-\beta$, then there is a constant M such that*

$$\left|r_0^2\log\frac{1-zr_0e^{-i\beta}}{1-zr_0e^{i\beta}}\right| < M$$

for all z in

$$Q(\beta) = \Big\{re^{i\theta} : r_0 < r < 1 \text{ and } |\theta-\beta| < \frac{\beta}{20}\Big\}.$$

Proof. Set $h(\theta,r) = \left|\frac{1-zr_0e^{-i\beta}}{1-zr_0e^{i\beta}}\right|^2$. If we write $z = re^{i\theta}$, then

$$h(\theta,r) = \frac{1+(rr_0)^2-2rr_0\cos(\theta-\beta)}{1+(rr_0)^2-2rr_0\cos(\theta+\beta)}.$$

It is easy to see that $h(\theta,r)$ is decreasing with respect to r in $Q(\beta)$. So $h(\theta,r)\geqslant h(\theta,1)$. On the other hand

$$h(\theta,1)=1-\frac{4r_0\Big(\sin^2\frac{\theta+\beta}{2}-\sin\frac{\theta-\beta}{2}\Big)}{(1-r_0)^2+4r_0\sin^2\frac{\theta+\beta}{2}}=1-\frac{4r_0\sin\theta\sin\beta}{\beta^2+4r_0\sin^2\frac{\theta+\beta}{2}}$$
$$\geqslant 1-\frac{4r_0\sin\frac{21}{20}\beta\sin\beta}{\beta^2+4r_0\sin^2\frac{39}{40}\beta}$$

if $|\theta-\beta|\leqslant\frac{\beta}{20}$ with β small enough. Since

$$\lim_{\beta\to 0}\Big(1-\frac{4r_0\sin\frac{21}{20}\beta\sin\beta}{\beta^2+4r_0\sin^2\frac{39}{40}\beta}\Big)=\frac{241}{1921}>0,$$

then $h(\theta,1)>\frac{240}{1921}$ if $|\theta-\beta|\leqslant\frac{\beta}{20}$ with small enough β. So there is a constant M such that

$$\Big|r_0^2\log\frac{1-zr_0\mathrm{e}^{-\mathrm{i}\beta}}{1-zr_0\mathrm{e}^{\mathrm{i}\beta}}\Big|<M$$

uniformly for z in $Q(\beta)$. ∎

LEMMA 5.3. *There are a positive number t_0 and a measurable function $g(t)>0$ such that*

$$\big|\{z\in\mathbb{D}:|P\chi_{1-\beta,\beta}(z)|>t\}\big|>\beta^2g(t)$$

for $t>t_0$ whenever β is small enough.

Proof. Suppose that $t>0$. We consider the distribution function $(P\chi_{1-\beta,\beta})_*(t)$ of $P\chi_{1-\beta,\beta}$ $(P\chi_{1-\beta,\beta})_*(t)=|\{z\in\mathbb{D}:|P\chi_{1-\beta,\beta}(z)|>t\}|$. It follows from Lemmas 5.1 and 5.2 that there is a constant $C>0$ such that

$$|P\chi_{1-\beta,\beta}(z)|\geqslant\frac{1}{\pi}\Big|\log\Big|\frac{1-z\mathrm{e}^{-\mathrm{i}\beta}}{1-z\mathrm{e}^{\mathrm{i}\beta}}\Big|\Big|-C$$

for z in $Q(\beta)$ whenever β is small enough. Clearly

$$(P\chi_{1-\beta,\beta})_*(t)\geqslant\Big|\Big\{z\in\mathbb{D}:\Big|\log\Big|\frac{1-z\mathrm{e}^{-\mathrm{i}\beta}}{1-z\mathrm{e}^{\mathrm{i}\beta}}\Big|\Big|>(t+C)\pi\Big\}\bigcap Q(\beta)\Big|.$$

It is easy to see that the set

$$\Big\{z\in\mathbb{D}:\Big|\log\Big|\frac{1-z\mathrm{e}^{-\mathrm{i}\beta}}{1-z\mathrm{e}^{\mathrm{i}\beta}}\Big|\Big|>(t+C)\pi\Big\}$$

contains a disk with center $\Big(\cos\beta,\frac{1+\mathrm{e}^{-2\pi(t+C)}}{1-\mathrm{e}^{-2\pi(t+C)}}\sin\beta\Big)$ and radius $\frac{2\sin\beta\mathrm{e}^{-\pi(t+C)}}{1-\mathrm{e}^{-2\pi(t+C)}}$. Then there are positive constants t_0 and C_1 such that for $t>t_0$

$$\Big|\Big\{z\in\mathbb{D}:\Big|\log\Big|\frac{1-z\mathrm{e}^{-\mathrm{i}\beta}}{1-z\mathrm{e}^{\mathrm{i}\beta}}\Big|\Big|>t+C\Big\}\bigcap Q(\beta)\Big|\geqslant\frac{C_1\sin^2\beta\mathrm{e}^{-2\pi(t+C)}}{(1-\mathrm{e}^{-2\pi(t+C)})^2}.$$

Since $\lim_{\beta\to 0}\frac{\sin\beta}{\beta}=1$, we have $\lambda_\beta(t)\geqslant g(t)\beta^2$ if β is small enough and $t>t_0$, where

$$g(t)=\frac{c_1\mathrm{e}^{-2\pi(t+C)}}{2(1-\mathrm{e}^{-2\pi(t+C)})^2}.\quad ∎$$

Now we are going to prove the main result of the section.

THEOREM 5.4. *Let φ and ψ be complementary Young functions. If the Bergman projection is bounded on the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$, then both φ and ψ satisfy the Δ_2-condition.*

Proof. Suppose that the Bergman projection is bounded on the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$. Then there is a constant C such that $\|Pf\|_{(\varphi)} \leqslant C\|f\|_{(\varphi)}$ for all f in $L^\varphi(\mathbb{D}, \mathrm{d}A)$. In particular, for any $\beta > 0$ we have $\|P\chi_{1-\beta,\beta}\|_{(\varphi)} \leqslant C\|\chi_{1-\beta,\beta}\|_{(\varphi)}$. By the definitions of the Luxemberg norm, this means that

$$\int_{\mathbb{D}} \varphi\Big(\frac{P\chi_{1-\beta,\beta}}{C\|\chi_{1-\beta,\beta}\|_{(\varphi)}}\Big)\,\mathrm{d}A(z) \leqslant 1.$$

However, by the definitions of the distribution function and the integration, we see that

$$\int_{\mathbb{D}} \varphi\Big(\frac{P\chi_{1-\beta,\beta}}{C\|\chi_{1-\beta,\beta}\|_{(\varphi)}}\Big)\,\mathrm{d}A(z) = \frac{1}{\pi}\int_0^\infty (P\chi_{1-\beta,\beta})_*(t)p\Big(\frac{t}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\frac{\mathrm{d}t}{C\|\chi_{1-\beta,\beta}\|_\varphi},$$

where $p(t)$ is the right derivative of the Young function φ. So we have

$$\int_0^\infty (P\chi_{1-\beta,\beta})_*(t)p\Big(\frac{t}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\frac{\mathrm{d}t}{C\|\chi_{1-\beta,\beta}\|_\varphi} \leqslant \pi.$$

Since $p(t)$ is nondecreasing, for any $t_0 > 0$

$$p\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\int_{t_0}^\infty (P\chi_{1-\beta,\beta})_*(t)\,\mathrm{d}t \leqslant \pi t_0.$$

On the other hand,

$$\varphi\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big) \leqslant p\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}.$$

Hence

$$\varphi\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\int_{t_0}^\infty (P\chi_{1-\beta,\beta})_*(t)\,\mathrm{d}t \leqslant \pi t_0.$$

It follows from Lemma 5.3 that

$$\varphi\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big)\int_{t_0}^\infty \beta^2 g(t)\,\mathrm{d}t \leqslant \pi t_0$$

or

$$\varphi\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big) \leqslant \frac{C_1 t_0}{\beta^2}. \tag{5.1}$$

Combining (5.1) and (9.23) of [10] it follows that

$$\varphi\Big(\frac{t_0}{C\|\chi_{1-\beta,\beta}\|_\varphi}\Big) \leqslant C t_0 \varphi\Big(\frac{1}{\|\chi_{1-\beta,\beta}\|_\varphi}\Big).$$

Now let $s = \frac{1}{\|\chi_{1-\beta,\beta}\|_\varphi}$; then s goes to ∞ as β goes to 0. If we choose t_0 so that $a = t_0/C = 2$, then $\varphi(2s) \leqslant C\varphi(s)$ for large enough s, which means that φ satisfies the Δ_2-condition.

Since we have shown above that φ satisfies the Δ_2-condition, it follows from Lemma 2.5 that the dual space of the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is the Orlicz space $L^\psi(\mathbb{D}, \mathrm{d}A)$. On the other hand, since the adjoint operator P^* on $L^\psi(\mathbb{D}, \mathrm{d}A)$ of P is also P, the boundedness of P on $L^\varphi(\mathbb{D}, \mathrm{d}A)$ implies that P is bounded on $L^\psi(\mathbb{D}, \mathrm{d}A)$ as well. Using the same argument as above we can get that $\psi(t)$ must satisfy the Δ_2-condition. The theorem is thus proved. ∎

Since $L^1(\mathbb{D}, \mathrm{d}A)$ and $L^\infty(\mathbb{D}, \mathrm{d}A)$ are nonreflexive, as an immediate consequence of the theorem, we see that the Bergman projection P is not bounded on $L^1(\mathbb{D}, \mathrm{d}A)$ or $L^\infty(\mathbb{D}, \mathrm{d}A)$.

6. THE DUALS OF BERGMAN SPACES L_a^φ

As we see in Section 1, we have

$$f(z) = \int_{\mathbb{D}} \frac{f(z)}{(1 - z\overline{w})^2} \,\mathrm{d}A(w) \tag{6.1}$$

for all f in L_a^2. It is natural to ask whether the integral formula (6.1) holds for all Bergman spaces L_a^φ. The following theorem says that this is the case.

THEOREM 6.1. *Let φ be a Young function and z in $\mathbb{D}$. Then*

$$f(z) = \int_{\mathbb{D}} \frac{f(z)}{(1 - z\overline{w})^2} \,\mathrm{d}A(w)$$

for all f in L_a^φ.

Proof. By the definition of the Young function, it is clear that $1/t \leqslant p(t)/\varphi(t)$ for all $t > 0$. This implies that

$$t \leqslant C\varphi(t) \tag{6.2}$$

for $t > t_0$ with a suitable constant C. So L_a^φ is a subset of L_a^1. Then (6.1) follows from Proposition 1.7 in [1]. ∎

Now we are going to prove the converse of Theorem 5.4.

THEOREM 6.2. *Suppose that φ and ψ are a pair of complementary Young functions. If both φ and ψ satisfy the Δ_2-condition, then the Bergman projection is bounded on the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$.*

Proof. Since P is bounded on $L^2(\mathbb{D}, \mathrm{d}A)$ and of weak type $(1,1)$ by Theorem 1.1, it follows from Theorem 4.3 that P is of mean strong type (φ, φ) if $1 < \underline{\alpha}_\varphi \leqslant \overline{\alpha}_\varphi < 2$. In particular, it is bounded on $L^\varphi(\mathbb{D}, \mathrm{d}A)$.

On the other hand, the complementary Young function ψ of φ satisfies $2 < \underline{\alpha}_\psi \leqslant \overline{\alpha}_\psi < \infty$ if $1 < \underline{\alpha}_\varphi \leqslant \overline{\alpha}_\varphi < 2$.

Using the duality, we also have that P is bounded on $L^\varphi(\mathbb{D}, \mathrm{d}A)$ if $2 < \underline{\alpha}_\varphi \leqslant \overline{\alpha}_\varphi < \infty$.

Now let $1 < \underline{\alpha}_\varphi \leqslant 2 \leqslant \overline{\alpha}_\varphi$. Take $\varphi_0(t) = t^{\underline{\alpha}_\varphi - \mathrm{e}}$ with $1 < \underline{\alpha}_\varphi - \mathrm{e}$ and $\varphi_1(t) = t^{\overline{\alpha}_\varphi + \mathrm{e}}$. It follows from the above argument and the remark following Definition 4.1 that P is of mean strong (φ_0, φ_0) and (φ_1, φ_1). So Theorem 4.3 tells us that P is of mean strong type (φ, φ). By Lemma 4.2, P is bounded on $L^\varphi(\mathbb{D}, \mathrm{d}A)$. ∎

Proof of Theorem 1.6. Suppose that l is a bounded linear functional on L^φ_{a}. By the Hahn-Banach theorem, l can be extended to a linear functional L on the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ so that $\|L\|$ is equal to $\|l\|$. Since the Orlicz space $L^\varphi(\mathbb{D}, \mathrm{d}A)$ is reflexive, both φ and ψ satisfy the Δ_2-condition by Lemma 2.6. It follows from Lemma 2.5 that there is a function h in $L^\psi(\mathbb{D}, \mathrm{d}A)$ such that $L(f) = \int_{\mathbb{D}} f\overline{h}\,\mathrm{d}A(z)$ for f in $L^\varphi(\mathbb{D}, \mathrm{d}A)$. Let $g = P(h)$. We see from Theorem 6.2 that g is in L^ψ_{a} and $\|g\|_\psi \leqslant C\|h\|_\psi$. If f is in L^φ_{a}, then

$$\int_{\mathbb{D}} f\overline{g}\,\mathrm{d}A(z) = \int_{\mathbb{D}} f\overline{P(h)}\,\mathrm{d}A(z) = \int_{\mathbb{D}} P(f)\overline{h}\,\mathrm{d}A(z).$$

It follows from Theorem 6.1 that $\int_{\mathbb{D}} f\overline{g}\,\mathrm{d}A(z) = \int_{\mathbb{D}} f\overline{h}\,\mathrm{d}A(z)$. So $l(f) = L(f) = \int_{\mathbb{D}} f\overline{g}\,\mathrm{d}A(z)$ for f in L^φ_{a}. It is clear, by the Hölder inequality, that $\|l\| \leqslant \|g\|_\psi$. On the other hand, since $\|g\|_\psi = \|Ph\|_\psi \leqslant C\|h\|_\psi$, by Lemma 2.5 we conclude that $\|l\|$ is equivalent to $\|g\|_\psi$. ∎

7. SOME CLASSICAL ESTIMATES

In this section we will establish several counterparts for the Bergman projection of the well-known inequalities in the theory of the Hardy spaces.

In the classical theory, it is well-known that even if for $0 < p < 1$ the Hardy space H^p fails to be a Banach space, there is the Kolmogorov inequality $\|Hf\|_p \leqslant c_p\|f\|_1$ for all f in $L^1(\partial\mathbb{D})$. Theorem 1.2 asserts that the Kolmogorov inequality holds for the Bergman projection as well.

Proof of Theorem 1.2. It follows from Theorem 1.1 that

$$\begin{aligned}\int_{\mathbb{D}} |Pf(z)|^p\,\mathrm{d}A(z) &= \int_0^\infty ps^{p-1}(Pf)_*(s)\,\mathrm{d}s \leqslant (Pf)_*(0)t^p + p\int_t^\infty s^{p-1}C\frac{\|f\|_1}{s}\,\mathrm{d}s \\ &\leqslant t^p + C\|f\|_1\frac{p}{1-p}t^{p-1}\end{aligned}$$

for all $t > 0$. Set $g(t) = t^p + C\|f\|_1\frac{p}{1-p}t^{p-1}$. ∎

It is easy to show that $g(t)$ assumes the minimun value at $t = C\|f\|_1$. So we have

$$\int_{\mathbb{D}} |Pf(z)|^p\,\mathrm{d}A(z) \leqslant \frac{1}{1-p}C^p\|f\|_1^p.$$

This gives $\|Pf\|_p \leqslant \left(\frac{1}{1-p}\right)^p C\|f\|_1$.

Before going to the proofs of Theorems 1.3 and 1.4, we recall that the Zygmund space $L\log^+ L$ is the set of complex valued measurable functions f on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f(z)| \log^+ |f(z)| \,\mathrm{d}A(z) < \infty$$

and the Zygmund space $L^{\exp}$ consists of all complex-valued functions f on $\mathbb{D}$ such that for some constant $k(f)$

$$\int_{\mathbb{D}} \exp\Big(\Big|\frac{f(z)}{k(f)}\Big|\Big) \,\mathrm{d}A(z) < \infty.$$

We have mentioned before that the spaces $L\log^+ L$ and $L^{\exp}$ are Orlicz spaces.

Proof of Theorem 1.3. From the integral formula (1.1) of the Bergman projection we see that $Pf(z) = \int_{\mathbb{D}} f(w)K(w,z)\,\mathrm{d}A(w)$ is analytic on $\mathbb{D}$ if f is in $L\log^+ L$. So we need only to show that the Bergman projection is a bounded map from $L\log^+ L$ into L^1. To do this, let

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < 1, \\ t-1 & \text{otherwise;} \end{cases}$$

and $\varphi_1(t) = t\log^+ t$, the Young function of the Orlicz space $L\log^+ L$. It is easy to see that $\varphi(t)$ is equivalent to the Young function of L^1. Thus, by Theorems 13.2 and 13.3 of [10], it suffices to verify that there is a constant C such that $\|Pf\|_\varphi \leqslant C\|f\|_{L\log^+ L}$. From the definition of the Bergman projection P and Theorem 1.1, we know that P is bounded on $L^2(\mathbb{D}, \mathrm{d}A)$ and of weak type $(1,1)$, so a standard argument as in the proof of Theorem 1.1 shows that

$$(Pf)_*(t) \leqslant \frac{c_1 \int_t^\infty f_*(s)\,\mathrm{d}s}{t} + \frac{c_2 \int_0^t f_*(s)\,\mathrm{d}s}{t^2}.$$

Now, let p be the right derivative of φ. Multiplying both sides of the last inequality by $p(t)$ and then integrating with respect to t, we infer that

$$\int_0^\infty (Pf)_*(2t)p(t)\,\mathrm{d}t \leqslant c_1 \int_1^\infty\!\!\int_t^\infty \frac{f_*(s)}{t}\,\mathrm{d}s\,\mathrm{d}t + c_2 \int_1^\infty\!\!\int_0^t \frac{f_*(s)}{t^2}\,\mathrm{d}s\,\mathrm{d}t = c_1 I_1 + c_2 I_2.$$

Since the domains of the integrals I_1 and I_2 are the trapezoids $\{(t,s) : 1 < t < \infty, t < s < \infty\}$ and $\{(t,s) : 1 < t < \infty, 0 < s < t\}$ respectively, interchanging the order of the integrations, we obtain

$$I_1 = \int_1^\infty f_*(s)\log s\,\mathrm{d}s \quad \text{and} \quad I_2 = \int_1^\infty \frac{f_*(s)}{s}\,\mathrm{d}s \leqslant \int_1^\infty f_*(s)\,\mathrm{d}s.$$

So

$$\int_1^\infty (Pf)_*(2t)p(t)\,\mathrm{d}t \leqslant C\int_1^\infty f_*(s)(1+\log s)\,\mathrm{d}s.$$

Since $\varphi(t)$ satisfies the Δ_2-condition, we have

$$\int_1^\infty (Pf)_*(t)p(t)\,\mathrm{d}t \leqslant C\int_1^\infty (Pf)_*(2t)p(t)\,\mathrm{d}t.$$

Thus $\|Pf\|_\varphi \leqslant C\|f\|_{L\log L}$.

In addition, the Young function $\varphi(t)$ is equivalent to the Young function of $L^1(\mathbb{D}, \mathrm{d}A)$. So we conclude that there is a constant C such that $\|Pf\|_1 \leqslant C\|f\|_{L\log L}$, which completes the proof of Theorem 1.3. ∎

Proof of Theorem 1.4. As mentioned in Example (iii) in Section 2, the space $L^{\exp}(\mathbb{D}, \mathrm{d}A)$ is the associate space of the Orlicz space $L\log^+ L$. Now the Young functions of $L^1(\mathbb{D}, \mathrm{d}A)$ and $L\log^+ L$ satisfy the Δ_2-condition, so the dual spaces of $L^1(\mathbb{D}, \mathrm{d}A)$ and $L\log^+ L$ are respectively $L^\infty(\mathbb{D}, \mathrm{d}A)$ and $L^{\exp}$ by Lemma 2.5. On the other hand, since the Bergman projection P is self-adjoint, from Theorem 3.1 we conclude that P is a bounded map from $L^{\exp}(\mathbb{D}, \mathrm{d}A)$ to $L^1(\mathbb{D}, \mathrm{d}A)$. ∎

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