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EXTENSIONS OF SEMIGROUPS OF OPERATORS

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Abstract. Let T be a representation of an abelian semigroup S on a Banach space X. We identify a necessary and sufficient condition, which we name superexpansiveness, for T to have an extension to a representation U on a Banach space Y containing X such that each $U(t)$ $(t \in S)$ has a contractive inverse. Although there are many such extensions (Y, U) in general, there is a unique one which has a certain universal property. The spectrum of this extension coincides with the unitary part of the spectrum of T, so various results in spectral theory of group representations can be extended to superexpansive representations.

Keywords: *Extension, semigroup, representation, expansive, unitary spectrum, approximate eigenvalue, generic, isometry, superreflexive.*

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1. INTRODUCTION

The spectral theory of bounded or non-quasianalytic representations of locally compact, abelian groups on Banach spaces, as developed by Arveson ([2]) and Lyubich ([17]), is very powerful. In particular, there are spectral subspaces associated with the representations and (weak) spectral mapping theorems hold. In general, the spectral theory of representations of semigroups is much less powerful. However, Douglas ([10]) showed that any semigroup of isometries can be extended to a group of isometries on a larger space. Since this construction preserves the unitary part of the spectrum of the representation, it is immediately possible to make certain deductions about semigroups of isometries ([4]). These results have played an important role in the stability theory of bounded one-parameter, and (especially) more general, semigroups of operators (see the survey articles [3], Section 5, [24], Section 4).

In this paper, we consider the question when an abelian semigroup of operators can be extended to a semigroup of invertible operators with contractive inverses on a larger space. An obvious necessary condition is that each operator is expansive in the sense that $\|Tx\| \geqslant \|x\|$ for all x. We shall see that this condition is sufficient as well as necessary in the case of a single operator or a one-parameter semigroup of operators (see [19] and [21] for a related result about extending a single operator). However, the necessary and sufficient condition, superexpansiveness, for a general semigroup, is a little stronger as it has to reflect the structure of the semigroup. This condition is closely related to Arens's condition for the existence of inverses in an extension of a Banach algebra ([1]). Indeed, it is possible to use Arens's result, together with a construction of Müller ([19]) (we are grateful to the referee for bringing [19] to our attention) to show that superexpansiveness is sufficient, although we give a more direct construction of an extension. Unlike the isometric case, the extension is not unique, but as in the case of isometric dual representations ([5]), there is a unique extension which has a certain universal property. For results about dilations of operators which are somewhat parallel to ours, see [23].

Section 2 contains the definition and background material on superexpansiveness, and the construction of the extension of a superexpansive representation is in Section 3. In Section 4, we show that the extension of a bounded semigroup of operators on a superreflexive Banach space acts on a superreflexive space.

In Section 5, we give some applications to spectral theory of non-quasianalytic representations of semigroups. We extend the notion of unitary spectrum to these representations, and in many cases it consists exactly of the unimodular approximate eigenvalues. For a superexpansive representation, the spectrum is preserved by the generic extension, so we are able to deduce various results for superexpansive, non-quasianalytic semigroup representations from known results for group representations.

As in the case of bounded representations, our results have applications to the (weighted) stability theory of representations of semigroups. These results will be given in a separate paper ([7]).

Throughout this paper, S will be a measurable subsemigroup of a locally compact abelian group G. We assume that $G = S - S$, and that S has non-empty interior S°, so that $G = S^\circ - S^\circ$. We define a relation on G by: $t_1 \preceq t_2$ if and only if $t_1 - t_2 \in S \cup \{0\}$. This relation is reflexive and transitive; it is antisymmetric if and only if $S \cap (-S) \subseteq \{0\}$.

A *weight* on S is a map $w : S \to [1, \infty)$ satisfying $w(s+t) \leqslant w(s)w(t)$ for all $s, t \in S$. The *exponential type* of a weight w is defined to be

$$\sup_{t\in S} \lim_{n\to\infty} \frac{\log w(nt)}{n}.$$

A weight w is said to be *non-quasianalytic* if

$$\sum_{n=1}^{\infty} \frac{\log w(nt)}{n^2} < \infty$$

for all $t \in S$. The exponential type of a non-quasianalytic weight is 0. We shall assume throughout that our weights are measurable and bounded on compact subsets of S.

We consider S to be equipped with the restriction of Haar measure. For a weight w, we let $L^1_w(S)$ be the Banach algebra of all (equivalence classes of) measurable functions $f : S \to \mathbb{C}$ such that $\|f\| := \int_S |f(t)|w(t)\,\mathrm{d}t < \infty$.

A *representation* of S on a Banach space X is a map $T : S \to \mathcal{B}(X)$ which is continuous in the strong operator topology and which is a semigroup homomorphism, so that $T(t_1 + t_2) = T(t_1)T(t_2)$ $(t_1, t_2 \in S)$. For a representation T, there is an associated weight w_T defined by $w_T(t) = \max(\|T(t)\|, 1)$. We refer to the exponential type of T, or T being non-quasianalytic, according to the corresponding notions for w_T.

We remark that our results include the following special cases:

(a) $S = \mathbb{Z}_+$, $G = \mathbb{Z}$, T is the representation generated by any single operator;

(b) $S = \mathbb{R}_+ = [0, \infty)$, $G = \mathbb{R}$, T is any C_0-semigroup;

(c) $S = \mathbb{Z}^n_+$, $G = \mathbb{Z}^n$, T is generated by any n commuting operators;

(d) $S = \mathbb{R}^n_+$, $G = \mathbb{R}^n$, T is generated by any n commuting C_0-semigroups;

(e) S any abelian (multiplicative) semigroup of injective bounded linear operators on X, G the (discrete) enveloping group of S, T the identity representation.

2. EXPANSIVE REPRESENTATIONS

We are seeking conditions under which a representation T of S on X can be extended to a representation U of G on a larger space Y in such a way that $U(-t)$ is a contraction, for each $t \in S$. It is clear that a necessary condition for this is that the representation should be expansive in the following sense.

We say that an operator T on a normed space X is *expansive* if $\|Tx\| \geqslant \|x\|$ for all $x \in X$, and a semigroup representation $T : S \to \mathcal{B}(X)$ is *expansive* if each of the operators $T(t)$ is expansive.

We shall see in Example 2.3 that expansiveness is not the condition that we seek. The problem with this definition is that when S is not a simple semigroup, expansiveness does not tell us enough about the way T acts. We need a definition which involves the whole representation rather than just its individual members. Such a condition is given in the following definition.

We say that the representation T is *superexpansive* if for every $n \in \mathbb{N}$, $t, t_1, \ldots, t_n \in S$ and $x, x_1, \ldots, x_n \in X$ which satisfy $t_i \preceq t$ for all i and $T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$, we have that $\|x\| \leqslant \|x_1\| + \cdots + \|x_n\|$.

As a justification for the terminology we note that T being superexpansive implies that $T(s)$ is an expansive operator for all $s \in S$. To see this take any $s_0 \in S$, and let $t = s + s_0$, $t_1 = s_0$ and $x_1 = T(s)x$. Example 2.3 will show that the converse is not true in general, but Proposition 2.2 will show that it is true in some cases, notably when $S = \mathbb{Z}_+$.

It is easy to verify that if $T(s)$ is isometric for all $s \in S$, then T is superexpansive. We now give another important example of a superexpansive representation.

EXAMPLE 2.1. Let w be a $\preceq$-increasing, continuous weight on S. There is a representation R of S on $L^1_w(S)$, given by right translations:

$$(R(t)f)(s) = \begin{cases} 0 & \text{if } s - t \notin S, \\ f(s-t) & \text{if } s - t \in S, \end{cases}$$

for all $s, t \in S$. Then $R(t)$ is a bounded operator for each $t \in S$ and $\|R(t)\| \leqslant w(t)$. The mapping $t \mapsto R(t)$ is easily seen to be strongly continuous.

We claim that R is superexpansive. Suppose that $f, f_1, \ldots, f_n \in L^1_w(S)$, $t, t_1, \ldots, t_n \in S$ with $t_i \preceq t$ and $R(t)f = R(t_1)f_1 + \cdots + R(t_n)f_n$. Then

$$\begin{aligned} \|f\| &= \int_{t+S} |(R(t)f)(s)| w(s-t) \,\mathrm{d}s = \int_{t+S} \left| \sum_{i=1}^n (R(t_i)f_i)(s) \right| w(s-t)\,\mathrm{d}s \\ &\leqslant \sum_{i=1}^n \int_{t+S} |(R(t_i)f_i)(s)| w(s-t)\,\mathrm{d}s \\ &\leqslant \sum_{i=1}^n \int_{t_i+S} |f_i(s-t_i)| w(s-t_i)\,\mathrm{d}s = \sum_{i=1}^n \|f_i\|. \end{aligned}$$

PROPOSITION 2.2. *Suppose that $G = S \cup (-S)$ and that $T(s)$ is expansive for all $s \in S$. Then T is superexpansive.*

Proof. Suppose that $n \in \mathbb{N}$, $t, t_1, \ldots, t_n \in S$, $x, x_1, \ldots, x_n \in X$, $t_i \preceq t$ for all i and $T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$. Since the relation $\preceq$ on G is total we can assume, without loss of generality, that $t_1 \succeq t_2 \succeq \cdots \succeq t_n$.

Since T is expansive, each of the operators $T(s)$ is injective, so

$$T(t - t_n)x = T(t_1 - t_n)x_1 + \cdots + T(t_{n-1} - t_n)x_{n-1} + x_n.$$

Hence

$$\begin{aligned} \|x_n\| &= \|T(t-t_n)x - T(t_1 - t_n)x_1 - \cdots - T(t_{n-1} - t_n)x_{n-1}\| \\ &= \|T(t_{n-1} - t_n)\big(T(t - t_{n-1})x - T(t_1 - t_{n-1})x_1 - \cdots - x_{n-1}\big)\| \\ &\geqslant \|T(t - t_{n-1})x - T(t_1 - t_{n-1})x_1 - \cdots - x_{n-1}\| \\ &\geqslant \big\|T(t - t_{n-1})x - T(t_1 - t_{n-1})x_1 - \cdots - T(t_{n-2} - t_{n-1})x_{n-2}\big\| - \|x_{n-1}\| \\ &= \|T(t_{n-2} - t_{n-1})\big(T(t - t_{n-2})x - T(t_1 - t_{n-2})x_1 - \cdots - x_{n-2}\big)\| - \|x_{n-1}\| \\ &\geqslant \big\|T(t - t_{n-2})x - T(t_1 - t_{n-2})x_1 - \cdots - T(t_{n-3} - t_{n-2})x_{n-3}\big\| \\ &\qquad\qquad - \|x_{n-2}\| - \|x_{n-1}\| \\ &\ \ \vdots \\ &\geqslant \|T(t - t_1)x\| - \|x_1\| - \|x_2\| - \cdots - \|x_{n-1}\| \\ &\geqslant \|x\| - \|x_1\| - \|x_2\| - \cdots - \|x_{n-1}\|. \end{aligned}$$

Reordering the terms, this says that $\|x\| \leqslant \|x_1\| + \|x_2\| + \cdots + \|x_n\|$. Hence T is super expansive. ∎

Our definitions of expansive and superexpansive representations are related to properties identified by Arens ([1]) when considering the possibility of extending a Banach algebra in such a way as to make one or more given elements invertible (see the remarks following Theorem 3.3). Arens asked whether his two properties were equivalent. Bollobás ([8]) gave an example to answer Arens's question and this also serves to show that expansive representations are not necessarily superexpansive.

EXAMPLE 2.3. In [8], Bollobás constructed a commutative Banach algebra, B, containing elements b_0, b_1, b_2, g_1, g_2 such that:

(i) $\|g_i x\| \geqslant \|x\|$ for all $x \in B$ and $i = 1, 2$;
(ii) $g_1 g_2 b_0 = g_1 b_1 + g_2 b_2$;
(iii) $\|b_1\| = \|b_2\| = 1$;
(iv) and $\|b_0\|$ is as large as desired.

So we can arrange that $\|b_0\| > \|b_1\| + \|b_2\| = 2$.

Taking the multiplication operators T_i on B given by $T_i x := g_i x$ for $i = 1, 2$, we obtain two commuting expansive operators, T_i, such that the representation of $\mathbb{Z}_+^2$ given by $(n, m) \mapsto T_1^n T_2^m$ is not superexpansive.

3. THE CONSTRUCTION OF AN EXTENSION

Let T be a representation of S on X. We say that (Y, U, π) is an *extension* of (X, T) if Y is a Banach space, U is a representation of G on Y, and $\pi : X \to Y$ is a linear map such that

(1) π is isometric,
(2) $U(t) \circ \pi = \pi \circ T(t)$ for all $t \in S$, and
(3) $U(t)$ is invertible for all $t \in S$.

Since $U : G \to \mathcal{B}(Y)$ is a homomorphism, it follows that $U(0)$ is an idempotent in $\mathcal{B}(Y)$, i.e., a projection. Therefore the condition "$U(t)$ is invertible", could be replaced by the equivalent condition "$U(0) = I$".

We say that an extension (Y, U, π) of (X, T) is *minimal* if the subspace $\{U(t)\pi(x) : t \in G,\, x \in X\}$ is dense in Y. Given any extension (Y, U, π) of (X, T), one can form a minimal extension by replacing Y by the closure of $\{U(t)\pi(x) : t \in G,\, x \in X\}$.

We say that a representation (Y, U) of G is *S-expansive* if $\|U(s)y\| \geqslant \|y\|$ for all $y \in Y$ and $s \in S$, or equivalently, $\|U(-s)\| \leqslant 1$ for all $s \in S$.

We say that a minimal S-expansive extension (Y, U, π) of (X, T) is *universal* if for all minimal S-expansive extensions (Z, V, ρ), all $x \in X$ and all $t \in G$,

$$\|V(t)\rho(x)\| \leqslant \|U(t)\pi(x)\|.$$

Since $V(s_1 - s_2)\rho(x) = V(-s_2)\rho(T(s_1)x)$, it suffices to check this condition when $t \in -S$. Furthermore, it is easy to see that this is equivalent to the existence of a (unique) linear contraction $i : Y \to Z$ such that $i \circ \pi = \rho$ and $V(t) \circ i = i \circ U(t)$ for all $t \in G$.

We first show that when T is isometric, then this universal property corresponds to U being isometric.

EXAMPLE 3.1. Let T be a representation of S by isometries on X, and let (Y,U,π) be the (unique) minimal extension of (X,T) to a representation of G by isometries ([10]). It is clear that (Y,U,π) is S-expansive, and we now show that it is universal.

Let (Z,V,ρ) be any minimal S-expansive extension of (X,T). For $x \in X$ and $s \in S$,

$$\|V(-s)\rho(x)\| \leqslant \|\rho(x)\| = \|x\| = \|U(-s)\pi(x)\|.$$

Conversely, suppose that (Z,V,ρ) is a universal minimal S-expansive extension of (X,T). For $x \in X$, $t \in G$ and $s \in S$,

$$\|V(t)V(-s)\rho(x)\| = \|V(t-s)\rho(x)\| \geqslant \|U(t-s)\pi(x)\| = \|x\| \geqslant \|V(-s)\rho(x)\|.$$

The minimality now implies that $V(t)$ is expansive. Similarly, $V(-t) = V(t)^{-1}$ is expansive, so $V(t)$ is an isometry.

PROPOSITION 3.2. *Let (X,T) be a representation of S, and suppose that there is an S-expansive extension (Y,U,π) of (X,T). Then (X,T) is superexpansive.*

Proof. Suppose that $n \in \mathbb{N}$, $t, t_1, \dots, t_n \in S$, $x, x_1, \dots, x_n \in X$, $t_i \preceq t$ for all i and $T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$. Since $t - t_i \in S \cup \{0\}$ and $U(t-t_i)^{-1}$ is a contraction for all i,

$$\begin{aligned}\|x\| &= \|U(t)^{-1}U(t)\pi(x)\| = \|U(t)^{-1}\pi(T(t)x)\| \\ &= \|U(t-t_1)^{-1}\pi(x_1) + \cdots + U(t-t_n)^{-1}\pi(x_n)\| \\ &\leqslant \|U(t-t_1)^{-1}\pi(x_1)\| + \cdots + \|U(t-t_n)^{-1}\pi(x_n)\| \\ &\leqslant \|x_1\| + \cdots + \|x_n\|. \quad \blacksquare\end{aligned}$$

We now give the main result of this section, the extension theorem, which is the converse of Proposition 3.2, and which generalises Douglas's construction ([10]) when T is isometric (see Example 3.1).

THEOREM 3.3. *Let (X,T) be a superexpansive representation of S. There is a universal minimal S-expansive extension (Y,U,π) of (X,T). Moreover, if (Z,V,ρ) is any other universal minimal S-expansive extension of (X,T) then there is an isometric isomorphism i of Y onto Z such that $i \circ \pi = \rho$ and $i \circ U(t) = V(t) \circ i$ for all $t \in G$.*

Proof. Let $X_0 = X \times G$. We define an equivalence relation on X_0 by $(x,t) \sim (y,s)$ if there exists $u \in S$ such that $s+u$, $t+u \in S$ and $T(s+u)x = T(t+u)y$. Since each operator $T(v)$ for $v \in S$ is expansive, it is injective. Hence, if $(x,t) \sim (y,s)$, then $T(s+u)x = T(t+u)y$ for *all* $u \in S$ with $s+u$, $t+u \in S$.

We let X_1 be the space of equivalence classes, $X_1 = X_0/\sim$, and we denote the equivalence class containing (x,t) by $[x,t]$. Since $G = S - S$, each equivalence class contains a member (x,t) with $t \in S$. Now X_1 is a vector space under the operations

$$\begin{aligned}[x,t] + [y,s] &= [T(s)x + T(t)y, s+t], \qquad \text{where } s,t \in S \\ \alpha[x,t] &= [\alpha x, t].\end{aligned}$$

The idea behind this is that $(T(s)x, t) \sim (x, t-s)$, and so we have converted the semigroup to a translation, which is clearly an invertible operation on this space. We have to define a norm on the space X_1, so that X is isometrically embedded into it.

For $[x, t] \in X_1$ we define

$$\|[x,t]\| = \inf\left\{\sum_{i=1}^{n} \|x_i\| : n \in \mathbb{N},\, x_i \in X,\, t_i \in S \cup \{0\},\, \sum_{i=1}^{n} [x_i, t_i] = [x,t]\right\}. \tag{3.1}$$

Since this definition is in terms of the equivalence class, and not a representative of the class, it is well defined.

The next step is to prove that $\|\cdot\|$ is a norm on X_1, and that there is an isometric embedding of X into X_1. The fact that $\|\cdot\|$ satisfies the triangle inequality is built into the definition since, if $[x,t], [y,s] \in X_1$,

$$\begin{aligned}
&\|[x,t] + [y,s]\| \\
&\quad = \inf\left\{\sum_{i=1}^{n} \|x_i\| : n \in \mathbb{N},\, x_i \in X,\, t_i \in S \cup \{0\}, \sum_{i=1}^{n} [x_i, t_i] = [x,t] + [y,s]\right\} \\
&\quad \leqslant \inf\left\{\sum_{i=1}^{n} \|x_i\| + \sum_{i=1}^{m} \|y_i\| : n, m \in \mathbb{N},\, x_i, y_i \in X,\, t_i, s_i \in S \cup \{0\},\right. \\
&\qquad\qquad \left.\sum_{i=1}^{n} [x_i, t_i] = [x,t],\, \sum_{i=1}^{m} [y_i, s_i] = [y,s]\right\} \\
&\quad = \|[x,t]\| + \|[y,s]\|.
\end{aligned}$$

Since $\lambda[x,t] = [\lambda x, t]$ and, for non-zero λ, $\sum_{i=1}^{n} [x_i, t_i] = [x,t]$ if and only if $\sum_{i=1}^{n} [\lambda x_i, t_i] = [\lambda x, t]$, it is easy to see that $\|\lambda[x,t]\| = |\lambda|\, \|[x,t]\|$.

To complete the proof that $\|\cdot\|$ is a norm we need to show that $\|[x,t]\| \neq 0$ whenever $[x,t] \neq 0$. It is clear that $[x,t] = 0$ if and only if $x = 0$. Let $x \neq 0$ and $t \in S \cup \{0\}$. Suppose that $n \in \mathbb{N}$, $x_i \in X$, $t_i \in S \cup \{0\}$ and $\sum_{i=1}^{n} [x_i, t_i] = [x,t]$. Choose $s \in S$ such that $t_i \preceq s$ for each i. Then

$$[x,t] = \sum_{i=1}^{n} [x_i, t_i] = \sum_{i=1}^{n} [T(s-t_i)x_i, s] = \left[\sum_{i=1}^{n} T(s-t_i)x_i, s\right].$$

Hence

$$T(s)x = T(t)\left(\sum_{i=1}^{n} T(s-t_i)x_i\right) = \sum_{i=1}^{n} T(s-t_i)(T(t)x_i),$$

where $T(0) := I$ if $0 \notin S$. Since $s - t_i \preceq s$ for each i, the superexpansiveness implies that

$$\|x\| \leqslant \sum_{i=1}^{n} \|T(t)x_i\| \leqslant \|T(t)\| \sum_{i=1}^{n} \|x_i\|.$$

Since this is true for all such x_i and t_i and $\|T(t)\| > 0$, we have $\|[x,t]\| \geqslant \|x\|/\|T(t)\| > 0$.

Next we need to embed X isometrically into X_1. The embedding is given by $\pi : x \mapsto [x,0]$, and it is immediate that $\|[x,0]\| \leqslant \|x\|$. The previous paragraph, with $t = 0$, shows that $\|x\| \leqslant \|[x,0]\|$.

So we have now shown that equation (3.1) defines a norm on the space X_1, under which the map $\pi : x \mapsto [x,0]$ is an isometric embedding of X into X_1. We now define $U(t)[x,s] = [x,s-t]$ ($x \in X$, $s,t \in G$). It is clear that U is a well-defined homomorphism of G into the space of all linear operators on X_1, and $U(t) \circ \pi = \pi \circ T(t)$ for all $t \in S$.

Next we check that $U(-s)$ is a contraction for all $s \in S$. Suppose that $s \in S$, $x_i \in X$, $t_i \in S \cup \{0\}$ and $[x,t] = \sum\limits_{i=1}^{n} [x_i, t_i]$. Then $[x, t+s] = \sum\limits_{i=1}^{n} [x_i, t_i + s]$, and $t_i + s \in S$, hence $\|[x,t+s]\| \leqslant \sum\limits_{i=1}^{n} \|x_i\|$ for any such x_i. Therefore $\|[x,t+s]\| \leqslant \|[x,t]\|$ for all $x \in X$, $t \in G$ and $s \in S$, so $U(-s)$ is a contraction for all $s \in S$.

Next we show that $U(s)$ is a bounded operator for all s. If $s \in S$, we have $U(s)[x,t] = [T(s)x,t]$. Suppose that $x_i \in X$ and $t_i \in S \cup \{0\}$ are such that $[x,t] = \sum\limits_{i=1}^{n} [x_i,t_i]$. Then $[T(s)x,t] = \sum\limits_{i=1}^{n} [T(s)x_i, t_i]$ hence $\|[T(s)x,t]\| \leqslant \sum\limits_{i=1}^{n} \|T(s)x_i\| \leqslant \sum\limits_{i=1}^{n} \|T(s)\| \, \|x_i\|$ for all such x_i and t_i. Therefore $\|[T(s)x,t]\| \leqslant \|T(s)\| \, \|[x,t]\|$. We shall prove a more general version of this inequality in Lemma 3.4. Any $s \in G$ can be written as $s = s_1 - s_2$ where $s_i \in S$ for $i = 1,2$. Then $U(s) = U(s_1)U(-s_2)$ and we have just shown that both $U(s_1)$ and $U(-s_2)$ are bounded operators, so $U(s)$ is a bounded operator.

We now take the Banach space Y to be the completion of X_1 with the norm given by (3.1), and extend each $U(t)$ continuously to an operator (also denoted by $U(t)$) on Y. To show that U is strongly continuous, we first observe that the maps $x \mapsto [(x,t)]$ from X into Y are injective, bounded linear maps for each $t \in G$, since $\|[(x,t)]\| \leqslant \|U(-t)\| \, \|x\|$. Let $s \in G$ with $s = s_1 - s_2$ where $s_i \in S^\circ$. Then, whenever $u \in G$ is close enough to s, we have $u + s_2 \in S$, hence

$$\begin{aligned}
\lim_{u \to s} \|U(u)[x,t] - U(s)[x,t]\| &= \lim_{u \to s} \|U(u-t)[x,0] - U(s-t)[x,0]\| \\
&\leqslant \lim_{u \to s} \|U(-s_2 - t)\| \, \|U(u+s_2)[x,0] - U(s_1)[x,0]\| \\
&= \lim_{u \to s} \|U(-s_2 - t)\| \, \|[T(u+s_2)x,0] - [T(s_1)x,0]\| \\
&= \lim_{u \to s} \|U(-s_2 - t)\| \, \|T(u+s_2)x - T(s_1)x\| = 0.
\end{aligned}$$

Hence U is strongly continuous on a dense subset of Y.

Now let $y \in Y$ and $s = s_1 - s_2 \in G$, with $s_1, s_2 \in S^\circ$. Since S° is locally compact and T is strongly continuous, there is a neighbourhood W of s_1 in G, contained in S, on which the map $\|T(\cdot)\|$ is bounded, say by C. Then $\|U(\cdot)\|$ is bounded on $W - s_2$ by $C\|U(-s_2)\|$ and $W - s_2$ is a neighbourhood of s in G. Thus U is locally bounded, and it follows that U is strongly continuous on Y.

It is clear that (Y,U,π) is a minimal S-expansive extension of (X,T). To show that (Y,U,π) is universal, let (Z,V,ρ) be any S-expansive extension of (X,T).

Suppose that $x \in X$, $t \in S \cup \{0\}$ and that $n \in \mathbb{N}$, $x_i \in X$, $t_i \in S \cup \{0\}$ $(1 \leqslant i \leqslant n)$ are such that $\sum_{i=1}^{n} [x_i, t_i] = [x, t]$. Choose $s \in S$ such that $t \preceq s$ and $t_i \preceq s$ for each i. Then

$$[T(s-t)x, s] = \sum_{i=1}^{n} [T(s-t_i)x_i, s] = \Big[\sum_{i=1}^{n} T(s-t_i)x_i, s\Big].$$

Hence $T(s-t)x = \sum_{i=1}^{n} T(s-t_i)x_i$. Therefore

$$\begin{aligned} V(-t)\rho(x) &= V(-s)V(s-t)\rho(x) = V(-s)\rho(T(s-t)x) \\ &= V(-s)\rho\Big(\sum_{i=1}^{n} T(s-t_i)x_i\Big) = \sum_{i=1}^{n} V(-s)V(s-t_i)\rho(x_i) \\ &= \sum_{i=1}^{n} V(-t_i)\rho(x_i). \end{aligned}$$

But $V(-t_i)$ is a contraction and ρ is isometric, therefore

$$\|V(-t)\rho(x)\| \leqslant \sum_{i=1}^{n} \|\rho(x_i)\| = \sum_{i=1}^{n} \|x_i\|.$$

Thus $\|V(-t)\rho(x)\| \leqslant \|[x,t]\| = \|U(-t)\pi(x)\|$. This suffices to show that (Y, U, π) is universal.

Finally, suppose that (Z, V, ρ) is any universal minimal S-expansive extension of (X, T). Then

$$\|V(t)\rho(x)\| = \|U(t)\pi(x)\|$$

for all $x \in X$ and $t \in G$. Let $i(U(t)\pi(x)) = V(t)\rho(x)$. It is easy to verify that i is well-defined and linear. Since i is isometric with dense range, it extends to an isometry of Y onto Z, and it is easy to verify that $i \circ \pi = \rho$ and $i \circ U(t) = V(t) \circ i$. ∎

REMARKS. (1) Suppose that (Z, V, ρ) is an extension of the superexpansive representation (X, T) such that

$$\|V(-t)\rho(x)\| \leqslant \|x\| \tag{3.2}$$

for all $x \in X$ and $t \in S$, a condition which is weaker than being S-expansive. Let (Y, U, π) be the extension constructed in Theorem 3.3. The proof above shows that

$$\|V(-t)\rho(x)\| \leqslant \|U(-t)x\|$$

for all $x \in X$ and $t \in S$. Hence there is a linear contraction $i : Y \to Z$ such that $i \circ \pi = \rho$ and $V(t) \circ i = i \circ U(t)$ for all $t \in G$.

(2) Let $x \in X$, and $t \in S$. Suppose that $y_1, y_2, \dots, y_n \in X$, $s_1, s_2, \dots, s_n \in S$, and $\sum_{i=1}^{n} s_i = t$. Put

$$\begin{aligned} &x_1 = y_1, \\ &x_i = y_i - T(s_{i-1})y_{i-1} \qquad (i = 2, \dots, n), \\ &x_{n+1} = x - T(s_n)y_n, \\ &t_1 = 0, \\ &t_i = s_1 + \cdots + s_{i-1} \qquad (i = 2, \dots, n+1). \end{aligned}$$

Then $\sum_{i=1}^{n+1} [x_i, t_i] = [x, t]$. Hence

$$\begin{aligned} \|[x,t]\| \leqslant \inf\Big\{ &\|y_1\| + \|y_2 - T(s_1)y_1\| + \cdots + \|x - T(s_n)y_n\| : \\ &n \in \mathbb{N},\, y_i \in X,\, s_i \in S,\, \sum_{i=1}^{n} s_i = t \Big\}. \end{aligned} \tag{3.3}$$

Equality holds when $G = S \cup (-S)$. In particular, the right-hand side of (3.3) then depends only on the equivalence class of (x, t) and satisfies the triangle inequality. As this fact plays no role in our results, we omit the proof.

(3) Read ([21]) and Müller ([19]) independently showed that a single operator T can be extended to an operator S whose spectrum is the approximate point spectrum of T. Although this extension is different from ours even in the case of a single operator, Müller's technique is relevant to us. He found a way to deduce the result for operators from the corresponding result in Banach algebras obtained earlier by Read ([20]). It is possible to use Müller's technique, together with the construction of Arens ([1]) for Banach algebras, to obtain the existence of an S-expansive extension in Theorem 3.3. Moreover the minimal part of that extension is universal, and so it coincides with our extesions, up to isometric isomorphism. However, the constructions of Arens and Müller produce a complicated description of the space Y, and our more direct construction is simpler to present for the purpose of establishing further properties. (We are grateful to the referee for bringing the work of Read and Müller to our attention.)

(4) Stroescu ([23]) has shown that, if G is a group and T is a strongly continuous mapping $T : G \to \mathcal{B}(X)$ satisfying $\|T(t)\| \leqslant w(t)$ for a weight w on G, then there is a dilation of T to a representation of G. She also applied this result to one-parameter semigroups of operators. There is no immediate connection between our results and hers, although there are some similarities in the constructions. (We are grateful to B. Nagy for bringing [23] to our attention.)

In the light of the uniqueness statement of Theorem 3.3, and to simplify the terminology, we call the extension (Y, U, π) constructed in Theorem 3.3 the *generic* extension of (X, T). Moreover, we will often suppress the map π and regard X as a subspace of Y. For the rest of this section, we consider further properties of the generic extension.

Let A be a bounded linear operator which commutes with $T(t)$ for all $t \in S$. Then we can define an operator $\widetilde{A}$ on X_1 by $\widetilde{A}[x,t] = [Ax,t]$. This is well-defined since if $[x,t] = [y,s]$ then there exists $u \in S$ such that $t+u, s+u \in S$ and $T(s+u)x = T(t+u)y$. Then

$$T(s+u)Ax = AT(s+u)x = AT(t+u)y = T(t+u)Ay,$$

hence $[Ax,t] = [Ay,s]$. Now we claim that $\widetilde{A}$ is a bounded linear operator and $\|\widetilde{A}\| = \|A\|$. Then we can extend $\widetilde{A}$ continuously to the completion of X_1. The linearity of $\widetilde{A}$ is obvious and it is clear that $\widetilde{A}$ agrees with A on the embedded copy of X, so that $\|\widetilde{A}\| \geqslant \|A\|$, provided that $\widetilde{A}$ is bounded.

LEMMA 3.4. *Let A be a bounded linear operator which commutes with $T(t)$ for all $t \in S$. Then $\|[Ax,t]\| \leqslant \|A\|\,\|[x,t]\|$.*

Proof. Since A commutes with all the $T(t)$, whenever $[x,t] = \sum_{i=1}^{n}[x_i,t_i]$ we have that $[Ax,t] = \sum_{i=1}^{n}[Ax_i,t_i]$. Therefore

$$\begin{aligned}
\|[Ax,t]\| &= \inf\Big\{\sum_{i=1}^{n}\|x_i\| : x_i \in X,\ t_i \in S\cup\{0\},\ [Ax,t] = \sum_{i=1}^{n}[x_i,t_i]\Big\}\\
&\leqslant \inf\Big\{\sum_{i=1}^{n}\|Ax_i\| : x_i \in X,\ t_i \in S\cup\{0\},\ [Ax,t] = \sum_{i=1}^{n}[Ax_i,t_i]\Big\}\\
&\leqslant \inf\Big\{\sum_{i=1}^{n}\|Ax_i\| : x_i \in X,\ t_i \in S\cup\{0\},\ [x,t] = \sum_{i=1}^{n}[x_i,t_i]\Big\}\\
&\leqslant \|A\| \inf\Big\{\sum_{i=1}^{n}\|x_i\| : x_i \in X,\ t_i \in S\cup\{0\},\ [x,t] = \sum_{i=1}^{n}[x_i,t_i]\Big\}. \quad ∎
\end{aligned}$$

PROPOSITION 3.5. *Let (Y,U) be the generic extension of (X,T). If $A \in \mathcal{B}(X)$ and A commutes with $T(t)$ for all $t \in S$, then A extends to a bounded linear operator $\widetilde{A}$ on Y with the properties $\|A\| = \|\widetilde{A}\|$ and the spectrum, $\sigma(\widetilde{A})$, of $\widetilde{A}$ is contained in $\sigma(A)$. Moreover, $\widetilde{A}$ is the unique bounded linear map on Y such that $U(t)Ax = \widetilde{A}U(t)x$ for all $t \in G$ and $x \in X$. The map $A \mapsto \widetilde{A}$ is an algebra homomorphism from the subalgebra $\mathcal{B}_T(X)$ of all bounded linear operators on X which commute with T into the algebra $\mathcal{B}_U(Y)$ of all bounded linear operators on Y which commute with U.*

Proof. We have already shown that $\widetilde{A}$ defines a bounded linear operator on Y and that $\|A\| = \|\widetilde{A}\|$. The fact that $U(t)Ax = \widetilde{A}U(t)x$ is immediate from the definition of $\widetilde{A}$, and the uniqueness follows from the minimality of the extension. It is clear that $\varphi : A \mapsto \widetilde{A}$ is an algebra homomorphism from $\mathcal{B}_T(X)$ into $\mathcal{B}_U(Y)$, with $\varphi(I_X) = I_Y$. Therefore if $\lambda \in \mathbb{C}\setminus\sigma(A)$, then $I_X = (\lambda I_X - A)^{-1}(\lambda I_X - A)$, so $I_Y = \varphi(I_X) = \varphi((\lambda I_X - A)^{-1})(\lambda I_Y - \widetilde{A})$, hence $\lambda \notin \sigma(\widetilde{A})$. ∎

COROLLARY 3.6. *Let (Y,U) be the generic extension of (X,T). Then*
(i) $U(t) = \widetilde{T(t)}$ *for all* $t \in S$;
(ii) $\|U(t)\| = \|T(t)\|$ *for all* $t \in S$;
(iii) $\|U(t)\| \leqslant \inf\{\|T(s)\| : s \in S,\, t \preceq s\}$ *for all* $t \in G$;
(iv) *if* T *is norm-continuous, then* U *is norm-continuous.*

Proof. It is clear that (i) holds, and (ii) is then a consequence of Proposition 3.5. If $t \in G$, $s \in S$ and $t \preceq s$, then

$$\|U(t)\| \leqslant \|U(-(s-t))\|\,\|U(s)\| \leqslant \|T(s)\|,$$

so (iii) follows.

By Proposition 3.5, $\|U(t) - U(s)\| = \|T(t) - T(s)\|$ for all $s, t \in S$. Now, for $s_1, s_2, t_1, t_2 \in S$,

$$\begin{aligned}\|U(t_1 - t_2) - U(s_1 - s_2)\| &= \|U(-t_2 - s_2)(U(t_1 + s_2) - U(s_1 + t_2))\| \\ &\leqslant \|T(t_1 + s_2) - T(s_1 + t_2)\|.\end{aligned}$$

Now (iv) follows. ∎

EXAMPLE 3.7. Consider $S = \mathbb{R}^n_+$ as a subsemigroup of $G = \mathbb{R}^n$; the relation $\preceq$ is then the usual coordinatewise lattice ordering. Let w be an increasing, continuous weight on $\mathbb{R}^n_+$, and consider the representation R of $\mathbb{R}^n_+$ on $L^1_w(\mathbb{R}^n_+)$ as in Example 2.1. Since the representation is superexpansive, Theorem 3.3 shows that there is a generic extension (Y,U) of this representation, and Corollary 3.6 shows that

$$\|U(t)\| \leqslant w'(t) := \inf\{w(s) : s \in \mathbb{R}^n_+,\, t \preceq s\} = w(t \vee 0),$$

where $t \vee 0$ is the supremum of $\{t, 0\}$ in the lattice $\mathbb{R}^n$. If w_1 is any weight on $\mathbb{R}^n$ which extends w, then there is a minimal extension given by right translations on $L^1_{w_1}(\mathbb{R}^n)$. This extension satisfies (3.2) if and only if $w_1 \leqslant w'$, and it is $\mathbb{R}^n_+$-expansive if and only if w_1 is increasing.

The expectation is that the generic extension should be translation on some space of functions on $\mathbb{R}^n$, and the universal property roughly says that the norm on the generic extension is as large as possible. Thus it is no surprise that the generic extension is given by right translations on the space $Y = L^1_{w'}(\mathbb{R}^n)$. The calculations for this are carried out in [25], Section 4.3.

The corresponding result holds for the representation of $\mathbb{Z}^n_+$ by right translations on a weighted space $\ell^1_w(\mathbb{Z}^n_+)$. However, it is not true in general for representations by right translations on $L^p_w(\mathbb{R}^n_+)$ or $\ell^p_w(\mathbb{Z}^n_+)$ for $p > 1$ ([25], Section 4.4.5).

4. ISOMETRIC REPRESENTATIONS ON SUPERREFLEXIVE SPACES

For many representations (X,T), the space X or the operators $T(t)$ have some special properties, which it would be desirable to preserve while extending the representation. In general, a superexpansive representation has many different minimal S-expansive extensions, and the universal property which identifies a unique generic extension is a general Banach space property (roughly speaking, it identifies the extension with the largest possible norm). Therefore one should not expect this construction to preserve other structure in general, but it is sometimes possible to construct a space which has a universal property related to the structure. This can be done for example, for representations by homomorphisms of Banach algebras and Banach lattices (see [25], Section 4.4 for details).

When T is a superexpansive representation of S on a Hilbert space X with generic extension (Y,U), the space Y may not be a Hilbert space ([25], Section 4.4.5), although Y is a Hilbert space when T is isometric ([15]). We refer the reader to [16] for further results in this area.

The following example taken from [11], Example 2.3.9 shows that Y may not be reflexive when X is reflexive and T is isometric.

EXAMPLE 4.1. Let ℓ_n^∞ denote the space $\mathbb{C}^n$ equipped with the supremum norm, and let X be the reflexive space

$$X = \ell^2\text{-}\bigoplus_{n=1}^{\infty} \ell_n^\infty.$$

Define $R_n : \ell_n^\infty \to \ell_{n+1}^\infty$ by

$$R_n : (\zeta_1, \zeta_2, \ldots, \zeta_n) \mapsto (\zeta_1, \zeta_2, \ldots, \zeta_n, 0),$$

and define $T : X \to X$ by

$$T : (x_n)_{n=1}^\infty \mapsto (0, R_1 x_1, R_2 x_2, \ldots, R_{n-1} x_{n-1}, \ldots),$$

where $x_n \in \ell_n^\infty$. Then T is an isometry, and induces a representation $n \mapsto T^n$ of $\mathbb{Z}_+$ on X.

The generic extension of (X,T) is given as follows:

$$Y = \ell^2\text{-}\bigoplus_{-\infty}^{\infty} c_0, \quad U : (y_n)_{-\infty}^\infty \mapsto (y_{n-1})_{-\infty}^\infty, \quad \pi : (x_n)_1^\infty \mapsto (x_n^*)_{-\infty}^\infty,$$

where c_0 is the space of all sequences $(y(k))_{k\geqslant 1}$ which converge to 0 with the supremum norm, and

$$x_n^*(k) = \begin{cases} x_n(k) & (1 \leqslant k \leqslant n), \\ 0 & \text{otherwise.} \end{cases}$$

The space Y is not reflexive.

In Example 4.1, X is not superreflexive. Theorem 4.2 will show that this is necesary in such an example. We recall now the notion of superreflexivity.

Let X be a Banach space, I be a set, and $\mathcal{U}$ be an ultrafilter on I. Consider the Banach spaces $l^\infty(I,X)$ of all bounded functions from I to X, and $c_0(I,X;\mathcal{U})$ of all bounded functions from I to X which converge to zero through the ultrafilter, both with the supremum norm. The space $l^\infty(I,X)/c_0(I,X;\mathcal{U})$ is an ultrapower of

X, which we denote by $(X)_{\mathcal{U}}$. Then X is *superreflexive* if all of its ultrapowers are reflexive, that is, for all possible choices of I and $\mathcal{U}$, the space $(X)_{\mathcal{U}}$ is reflexive. For equivalent definitions of superreflexivity, see [9], Section VII.4.B, [12], [13]. Note that closed subspaces and ultrapowers of superreflexive spaces are superreflexive ([22], Proposition 2.1).

THEOREM 4.2. *Suppose that T is an isometric representation on a superreflexive Banach space X, and that (Y, U, π) is the generic extension. Then Y is superreflexive.*

Proof. To prove this result it is sufficient to exhibit the extension space as a subspace of some ultrapower of X. We shall take the set $I = S$ and the ultrafilter $\mathcal{U}$ will be related to the ordering on S. To be precise consider the filter $\mathcal{F}$ on S which is generated by the sets

$$I_t := \{s \in S : s \succeq t\}$$

for $t \in S$. It is easy to check that the above sets do form a filter base, and hence generate a filter. We now take $\mathcal{U}$ to be any ultrafilter containing $\mathcal{F}$.

Given $x \in X$ and $t \in S$, choose a function $f \in l^\infty(S, X)$ such that $f(t+s) = T(s)x$ for all $s \in S$, and set $\theta(U(-t)\pi(x)) = f + c_0(S, X; \mathcal{U})$. Since $I_t \in \mathcal{U}$, it is easy to verify that this map is well-defined (independent of the choice of f and of different representations of a vector as $U(-t)\pi(x)$) and linear. Moreover,

$$\begin{aligned}\|\theta(U(-t)\pi(x))\| &= \|f + c_0(S, X; \mathcal{U})\| = \inf_{A \in \mathcal{U}} \sup_{s \in A} \|f(s)\| \\ &= \inf_{A \in \mathcal{U}} \sup_{s \in A \cap I_t} \|T(s-t)x\| = \|x\| = \|U(-t)\pi(x)\|,\end{aligned}$$

since T and U are isometric (Example 3.1). Hence θ extends by continuity to an isometry of Y onto a closed subspace of $(X)_{\mathcal{U}}$, so Y is a superreflexive space. ∎

We have been unable to establish a version of Theorem 4.2 for non-isometric representations. The primary obstruction to establishing such a result is not, as it may appear at first, the use of bounded functions in the definition of superreflexivity, but rather the representation of the extension space as a space of functions. In the isometric case the extension space can be naturally identified with a space of functions, indeed this is how Douglas ([10]) proved his result, however such a natural identification does not appear to exist for extensions of slowly growing representations. Indeed, the obstruction is the same as that to using translations on the space $BUC(S, X)/C_0(S, X)$ to construct the extension space, which is essentially the method Douglas uses in his extension theorem. This method is not applicable unless T has the property that $\|T(t)x\|/\|T(t)\| \to 0$ only if $x = 0$. Some operators of this type are considered in [16].

5. SPECTRA AND ISOMETRIES

In this section we let T be a representation of S on a complex Banach space X, and we let $\widehat{G}$ be the dual group of G. The *unitary spectrum* $\mathrm{Sp_u}(T)$ of T is the set of all $\chi \in \widehat{G}$ such that

$$|\widehat{f}(\chi)| \leqslant \|\widehat{f}(T)\| \tag{5.1}$$

whenever $f \in C_c(S)$, the space of continuous complex-valued functions on S with compact support. Here,

$$\widehat{f}(\chi) = \int_S f(t)\chi(t)\,\mathrm{d}t, \quad \widehat{f}(T)x = \int_S f(t)T(t)x\,\mathrm{d}t.$$

If w is a weight on S such that $\|T(t)\| \leqslant w(t)$ for all $t \in S$ (for example, $w = w_T$), then it follows that (5.1) holds for all f in the closure of $C_c(S)$ in $L^1_w(S)$, in particular for all $f \in L^1_w(S^\circ)$. When w is of exponential type 0, it then follows that (5.1) also holds for all $f \in L^1_w(S)$, by an argument involving translations and convolutions (see the proof of Proposition 5.3).

When T is a bounded representation of S, this definition of $\mathrm{Sp_u}(T)$ agrees with that in [6]. In particular, if (X, U) is a bounded representation of G, then $\mathrm{Sp_u}(U)$ is the Arveson spectrum (or finite L-spectrum) of U. If U is a non-quasianalytic representation of G, then $\mathrm{Sp_u}(U)$ agrees with the spectrum defined in [14], and also with the L-spectrum of U. In this case, we shall write $\mathrm{Sp}(U)$ instead of $\mathrm{Sp_u}(U)$.

This definition of unitary spectrum may not agree with the usual concept of spectrum for representations of $\mathbb{Z}_+$ and $\mathbb{R}_+$ with positive exponential type (see [6], Example 2.1). However, for representations of exponential type 0, the definition relates only to the peripheral part of the spectrum, and the following example, proposition, and subsequent remarks shows that this definition of $\mathrm{Sp_u}(T)$ is natural in many cases.

EXAMPLE 5.1. (i) Let $S = \mathbb{Z}_+$ and identify $\widehat{\mathbb{Z}}$ with the unit circle Γ. Let T be a representation of $\mathbb{Z}_+$ on X with exponential type 0, so $T(1)$ has spectral radius at most 1. Then $\mathrm{Sp_u}(T) = \sigma(T(1)) \cap \Gamma$.

To see this, let $\mathcal{A}$ be the norm-closed linear span of $\{T(r) : r \in \mathbb{Z}_+\}$, and let $\lambda \in \Gamma$. Then $\mathcal{A}$ is a commutative Banach algebra, and $\lambda \in \mathrm{Sp_u}(T)$ if and only if

$$\Big|\sum_{r=0}^{n} \alpha_r \lambda^r\Big| \leqslant \Big\|\sum_{r=0}^{n} \alpha_r T(1)^r\Big\|$$

for all $n \in \mathbb{N}$ and $\alpha_r \in \mathbb{C}$. Thus $\lambda \in \mathrm{Sp_u}(T)$ if and only if there is a character ψ of $\mathcal{A}$ such that $\psi(T(1)) = \lambda$, or equivalently, $\lambda \in \sigma_{\mathcal{A}}(T(1))$. However, the topological boundaries of $\sigma_{\mathcal{A}}(T(1))$ and $\sigma(T(1))$ coincide, and the result follows.

(ii) Let $S = \mathbb{R}_+$ and identify $\widehat{\mathbb{R}}$ with $\mathbb{R}$. Let T be a representation of $\mathbb{R}_+$ on X with zero exponential type. Let A be the generator of the C_0-semigroup T, so that $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leqslant 0\}$. Then $\mathrm{i}\,\mathrm{Sp_u}(T) = \sigma(A) \cap \mathrm{i}\mathbb{R}$. This can be proved by a similar argument to (i) above, taking $\mathcal{A}$ to be the norm-closure of $\{\widehat{f}(T) : f \in C_c(S)\}$, and considering the spectra in $\mathcal{A}$ and $\mathcal{B}(X)$ of the element $(I - A)^{-1}$ of $\mathcal{A}$.

In both parts of Example 5.1, $\mathrm{Sp_u}(T)$ consists of approximate eigenvalues of the representation T, and we now see that this remains true for many non-quasianalytic representations of general semigroups. Thus, we say that $\chi \in \widehat{G}$ is an *approximate eigenvalue*, and a net (x_α) of unit vectors in X is an *approximate eigenvector*, of T if $\lim\limits_{\alpha} \|T(t)x_\alpha - \chi(t)x_\alpha\| = 0$ uniformly for t in compact subsets of S.

PROPOSITION 5.2. *Let T be a representation of S on X, and let $\chi \in \widehat{G}$. Suppose that there is a weight w such that the following conditions are satisfied:*

(i) $\|T(t)\| \leqslant w(t)$ *for all* $t \in S$*;*

(ii) w *is non-quasianalytic;*

(iii) *For all* $s \in S$, $\sup\limits_{n \geqslant 1} \inf\limits_{t \in S} \big(\frac{w(ns+t)}{w(t)}\big)^{1/n} \geqslant 1$.

Then $\mathrm{Sp_u}(T)$ *is the set of all approximate eigenvalues for* T *in* $\widehat{G}$.

REMARKS. Condition (ii) of Proposition 5.2 is satisfied if w is continuous or if w is $\preceq$-increasing. Condition (iii) is satisfied if w is $\preceq$-increasing or if w extends to a weight on G with zero exponential type.

For a given non-quasianalytic representation T, it is possible to choose a weight w to satisfy all the conditions of Proposition 5.2 in each of the following cases:

(a) T is expansive (put $w(t) = \|T(t)\|$);

(b) (X, T) has an extension to a representation (Y, U, π) of G with zero exponential type (put $w(t) = \|U(t)\|$).

For some semigroups, including $\mathbb{Z}_+^n$ and $\mathbb{R}_+^n$, it is possible to define a weight satisfying conditions (i) and (iii) by

$$w(t) = \max(\sup\{\|T(s)\| : s \in S,\ s \preceq t\}, 1).$$

However it is not clear (even for $S = \mathbb{Z}_+$) that this weight is non-quasianalytic whenever T is non-quasianalytic.

Proof of Proposition 5.2. Suppose that χ is an approximate eigenvalue of T, with approximate eigenvector (x_α). Then $\lim\limits_{\alpha} \|\widehat{f}(T)x_\alpha - \widehat{f}(\chi)x_\alpha\| = 0$, so $\|\widehat{f}(T)\| \geqslant |\widehat{f}(\chi)|$ whenever $f \in C_{\mathrm{c}}(S)$. Thus $\chi \in \mathrm{Sp_u}(T)$.

The converse has been proved in [14], Lemma 1.2.8 and [18] (see [17], p. 203) for non-quasianalytic representations of groups. The general result is proved in [25], Theorem 3.5.2 using a similar strategy to [14], but with additional technical complications. Since we shall not use this result here, we omit the lengthy proof. ∎

PROPOSITION 5.3. *Let (X, T) be a superexpansive, non-quasianalytic representation of S. Then the generic extension (Y, U) is non-quasianalytic and* $\mathrm{Sp}(U) = \mathrm{Sp_u}(T)$.

Proof. The fact that U is non-quasianalytic follows from Corollary 3.6 (iii). Let $f \in C_{\mathrm{c}}(S)$, $t \in G$, $x \in X$. Then

$$\widehat{f}(U)U(t)x = U(t)\widehat{f}(U)x = U(t)\int\limits_S f(s)T(s)x\,\mathrm{d}s = U(t)\widehat{f}(T)x.$$

By Proposition 3.5, $\widehat{f}(U) = (\widehat{f}(T))^{\sim}$ and $\|\widehat{f}(U)\| = \|\widehat{f}(T)\|$. Now, for $\chi \in \mathrm{Sp}(U)$, $|\widehat{f}(\chi)| \leqslant \|\widehat{f}(U)\| = \|\widehat{f}(T)\|$, so $\chi \in \mathrm{Sp_u}(T)$.

Conversely, let $\chi \in \mathrm{Sp_u}(T)$, and let $f \in C_c(G)$ with support K. Since K is compact and $G = S^\circ - S^\circ$, there exists $s \in S$ such that $s + K \subseteq S$. Define $f_s : S \to \mathbb{C}$ by

$$f_s(t) = f(t-s).$$

Then f_s is supported by the compact subset $s+K$ of S, so

$$|\widehat{f_s}(\chi)| \leqslant \|\widehat{f_s}(U)\|.$$

But $\widehat{f_s}(\chi) = \chi(s)\widehat{f}(\chi)$ and $\widehat{f_s}(U) = U(s)\widehat{f}(U)$, so

$$|\widehat{f}(\chi)| \leqslant \|T(s)\|\,\|\widehat{f}(U)\|.$$

Replacing f by its n-fold convolution gives

$$|\widehat{f}(\chi)| \leqslant \|T(ns)\|^{1/n}\|\widehat{f}(U)\| \to \|\widehat{f}(U)\|$$

as $n \to \infty$, since T is non-quasianalytic. Thus, $\chi \in \mathrm{Sp}(U)$. ∎

REMARK. In Proposition 5.3, it is possible to prove that $\mathrm{Sp_u}(T) \subseteq \mathrm{Sp}(U)$ via Proposition 5.2 by the same method as in [4], Proposition 2.1, but the argument given there that $\mathrm{Sp}(U) \subseteq \mathrm{Sp_u}(T)$ does not work in non-isometric cases. It is the latter inclusion which we shall need in the following results.

COROLLARY 5.4. *Let T be a superexpansive, non-quasianalytic representation of S on a Banach space X.*

(i) *If $X \neq \{0\}$, then $\mathrm{Sp_u}(T)$ is non-empty.*

(ii) *T is norm-continuous if and only if $\mathrm{Sp_u}(T)$ is compact.*

Proof. Part (i) follows from Proposition 5.3 and the corresponding result for non-quasianalytic representations of groups ([17], Theorem 3, p. 203, [14], Proposition 1.2.6). Part (ii) follows from Proposition 5.3, Corollary 3.5, and the corresponding result for groups [17], Lemma 3, p. 201, [14], Proposition 1.3.5. ∎

We shall now consider the situation when $\mathrm{Sp_u}(T)$ is countable, and $\log\|T(nt)\| = \mathrm{o}(\sqrt{n})$ as $n \to \infty$, for all $t \in S$. We use our extension theorem, and a result of Zarrabi ([26]) for group representations, to show that such a representation is always a group representation. We need two preliminary results.

PROPOSITION 5.5. ([14], Theorem 4.5.1) *Let w be a weight on G such that, for all $t \in S$, $w(-t) = 1$ and $\log w(nt) = \mathrm{o}(\sqrt{n})$ as $n \to \infty$. Then each countable closed subset of $\widehat{G}$ is a set of spectral synthesis for $L^1_w(G)$.*

THEOREM 5.6. *Let U be a representation of G which is bounded by a weight w, and suppose that the following conditions are satisfied:*

(i) *$w(-t) = 1$ for all $t \in S$;*

(ii) *$\log w(nt) = \mathrm{o}(\sqrt{n})$ as $n \to \infty$, for each $t \in S$;*

(iii) *$\mathrm{Sp}(U)$ is countable.*

Then $U(t)$ is an isometry for each $t \in G$.

Proof. This was proved by Zarrabi ([26]), when $G = \mathbb{Z}$ and $S = \mathbb{Z}_-$. The general case is similar, using Proposition 5.5; a full proof is given in [25], Theorem 3.7.2. ∎

THEOREM 5.7. *Let T be a superexpansive representation of S on X, and suppose that* $\mathrm{Sp_u}(T)$ *is countable and* $\|T(nt)\| = \mathrm{o}(\sqrt{n})$ *as* $n \to \infty$*, for each* $t \in S$. *Then* $T(t)$ *is an invertible isometry, for each* $t \in S$.

Proof. Let (Y, U) be the generic extension of (X, T), and let $w(t) = \max(\|U(t)\|, 1)$ $(t \in G)$. By Proposition 5.3 and Theorem 5.6, $U(t)$, and hence $T(t)$, is an isometry, for each $t \in S$. The invertibility of $T(t)$ now follows from [4], Theorem 5.1. ∎

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