SP-PROPERTY FOR A PAIR OF C*-ALGEBRAS

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Abstract. Recall that a C*-algebra $A$ has the SP-property if every non-zero hereditary C*-subalgebra of $A$ has a non-zero projection. Let $1 \in A \subset B$ be a pair of unital C*-algebras. In this paper we investigate a sufficient condition for $B$ to have the SP-property, given that $A$ has it. In particular, if there exists a faithful conditional expectation $E$ from $B$ to $A$ of index-finite type in the sense of Watatani, then $B$ has the SP-property under the condition that $A$ is simple with the SP-property. As an application, we have the structure theory of purely infinite simple C*-algebras.

Keywords: C*-index theory, SP-property, conditional expectation.

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1. INTRODUCTION

A C*-algebra $A$ has the SP-property if every non-zero hereditary C*-subalgebra of $A$ has a non-zero projection. This concept has been studied by several mathematicians. For example, this concept is weaker than the real rank zero condition, which means that every hereditary C*-subalgebra of $A$ has an approximate identity of projections ([2], [19], and [9]). When $A$ is a simple unital C*-algebra, Jeong and the author ([13]) in the case of the integer group, and Kishimoto and Kumjian ([16]) in the case of a general discrete group $G$, proved that the reduced crossed product $A \times \alpha \Gamma$ has the SP-property if $A$ has the SP-property and $\alpha$ is a homomorphism from $G$ into the set of automorphisms on $A$ such that $\alpha_g$ is outer for all $g \in G$. In the case that $G$ is finite, Jeong and the author ([14]) proved that any crossed product algebra $A \times \alpha \Gamma$ has the SP-property when $A$ has the SP-property. As an application, we showed that any crossed product algebra $A \times \alpha \Gamma$ has the cancellation property under the additional condition that $A$ has stable rank one, that is, the set of invertible elements in $A$ is dense in it. Moreover, under the same condition if a given crossed product algebra has real rank zero, it also has stable rank one. Unfortunately, however, we do not know if a crossed product algebra of a UHF-algebra by $\mathbb{Z}_2$ has stable rank one, in general. Note that Elliott presented...
an example of a crossed product algebra of this type which does not have real rank zero ([8]).

In this paper, we consider a general condition for a pair of unital C*-algebras with the same unit to have the SP-property. In particular, we consider this problem in the case of conditional expectation from B to A of index-finite type in the sense of Watatani ([25]).

Our main theorem (Theorem 5.1) is that if there exists a faithful conditional expectation E from B to A of index-finite type, then B has the SP-property provided that A is simple with the SP-property. Before giving a proof of it, we consider the case that A is a purely infinite simple C*-algebra in Section 4. There, we point out the existence of one pair of elements as a quasi-basis for A such that B is a direct sum of purely infinite simple C*-algebras. This is a proof of an announcement by Izumi at the Fields Institute in 1995. We believe that our observation will be helpful in determining the stable rank of the crossed product algebra A ×_α G of a simple unital C*-algebra A with stable rank one by a finite group G.

2. THE SP-PROPERTY

In this section we present a sufficient condition for B to have the SP-property, given that A has it.

The argument in Lemma 10 of [16] gives the following general result.

**Theorem.** Let 1 ∈ A ⊂ B be a pair of C*-algebras. Suppose that A has the SP-property and there is a faithful conditional expectation E from B to A. If for any non-zero positive element x in B and an arbitrary positive number ε > 0 there is an element y in A such that

\[ \|y^*(x - E(x))y\| < \varepsilon, \quad \|y^*E(x)y\| \geq \|E(x)\| - \varepsilon \]

then B has the SP-property. Moreover, every non-zero hereditary C*-subalgebra of B has a projection which is equivalent to some projection in A in the sense of Murray-von Neumann.

**Proof.** Set \( a = y^*E(x)y \). Consider the continuous functions f and g defined by

\[ f(t) = \max(0, t - (1 - \varepsilon)\|a\|), \quad g(t) = \min(t, (1 - \varepsilon)\|a\|). \]

Note that \( fg = (1 - \varepsilon)\|a\|f \).

Since A has the SP-property, there is a non-zero projection p in \( f(a)Af(a) \). Then, there is an element \( d_1 \in f(a)A \) such that \( \|p - f(a)d_1\| < \frac{1}{2} \). So, \( \|p - d_1f(a)^2d_1\| < 1 \), and \( \|p - pd_1f(a)^2d_1p\| < 1 \). So, \( pd_1f(a)^2d_1p \) is invertible in \( pAp \). Hence, there is an element \( d_2 \in pAp \) such that \( p = d_2pd_1f(a)^2d_1pd_2 \). Set \( w = d_2pd_1f(a)^{1/2} \). Then, \( w \) in \( f(a)Af(a) \) such that \( p = wf(a)w^* \).

Let \( z_0 = (1 - \varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}}wf(a)^{1/2} \). Then \( \|z_0\| = (1 - \varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}} \) and \( z_0g(a)z_0^* = p \). Since \( g(a) \leq a \), we have \( p = z_0g(a)z_0^* \leq z_0az_0^* \). Thus, there exists an element \( z \in pA \) such that

\[ zaz^* = p, \quad \|z\| \leq \|z_0\| = (1 - \varepsilon)^{-\frac{1}{2}}\|a\|^{-\frac{1}{2}}. \]
Hence, we have
\[ \| zy^*xyz^* - p \| = \| zy^*(x - E(x))yz^* \| < \frac{\varepsilon}{1 - \varepsilon} \times \frac{1}{\| E(x) \| - \varepsilon}. \]
Note that the last inequality follows from the fact that
\[ \| a \|^{-1} \leq \frac{1}{\| E(x) \| - \varepsilon}. \]
We may assume that \( \| zy^*xyz^* - p \| < 1 \). Since \( zy^*xyz^* \in pBp \), \( zy^*xyz^* \) is invertible in \( pBp \); that is, there exists an element \( z_1 \in B \) such that \( z_1 y^*xyz_1^* = p \).

Therefore, since \( z_1 y^*xyz_1^* \in \mathbb{R}^+ \) by Section 1.4 of [6], \( xBx \) has a projection which is equivalent to a projection \( p \) in \( A \). Indeed, \( p = z_1 y^*xyz_1^* \sim x^*y_1^*z_1y^*x \frac{1}{x^*y_1^*z_1y^*x} \subseteq xBx. \)

Next, we consider the following stronger assumption on a conditional expectation \( E \) from \( B \) to \( A \).

**Definition.** Let \( 1 \in A \subset B \) be a pair of \( C^* \)-algebras. A conditional expectation \( E \) from \( B \) to \( A \) is called *outer* if for any element \( x \in B \) with \( E(x) = 0 \) and any non-zero hereditary \( C^* \)-subalgebra \( C \) of \( A \),
\[ \inf \{ \| cEc \| : c \in C^+, \| c \| = 1 \} = 0. \]

The following result comes from the same argument as in Lemma 3.2 of [15] and Theorem 2.1.

**Corollary.** Let \( 1 \in A \subset B \) be a pair of \( C^* \)-algebras. Suppose that \( A \) has the \( SP \)-property and there is a faithful conditional expectation \( E \) from \( B \) to \( A \). If \( E \) is outer, then \( B \) has the \( SP \)-property.

**Proof.** For the reader we write a sketch of the proof. Let \( x \) be a non-zero element in \( B \), and let \( \varepsilon > 0 \) be an arbitrary positive number. Consider a continuous function \( f : \mathbb{R} \to \mathbb{R}^+ \) given by
\[ f(t) = \begin{cases} 1, & t \geq \| E(x) \|; \\
\text{linear}, & \| E(x) \| - \varepsilon \leq t < \| E(x) \|; \\
0, & t < \| E(x) \| - \varepsilon. \end{cases} \]

Let \( C \) be a hereditary \( C^* \)-subalgebra of \( A \) generated by \( f(E(x)) \). Then, since \( \| E(x)c - E(x)c \| < \varepsilon \), for any positive \( c \in C \) with norm one, \( \| cE(x)c \| > \| E(x) \| - \varepsilon \). Indeed, since \( \| cE(x)c \| - \| E(x) \| \| c \| < \varepsilon \), we have \( \| cE(x)c \| \| c^2 \| < \varepsilon \). Hence,
\[ \| E(x) \| = \| \| E(x)c \| c \| = \| cE(x)c + \| E(x) \| c^2 - cE(x)c \| < \| cE(x)c \| + \varepsilon. \]

From the outerness of \( E \), for any arbitrary positive number \( \varepsilon > 0 \) there is a positive element \( y \in C \) with norm one such that
\[ \| y(x - E(x))y \| < \varepsilon, \quad \| yE(x)y \| > \| E(x) \| - \varepsilon. \]
Hence, \( B \) has the \( SP \)-property by Theorem 2.1.

We present some examples of a pair of \( C^* \)-algebras with an outer conditional expectation.
Example ([15]) Let $G$ be a discrete group and let $\alpha$ be a representation of $G$ by automorphisms of a simple unital $C^*$-algebra $A$. Suppose $\alpha$ is outer. Then, the canonical conditional expectation from the reduced crossed product $A \times_\alpha G$ to $A$ is outer.

Proof. Let $u_g, g \in G$ be the standard unitaries in the multiplier algebra of $A \times_\alpha G$ implementing $\alpha$. Let $e$ be the identity of $G$. Let $x$ be an element in $A \times_\alpha G$ with $E(x) = 0$. We approximate $x$ by an element of the dense $*$-algebra spanned by $Au_g, g \in G$, and hence we may assume that $x = \sum_{i=1}^n c_i u_{g_i}$, where $c_i \in A$, and $g_1, \ldots, g_n$ are distinct elements of $G \setminus \{e\}$.

By Lemma 3.2 of [15], for any $\varepsilon > 0$ there is a positive element $c \in C$ such that $\|c\| = 1, \|cc_i u_{g_i}c\| < \varepsilon, i = 1, \ldots, n$.

Hence,

$$\|cxc\| = \left\|c \left( \sum_{i=1}^n c_i u_{g_i} \right) c \right\| \leq \sum_{i=1}^n \|cc_i u_{g_i}c\| < \varepsilon.$$ 

This completes the proof. 

Example Let $\rho$ be a corner endomorphism of a unital $C^*$-algebra $A$, and let $E$ be the canonical conditional expectation from the crossed product $A \times_\rho \mathbb{N}$ by $\rho$ to $A$. Suppose that $\tilde{T}(\rho) = \{ \lambda \in \mathbb{T} : \tilde{\rho}(I) = I \text{ for } \forall I \in \text{Prime}(A \times_\rho \mathbb{N}) \} = \mathbb{T}$.

Then $E$ is outer.

Proof. This comes from the same argument as in Example 2.4 modifying Proposition 2.2 of [13]. 

We now have the structure theorem for the pure infiniteness of a simple crossed product algebra of a purely infinite simple $C^*$-algebra by a discrete group.

Corollary ([12], [16]) Let $A$ be a purely infinite simple $C^*$-algebra, $G$ a discrete group, and $\alpha$ an action of $G$ on $A$. Suppose that $\alpha$ is outer. Then the reduced crossed product $A \times_\alpha G$ is a purely infinite simple $C^*$-algebra.

Proof. In the case of a countable abelian group $G$ see [12], Corollary 3.3. In the case of a general discrete group $G$ see [16], Lemma 10. 

3. C*-INDEX THEORY

In this section, we summarize the C*-index theory of Watatani ([25]). Let $1 \in A \subseteq B$ be a pair of C*-algebras, and let $E : B \rightarrow A$ be a faithful conditional expectation from $B$ to $A$.

A finite family $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ in $B \times B$ is called a quasi-basis for $E$ if

$$\sum_{i=1}^{n} u_i E(v_i b) = \sum_{i=1}^{n} E(b u_i) v_i = b \quad \text{for } b \in B.$$ 

We say that a conditional expectation $E$ is of \textit{index-finite type} if there exists a quasi-basis for $E$. In this case the index of $E$ is defined by

$$\text{Index}(E) = \sum_{i=1}^{n} u_i v_i.$$ 

Note that $\text{Index}(E)$ does not depend on the choice of a quasi-basis and every conditional expectation $E$ of index-finite type on a C*-algebra has a quasi-basis of the form $\{(u_1, u_1^*), \ldots, (u_n, u_n^*)\}$ (Lemma 2.1.6, [25]). Moreover, $\text{Index}(E)$ is always contained in the centre of $B$, so that it is a scalar whenever $B$ has a trivial centre, in particular when $B$ is simple.

Let $E : B \rightarrow A$ be a faithful conditional expectation. Then $B_A(\ast B)$ is a pre-Hilbert module over $A$ with an $A$-valued inner product

$$\langle x, y \rangle = E(x^* y), \quad x, y \in B_A.$$ 

Let $\mathcal{E}$ be the completion of $B_A$ with respect to the norm on $B_A$ defined by

$$\|x\|_{B_A} = \|E(x^* x)\|^\frac{1}{2}_{A}, \quad x \in B_A.$$ 

Then $\mathcal{E}$ is a Hilbert C*-module over $A$. Since $E$ is faithful, the canonical map $B \rightarrow \mathcal{E}$ is injective. Let $L_A(\mathcal{E})$ be the set of all (right) $A$-module homomorphisms $T : \mathcal{E} \rightarrow \mathcal{E}$ with an adjoint $A$-module homomorphism $T^* : \mathcal{E} \rightarrow \mathcal{E}$ such that

$$\langle T \xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle \quad \xi, \zeta \in \mathcal{E}.$$ 

Then $L_A(\mathcal{E})$ is a C*-algebra with the operator norm $\|T\| = \sup \{\|T \xi\| : \|\xi\| = 1\}$. There is an injective $*$-homomorphism $\lambda : B \rightarrow L_A(\mathcal{E})$ defined by

$$\lambda(b)x = bx$$ 

for $x \in B_A$, $b \in B$, so that $B$ can be viewed as a C*-subalgebra of $L_A(\mathcal{E})$. Note that the map $e_A : B_A \rightarrow B_A$ defined by

$$e_A x = E(x), \quad x \in B_A$$ 

is bounded and thus it can be extended to a bounded linear operator, denoted by $e_A$ again, on $\mathcal{E}$. Then $e_A \in L_A(\mathcal{E})$ and $e_A = e_A^2 = e_A^*$; that is, $e_A$ is a projection in $L_A(\mathcal{E})$.

The (reduced) C*-basic construction is a C*-subalgebra of $L_A(\mathcal{E})$ defined to be

$$C^*(B, e_A) = \overline{\text{span}} \{\lambda(x)e_A \lambda(y) : x, y \in B\}$$ 

([25], Definition 2.1.2).

Then,
Lemma (25), Lemma 2.1.4) (i) $e_A C^*(B, e_A) e_A = \lambda(A) e_A$.
  (ii) $\psi : A \to e_A C^*(B, e_A) e_A$, $\psi(a) = \lambda(a) e_A$, is a $*$-isomorphism (onto).

Lemma (25), Lemma 2.1.5) The following are equivalent:
  (i) $E : B \to A$ is of index-finite type.
  (ii) $C^*(B, e_A)$ has an identity and there exists a number $c$ with $0 < c < 1$ such that
      $$E(x^* x) \geq c(x^* x), \quad x \in B.$$ 

The above inequality was shown first in [20] by Pimsner and Popa for the conditional expectation $E_N : M \to N$ from a type II$_1$ factor $M$ onto its subfactor $N$ (c can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \to B$ defined by
  $$E_B(\lambda(x)e_A\lambda(y)) = (\text{Index}(E))^{-1}xy, \quad x, y \in B$$
is called the dual conditional expectation of $E : B \to A$. If $E$ is of index-finite type, so is $E_B$ with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\text{Index}(E)} u_i e_A$, and $\{(u_i, u_i^*)\}$ is a quasi-basis for $E$ ([25], Proposition 2.3.4).

Even if $\text{Index}(E)$ is scalar, we do not know the relation between the number of pairs in a quasi-basis and $\text{Index}(E)$. Izumi, however, showed recently that if we extend a conditional expectation $E$ from $\sigma$-unital $C^*$-algebra $D$ to a stable simple $C^*$-algebra $C$ with $D/C = D$ to the multiplier algebra $M(D)$, then it has only one pair as a quasi-basis. In the case that $C$ and $D$ are stable, we have the following result.

Theorem ([11]) Let $1 \in A \subseteq B$ be a pair of unital $C^*$-algebras, and let $E$ be a faithful conditional expectation from $B$ on $A$ of index-finite type. Suppose that $A$ is simple. Let $\tilde{E}$ be the restriction of $(E \otimes \text{id})^{**}$ to the multiplier algebra $M(B \otimes K)$ of $B \otimes K$, where $K$ denotes a $C^*$-algebra of compact operators on some separable infinite-dimensional Hilbert space. Then, $\tilde{E}$ is a conditional expectation from $M(B \otimes K)$ to $M(A \otimes K)$. Moreover, there exists an isometry $W$ in $M(B \otimes K)$ such that $\{(\sqrt{\text{Index}(E)} W^*, \sqrt{\text{Index}(E)} W)\}$ is a quasi-basis for $\tilde{E}$.

Proof. For completeness, we will give a sketch of the proof.

Let $e_A$ be the projection on the right $A$-Hilbert module $B_A$ defined by $e_A x = E(x)$. Then, $e_A \otimes \text{id}$ is a projection in the multiplier algebra $M(C^*(B, e_A) \otimes K)$ of $C^*(B, e_A) \otimes K$. Since $C^*(B, e_A)$ is isomorphic to a full hereditary algebra of some matrix algebra over $A$ (Lemma 3.1 (ii)), $C^*(B, e_A) \otimes K$ is simple, so $e_A \otimes \text{id}$ is a full projection in $M(C^*(B, e_A) \otimes K)$. By Lemma 2.5 of [3] there is an isometry $V$ in $M(C^*(B, e_A) \otimes K)$ such that $V^* V = 1$ and $V V^* = e_A \otimes \text{id}$.

Let $E_B$ be the dual conditional expectation of $E$ from $C^*(B, e_A)$ to $B$, and $\tilde{E}_B$ be the restriction of $(E_B \otimes \text{id})^{**}$ to $M(C^*(B, e_A) \otimes K)$. Then, for any $x \in M(C^*(B, e_A) \otimes K)$ we have
  $$(e_A \otimes \text{id}) \tilde{E}_B((e_A \otimes \text{id}) x) = \frac{1}{\text{Index}(E)} (e_A \otimes \text{id}) x.$$ 

Set $W = \sqrt{\text{Index}(E)} \tilde{E}_B(V)$. Then, $W \in M(B \otimes K)$, and $V = \sqrt{\text{Index}(E)} (e_A \otimes \text{id}) W$. Hence, we have
  $$1 = V^* V = \text{Index}(E) W^* (e_A \otimes \text{id}) W.$$ 

Therefore $\{\sqrt{\text{Index}(E)} W^*, \sqrt{\text{Index}(E)} W\}$ is a quasi-basis for $\tilde{E}$. $\quad \blacksquare$
However, in the case that $A$ is purely infinite simple, we can find only one pair as a quasi-basis for $E$ as follows:

**Lemma.** Let $1 \in A \subset B$ be a pair of $C^*$-algebras, and let $E$ be a conditional expectation from $B$ to $A$ of index-finite type. Suppose that $A$ is a purely infinite simple $C^*$-algebra. Then there is a co-isometry $\frac{1}{\sqrt{\text{Index}(E)}} v$ such that the pair $\{v, v^*\}$ is a quasi-basis for $E$ in $B$. That is, for any element $x$ in $B$

$$x = E(xv)v^*.$$ 

**Proof.** Let $e_A$ be the projection on the right $A$-Hilbert module $B_A$ defined by $e_A(x) = E(x)$. Then the basic extension $C^*(B, e_A)$ of $B$ is isomorphic to a hereditary subalgebra of some matrix algebra over $A$. Since $A$ is a purely infinite simple $C^*$-algebra, so is $C^*(B, e_A)$. Thus, there is a co-isometry $w$ in $C^*(B, e_A)$ such that $w^*w \leq e_A$ and $ww^* = 1$ ([7]). Note that $we_A = w$.

Using a similar argument as in Lemma 1.2 of [20], there exists a non-zero element $v \in B$ such that $we_A = ve_A$. Then we have $ve_A^*v = 1$. Since $E_B(ve_A^*v) = \frac{1}{\sqrt{\text{Index}(E)}} v^*v = 1$, $\sqrt{\text{Index}(E)} v$ is a co-isometry.

Moreover, we know that $\{v, v^*\}$ is a quasi-basis for $E$. Indeed, for any $b \in B$

$$b = (ve_Av^*)(b) = vE(v^*b) = E(bv)v^*.$$ 

4. THE STRUCTURE THEORY FOR PURELY INFINITE SIMPLE $C^*$-ALGEBRAS

At the end of Section 2, we claimed that if $A$ is a purely infinite simple $C^*$-algebra and $\alpha$ is an outer action from a discrete group $G$ on $A$, then the reduced crossed product $A \times_{\alpha} G$ is also purely infinite simple. In this section, we consider the case of a pair of unital simple $C^*$-algebras $1 \in A \subset B$ with a finite $C^*$-index, and deduce the pure infiniteness of $B$ under the condition that $A$ is purely infinite, which was announced by Izumi at the Fields Institute in 1995. To conclude this result, we use the characterization of the simplicity of the corona algebra of a stable $C^*$-algebra by Rørdam ([24]).

First, we discuss a special case of the SP-property for a pair of $C^*$-algebras, which will help the reader to understand the general case.

**Proposition** Let $1 \in A \subset B$ be a pair of $C^*$-algebras, and let $E$ be a faithful conditional expectation from $B$ to $A$ of index-finite type. Suppose that $A$ is a purely infinite simple $C^*$-algebra. Then $B$ has the SP-property.

We need several lemmas.

**Lemma.** Let $A$ be a unital $C^*$-algebra with the SP-property. Let $a$ be a non-zero, non-invertible positive element in $A$. Then for any positive number $\varepsilon > 0$ there is a projection $e$ in $A$ such that $\|ea\| < \varepsilon$.

**Proof.** Choose a continuous function $f : [0, \|a]\| \to \mathbb{R}^+$ so that $f(t) = 0$ if $t \geq \varepsilon$ and $f(a) \neq 0$. Since $A$ has the SP-property, there is a non-zero projection $e$ in the hereditary subalgebra $f(a)A f(a)$. Then we have $\|ea\| < \varepsilon$. 

\[ \]
Lemma (18, 21) Let $C^*(p, q)$ be the universal unital $C^*$-algebra generated by two projections.

(i) There is an isomorphism from $C^*(p, q)$ onto

$$D = \{ f \in C([0, 1], M_2(\mathbb{C})) : f(0), f(1) \text{ are diagonal} \}$$

which carries the generating projections into the functions

$$p_D(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_D(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$ 

(ii) The spectrum of $C^*(p, q)$ is homeomorphic to the quotient of two copies of $[0, 1]$ in which the corresponding points in $(0, 1)$ have been identified.

Proof of Proposition 4.1. Let $x$ be a non-zero positive element in $B$ with $\|x\| = 1$. Since $A$ is purely infinite simple, there is an element $z \in A$ such that $E(z^*xz) = 1$. Set $y = z^*(x - E(x))z$.

From Lemma 3.4 there is a quasi-basis $\{v, v^*\}$ for $E$ so that $b = E(bv)v^*$ for $b \in B$. So, there is $a \in A$ such that $y = av^*$, $a \in A$. Then $a$ is not invertible in $A$. Indeed, if $a$ is invertible, $v^* = a^{-1}y$, hence $E(v^*) = E(a^{-1}y) = 0$. On the contrary, $1 = E(v)v^* = 0$. This is a contradiction. Hence, either $|a|$ or $|a^*|$ is not invertible.

Suppose that $|a^*|$ is not invertible. Let $\varepsilon$ be an arbitrary positive number. Then, from Lemma 4.2, there is a projection $e$ such that $\|e[a^*]\| < \frac{\varepsilon}{\|v\|}$. Write $a = u|a|$ by a polar decomposition in $A^{**}$. Then

$$\|ea\| = \|e|a^*|u\| < \frac{\varepsilon}{\|v\|}.$$ 

So we have $\|ey\| < \varepsilon$, hence $\|ez^*(x - E(x))ze\| = \|eye\| < \varepsilon$. Since $\|e\| = 1 > \|E(x)\| - \varepsilon$, $xB\varepsilon$ has a non-zero projection from the proof of Theorem 2.1.

Suppose that $|a^*|$ is invertible. Set $u^* = a^*|a^*|^{-1}$. Then, $u$ is a co-isometry in $A$, and $a = |a^*|u$. Then, $y = |a^*|uv^*$. Set $p = u^*u$ and $q = \frac{1}{\text{index}(Ev^*v)}. Then $p$ and $q$ are non-zero projections. Let $\pi$ be a homomorphism from the $C^*$-algebra $D$ in Lemma 4.3 (i) to a $C^*$-algebra $C^*(p, q)$ generated by $p, q$ such that $\pi(p_D) = p$ and $\pi(q_D) = q$. From Lemma 4.3 (ii) we can write the spectrum $\hat{D}$ of $D$ as follows:

$$\begin{array}{cccccccc}
t_0 & t_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
s_0 & s_1,
\end{array}$$

where $t_0, s_0, t_1,$ and $s_1$ are end points of $\hat{D}$.

We may assume that

$$p_D\hat{D}p_D \subseteq \begin{array}{cccccccc}
t_0 & t_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ
\end{array}$$

Then, we consider two cases.

Case 1. $\ker(\pi) \not\ni t_0$. 
Let $\eta > 0$ be an arbitrary positive number. Consider a continuous function $g : [0, 1] \to [0, 1]$ such that $g(0) = 1$, $g(t) = \eta$ for $t \geq \eta$, and linear on $[0, \eta]$. Define

$$T(t) = \begin{pmatrix} g(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in p_D Dp_D.$$ 

Then $T \in p_D Dp_D$ and $\|T\| = 1$. Note that

$$p_D q Dp_D(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in p_D Dp_D.$$ 

Hence,

$$T(t)(p_D q Dp_D)(t) = \begin{pmatrix} g(t)t & 0 \\ 0 & 0 \end{pmatrix}$$

and $\|T p_D q Dp_D\| \leq \eta$.

Set $c = \pi(T)$. Since $t_0 \notin \ker(\pi)$,

$$\|c\| = \|\pi(T)\| = 1.$$ 

Set $b = \alpha^{-1/2} cu^* |a^*|^{-1}$, where $\alpha$ is a positive number and $(aa^*)^{-1} \geq \alpha$. Then,

$$bb^* \geq cu^* uc = cpc = \pi(T^2),$$

so $\|b\| \geq 1$. Note that $by = \alpha^{-1/2} cu^* |a^*|^{-1} y = \alpha^{-1/2} cu^* av^* = \alpha^{-1/2} cu^* av^*$. Hence,

$$\|by\|^2 = \|byy^* b^*\| = \|\alpha^{-1} \text{Index}(E) cu^* av^*\| \leq \alpha^{-1} \text{Index}(E) \eta.$$ 

So, if we take $\eta$ sufficiently small, then $\|by\|^2 < \varepsilon$. Hence, $\|bz^* (x - E(x)) zb\| = \|by\| \leq \varepsilon$. Since $\|bz^* (x - E(x)) zb\| = \|by\|^2 \geq 1 > \|E(x)\| - \varepsilon$, from the proof of Theorem 2.1 $\overline{xBx}$ has a non-zero projection.

(Suppose that $\sigma(z^* x z)$ is not invertible. We may assume that $z^* x z$ is not invertible. Since $t_0 \in \ker(\pi)$, $pqp = \pi(p_D q Dp_D)$ is invertible in $pC^*(p, q)p$. So, there is a positive number $\lambda$ such that $pqp \geq \lambda p$. Since $y = y^*$, we have

$$y^2 = yy^* = |a^*| up qu^* a^* \geq \lambda |a^*| up qu^* a^* = \lambda |a^*| (aa^*)^2 |a^*| = \lambda |a^*|^2.$$ 

Hence $y$ is invertible, since $|a^*|$ is invertible. Note that $\|z^* x z\| \geq \|E(z^* x z)\| = 1$.

Claim. 1 $\notin \sigma(z^* x z)$ ($= \text{the set of spectrum of } z^* x z$).

Suppose that $1 \in \sigma(z^* x z)$. Then, $z^* x z - 1$ is not invertible. So, $y = z^* (x - E(x)) z = z^* x z - 1$ is not invertible. This is a contradiction to the fact that $y$ is invertible.

So, since $\|z^* x z\| \geq 1$ and $0 \in \sigma(z^* x z)$, there is a non-zero spectral projection $\chi(z^* x z)$ of $z^* x z$ which is not equal to one. Then, $\chi(z^* x z) \in \overline{z^* x B x z}$. Since $\overline{z^* x B x z} \cong \overline{x z B x z}$ by [6], 1.4, $\overline{x B x}$ has a non-zero projection. \qed
Remark In Proposition 4.1, since $A$ is simple and $E$ is of index-finite type, we know that $B$ is a direct sum of finitely simple $C^*$-algebras $B_i$ by the next lemma. So, if we conclude that the last projection $\chi(x^*x)$ is infinite, then we could conclude that each simple $C^*$-algebra $B_i$ is purely infinite. In fact, in the case that $B$ is a crossed product algebra of $A$ by a finite group, we can conclude that any non-zero hereditary $C^*$-subalgebra of $B$ has a non-zero projection that is equivalent to some projection in $M_n(A)$ for some $n$, using the same method as in Theorem 1.1 ([13], Theorem 4.2).

Lemma ([11]) Let $1 \in A \subset B$ be a pair of $C^*$-algebras, and let $E$ be a faithful conditional expectation from $B$ to $A$ of index-finite type.

(i) If $A$ is simple, then $B$ can be written as a finite direct sum of simple $C^*$-algebras.

(ii) If $B$ is simple, then $A$ can be written as a finite direct sum of simple $C^*$-algebras.

Proof. We have only to show case (i). Case (ii) can be deduced from the fact that $A$ is isomorphic to a corner hereditary $C^*$-subalgebra of $C^*(B,e_A)$, by Lemma 3.1 (ii).

Let $J$ be a non-zero closed two-sided ideal of $B$, and $\{u_\lambda\}$ be an approximate identity for $J$. Then $\{u_\lambda\}$ converges strongly to a positive element $z$ in the centre of $B^{**}$. We claim that $z \in B$. Let $E^{**}$ be a faithful conditional expectation from $B^{**}$ to $A^{**}$, which is derived from $E$. Note that $\text{Index}(E^{**}) = \text{Index}(E)$. Then $E(u_\lambda)$ converges strongly to $E^{**}(z)$ in the centre of $A^{**}$. Set $c = \sup\{\|E(u_\lambda)\| : \lambda\}$. Since $A$ is simple, $c = E^{**}(z)$. In fact, assume that $c \neq E^{**}(z)$. Then, for an arbitrary positive number $\varepsilon$ there exists a non-zero projection $r$ in the centre of $A^{**}$ such that

$$rE^{**}(z) < (c - \varepsilon)r.$$  

Since $A$ is simple, $\|E(u_\lambda)r\| = \|E(u_\lambda)\|$, hence $\|rE^{**}(z)\| = c$, which is a contradiction.

Set $Q(A) = \{\varphi \in A_+^* : \|\varphi\| = 1\}$. Then $Q(A)$ is weak-$*$-compact. Since $\varphi(c - E(u_\lambda))$ converges to 0 for $\varphi \in Q(A)$, $c - E(u_\lambda)$ converges uniformly to 0 by Dini’s Theorem. Hence,

$$\|E(u_\lambda) - E^{**}(z)\| \longrightarrow 0, \quad \lambda \not\rightarrow.$$  

On the contrary, since $E^{**}$ is of index-finite type, there is a positive number $d$ such that

$$E^{**}(x) \geq dx, \quad x \in B_+^{**}$$  

from Lemma 3.2 (ii). Since $\|E^{**}(u_\lambda - z)\|$ converges to 0 uniformly, $\|u_\lambda - z\|$ converges uniformly to 0. Hence $z$ is in the centre of $B$, and $J = zB$. Since the centre of $B$ is finite-dimensional by Proposition 2.7.3 of [25], $B$ can be written as a direct sum of finitely simple $C^*$-algebras.

A proof of the following theorem is given more directly by Theorem 4.5 of [13] in the case that $B$ is a crossed product algebra of $A$ by a finite group.
5. MAIN THEOREM

In this section we present the main theorem. The proof is almost the same as in Proposition 4.1 using Theorem 3.3.

**Theorem**. Let \( 1 \in A \subseteq B \) be a pair of separable \( \mathrm{C}^\ast \)-algebras, and let \( E \) be a faithful conditional expectation from \( B \) to \( A \) of index-finite type. Suppose that \( A \) is a purely infinite simple \( \mathrm{C}^\ast \)-algebra. Then \( B \) is a finite direct sum of purely infinite simple \( \mathrm{C}^\ast \)-algebras.

**Proof.** By Lemma 4.5 (i), \( B \) is a finite direct sum of simple \( \mathrm{C}^\ast \)-algebras \( B_i \). Take central projections \( p_i \) in \( B \) such that \( p_iB = B_i \). Then there are conditional expectations \( E_i \) of index-finite type: \( B_i \rightarrow p_iAp_i \). Since each \( p_iAp_i \) is purely infinite simple, we may assume that \( B \) is simple.

Consider a conditional expectation \( E \) of index-finite type from \( M(B \otimes K) \) to \( M(A \otimes K) \). Then there is a conditional expectation \( F \) of index-finite type from \( M(B \otimes K)/(B \otimes K) \) to a \( \mathrm{C}^\ast \)-algebra \( \{ x + B \otimes K : x \in M(A \otimes K) \} \) (= \( D \)). Since \( A \) is purely infinite simple, \( M(A \otimes K)/(A \otimes K) \) is simple by Theorem 3.2 of [24]. So, \( M(A \otimes K)/(A \otimes K) \) is isomorphic to \( D \). Since \( F \) is of index-finite type, \( M(B \otimes K)/(B \otimes K) \) is a direct sum of some simple \( \mathrm{C}^\ast \)-algebras by Lemma 4.5 (i).

We claim that \( M(B \otimes K)/(B \otimes K) \) is simple. Indeed, since \( B \) is a separable simple \( \mathrm{C}^\ast \)-algebra with the \( \text{SP} \)-property by Proposition 4.1, this corona algebra is prime by Theorem 2.7 of [17]. So it should be simple. Again, by Theorem 3.2 of [24], \( B \) is purely infinite.

**5. MAIN THEOREM**

In this section we present the main theorem. The proof is almost the same as in Proposition 4.1 using Theorem 3.3.

**Theorem.** Let \( 1 \in A \subseteq B \) be a pair of unital \( \mathrm{C}^\ast \)-algebras, and let \( E \) be a faithful conditional expectation from \( B \) to \( A \) of index-finite type. Suppose that \( A \) is simple and has the \( \text{SP} \)-property. Then \( B \) has the \( \text{SP} \)-property.

Note that in the case that \( B = A \times \alpha G \) is a crossed product algebra of \( A \) by a finite group \( G \), Jeong and Osaka concluded the statement more directly ([13], Theorem 4.2).

**Proof.** We show that \( B \otimes K \) has the \( \text{SP} \)-property, where \( K \) denotes a \( \mathrm{C}^\ast \)-algebra of compact operators on some separable infinite-dimensional Hilbert spaces. From Theorem 3.3 there is a conditional expectation \( \tilde{E} \) from \( M(B \otimes K) \) to \( M(A \otimes K) \), and an isometry \( W \) in \( M(B \otimes K) \) such that \( \{ (\sqrt{\text{Index}(E)}W^*, \sqrt{\text{Index}(E)}W) \} \) is a quasi-basis for \( \tilde{E} \). Set \( v = \sqrt{\text{Index}(E)}W^* \).

Take \( x \in (B \otimes K)_\oplus \) with \( \|x\| = 1 \). As in the proof of Theorem 2.1 there is a continuous function \( f : [0, 1] \rightarrow [0, 1] \) such that \( f(\tilde{E}(x))(A \otimes K)f(\tilde{E}(x)) \) has a non-zero projection \( r \) and \( z \in r(A \otimes K)f(\tilde{E}(x)) \) such that \( z\tilde{E}(x)z^* = r \). Set \( y = z(x - \tilde{E}(x))z^* \). Then, \( \tilde{E}(y) = 0 \) and \( ry = yg = yr \). So, \( y \in r(B \otimes K)r \). Write \( y = \tilde{E}(yv)v^* = av^* \). Since

\[ ra = r\tilde{E}(yv) = \tilde{E}(ryv) = \tilde{E}(yv) = a, \]

\( |a^*| \in r(A \otimes K)r \). Note that \( r(A \otimes K)r \) has the \( \text{SP} \)-property.

Let \( \varepsilon \) be an arbitrary positive number.
Suppose that $|a^*|$ is not invertible. Then, since $r(A \otimes K)r$ is a unital $C^*$-algebra with the SP-property, from Lemma 4.2 there is a projection $e$ in $r(A \otimes K)r$ such that $\|e|a^*|\| < \frac{\varepsilon}{\|v\|}$. So we can conclude that $x(B \otimes K)x$ has a non-zero projection, by the same argument as in Proposition 4.1.

Suppose that $|a^*|$ is invertible. Then using Lemma 4.3 we can conclude that $x(B \otimes K)x$ has a non-zero projection, by the same steps in the proof of Proposition 4.1.

Therefore, $B \otimes K$ has the SP-property, and so has $B$. □

**Corollary** Let $1 \in A \subset B$ be a pair of simple unital $C^*$-algebras, and let $E$ be a conditional expectation from $B$ to $A$ of index-finite type. Then $A$ has the SP-property if and only if $B$ has the SP-property.

**Proof.** Suppose that $B$ has the SP-property. Consider the basic construction:

$$1 \in A \subset B \subset C^*(B, e_A).$$

Since $A$ is simple, $C^*(B, e_A)$ is simple from Corollary 2.2.14 in [25]. So we know that $C^*(B, e_A)$ has the SP-property from Theorem 5.1. Hence from Lemma 3.1 (ii) we know that $A$ has the SP-property. □

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**REFERENCES**


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