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THE EXT CLASS OF AN APPROXIMATELY INNER AUTOMORPHISM. II

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Abstract. Let A be a simple unital AT algebra of real rank zero and $\mathrm{Inn}(A)$ the group of inner automorphisms of A. In the previous paper we have shown that the natural map of the group $\overline{\mathrm{Inn}}(A)$ of approximately inner automorphisms into $\mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$ is surjective; the kernel of this map includes the subgroup of automorphisms which are homotopic to $\mathrm{Inn}(A)$. In this paper we consider the quotient of $\overline{\mathrm{Inn}}(A)$ by the smaller normal subgroup $\mathrm{AInn}(A)$ which consists of asymptotically inner automorphisms and describe it as $\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$, where $\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ is a kind of extension group which takes into account the fact that $\mathrm{K}_0(A)$ is an ordered group and has the usual Ext as a quotient.

Keywords: *C^*-algebra, automorphism, K-theory, extension, trace, asymptotically inner, real rank zero.*

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1. INTRODUCTION

An automorphism α of a unital C^*-algebra A is called inner if there is a unitary $u \in A$ such that $\alpha(a) = \mathrm{Ad}\, u(a) = uau^*$, $a \in A$. We denote by $\mathrm{Inn}(A)$ the group of inner automorphisms of A, which is a normal subgroup of the group $\mathrm{Aut}(A)$ of all automorphisms of A. The topology on $\mathrm{Aut}(A)$ is determined by pointwise convergence on A. The closure $\overline{\mathrm{Inn}}(A)$ of $\mathrm{Inn}(A)$ in $\mathrm{Aut}(A)$ is, by definition, the group of approximately inner automorphisms.

There are two distinguished normal subgroups of $\overline{\mathrm{Inn}}(A)$ containing $\mathrm{Inn}(A)$. One is the group $\mathrm{HInn}(A)$ of automorphisms which are homotopic to $\mathrm{Inn}(A)$, i.e., $\alpha \in \mathrm{HInn}(A)$ if and only if there is a continuous map $\alpha. : [0, 1] \to \overline{\mathrm{Inn}}(A)$ such that

$$\alpha_0 \in \mathrm{Inn}(A), \quad \alpha_1 = \alpha\,.$$

The other is the group $\mathrm{AInn}(A)$ of asymptotically inner automorphisms, i.e., $\alpha \in \mathrm{AInn}(A)$ if and only if there is a continuous map $\alpha.: [0,1] \to \overline{\mathrm{Inn}}(A)$ and a continuous map $u.: [0,1) \to U(A)$ with $U(A)$ the unitary group of A such that

$$\alpha_t = \mathrm{Ad}\, u_t \quad \text{for } t \in [0,1),\ \alpha_1 = \alpha\,.$$

It is easy to show that they are indeed normal subgroups and that

$$\mathrm{Inn}(A) \subset \mathrm{AInn}(A) \subset \mathrm{HInn}(A) \subset \overline{\mathrm{Inn}}(A).$$

In this paper we describe the quotient

$$\overline{\mathrm{Inn}}(A)/\mathrm{AInn}(A)$$

in terms of K-theoretic data when A is a simple unital AT algebra of real rank zero.

Recall that a unital C^*-algebra A is said to be a unital AT algebra if it is expressible as the inductive limit of T algebras, i.e., finite direct sums of matrix algebras over $C(\mathbb{T})$, with unital embeddings. Note that a unital AT algebra A is stably finite and we denote by T_A the convex set of tracial states of A.

Let A be a simple unital AT algebra of real rank zero and $\alpha \in \overline{\mathrm{Inn}}(A)$. (In this case $\alpha \in \mathrm{Aut}(A)$ belongs to $\overline{\mathrm{Inn}}(A)$ if and only if $\alpha_* = \mathrm{id}$ on $\mathrm{K}_*(A)$ ([7]).) The mapping torus of α is the C^*-algebra:

$$M_\alpha = \{x \in C[0,1] \otimes A;\ \alpha(x(0)) = x(1)\}.$$

The suspension of A, $\mathrm{S}A$, is identified with the ideal of M_α:

$$\mathrm{S}A = \{x \in C[0,1] \otimes A;\ x(0) = 0 = x(1)\}.$$

From the short exact sequence:

$$0 \longrightarrow \mathrm{S}A \longrightarrow M_\alpha \longrightarrow A \longrightarrow 0,$$

one obtains the usual six-term exact sequence in K-theory, which, since $\alpha \in \overline{\mathrm{Inn}}(A)$, splits into two short exact sequences:

$$0 \longrightarrow \mathrm{K}_i(A) \longrightarrow \mathrm{K}_{i+1}(M_\alpha) \longrightarrow \mathrm{K}_{i+1}(A) \longrightarrow 0$$

for $i = 0, 1$, where $\mathrm{K}_{i+1}(\mathrm{S}A)$ has been identified with $\mathrm{K}_i(A)$. Let $\eta_i(\alpha)$ denote the class of this sequence in $\mathrm{Ext}(\mathrm{K}_{i+1}, \mathrm{K}_i(A))$ and let η denote the map of $\overline{\mathrm{Inn}}(A)$ into

$$\bigoplus_{i=0}^{1} \mathrm{Ext}(\mathrm{K}_{i+1}(A), \mathrm{K}_i(A))$$

defined by $\alpha \mapsto (\eta_0(\alpha), \eta_1(\alpha))$, which is a group homomorphism. (By using KK theory and the universal coefficient theorem ([13]), $\eta(\alpha)$ is also described as $\mathrm{KK}(\alpha) - \mathrm{KK}(\mathrm{id})$.) In the previous paper ([11]) we showed that η induces a surjective homomorphism:

$$\overline{\mathrm{Inn}}(A)/\mathrm{HInn}(A) \longrightarrow \mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A)).$$

To state the main result of this paper we proceed to describe a natural map R_α of $\mathrm{K}_1(M_\alpha)$ into $\mathrm{Aff}(T_A)$, which is the real Banach space of affine continuous functions on the compact tracial state space T_A of A. Note that, since we assume that A has real rank zero, T_A is isomorphic to the state space of $\mathrm{K}_0(A)$ ([1]). If

$u \in M_\alpha$ is a unitary given as a piecewise smooth function of $[0,1]$ into A, then $R_\alpha([u])$ is defined by

$$R_\alpha([u])(\tau) = \frac{1}{2\pi \mathrm{i}} \int_0^1 \tau(\dot{u}(t)u(t)^*)\,\mathrm{d}t$$

for $\tau \in T_A$. The map R_α is a group homomorphism of $\mathrm{K}_1(M_\alpha)$ into $\mathrm{Aff}(T_A)$ and extends the natural map D of $\mathrm{K}_0(A)$ into $\mathrm{Aff}(T_A)$ when $\mathrm{K}_0(A)$ is regarded as a subgroup of $\mathrm{K}_1(M_\alpha)$.

We take the set of pairs (E, R) where E is an abelian group such that

$$0 \longrightarrow \mathrm{K}_0(A) \xrightarrow{\iota} E \xrightarrow{q} \mathrm{K}_1(A) \longrightarrow 0$$

and R is a homomorphism:

$$R : E \longrightarrow \mathrm{Aff}(T_A)$$

such that $R \circ \iota = D$. We can form a group $\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ from this set in much the same way as we do $\mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ from the set of E alone. From the previous paragraph we can associate $\widetilde{\eta}_0(\alpha) \in \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ with each $\alpha \in \overline{\mathrm{Inn}}(A)$ and show that $\widetilde{\eta}_0$ is a homomorphism. Our main result is

$$\overline{\mathrm{Inn}}(A)/\mathrm{AInn}(A) \cong \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$$

where the isomorphism is induced by the map $\alpha \mapsto (\widetilde{\eta}_0(\alpha), \eta_1(\alpha))$ (see Theorem 4.4).

In Section 2 we will define $\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ and the homomorphism

$$\widetilde{\eta} : \overline{\mathrm{Inn}}(A) \to \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$$

in detail and in Section 3 we will show that

$$\ker \widetilde{\eta} = \mathrm{AInn}(A).$$

In Section 4 we will show that $\widetilde{\eta}$ is surjective; thus proving the main result.

2. ORDEREXT

Let A be a simple unital C^*-algebra and let T_A be the set of tracial states of A. Let $\alpha \in \overline{\mathrm{Inn}}(A)$ and let M_α be the mapping torus of α. For a unitary $u \in M_\alpha$ such that $t \mapsto u(t)$ is (piecewise) C^1 and for $\tau \in T_A$, we define

$$\tau(u) = \frac{1}{2\pi \mathrm{i}} \int_0^1 \tau(\dot{u}(t)u(t)^*)\,\mathrm{d}t.$$

In [16] this is denoted by $\widetilde{\Delta}_\tau(u)$. Since $\tau(\dot{u}(t)u(t)^*) = -\tau(u(t)\dot{u}(t)^*)$, it follows that $\tau(u) \in \mathbb{R}$. If $u, v \in M_\alpha$ are C^1-unitaries, we obtain that

$$\tau(uv) = \tau(u) + \tau(v).$$

If $h = h^* \in M_\alpha$ is C^1, then we have for $u = \mathrm{e}^{2\pi \mathrm{i} h}$

$$\tau(u) = \int_0^1 \tau(\dot{h}(t))\mathrm{d}t = \tau(h(1)) - \tau(h(0)) = 0,$$

where we have used that $\tau \circ \alpha = \tau$, which follows since $\alpha \in \overline{\mathrm{Inn}}(A)$. Thus it follows that $\tau(u)$ is constant on each connected component of the C^1-unitary group of M_α. By taking the matrix algebras over M_α and using the density of C^1-unitaries in the unitary group, we obtain a homomorphism $\tau : \mathrm{K}_1(M_\alpha) \to \mathbb{R}$ by $[u] \mapsto \tau(u)$ for each $\tau \in T_A$. Since $\tau \in T_A \mapsto \tau(u)$ is affine and continuous, we thus obtain:

Lemma 2.1. *For any $\alpha \in \overline{\mathrm{Inn}}(A)$ there exists a homomorphism*

$$R_\alpha : \mathrm{K}_1(M_\alpha) \longrightarrow \mathrm{Aff}(T_A)$$

defined by $R_\alpha([u])(\tau) = \tau(u)$, which is called the rotation map *for α.*

Since $\alpha_* = \mathrm{id}$ on $\mathrm{K}_i(A)$, we have the short exact sequence:

$$0 \longrightarrow \mathrm{K}_0(A) \xrightarrow{\iota_*} \mathrm{K}_1(M_\alpha) \xrightarrow{q_*} \mathrm{K}_1(A) \longrightarrow 0$$

from the short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathrm{S}A \xrightarrow{\iota} M_\alpha \xrightarrow{q} A \longrightarrow 0.$$

If p is a projection in A, we have that $\iota_*([p]) = [u]$ where $u \in M_\alpha$ is the unitary defined by

$$u(t) = \mathrm{e}^{2\pi \mathrm{i} t} p + 1 - p.$$

Thus we obtain:

Lemma 2.2. *For $\alpha \in \overline{\mathrm{Inn}}(A)$ the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{K}_0(A) & & \xrightarrow{\iota_*} & & \mathrm{K}_1(M_\alpha) \\ & D \searrow & & \swarrow R_\alpha & \\ & & \mathrm{Aff}(T_A) & & \end{array}$$

where D is the homomorphism of $\mathrm{K}_0(A)$ into $\mathrm{Aff}(T_A)$ defined by $D([p])(\tau) = \tau(p)$, which is called the dimension map *for A.*

Let $G_i = \mathrm{K}_i(A)$. If

$$0 \longrightarrow G_0 \xrightarrow{\iota} E \xrightarrow{q} G_1 \longrightarrow 0$$

is exact, we denote this short exact sequence by E, the same symbol at the middle. Let R be a homomorphism of E into $\mathrm{Aff}(T_A)$ such that $R \circ \iota = D$. We consider the set of all pairs (E, R), which we call order-extensions for (G_1, G_0).

If (E', R') is another order-extension, we say that (E, R) and (E', R') are isomorphic if there is an isomorphism φ of E into E' such that $R = R' \circ \varphi$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\iota} & E & \xrightarrow{q} & G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & G_0 & \xrightarrow{\iota'} & E' & \xrightarrow{q'} & G_1 & \longrightarrow & 0 \end{array}$$

is commutative. Note that if (E, R) and (E', R') are isomorphic, E and E' are isomorphic as extensions. We define an addition for such pairs by extending that for extensions as follows. If (E, R) and (E', R') are given, define

$$\begin{aligned} E'' &= \{(x, y) \in E \oplus E' \mid q(x) = q'(y)\}/\{(\iota(a), -\iota'(a)) \mid a \in G_0\} \\ \iota'' &: G_0 \longrightarrow E'', \quad a \longmapsto [(\iota(a), 0)] \\ q'' &: E'' \longrightarrow G_1, \quad [(x, y)] \longmapsto q(x) \\ R'' &: E'' \longrightarrow \mathrm{Aff}(T_A), \quad [(x, y)] \longmapsto R(x) + R'(y). \end{aligned}$$

It is easy to show that these objects are well defined,

$$0 \longrightarrow G_0 \xrightarrow{\iota''} E'' \xrightarrow{q''} G_1 \longrightarrow 0$$

is exact, and $R'' \circ \iota'' = D$. The sum of (E, R) and (E', R') is defined to be (E'', R''). Again it is easy to show that the isomorphism classes of those orderextensions form an abelian semigroup. Then the identity element for this semigroup is given by the isomorphism class $[(E_0, R_0)]$ of the trivial order-extension (E_0, R_0) given by:

$$\begin{aligned} E_0 &= G_0 \oplus G_1 \\ \iota_0 &: G_0 \longrightarrow E_0, \quad a \longmapsto (a, 0) \\ q_0 &: E_0 \longrightarrow G_1, \quad (a, b) \longmapsto b \\ R_0 &: E_0 \longrightarrow \mathrm{Aff}(T_A), \quad (a, b) \longmapsto D(a). \end{aligned}$$

The inverse of $[(E, R)]$ is given by $[(E', R')]$ where

$$E' = E, \quad \iota' = -\iota, \quad q' = q, \quad R' = -R.$$

Thus this semigroup is a group, which we denote by $\mathrm{OrderExt}(G_1, G_0)$. Note that $\mathrm{OrderExt}(G_1, G_0)$ depends also on the dimension map $D : G_0 \to \mathrm{Aff}(T_A)$.

Lemma 2.3. *The map*

$$\begin{aligned} \widetilde{\eta}_0 : \overline{\mathrm{Inn}}(A) &\longrightarrow \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \\ \alpha &\longmapsto [(\mathrm{K}_1(M_\alpha), R_\alpha)] \end{aligned}$$

is a homomorphism.

Proof. By Lemma 2.2, $\widetilde{\eta}_0$ is well-defined.

Let $\alpha, \beta \in \overline{\mathrm{Inn}}(A)$ and (E, R) be the sum of $(\mathrm{K}_1(M_\alpha), R_\alpha)$ and $(\mathrm{K}_1(M_\beta), R_\beta)$. We have to show that (E, R) is isomorphic to $(\mathrm{K}_1(M_{\alpha\beta}), R_{\alpha\beta})$.

Let $g \in \mathrm{K}_1(M_\alpha)$ and $h \in \mathrm{K}_1(M_\beta)$ such that $q(g) = q(h)$. Let $v \in M_n \otimes M_\alpha$ and $w \in M_n \otimes M_\beta$ be unitaries such that $[v] = g$, $[w] = h$, and $v(0) = w(0)$. Then we define a unitary $u \in M_n \otimes M_{\alpha\beta}$ by

$$u(t) = \begin{cases} v(2t) & 0 \leqslant t \leqslant 1/2, \\ \alpha(w(2t-1)) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $[u] \in \mathrm{K}_1(M_{\alpha\beta})$ depends only on $[v]$ and $[w]$. Thus we have a map φ from

$$\{(g, h) \in \mathrm{K}_1(M_\alpha) \oplus \mathrm{K}_1(M_\beta) \mid q(g) = q(h)\}$$

to $\mathrm{K}_1(M_{\alpha\beta})$. It is easy to show that φ is a surjective homomorphism and the kernel of φ equals $\{(\iota(a), -\iota(a)) \mid a \in \mathrm{K}_0(A)\}$. Hence φ induces an isomorphism $\phi : E \to \mathrm{K}_1(M_{\alpha\beta})$. Since

$$R_{\alpha\beta}([u]) = R_\alpha([v]) + R_\beta([w])$$

for the above u, (E, R) is isomorphic to $(\mathrm{K}_1(M_{\alpha\beta}), R_{\alpha\beta})$. ∎

LEMMA 2.4. *If* (E,R) *is an order-extension for* (G_1,G_0) *and* $\operatorname{Range} R = \operatorname{Range} D$*, then*

$$0 \longrightarrow \ker D \overset{\iota_*|\ker D}{\longrightarrow} \ker R \overset{q_*|\ker R}{\longrightarrow} G_1 \longrightarrow 0$$

is exact.

Proof. It is obvious that the above sequence is well-defined, the compositions of two consecutive maps vanish, and it is exact at $\ker D$. Let $g \in \ker R$ with $q_*(g) = 0$. Then there is a $g' \in G_0$ such that $\iota_*(g') = g$. But, since $D(g') = R(g) = 0$, we have that $g' \in \ker D$, which implies that it is exact at $\ker R$. Let $g \in G_1$. Then there is a $g' \in E$ with $q_*(g') = g$ and there must be a $g'' \in G_0$ such that $D(g'') = R(g')$. Since $q_*(g' - \iota_*(g'')) = g$ and $R(g' - \iota_*(g'')) = 0$, we have that $g \in \operatorname{Range}(q_*|\ker R)$. ∎

PROPOSITION 2.5. *If* (E,R) *is an order-extension for* (G_1,G_0)*, the following conditions are equivalent:*

(i) $[(E,R)] = 0$;

(ii) (a) $0 \to G_0 \to E \to G_1 \to 0$ is trivial,

(b) $\operatorname{Range} R = \operatorname{Range} D$,

(c) $0 \to \ker D \to \ker R \to G_1 \to 0$ is trivial;

(iii) $0 \to \ker D \to \ker R \to G_1 \to 0$ is exact and trivial.

Proof. If (E_0,R_0) is the trivial order-extension, it satisfies (ii). Any order-extension isomorphic to (E_0,R_0) also satisfies (ii). Thus (i) implies (ii).

Suppose that (E,R) satisfies (ii). Note that the sequence in (c) is exact by Lemma 2.4. By (c) there is a homomorphism ν of G_1 into $\ker R$ such that $q \circ \nu = \mathrm{id}$. Hence $E = \iota(G_0) \oplus \nu(G_1)$ and R is given by

$$\iota(G_0) \oplus \nu(G_1) \to \operatorname{Aff}(T_A), \quad a + b \mapsto D(a).$$

Thus (E,R) is isomorphic to the trivial order-extension, i.e., (ii) implies (i).

It follows from Lemma 2.4 that (ii) implies (iii). The converse also follows from the arguments in the previous paragraph. ∎

REMARK 2.6. By the Thom isomorphism ([5]), $\mathrm{K}_i(M_\alpha)$ is isomorphic to $\mathrm{K}_{i+1}(A \times_\alpha \mathbb{Z})$ as an abelian group. By extending $\tau \in T_A$ to a tracial state of $A \times_\alpha \mathbb{Z}$ and defining a natural map $D_\alpha : \mathrm{K}_0(A \times_\alpha \mathbb{Z}) \to \operatorname{Aff}(T_A)$, it follows that $(\mathrm{K}_1(M_\alpha), R_\alpha)$ is isomorphic to $(\mathrm{K}_0(A \times_\alpha \mathbb{Z}), D_\alpha)$ ([5]). See also [6], [12], [1].

3. ASYMPTOTICALLY INNER AUTOMORPHISMS

From now on we will assume that the C^*-algebra A is a simple unital AT algebra of real rank zero. In this case by Elliott's result ([7]) A is determined by $(\mathrm{K}_0(A), [1], \mathrm{K}_1(A))$ up to isomorphism, where $\mathrm{K}_0(A)$ is a dimension group, $\mathrm{K}_1(A)$ is a torsion-free abelian group, and $[1] \in \mathrm{K}_0(A)^+$. Note that the tracial state space T_A of A is identified with the compact convex set of order-preserving homomorphisms $f : \mathrm{K}_0(A) \to \mathbb{R}$ with $f([1]) = 1$.

Let $\alpha \in \overline{\mathrm{Inn}}(A)$. We recall that α is asymptotically inner if there exists a continuous map $v : [0, 1) \to U(A)$ such that

$$\alpha(a) = \lim_{t\to 1} \mathrm{Ad}\, v_t(a), \quad a \in A.$$

We denote by $\mathrm{AInn}(A)$ the group of asymptotically inner automorphisms of A. We also recall that $\widetilde{\eta}$ is the homomorphism of $\overline{\mathrm{Inn}}(A)$ into

$$\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$$

defined by $\alpha \mapsto \widetilde{\eta}_0(\alpha) \oplus \eta_1(\alpha)$.

Before stating the main theorem of this section, let us recall the notion of Bott element for pairs of almost commuting unitaries in a unital C^*-algebra A ([10], [11]): Given $u, v \in U(A)$ with $[u, v] \equiv uv - vu \approx 0$, we associate $B(u, v) \in \mathrm{K}_0(A)$, which is the equivalence class of a projection close to the image of the Bott projection in $M_2 \otimes C(\mathbb{T}^2)$ under the *quasi-homomorphism* from $M_2 \otimes C(\mathbb{T}^2)$ into $M_2 \otimes A$ mapping the two canonical unitaries of $C(\mathbb{T}^2)$ into u, v respectively. If $A = M_n$, this can also be given by

$$B(u, v) = \frac{1}{2\pi \mathrm{i}} \mathrm{Tr}(\log vuv^*u^*) \in \mathbb{Z} = \mathrm{K}_0(M_n),$$

where log is the logarithm with values in $\{z; \mathrm{Im}(z) \in (-\pi, \pi)\}$. (That $B(u, v)$ is an integer follows from the fact that the determinant of vuv^*u^* is 1.) We note that $B(u, v)$ is invariant under homotopy of pairs of almost commuting unitaries and that $B(u, v) = -B(u^*, v) = -B(v, u)$, $B(u, v_1v_2) = B(u, v_1) + B(u, v_2)$. We quote [4] for another characterization of the Bott element, which is used to prove the following result we will need later: If A is a simple unital AT algebra of real rank zero and $u, v \in U(A)$ satisfy that $[u, v] \approx 0$, $B(u, v) = 0$, $\mathrm{Sp}(v)$ is almost dense in $\mathbb{T}$, and $[u] = 0$, then there is a path u_t, $t \in [0, 1]$ in $U(A)$ such that $[u_t, v] \approx 0$, $u_0 = 1$, and $u_1 = u$.

Theorem 3.1. *Let A be a simple unital AT algebra of real rank zero and let $\alpha \in \overline{\mathrm{Inn}}(A)$. Then the following conditions are equivalent:*
(i) $\widetilde{\eta}(\alpha) = 0$,
(ii) $\alpha \in \mathrm{AInn}(A)$.

Proof of (ii) $\Rightarrow$ (i). Since η is homotopy invariant, $\eta(\alpha) = (\eta_0(\alpha), \eta_1(\alpha)) = 0$ in $\mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$.

We may suppose that we have a piecewise C^1 map v of $[0, 1)$ into $U(A)$ such that

$$\alpha(a) = \lim_{t\to 1} \mathrm{Ad}\, v_t(a), \quad a \in A.$$

Let $u \in U(A)$. We define a unitary $\widehat{u} \in M_\alpha \otimes M_2$ by composing the following paths:

$$[0,1] \ni t \mapsto R_t \begin{pmatrix} 1 & 0 \\ 0 & v_0 \end{pmatrix} R_t^{-1} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t \begin{pmatrix} 1 & 0 \\ 0 & v_0^* \end{pmatrix} R_t^{-1}$$

and

$$[0,1) \ni t \mapsto \begin{pmatrix} v_t u v_t^* & 0 \\ 0 & 1 \end{pmatrix}$$

with

$$1 \mapsto \begin{pmatrix} \alpha(u) & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$R_t = \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix}.$$

Then it follows that $\tau(\dot{\widehat{u}}(t)\widehat{u}(t)^*) = 0$ for $\tau \in T_A$. In particular, $R_\alpha([\widehat{u}]) = 0$. Since $q_*([\widehat{u}]) = [u]$, the map $[u] \mapsto [\widehat{u}]$ defines a homomorphism φ of $\mathrm{K}_1(A)$ into $\ker R_\alpha$ such that $q_* \circ \varphi = \mathrm{id}$. This implies that

$$0 \longrightarrow \ker D \longrightarrow \ker R_\alpha \longrightarrow \mathrm{K}_1(A) \longrightarrow 0$$

is exact and trivial, and thus concludes the proof by Proposition 2.5. ∎

The rest of this section will be devoted to the proof of (i) ⇒ (ii).

Let $\{A_n\}$ be an increasing sequence of T subalgebras of A such that $A_1 \ni 1$ and $A = \overline{\bigcup_{n=1}^\infty A_n}$. We express A_n as

$$A_n = \bigoplus_{i=1}^{k_n} B_{n,i} \otimes C(\mathbb{T})$$

where $B_{n,i}$ is isomorphic to the full matrix algebra $M_{[n,i]}$. By identifying $\mathrm{K}_i(A)$ with $\mathbb{Z}^{k_n}$ in a natural way we obtain a homomorphism $\mathrm{K}_i(A_n)$ into $\mathrm{K}_i(A_{n+1})$ as the multiplication of a matrix χ_n^i. We always assume that $\chi_n^0(i,j)$ is big and $|\chi_n^1(i,j)|/\chi_n^0(i,j)$ is small compared with 1 and that the embedding of A_n into A_{n+1} is in standard form, i.e., $B_n = \bigoplus_{i=1}^{k_n} B_{ni} \subset B_{n+1}$ and the canonical unitary z_n of $1 \otimes C(\mathbb{T}) \subset A_n$ in $B_{n+1} \cap B_n' \otimes C(\mathbb{T})$ is a direct sum of elements of the form:

$$\begin{pmatrix} 0 & & & z_{n+1}^L \\ 1 & \cdot & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

with $L = \pm 1$; e.g., if $\chi_n^1(i,j) > 0$, $z_n p_{n+1\,i} p_{nj}$ is a direct sum of $\chi_n^1(i,j)$ matrices of the above form with $L = 1$ in $B_{n+1} \cap B_n' \otimes C(\mathbb{T}) p_{n+1\,i} p_{nj} \cong M_{\chi_n^0(i,j)} \otimes C(\mathbb{T})$ ([7], [11]).

For each $n = 1, 2, \ldots$ let

$$M_{\alpha,n} = \{x \in C[0,1] \otimes A \mid x(0) \in A_n,\ \alpha(x(0)) = x(1)\}.$$

Then we obtain the exact sequence of C^*-algebras:

$$0 \longrightarrow \mathrm{S}A \xrightarrow{\iota_n} M_{\alpha,n} \xrightarrow{q_n} A_n \longrightarrow 0$$

from which follow the exact sequences of abelian groups:

$$0 \longrightarrow \mathrm{K}_i(A) \longrightarrow \mathrm{K}_{i+1}(M_{\alpha,n}) \longrightarrow \mathrm{K}_{i+1}(A_n) \longrightarrow 0.$$

Since $\mathrm{K}_i(A_n) \cong \mathbb{Z}^{k_n}$, the above extensions are all trivial.

Let $R = R_\alpha$ and $R_n = R \circ j_{n*} : \mathrm{K}_1(M_{\alpha,n}) \to \mathrm{Aff}(T_A)$, where j_n is the embedding of $M_{\alpha,n}$ into M_α. Since $\mathrm{Range}\, D = \mathrm{Range}\, R_n$, we obtain by Lemma 2.4 that

$$0 \longrightarrow \ker D \xrightarrow{\iota_{n*}} \ker R_n \xrightarrow{q_{n*}} \mathrm{K}_1(A_n) \longrightarrow 0$$

is exact. Note that the inductive limit of these extensions is naturally isomorphic to the exact sequence:

$$0 \to \ker D \to \ker R \to \mathrm{K}_1(A) \to 0.$$

We shall specify a homomorphism φ_n of $\mathrm{K}_1(A_n)$ into $\ker R_n$ such that

$$q_{n*} \circ \varphi_n = \mathrm{id}.$$

Since $\alpha \in \overline{\mathrm{Inn}}(A)$, there exists a $u_n \in U(A)$ for each n such that

$$\alpha | B_n = \mathrm{Ad}\, u_n | B_n, \quad \alpha(z_n) \approx \mathrm{Ad}\, u_n(z_n),$$

where $B_n = \bigoplus_{i=1}^{k_n} B_{n,i} \subset A_n$ and z_n is the canonical unitary of $C(\mathbb{T}) \subset A_n$. Define

$$h_{n,i} = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \mathrm{Ad}\, u_n(z_{n,i})^*$$

where $z_{n,i} = z_n p_{n,i} + 1 - p_{n,i}$ with $p_{n,i}$ the identity of $B_{n,i}$ and $h_{n,i} = h_{n,i}^*$ is defined uniquely as $\|h_{n,i}\| \approx 0$ since $\alpha(z_{n,i}) \mathrm{Ad}\, u_n(z_{n,i}^*) \approx 1$. Define $\zeta_{n,i} \in U(M_{\alpha,n} \otimes M_2)$ by composing two paths of unitaries:

$$[0,1] \ni t \mapsto R_t(u_n^* \oplus 1) R_t^{-1} (u_n \oplus 1)(z_{n,i} \oplus 1)(u_n^* \oplus 1) R_t (u_n \oplus 1) R_t^{-1}$$

and

$$[0,1] \ni t \mapsto \mathrm{e}^{2\pi \mathrm{i} t h_{n,i}} \mathrm{Ad}\, u_n(z_{n,i}) \oplus 1.$$

Then we have that

$$q_n(\zeta_{n,i}) = z_{n,i} \oplus 1, \quad R_n([\zeta_{n,i}]) = \widehat{h}_{n,i},$$

where $\widehat{h}_{n,i} \in \mathrm{Aff}(T_A)$ is defined by

$$\widehat{h}_{n,i}(\tau) = \tau(h_{n,i}), \quad \tau \in T_A.$$

Since the above procedure applies to a unitary $z_n p + 1 - p$ with p a minimal projection in B_n, it follows that $[\zeta_{n,i}] \in \mathrm{K}_1(M_{\alpha,n})$ is divisible by $[n,i]$. Thus one obtains a homomorphism φ_n of $\mathrm{K}_1(A_n)$ into $\mathrm{K}_1(M_{\alpha,n})$ with $q_{n*} \circ \varphi = \mathrm{id}$ by setting

$$\varphi_n : [z_{n,i}] \longmapsto [\zeta_{n,i}].$$

LEMMA 3.2. $\mathrm{Range}\, D$ *is dense in* $\mathrm{Aff}(T_A)$.

Proof. Since A is a simple unital AT algebra of real rank zero, it is approximately divisible ([8]). Thus this is 3.14 (a) of [3]. (A unital C^*-algebra is approximately divisible if it has a central sequence $\{B_n\}$ of unital C^*-subalgebras with $B_n \cong M_2 \oplus M_3$ ([3]). Since A is obtained as the inductive limit of $\{A_n\}$ all being T algebras with unital embeddings and the embeddings need to satisfy only the K-theoretic conditions and the condition of real rank zero ([2]), thanks to Elliott's result [7], we can easily arrange the inductive system so that $A_{n+1} \cap A_n' \supset M_2 \oplus M_3$, which implies that A is approximately divisible.) ∎

Let

$$\delta_n = \min_i \inf\{\tau(p_{n,i});\ \tau \in T_A\},$$

where $p_{n,i}$ is the identity of $B_{n,i}$. Since A is simple, δ_n is strictly positive. We choose the unitary $u_n \in A$ so that $\|h_{n,i}\| < \delta_n$. Since $\mathrm{Range}\, R_n = \mathrm{Range}\, D$, we have, for any $\varepsilon > 0$ with $\|h_{n,i}\| + \varepsilon < \delta_n$, projections $p_\pm \in A$ such that

$$\frac{1}{[n,i]}\widehat{h}_{n,i} = D(p_+) - D(p_-), \quad \|D(p_\pm)\| < \frac{1}{[n,i]}(\|h_{n,i}\| + \varepsilon),$$

where D is also regarded as a map of the projections into $\mathrm{Aff}(T_A)$. (First we approximate $\widehat{h}_{ni+}/[n,i]$ by $D(p_+)$ with p_+ a projection such that $D(p_+) - \widehat{h}_{ni+}/[n,i] > 0$ (or strictly positive), where h_{ni+} is the positive part of $h_{n,i}$. We should note that $\|\widehat{h}_{ni+}/[n,i]\| \leqslant \|h_{n,i}\|/[n,i]$ and find a projection p_- such that $D(p_-) = D(p_+) - \widehat{h}_{n,i}/[n,i] \approx \widehat{h}_{ni-}/[n,i]$.) Since $D(p_\pm) < \delta_n/[n,i] \leqslant D(p_{n,i})/[n,i]$, we find projections $e_{i\pm} \in p_{n,i}Ap_{n,i} \cap B'_{n,i}$ such that

$$\widehat{h}_{n,i} = D(e_{i+}) - D(e_{i-}), \quad \|D(e_{i\pm})\| < \|h_{n,i}\| + \varepsilon.$$

Thus, by making $\|h_{n,i}\|$ small, we can make $\|D(e_{i\pm})\|$ arbitrarily small. Then, by using Lemma 3.4 below, we can find a unitary $w_{n,i} \in p_{n,i}Ap_{n,i} \cap B'_{n,i}$ such that $w_{n,i} = w_{n,i}p_{n,i} + 1 - p_{n,i}$, $\mathrm{Ad}\, w_{n,i}(z_{n,i}) \approx z_{n,i}$, (in the order of $\|h_{n,i}\|$), $\widehat{k}_{n,i} = \widehat{h}_{n,i}$, where

$$k_{n,i} = \frac{1}{2\pi \mathrm{i}} \log\, \mathrm{Ad}\, w_{n,i}(z_{n,i}) z^*_{n,i}.$$

Let $w_n = w_{n1}w_{n2}\cdots w_{nk_n}$. Note that

$$\begin{aligned}\alpha(z_{n,i})\mathrm{Ad}\, u_n w_n(z^*_{n,i}) &= \alpha(z_{n,i})\mathrm{Ad}\, u_n(z^*_{n,i})\mathrm{Ad}\, u_n(z_{n,i}\mathrm{Ad}\, w_n(z^*_{n,i}))\\ &= \mathrm{e}^{2\pi \mathrm{i} h_{n,i}}\mathrm{Ad}\, u_n(\mathrm{e}^{-2\pi \mathrm{i} k_{n,i}}).\end{aligned}$$

Then composing the two paths:

$$[0,1] \ni t \longmapsto \mathrm{Ad}\, u_n(\mathrm{e}^{-2\pi \mathrm{i} t k_{n,i}}) \quad \text{and} \quad [0,1] \ni t \longmapsto \mathrm{e}^{2\pi \mathrm{i} t h_{n,i}}\mathrm{Ad}\, u_n(\mathrm{e}^{-2\pi \mathrm{i} k_{n,i}})$$

multiplied with $\mathrm{Ad}\, u_n w_n(z_{n,i})$ to the right, we obtain a path U from $\mathrm{Ad}\, u_n w_n(z_{n,i})$ to $\alpha(z_{n,i})$ such that

$$\frac{1}{2\pi \mathrm{i}}\int_0^1 \tau(\dot{U}(t)U(t)^*)\mathrm{d}t = 0, \quad \tau \in T_A.$$

Since U is in a small neighbourhood of $\alpha(z_{n,i}) \approx \mathrm{Ad}\, u_n w_n(z_{n,i})$, it follows that the unitary $\zeta_{n,i}$ obtained from $z_{n,i}$ in the same way as before with $u_n w_n$ in place of u_n satisfies

$$R_n([\zeta_{n,i}]) = 0. \quad \blacksquare$$

Thus we have shown:

LEMMA 3.3. *Suppose that $\widetilde{\eta}_0(\alpha) = 0$. Then for any n and $\varepsilon \in (0,1)$ there exists a unitary $u_n \in A$ such that*

$$\alpha|B_n = \operatorname{Ad} u_n|B_n, \quad \|\alpha(z_{n,i}) - z_{n,i}\| < \varepsilon, \quad \widehat{h}_{n,i} = 0,$$

where

$$h_{n,i} = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \operatorname{Ad} u_n(z^*_{n,i}).$$

Hence defining a unitary $\zeta_{n,i} \in M_{\alpha,n} \otimes M_2$ by composing the two paths:

$$[0,1] \ni t \mapsto R_t(u^*_n \oplus 1)R_t^{-1}(\operatorname{Ad} u_n(z_{n,i}) \oplus 1)R_t(u_n \oplus 1)R_t^{-1}$$

and

$$[0,1] \ni t \mapsto \mathrm{e}^{2\pi \mathrm{i} t h_{n,i}} \operatorname{Ad} u_n(z_{n,i}) \oplus 1,$$

where R_t is defined as before, one can define a homomorphism φ_n of $\mathrm{K}_1(A_n)$ into $\ker R_n$ *by $\varphi([z_{n,i}]) = [\zeta_{n,i}]$, $i = 1, \ldots, k_n$.*

LEMMA 3.4. *If $e \in p_{n,i}Ap_{n,i} \cap B'_{n,i}$ is a projection such that $\|D(e)\|$ is sufficiently small, then for any $\varepsilon > 0$ there exists a unitary $w_\pm \in p_{n,i}Ap_{n,i} \cap B'_{n,i}$ such that*

$$\|\operatorname{Ad} w_\pm(z_{n,i}) - z_{n,i}\| < 2\pi\|D(e)\| + \varepsilon, \quad [w_\pm] = 0, \quad B(w_\pm, z_{n,i}) = \pm[e].$$

In particular if $k_\pm = (1/2\pi\mathrm{i}) \log \operatorname{Ad} z_{n,i} w_\pm(z^*_{n,i})$, it follows that $\widehat{k}_\pm = \pm D(e)$.

Proof. To simplify the notation we may suppose that $p_{n,i}A_{n,i}p_{n,i} \cap B'_{n,i}$ to be A and $z_{n,i}$ to be the canonical unitary $z_1 \in A_1 = C(\mathbb{T})$.

Since the projection e plays a role only through $[e]$, we may suppose that $e \in A_m$ for some $m > 1$. We will later assume that m is sufficiently large. Since $A_n \hookrightarrow A_{n+1}$ are in the standard form, $z_1 p_{mj}$ in $B_{mj} \otimes C(\mathbb{T})$ looks like a direct sum of elements of the form:

$$\begin{pmatrix} 0 & & & z_{mj}^{L_s} \\ 1 & \cdot & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} \in M_{M_s}(C(\mathbb{T}))$$

where $L_s = \pm 1$, $M_s \gg 1$ and

$$\sum_s L_s = \chi^1_{m1}(j,1), \quad \sum_s M_s = \chi^0_{m1}(j,1) = [m,j].$$

Note that $D(e)$ takes values in the convex hull of

$$\frac{\dim(ep_{mj})}{[m,j]}, \quad j = 1, \ldots, k_m,$$

which are all assumed to be much less than 1. Let t_m be the maximum of these k_m values. Then t_m decreases as $m \to \infty$ and the limit of t_m equals $\tau(e)$ for some $\tau \in T_A$ (or $\|D(e)\|$). Thus if m is sufficiently large, we may assume that

$t_m < \|D(e)\| + \varepsilon/4\pi$. We can obtain the required unitary w_j in $B_{mj} \otimes C(\mathbb{T})$ as the direct sum of elements of the form:

$$\begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{M_s-1} \end{pmatrix}$$

where $\omega = \mathrm{e}^{-2\pi\mathrm{i}N_s/M_s}$ and the integers N_s are chosen so that

$$\sum N_s = \dim(ep_{mj}), \quad \frac{N_s}{M_s} \approx \frac{\dim(ep_{mj})}{[m,j]}.$$

Note that by defining

$$k_j = \frac{1}{2\pi\mathrm{i}} \log\, z_1 p_{mj} \mathrm{Ad}\, w_j(z_1^* p_{mj}),$$

the Bott element $B(w_j, z_1 p_{mj}) \in \mathrm{K}_0(A_m p_{mj}) = \mathbb{Z}$ for the almost commuting pair $w_j, z_1 p_{mj}$ of unitaries in $A_m p_{mj} = B_{mj} \otimes C(\mathbb{T})$ is equal to

$$\mathrm{Tr}(k_j) = \mathrm{Tr}\Big(\bigoplus_s \frac{N_s}{M_s} 1_s\Big) = \sum N_s = \dim(ep_{mj}),$$

where $k_j \in B_{mj} \otimes C(\mathbb{T})$ should be evaluated at some (or any) point of $\mathbb{T}$ (see [10], [11], [4]). This shows that

$$B(w_j, z_1 p_{mj}) = [ep_{mj}],$$

and in particular that $\widehat{k}_j = D(ep_{mj})$.

If m is sufficiently large or all M_s are sufficiently large, we can assume that

$$\frac{N_s}{M_s} < \|D(e)\| + \varepsilon/2\pi.$$

Thus we obtain the norm estimate

$$\|\mathrm{Ad}\, w_j(z_1 p_{mj}) - z_1 p_{mj}\| < 2\pi\|D(e)\| + \varepsilon.$$

By taking $w_+ = w_1 + w_2 + \cdots + w_{k_m}$, this completes the proof for w_+. For w_- we just replace ω in the definition of w_j by $\overline{\omega} = \mathrm{e}^{2\pi\mathrm{i}N_s/M_s}$. ∎

By defining $\varphi_n : \mathrm{K}_1(A_n) \to \ker R_n$ as above, we identify $\ker R_n$ with $\ker D \oplus \mathrm{K}_1(A_n)$. We now have to translate the natural map $\ker R_n \to \ker R_{n+1}$ into the map $\psi_n : \ker D \oplus \mathrm{K}_1(A_n) \to \ker D \oplus \mathrm{K}_1(A_{n+1})$:

$$\begin{array}{ccccccccc} 0 & \to & \ker D & \to & \ker D & \oplus & \mathrm{K}_1(A_n) & \to & \mathrm{K}_1(A_n) & \to & 0 \\ & & \| & & \| & \psi_n^0 \swarrow & \downarrow \chi_n^1 & & \downarrow \chi_n^1 & & \\ 0 & \to & \ker D & \to & \ker D & \oplus & \mathrm{K}_1(A_{n+1}) & \to & \mathrm{K}_1(A_{n+1}) & \to & 0 \end{array}$$

where we have used that ψ_n must be of the form $\psi_n(a,b) = (a + \psi_n^0(b), \chi_n^1(b))$.

LEMMA 3.5. *If* u_n *is a unitary in* A *and* $\varepsilon \in (0,1)$ *such that*

$$\alpha|B_n = \mathrm{Ad}\, u_n|B_n, \quad \|\alpha(z_n) - \mathrm{Ad}\, u_n(z_n)\| < \varepsilon, \quad \widehat{h}_{n,i} = 0,$$

then for any $m \leqslant n$ *and* $j = 1, \ldots, k_m$,

$$\|\alpha(z_{mj}) - \mathrm{Ad}\, u_n(z_{mj})\| < \varepsilon, \tag{3.1}$$

$$\widehat{h}_{mj} = 0, \tag{3.2}$$

where

$$h_{mj} = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{mj}) \mathrm{Ad}\, u_n(z_{mj}^*).$$

Proof. By the assumption on the embedding of A_m into A_n, (3.1) follows immediately. Since the homomorphism $\varphi_n : \mathrm{K}_1(A_n) \to \ker R_n$ can be defined on $[z_{mj}]$ in the canonical way and $R_n \varphi_n([z_{mj}]) = \widehat{h}_{mj}$, (3.2) also follows immediately. ∎

LEMMA 3.6. *The homomorphism* $\psi_n^0 : \mathrm{K}_1(A_n) \to \ker D$ *is given by*

$$[z_{n,i}] \longmapsto B(u_{n+1}^* u_n, z_{n,i}),$$

where $[z_{n,i}] = [n,i] e_i$ *with* $(e_i)_i$ *the canonical basis for* $\mathbb{Z}^{k_n} = \mathrm{K}_1(A_n)$ *and* $B(u_{n+1}^* u_n, z_{n,i})$ *is divisible by* $[n,i]$.

Proof. First of all we shall show that $D(B(u_{n+1}^* u_n, z_{n,i})) = 0$. If we define the self-adjoint $h_i \in A$ by

$$h_1 = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \mathrm{Ad}\, u_{n+1}(z_{n,i}^*),$$
$$h_2 = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \mathrm{Ad}\, u_n(z_{n,i}^*),$$
$$h_3 = \frac{1}{2\pi \mathrm{i}} \log z_{n,i} \mathrm{Ad}\, (u_{n+1}^* u_n)(z_{n,i}^*),$$

then $\widehat{h}_2 = 0$ and $\widehat{h}_1 = 0$ by Lemma 3.5 and hence $\widehat{h}_3 = 0$ since

$$\mathrm{Ad}\, u_{n+1}(\mathrm{e}^{2\pi \mathrm{i} h_3}) = \mathrm{e}^{-2\pi \mathrm{i} h_1} \mathrm{e}^{2\pi \mathrm{i} h_2}.$$

(One way of proving that $\widehat{h}_3 = 0$ is to take a closed path w of unitaries:

$$w(t) = \begin{cases} \mathrm{e}^{-6\pi \mathrm{i} t h_1} & 0 \leqslant t \leqslant 1/3, \\ \mathrm{e}^{-2\pi \mathrm{i} h_1} \mathrm{e}^{2\pi \mathrm{i}(3t-1) h_2} & 1/3 \leqslant t \leqslant 2/3, \\ \mathrm{e}^{-2\pi \mathrm{i} h_1} \mathrm{e}^{2\pi \mathrm{i} h_2} \mathrm{Ad}\, u_{n+1}(\mathrm{e}^{2\pi \mathrm{i}(3t-2) h_3}) & 2/3 \leqslant t \leqslant 1 \end{cases}$$

in a neighbourhood of 1, and compute for any $\tau \in T_A$,

$$0 = 1/2\pi \mathrm{i} \int_0^1 \tau(\dot{w}(t) w(t)^*)\, \mathrm{d}t = -\tau(h_1) + \tau(h_2) - \tau(h_3).)$$

We may suppose that $u_{n+1}^* u_n \in A_m \cap B_n'$ for some $m > n$. In this case $B(u_{n+1}^* u_n, z_{n,i})$ in $\mathrm{K}_0(A_m)$ is defined by $\left(\mathrm{Tr}_{B_{mj}}(h_3 p_{mj})\right)_j$, where $h_3 p_{mj} \in B_{mj} \otimes C(\mathbb{T})$ is evaluated at a point of $\mathbb{T}$, and $\widehat{h}_3 = 0$ means that for any $\tau \in T_A$,

$$\sum_j \tau(p_{mj}) \frac{\mathrm{Tr}_{B_{mj}}(h_3 p_{mj})}{[m,j]} = 0.$$

Define a path v_{nt}, $t \in [0,1]$ of unitaries in $A \otimes M_2$ by

$$v_{nt} = R_t(u_n^* \oplus 1)R_t^{-1}(u_n \oplus 1).$$

Then to compute $\psi_n^0([z_{n,i}])$ we have to calculate

$$\begin{aligned}\psi_n^0([z_{n,i}]) &= \varphi_n([z_{n,i}]) - \varphi_{n+1}([z_{n,i}]) \\ &= [t \mapsto \operatorname{Ad} v_{n,t}(z_{n,i})] - [t \mapsto \operatorname{Ad} v_{n+1,t}(z_{n,i})]\end{aligned} \tag{3.3}$$

in $\mathrm{K}_1(M_{\alpha,n+1})$ where $z_{n,i}$ is identified with $z_{n,i} \oplus 1$ (see 2.8 of [11] for a similar computation). More precisely, we have to add a short path from $\operatorname{Ad} u_n(z_{n,i})$ (respectively $\operatorname{Ad} u_{n+1}(z_{n,i})$) to $\alpha(z_{n,i})$ to the path $t \mapsto \operatorname{Ad} v_{n,t}(z_{n,i})$ (respectively $t \mapsto \operatorname{Ad} v_{n+1,t}(z_{n,i})$) to get a unitary in $M_{\alpha,n+1} \otimes M_2$ and we always understand the formulae in this way. Note that (3.3) is equal to

$$[t \mapsto \operatorname{Ad} v_{n,t}(z_{n,i}) \operatorname{Ad} v_{n+1,t}(z_{n,i}^*)]$$

in $\mathrm{K}_1(\mathrm{S}A) \subset \mathrm{K}_1(M_{\alpha,n+1})$ or, by applying $t \mapsto \operatorname{Ad} v_{n+1,t}^*$, which induces the identity map on $\mathrm{K}_1(\mathrm{S}A)$, to $[t \mapsto v_{n+1,t}^* v_{n,t} z_{n,i} v_{n,t}^* v_{n+1,t} z_{n,i}^*]$. Since

$$v_{n+1,t}^* v_{n,t} = (u_{n+1}^* \oplus 1)R_t(u_{n+1}u_n^* \oplus 1)R_t^{-1}(u_n \oplus 1),$$

the above element is equal to the class of

$$t \mapsto (u_{n+1}z_{n,i}^*u_{n+1}^* \oplus 1)R_t(u_{n+1}u_n^* \oplus 1)R_t^{-1}(u_n z_{n,i} u_n^* \oplus 1)R_t(u_n u_{n+1}^* \oplus 1)R_t^{-1}$$

by applying $\operatorname{Ad}(u_{n+1}z_{n,i}^* \oplus 1)$. Again this is equal to the class of

$$t \mapsto (u_n^* u_{n+1} z_{n,i}^* u_{n+1}^* u_n \oplus 1)R_t(u_n^* u_{n+1} \oplus 1)R_t^{-1}(z_{n,i} \oplus 1)R_t(u_{n+1}^* u_n \oplus 1)R_t^{-1}$$

by applying $t \mapsto \operatorname{Ad}(u_n^* \oplus u_n^*)$. More precisely, we have to add a short path to connect the value at $t = 1$, $u_n^* u_{n+1} z_{n,i}^* u_{n+1}^* u_n z_{n,i} \oplus 1$ to 1. Since $u_{n+1}^* u_n \in A_m \cap B_n'$ by the assumption, the path can be taken in A_m. The above element in $\mathrm{K}_1(\mathrm{S}A_m) = \mathrm{K}_0(A_m)$ is equal to

$$\begin{aligned}&\left(-\frac{1}{2\pi \mathrm{i}} \mathrm{Tr}_{B_{mj}} \log(u_n^* u_{n+1} z_{n,i}^* u_{n+1}^* u_n z_{n,i} p_{mj})\right)_j \\ &\quad = \left(\frac{1}{2\pi \mathrm{i}} \mathrm{Tr}_{\mathrm{B_{mj}}} \log(z_{n,i}(u_{n+1}^* u_n) z_{n,i}^* (u_{n+1}^* u_n)^* p_{mj})\right)_j \\ &\quad = B_{A_m}(u_{n+1}^* u_n, z_{n,i}).\end{aligned}$$

Note also that since the non-trivial part of $z_{n,i}(u_{n+1}^* u_n) z_{n,i}^* (u_{n+1}^* u_n)^*$ belongs to $p_{n,i} A_m p_{n,i} \cap B_{n,i}'$, each component of $B_{A_m}(u_{n+1}^* u_n, z_{n,i})$ is divisible by $[n,i]$. Then we obtain that

$$\psi_n^0([z_{n,i}]) = B(u_{n+1}^* u_n, z_{n,i}), \quad i = 1, \ldots, k_n,$$

is a well-defined homomorphism of $\mathrm{K}_1(A_n)$ into $\ker D \subset \mathrm{K}_0(A)$. ∎

LEMMA 3.7. *Suppose that* $\widetilde{\eta}_0(\alpha) = 0$. *Then there exist unitaries* $u_n \in A$ *such that*

$$\begin{array}{ll} \alpha|B_n = \mathrm{Ad}\, u_n|B_n, & \\ \|\alpha(z_m) - \mathrm{Ad}\, u_n(z_m)\| < 2^{-n}, & m \leqslant n, \\ B(u_{n+1}^* u_n, z_{n,i}) = 0, & i = 1, \dots, k_n, \\ \widehat{h}_{n,i} = 0, & i = 1, \dots, k_n, \end{array}$$

where

$$h_{n,i} = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \mathrm{Ad}\, u_n(z_{n,i}^*).$$

Proof. By the assumption and Proposition 2.5, the sequence of trivial extensions:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ker D & \longrightarrow & \ker D & \oplus & \mathrm{K}_1(A_n) & \xrightarrow{q_n} & \mathrm{K}_1(A_n) & \longrightarrow & 0 \\ & & \| & & \| & \psi_n^0 \swarrow & \downarrow \chi_n^1 & & \downarrow \chi_n^1 & & \\ 0 & \longrightarrow & \ker D & \longrightarrow & \ker D & \oplus & \mathrm{K}_1(A_{n+1}) & \xrightarrow{q_{n+1}} & \mathrm{K}_1(A_{n+1}) & \longrightarrow & 0 \\ & & \| & & \| & \swarrow & \downarrow & & \downarrow & & \end{array}$$

defines the trivial extension in $\mathrm{Ext}(\mathrm{K}_1(A), \ker D)$. Hence we have a homomorphism $h_n^0 : \mathrm{K}_1(A_n) \to \ker D$ for each n such that

$$\psi_n^0 = h_n^0 - h_{n+1}^0 \chi_n^1.$$

(To see this we denote by E the inductive limit of the middle terms, and by φ a homomorphism of $\mathrm{K}_1(A)$ into E such that $q\varphi = \mathrm{id}$. If ξ_n denotes the natural homomorphism of $\mathrm{K}_1(A_n)$ into $\ker D \oplus \mathrm{K}_1(A_n)$ composed with $\ker D \oplus \mathrm{K}_1(A_n) \to E$, ψ_n^0 is given by $\psi_n^0 = \xi_n - \xi_{n+1}\chi_n^1$. We set $h_n^0 = \xi_n - \varphi_n$ where φ_n is the homomorphism $\mathrm{K}_1(A_n) \to \mathrm{K}_1(A)$ composed with $\varphi : \mathrm{K}_1(A) \to E$. Then it follows that

$$h_n^0 - h_{n+1}^0 \chi_n^1 = \xi_n - \varphi_n - \xi_{n+1}\chi_n^1 + \varphi_{n+1}\chi_n^1 = \xi_n - \xi_{n+1}\chi_n^1 = \psi_n^0,$$

where we have used that $\varphi_n = \varphi_{n+1}\chi_n^1$.)

Since $h_n^0(e_i^n) \in \ker D$, where $(e_i^n)_{i=1}^{k_n}$ is the canonical basis for $\mathbb{Z}^{k_n} = \mathrm{K}_1(A_n)$, we can find projections $e_{i\pm}^n \in p_{n,i} A p_{n,i} \cap B'_{n,i}$ such that

$$[n,i] h_n^0(e_i^n) = [e_{i+}^n] - [e_{i-}^n]$$

and $\|D(e_{i\pm}^n)\|$ is arbitrarily small. (We find a positive $g \in \mathrm{K}_0(A)$ with $\|D(g)\|$ sufficiently small and then find projections $e_{i\pm}^n$ such that $[e_{i+}^n] = [n,i](g + h_n^0(e_i^n))$ and $[e_{i-}^n] = [n,i]g$.) Then, by Lemma 3.4, we find a unitary $w_n \in A \cap B'_n$ such that

$$[w_n] = 0, \quad B(w_n, z_{n,i}) = -[e_{i+}^n] + [e_{i-}^n] = -[n,i] h_n^0(e_i^n)$$

and $\|[w_n, z_{n,i}]\|$ is arbitrarily small for $i = 1, \dots, k_n$. Since

$$\begin{aligned} B(w_{n+1}^*, z_{n,i}) &= \sum_j B(w_{n+1}^* p_{n+1,j}, z_{n,i} p_{n+1,j}) \\ &= \sum_j \chi_n^1(j,i)[n,i] B(w_{n+1}^*, z_{n+1,j})/[n+1,j] \\ &= \sum_j \chi_n^1(j,i)[n,i] h_{n+1}^0(e_j^{n+1}) = [n,i] h_{n+1}^0 \chi_n^1(e_i^n), \end{aligned}$$

we have that

$$B(w_{n+1}^* u_{n+1}^* u_n w_n, z_{n,i}) = 0.$$

Since $D(B(w_n, z_{n,i})) = 0$, we have that $\widehat{k}_i = 0$ for $k_i = (1/2\pi\mathrm{i}) \log z_{n,i} \mathrm{Ad}\, w_n(z_{n,i}^*)$, and hence that $\widehat{h}_i = 0$ for $h_i = (1/2\pi\mathrm{i}) \log \alpha(z_{n,i}) \mathrm{Ad}\, u_n w_n(z_{n,i}^*)$. Thus by replacing u_n by $u_n w_n$, we have the conclusion. ∎

Note that the exact sequence

$$0 \longrightarrow \mathrm{K}_1(A) \longrightarrow \mathrm{K}_0(M_\alpha) \longrightarrow \mathrm{K}_0(A) \longrightarrow 0$$

is obtained as the inductive limit of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_1(A) & \longrightarrow & \mathrm{K}_0(M_{\alpha,n}) & \longrightarrow & \mathrm{K}_0(A_n) & \longrightarrow & 0 \\ & & \| & & \downarrow \psi_n & & \downarrow \chi_n^0 & & \\ 0 & \longrightarrow & \mathrm{K}_1(A) & \longrightarrow & \mathrm{K}_0(M_{\alpha,n+1}) & \longrightarrow & \mathrm{K}_0(A_{n+1}) & \longrightarrow & 0. \\ & & \| & & \downarrow & & \downarrow & & \end{array}$$

By defining a homomorphism $\varphi_n : \mathrm{K}_0(A_n) \to \mathrm{K}_0(M_{\alpha,n})$ just as in Lemma 3.3, we identify $\mathrm{K}_0(M_{\alpha,n})$ with $\mathrm{K}_1(A) \oplus \mathrm{K}_0(A_n)$ and find a homomorphism $\psi_n^1 : \mathrm{K}_0(A_n) \to \mathrm{K}_1(A)$ as in the following diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_1(A) & \longrightarrow & \mathrm{K}_1(A) & \oplus & \mathrm{K}_0(A_n) & \longrightarrow & \mathrm{K}_0(A_n) & \longrightarrow & 0 \\ & & \| & & \| & \psi_n^1 \swarrow & \downarrow \chi_n^0 & & \downarrow \chi_n^0 & & \\ 0 & \longrightarrow & \mathrm{K}_1(A) & \longrightarrow & \mathrm{K}_1(A) & \oplus & \mathrm{K}_0(A_{n+1}) & \longrightarrow & \mathrm{K}_0(A_{n+1}) & \longrightarrow & 0. \\ & & \| & & \| & \swarrow & \downarrow & & \downarrow & & \end{array}$$

LEMMA 3.8. *The homomorphism* $\psi_n^1 : \mathrm{K}_0(A_n) \to \mathrm{K}_1(A)$ *is given by*

$$[p_{n,i}] \mapsto [u_{n+1}^* u_n p_{n,i}]$$

where $[p_{n,i}] = [n,i] e_i$ *with* (e_i) *the canonical basis for* $\mathbb{Z}^{k_n} = \mathrm{K}_0(A_n)$ *and* $[u_{n+1}^* u_n p_{n,i}]$ *is divisible by* $[n,i]$.

Proof. As in the proof of Lemma 3.6 we have to decide

$$[t \mapsto \mathrm{Ad}\, v_{n,t}(p_{n,i})] - [t \mapsto \mathrm{Ad}\, v_{n+1,t}(p_{n,i})] \tag{3.4}$$

in $\mathrm{K}_0(M_{\alpha,n+1})$, where $p_{n,i}$ denotes $p_{n,i} \oplus 0$ in $A \otimes M_2$. (Note that $\mathrm{Ad}\, u_n(p_{n,i}) = \alpha(p_{n,i})$ and $\mathrm{Ad}\, u_{n+1}(p_{n,i}) = p_{n,i}$.) Note that the identification of $\mathrm{K}_1(A)$ with $\mathrm{K}_0(\mathrm{S}A)$ is done in such a way that $[u_n]$ corresponds to

$$[t \mapsto \mathrm{Ad}\, v_{n,t}(1 \oplus 0)] - [(1 \oplus 0)]$$

([1], 8.2.2). Since

$$[t \mapsto \mathrm{Ad}\, v_{n,t}(p_{n,i})] = [t \mapsto \mathrm{Ad}\, v_{n,t}(1 \oplus 0)] - [t \mapsto \mathrm{Ad}\, v_{n,t}(1 - p_{n,i})],$$

(3.4) equals

$$\begin{aligned} &[t \mapsto \mathrm{Ad}\, v_{n,t}(1 \oplus 0)] - [t \mapsto \mathrm{Ad}\, (v_{n,t}(1 - p_{n,i}) + v_{n+1,t} p_{n,i})(1 \oplus 0)] \\ &\quad = [u_n] - [u_n(1 - p_{n,i}) + u_{n+1} p_{n,i}] = [u_{n+1}^* u_n p_{n,i}], \end{aligned}$$

where we have used the fact that

$$t \mapsto v_{n,t}((1 - p_{n,i}) \oplus (1 - \alpha(p_{n,i})) + v_{n+1,t}(p_{n,i} \oplus \alpha(p_{n,i}))$$

is a path of unitaries from $1 \oplus 1$ to

$$(u_n(1-p_{n,i}) + u_{n+1}p_{n,i}) \oplus (u_n^*(1-\alpha(p_{n,i})) + u_{n+1}^*\alpha(p_{n,i})). \quad \blacksquare$$

LEMMA 3.9. *Suppose that* $\widetilde{\eta}(\alpha) = 0$. *Then there is a unitary* $u_n \in A$ *for each* n *such that*

$$\alpha|B_n = \operatorname{Ad} u_n|B_n, \quad \|\alpha(z_m) - \operatorname{Ad} u_n(z_m)\| < 2^{-n}, \quad m \leqslant n,$$
$$B(u_{n+1}^* u_n, z_{n,i}) = 0, \quad [u_{n+1}^* u_n p_{n,i}] = 0, \quad \widehat{h}_{n,i} = 0$$

for $i = 1, \ldots, k_n$, *where*

$$h_{n,i} = \frac{1}{2\pi \mathrm{i}} \log \alpha(z_{n,i}) \operatorname{Ad} u_n(z_{n,i}^*).$$

Proof. Comparing with Lemma 3.7, the newly appeared conditions are only

$$[u_{n+1}^* u_n p_{n,i}] = 0.$$

We will find a unitary $w_n \in A \cap B_n'$ such that $[w_n, z_n] = 0$ and the above conditions are satisfied by replacing all u_n by $u_n w_n$. With the condition $[w_{n+1}, z_{n+1}] = 0$, it follows that $[w_{n+1}, z_n] = 0$ and that the other conditions are preserved.

From the assumption that

$$0 \longrightarrow \mathrm{K}_1(A) \longrightarrow \mathrm{K}_0(M_\alpha) \longrightarrow \mathrm{K}_0(A) \longrightarrow 0$$

is trivial, we have a homomorphism $h_n^1 : \mathrm{K}_0(A_n) \to \mathrm{K}_1(A)$ for each n such that

$$\psi_n^1 = h_n^1 - h_{n+1}^1 \chi_n^0.$$

We only have to find a unitary $w_n \in A \cap B_n'$ such that $[w_n, z_n] = 0$ and

$$[w_n p_{n,i}] = -[n,i] h_n^1(e_i), \quad i = 1, \ldots, k_n.$$

Since $z_n p_{n,i}$ in $p_{n,i} A_m p_{n,i} \cap B_{n,i}'$ for $m > n$ is a direct sum of elements of the form

$$\begin{pmatrix} 0 & & & z_{n+1}^L p_{n,i} \\ 1 & \cdot & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

with $L = \pm 1$, this follows immediately. $\blacksquare$

Proof of (i) $\Rightarrow$ (ii) *of Theorem* 3.1. Under the assumption (i) we have found a sequence $\{u_n\}$ of unitaries as in the previous lemma. Now we apply the homotopy lemma to the pair $u_{n+1}^* u_n p_{n,i}, z_n p_{n,i}$ of unitaries in $p_{n,i} A p_{n,i} \cap B_{n,i}'$ ([4], 8.1): From the conditions

$$B(u_{n+1}^* u_n, z_{n,i}) = 0, \quad [u_{n+1}^* u_n p_{n,i}] = 0$$

calculated in $p_{n,i} A p_{n,i} \cap B_{n,i}'$, that follow since $\mathrm{K}_*(p_{n,i} A p_{n,i} \cap B_{n,i}') \to \mathrm{K}_*(p_{n,i} A p_{n,i}) \to \mathrm{K}_*(A)$ are injective, and the condition $\|[u_{n+1}^* u_n, z_n]\| \to 0$ as $n \to \infty$, we obtain a continuous path $v_{n,i;t}$ of unitaries in $p_{n,i} A p_{n,i} \cap B_{n,i}'$ such that

$$v_{n,i;0} = p_{n,i}, \quad v_{n,i;1} = u_n^* u_{n+1} p_{n,i}$$

and

$$\max_t \|[v_{n,i;t}, z_{n,i}]\| \longrightarrow 0 \text{ as } n \to \infty.$$

Let $v_{n;t} = \sum_i v_{n,i;t}$, and define a continuous path v_t of unitaries for $t \in [1, \infty)$ by

$$v_1 = u_1,$$
$$v_{n+t} = u_n v_{n;t}, \quad 0 \leqslant t \leqslant 1$$

for $n = 1, 2, \ldots$. Then since $\max_t \|[v_{n;t}, z_m]\| \longrightarrow 0$ as $n \to \infty$, we obtain that for any m, $\lim_{t\to\infty} \operatorname{Ad} v_t(z_m) = \alpha(z_m)$. We also have that for $t \geqslant m$ and $a \in B_m$ $\operatorname{Ad} v_t(a) = \alpha(a)$. Thus it follows that for any $x \in A$ $\lim_{t\to\infty} \operatorname{Ad} v_t(x) = \alpha(x)$. This completes the proof. ∎

4. MAIN THEOREM

Proposition 4.1. *If* $\varphi \in \operatorname{Hom}(\mathrm{K}_1(A), \operatorname{Aff}(T_A))$, *there exists an automorphism* $\alpha \in \overline{\operatorname{Inn}}(A)$ *such that* $\eta(\alpha)$ *is trivial and the rotation map* $R_\alpha : \mathrm{K}_1(M_\alpha) \to \operatorname{Aff}(T_A)$ *is given by*

$$R_\alpha(a, b) = D(a) + \varphi(b)$$

for some identification of $\mathrm{K}_1(M_\alpha)$ *with* $\mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$.

To prove this we first prepare:

Lemma 4.2. *If* $\varphi \in \operatorname{Hom}(\mathrm{K}_1(A), \operatorname{Aff}(T_A))$, *there exists an inductive system*

$$\mathbb{Z}^{k_1} \xrightarrow{\chi_1^i} \mathbb{Z}^{k_2} \xrightarrow{\chi_2^i} \mathbb{Z}^{k_3} \longrightarrow \cdots$$

whose limit is isomorphic to $\mathrm{K}_i(A)$ *for* $i = 0, 1$ *and homomorphisms* $h_n : \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_{n+1}}$ *such that*

$$|\varphi \circ \chi^1_{\infty,n-1}(e_j^{n-1}) - D \circ \chi^0_{\infty,n} \circ h_{n-1}(e_j^{n-1})| < 2^{-n+1} \ell_{n-1}^{-1} D \circ \chi^0_{\infty,n-1}(e_j^{n-1}),$$
$$|h_{n-1} \circ \chi^1_{n-2}(e_j^{n-2}) - \chi^0_{n-1} \circ h_{n-2}(e_j^{n-2})| < 2^{-n+3} \ell_{n-2}^{-1} \chi^0_{n,n-2}(e_j^{n-2}),$$
$$|\chi^0_n(i,j)| \geqslant 2^{n+1} \max(|\chi^1_n(i,j)|, 1),$$

where that $|x| < y$ *for* $x, y \in \mathbb{Z}^{k_n}$ *means that* $|x_i| < y_i$ *for all* i, $(e_j^n)_j$ *is the canonical basis for* $\mathbb{Z}^{k_n}$,

$$\ell_n = \max\{[n, j] \mid j = 1, \ldots, k_n\},$$

and $([n, j])_j \in \mathbb{Z}^{k_n}$ *corresponds to* $[1] \in \mathrm{K}_0(A)$.

Proof. Suppose that we are given inductive systems

$$\mathbb{Z}^{k_1} \xrightarrow{\chi_1^i} \mathbb{Z}^{k_2} \longrightarrow \cdots$$

such that the limit is isomorphic to $\mathrm{K}_i(A)$ for $i = 0, 1$, and

$$\chi^0_n(i,j) \geqslant 2^{n+1} \max(|\chi^1_n(i,j)|, 1).$$

By passing to a subsequence we construct the homomorphisms h_n with the required properties.

Suppose that we have constructed $h_1, \ldots, h_{n-1}$ and fixed $\mathbb{Z}^{k_1}, \ldots, \mathbb{Z}^{k_n}$. Then we compute ℓ_n and find $\xi : \mathbb{Z}^{k_n} \to \mathrm{K}_0(A)$ such that

$$|\varphi\chi^1_{\infty,n}(e^n_j) - D\xi(e^n_j)| < 2^{-n}\ell_n^{-1} D\chi^0_{\infty,n}(e^n_j).$$

This is obviously possible by the density of $\operatorname{Range} D$ and

$$\inf_{\tau \in T_A} D\chi^0_{\infty,n}(e^n_j)(\tau) > 0.$$

Then we find an $m > n$ such that $\operatorname{Range}\xi \subset \operatorname{Range}\chi^0_{\infty,m}$, and $\eta : \mathbb{Z}^{k_n} \to \mathbb{Z}^m$ such that

$$\begin{array}{ccc} \mathbb{Z}^{k_n} & \xrightarrow{\eta} & \mathbb{Z}^m \\ & \xi \searrow \quad \swarrow \chi^0_{\infty,m} & \\ & \mathrm{K}_0(A) & \end{array}$$

is commutative. Note

$$\begin{aligned}
&|D\chi^0_{\infty,m}\eta\chi^1_{n-1}(e^{n-1}_j) - D\chi^0_{\infty,n}h_{n-1}(e^{n-1}_j)| \\
&\quad \leqslant |D\chi_{\infty,m}\eta\chi^1_{n-1}(e^{n-1}_j) - \varphi\chi^1_{\infty,n-1}(e^{n-1}_j)| \\
&\qquad + |\varphi\chi^1_{\infty,n-1}(e^{n-1}_j) - D\chi^0_{\infty,n}h_{n-1}(e^{n-1}_j)| \\
&\quad < 2^{-n}\ell_n^{-1}\sum_{i=1}^{k_n} D\chi^0_{\infty,n}(e^n_i)|\chi^1_{n-1}(i,j)| + 2^{-n+1}\ell_{n-1}^{-1}D\chi^0_{\infty,n-1}(e^{n-1}_j) \\
&\quad < 2^{-n}\ell_n^{-1}\sum_{i} D\chi^0_{\infty,n}(e^n_i)\chi^0_{n-1}(i,j) + 2^{-n+1}\ell_{n-1}^{-1}D\chi^0_{\infty,n-1}(e^{n-1}_j) \\
&\quad < (2^{-n}\ell_n^{-1} + 2^{-n+1}\ell_{n-1}^{-1})D\chi^0_{\infty,n-1}(e^{n-1}_j) \\
&\quad < 2^{-n+2}\ell_{n-1}^{-1}D\chi^0_{\infty,n-1}(e^{n-1}_j).
\end{aligned}$$

Thus by choosing a sufficiently large $\ell > m$ it follows that

$$|\chi^0_{\ell,m}\eta\chi^1_{n-1}(e^{n-1}_j) - \chi^0_{\ell,n}h_{n-1}(e^{n-1}_j)| < 2^{-n+2}\ell_{n-1}^{-1}\chi^0_{\ell,n-1}(e^{n-1}_j).$$

By taking $\mathbb{Z}^{k_\ell}$ for $\mathbb{Z}^{k_{n+1}}$ and $\chi^0_{\ell,m}\eta$ for h_n, the lemma is proved. ∎

Proof of Proposition 4.1. By the previous lemma we have the following diagram:

$$\begin{array}{cccccccccc}
\longrightarrow & \mathbb{Z}^{k_n} & \xrightarrow{\chi^1_n} & \mathbb{Z}^{k_{n+1}} & \longrightarrow & & & \cdots & \longrightarrow & \mathrm{K}_1(A) \\
& & \searrow h_n & & \searrow h_{n+1} & & & & & \\
\longrightarrow & \mathbb{Z}^{k_n} & \xrightarrow{\chi^0_n} & \mathbb{Z}^{k_{n+1}} & \longrightarrow & \mathbb{Z}^{k_{n+2}} & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0(A)
\end{array}$$

with the specified properties. Accordingly, we construct an increasing sequence $\{A_n\}$ of T algebras such that

$$A_n = B_n \otimes C(\mathbb{T}), \quad B_n = \bigoplus_{i=1}^{k_n} B_{n,i}, \quad B_{n,i} \cong M_{[n,i]}$$

and the embeddings of A_n into A_{n+1} are in the standard form. By Elliott's theory ([7]), we identify $\overline{\bigcup_{n=1}^{\infty} A_n}$ with A.

Define $\psi_n^0 : \mathrm{K}_1(A_n) \to \mathrm{K}_0(A_{n+2})$ by

$$\psi_n^0 = h_{n+1}\chi_n^1 - \chi_{n+1}^0 h_n.$$

By the properties specified in Lemma 4.2 we have that

$$|\psi_n^0(i,j)| < 2^{-n+1}\ell_n^{-1}\chi_{n+2,n}^0(i,j).$$

Then, by Lemma 3.4 (and its proof), we find a unitary $w_{nj} \in B_{n+2} \cap B_n'$ such that

$$\begin{aligned} &w_{nj} = w_{nj}p_{nj} + 1 - p_{nj}, \\ &\|\mathrm{Ad}\, w_{nj}(z_{nj}) - z_{nj}\| \leqslant 3\pi 2^{-n+1}, \\ &B(w_{nj}, z_{nj}) = -[n,j]\psi_n^0(e_j^n). \end{aligned}$$

(Because z_{nj} in $B_{n+2,i} \otimes C(\mathbb{T})$ is a direct sum of elements of the form as in the proof of Lemma 3.4 such that the matrix sizes M_s are at least 2^{2n}; hence the error introduced by choosing N_s in that proof will be of the order 2^{-2n}.) If w_n denotes $w_{n1}w_{n2}\cdots w_{nk_n}$, then we have that

$$\begin{aligned} &w_n \in B_{n+2} \cap B_n', \quad \|\mathrm{Ad}\, w_n(z_n) - z_n\| \leqslant 3\pi 2^{-n+1}, \\ &B(w_n, z_{nj}) = -[n,j]\psi_n^0(e_j^n), \quad [w_n p_{nj}] = 0. \end{aligned}$$

We define the following two automorphisms β_0, β_1 of A by

$$\beta_0 = \lim_{n\to\infty} \mathrm{Ad}\,(w_2 w_4 \cdots w_{2n}), \quad \beta_1 = \lim_{n\to\infty} \mathrm{Ad}\,(w_1 w_3 \cdots w_{2n-1}).$$

To show the limits exist, note that $[w_m, w_n] = 0$ if $|m-n| \geqslant 2$ and the limits obviously exist on $\bigcup_{n=1}^{\infty} B_n$. Since $\mathrm{Ad}\,(w_n w_{n+2} \cdots w_{n+2k})(z_n)$ in A_{n+2k+2} is a direct sum of elements of the form

$$\begin{pmatrix} 0 & & & z_{n+2k+2}^L \\ 1 & \cdot & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

with $L = \pm 1$, we have that

$$\|\mathrm{Ad}\,(w_n \cdots w_{n+2k} w_{n+2k+2})(z_n) - \mathrm{Ad}\,(w_n \cdots w_{n+2k})(z_n)\| < 3\pi 2^{-(n+2k+1)}.$$

Then it also follows that the limits exist on $z_1, z_2, \ldots$. Since the same reasoning applies to the inverses, we have shown that β_0, β_1 exist as automorphisms.

Now we shall show that the product $\beta_0\beta_1$ has the required properties.

By [11], 2.4, the extension $\eta_1(\beta_i)$

$$0 \longrightarrow \mathrm{K}_1(A) \longrightarrow \mathrm{K}_0(M_{\beta_i}) \longrightarrow \mathrm{K}_0(A) \longrightarrow 0$$

is trivial for $i = 0, 1$ and the extension $\eta_0(\beta_i)$

$$0 \longrightarrow \mathrm{K}_0(A) \longrightarrow \mathrm{K}_1(M_{\beta_i}) \longrightarrow \mathrm{K}_1(A) \longrightarrow 0$$

is given as the inductive limit of

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{Z}^{k_n} & \longrightarrow & \mathbb{Z}^{k_n} & \oplus & \mathbb{Z}^{k_n} & \longrightarrow & \mathbb{Z}^{k_n} & \longrightarrow & 0 \\ & & \chi_{n+2,n}^0 \downarrow & & \downarrow & \psi_n^0 \swarrow & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^{k_{n+2}} & \longrightarrow & \mathbb{Z}^{k_{n+2}} & \oplus & \mathbb{Z}^{k_{n+2}} & \longrightarrow & \mathbb{Z}^{k_{n+2}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \swarrow & \downarrow & & \downarrow & & \end{array}$$

with $n \equiv i \pmod 2$. Hence $\eta_1(\beta_0\beta_1) = \eta_1(\beta_0) + \eta_1(\beta_1) = 0$. We will compute $\eta_0(\beta_0) + \eta_0(\beta_1)$ below.

Define

$$E = \{(x, y) \in \mathrm{K}_1(M_{\beta_0}) \oplus \mathrm{K}_1(M_{\beta_1}) \mid q(x) = q(y)\}/\{(a, -a) \mid a \in \mathrm{K}_0(A)\}.$$

If $g \in \mathrm{K}_1(A)$ is the image of $x_{2n+1} \in \mathbb{Z}^{k_{2n+1}}$, define $\eta_n : \mathrm{Range}\,\chi^1_{\infty,2n+1} \to \mathrm{K}_1(M_{\beta_0}) \oplus \mathrm{K}_1(M_{\beta_1})$ by

$$\eta_n(g) = (h_{2n+1}(x_{2n+1}), x_{2n+2}) \oplus (0, x_{2n+1}),$$

where the right hand side should be regarded as an element of $\mathrm{K}_1(M_{\beta_0}) \oplus \mathrm{K}_1(M_{\beta_1})$. Then

$$\begin{aligned}
&\eta_{n+1}(g) - \eta_n(g)\\
&\quad = (h_{2n+3}(x_{2n+3}) - \psi^0_{2n+2}(x_{2n+2}) - \chi^0_{2n+4,2n+2} h_{2n+1}(x_{2n+1}),\, 0)\\
&\qquad \oplus (-\psi^0_{2n+1}(x_{2n+1}), 0)\\
&\quad = (\chi^0_{2n+3} h_{2n+2}(x_{2n+2}) - \chi^0_{2n+4,2n+2} h_{2n+1}(x_{2n+1}),\, 0)\\
&\qquad \oplus (-h_{2n+2}(x_{2n+2}) + \chi^0_{2n+2} h_{2n+1}(x_{2n+1}), 0).
\end{aligned}$$

Thus (η_n) gives a well-defined homomorphism $\eta : \mathrm{K}_1(A) \to E$ such that $q\eta = \mathrm{id}$. This shows that $\eta_0(\beta_0\beta_1) = 0$.

Let $u_n = w_n w_{n-2} \cdots$. We take a path $v(t)$ of unitaries in $A \otimes M_2$ from z_{nj} to $\beta_0(z_{nj})$ by composing the following two paths for even $m \geqslant n$:

$$v_1(t) = R_t(1 \oplus u_m)R_t^{-1}(z_{nj} \oplus 1)R_t(1 \oplus u_m^*)R_t^{-1},$$

and a short path v_2 from $\mathrm{Ad}\,u_m(z_{nj})$ to $\beta_0(z_{nj})$. For $\tau \in T_A$ we want to compute

$$\frac{1}{2\pi \mathrm{i}} \int_0^1 \tau(\dot{v}(t)v(t)^*)\mathrm{d}t.$$

We know the contribution from v_1 is zero and the contribution from v_2 is given by

$$\lim_{k\to\infty} \tau(B(w^*_{m+2}w^*_{m+4}\cdots w^*_{m+2k}, z_{nj})) = \lim \tau\Big(\sum_{i=1}^k \chi^0_{\infty,2m+2i+2}\psi^0_{m+2i}\chi^1_{m+2i,n}(e^n_j)\Big).$$

Thus we obtain that

$$R_{\beta_0}([v]) = \sum_{i=1}^\infty D\chi^0_{\infty,2m+2i+2}\psi^0_{m+2i}\chi^1_{m+2i,n}(e^n_j).$$

A similar computation applies to β_1. For an odd n we let $m = n-1$ for computing $r_0 = R_{\beta_0}([v])$ and let $m = n$ for computing the corresponding r_1, and obtain that

$$\begin{aligned}
r_0 + r_1 &= \sum_{i=1}^\infty D\chi^0_{\infty,n+i+2}\psi^0_{n+i}\chi^1_{n+i,n}(e^n_j)\\
&= \sum_{i=1}^\infty \big(D\chi^0_{\infty,n+i+2}h_{n+i+1}\chi^1_{n+i+1,n}(e^n_j) - D\chi^0_{\infty,n+i+1}h_{n+i}\chi^1_{n+i,n}(e^n_j)\big)\\
&= \varphi\chi^1_{\infty,n}(e^n_j) - D\chi^0_{\infty,n+1}h_{n+1}\chi^1_n(e^n_j).
\end{aligned}$$

Under the identification of $\mathrm{K}_1(M_{\beta_0\beta_1})$ with $\mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ specified above, the above element corresponds to $(-h_{n+1}\chi^1_n(e^n_j), [z_{nj}])$. This implies that $R_{\beta_0\beta_1}$ satisfies the required properties. ∎

Let Q be the homomorphism of $\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ into $\mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ defined by $[(E, R)] \mapsto [E]$. Then $\ker Q$ is the subgroup of the isomorphism classes of (E_0, R_φ) where E_0 is the trivial extension $\mathrm{K}_1(A) \oplus \mathrm{K}_0(A)$, and $R_\varphi : E_0 \to \mathrm{Aff}(T_A)$ is determined by $\varphi \in \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{Aff}(T_A))$ as in the previous proposition:

$$R_\varphi : (a, b) \mapsto D(a) + \varphi(b).$$

Proposition 4.3. *The following sequences of abelian groups are exact:*

$$0 \longrightarrow \ker Q \longrightarrow \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \xrightarrow{Q} \mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Hom}(\mathrm{K}_1(A), \ker D) \longrightarrow \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_0(A))$$
$$\longrightarrow \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{Aff}(T_A)) \longrightarrow \ker Q \longrightarrow 0.$$

Proof. For the first sequence we only have to show that Q is surjective. Given an extension

$$0 \longrightarrow \mathrm{K}_0(A) \longrightarrow E \longrightarrow \mathrm{K}_1(A) \longrightarrow 0,$$

we regard $\mathrm{K}_0(A)$ as a subgroup of E and have to extend $D : \mathrm{K}_0(A) \to \mathrm{Aff}(T_A)$ to a homomorphism $R : E \to \mathrm{Aff}(T_A)$. This can be done step by step by using the fact that $\mathrm{Aff}(T_A)$ is divisible.

For the second sequence we only have to show that (E_0, R_φ) and (E_0, R_ψ) are isomorphic if and only if $\varphi = \psi + D \circ h$ for some $h \in \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_0(A))$. This follows because an isomorphism $\mu : E_0 \to E_0$ is given by

$$\mu : (a, b) \mapsto (a + h(b), b)$$

for some $h \in \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_0(A))$ with $R_\psi \circ \mu = R_\varphi$. ∎

Theorem 4.4. *Let A be a simple unital* AT *algebra of real rank zero, $\overline{\mathrm{Inn}}(A)$ the group of approximately inner automorphisms of A, and $\mathrm{AInn}(A)$ the group of asymptotically inner automorphisms of A. Then $\mathrm{AInn}(A)$ is a normal subgroup of $\overline{\mathrm{Inn}}(A)$ and the quotient $\overline{\mathrm{Inn}}(A)/\mathrm{AInn}(A)$ is isomorphic to*

$$\mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$$

with isomorphism induced by $\widetilde{\eta}$.

Proof. Before Theorem 3.1 we have described the homomorphism

$$\widetilde{\eta} : \overline{\mathrm{Inn}}(A) \to \mathrm{OrderExt}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A)),$$

and showed in Theorem 3.1 that $\ker \widetilde{\eta} = \mathrm{AInn}(A)$. By 3.1 of [11] we have shown that $\eta = (\eta_0, \eta_1) = (Q\widetilde{\eta}_0, \eta_1)$ is surjective onto $\mathrm{Ext}(\mathrm{K}_1(A), \mathrm{K}_0(A)) \oplus \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(A))$. By Proposition 4.1 we know that $\mathrm{Range}\, \widetilde{\eta}$ contains $\ker Q$, which shows that $\widetilde{\eta}$ is surjective. This completes the proof. ∎

EXAMPLE 4.5. If A is the irrational rotation C^*-algebra generated by unitaries u, v with $uvu^*v^* = e^{2\pi i\theta}1$ for some irrational number $\theta \in (0,1)$, then A is a simple unital AT algebra of real rank zero by [9], and $K_i(A) \cong \mathbb{Z}^2$ and hence $\mathrm{Ext}(K_i(A), K_{i+1}(A)) = 0$. But since A has only one tracial state and Range $D = \mathbb{Z}+\theta\mathbb{Z}$, it follows that $\mathrm{Hom}(K_1(A), \mathrm{Aff}(T_A)) \cong \mathbb{R}^2$ and $\mathrm{OrderExt}(K_1(A), K_0(A)) \cong \mathbb{R}^2/(\mathbb{Z}+\theta\mathbb{Z})^2$ which is isomorphic to $\overline{\mathrm{Inn}}(A)/\mathrm{AInn}(A)$. Note also that $\mathrm{HInn}(A) = \overline{\mathrm{Inn}}(A)$ in this case since the natural $\mathbb{T}^2$ action on A exhausts all OrderExt.

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