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LOGARITHMIC SOBOLEV INEQUALITIES: CONDITIONS AND COUNTEREXAMPLES

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Abstract. Let M be any noncompact, connected, complete Riemannian manifold with Riemannian distance function (from a fixed point) ρ. Consider $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $\mathrm{d}\mu := \mathrm{e}^V\,\mathrm{d}x$ a probability measure. Define $\delta \geqslant 0$ as the smallest possible constant such that for any $K, \varepsilon > 0$, $\mu(\exp[(\delta K + \varepsilon)\rho^2]) < \infty$ implies the logarithmic Sobolev inequality (abbrev. LSI) for any M and V with $\mathrm{Ric}\text{-}\mathrm{Hess}_V \geqslant -K$. It is shown in the paper that $\delta \in [\frac{1}{4}, \frac{1}{2}]$.

Moreover, some differential type conditions are presented for the LSI. As a consequence, a result suggested by D. Stroock is proved: for $V = -r\rho^2$ with $r > 0$, the LSI holds provided the Ricci curvature is bounded below.

Keywords: *Logarithmic Sobolev inequality, Riemannian manifold, Ricci curvature.*

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1. INTRODUCTION

Let M be a noncompact, connected, complete Riemannian manifold of dimension d. We assume throughout the paper that M is convex for the case $\partial M \neq \emptyset$. Consider the operator $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $Z = \int \exp[V]\,\mathrm{d}x < \infty$. Let $\mathrm{d}\mu = Z^{-1}\exp[V]\,\mathrm{d}x$. We say that the logarithmic Sobolev inequality (abbrev. LSI) holds for L (or for V) if there exists $\alpha > 0$ such that

$$\mu(f^2 \log f^2) \leqslant \frac{2}{\alpha}\mu(|\nabla f|^2), \quad \mu(f^2) = 1. \tag{1.1}$$

The largest possible constant (denoted also by α) is called the logarithmic Sobolev constant.

Fix a point $o \in M$ and let ρ be the Riemannian distance function from o. According to Herbst's argument (see, e.g., [2]), if the LSI holds, then there exists $\varepsilon > 0$ such that

$$\mu(\mathrm{e}^{\varepsilon\rho^2}) < \infty. \tag{1.2}$$

Assume that there exists $K \geqslant 0$ such that Ric-Hess$_V \geqslant -K$, which means that (Ric-Hess$_V$)$(X, X) \geqslant -K$ for any $X \in TM$ with $|X| = 1$. Let P_t denote the semigroup of L (with reflecting boundary if $\partial M \neq \emptyset$). By [19] one has $\|P_t\|_{2\to p} < \infty$ for some $t > 0$ and $p > 2$ provided

$$\mu(\mathrm{e}^{(2K+\varepsilon)\rho^2}) < \infty \tag{1.3}$$

for some $\varepsilon > 0$. Here, $\|\cdot\|_{2\to p}$ denotes the operator norm from $L^2(\mathrm{d}\mu)$ to $L^p(\mathrm{d}\mu)$. It is well known from [11] that $\|P_t\|_{2\to p} < \infty$ for some $t > 0$ and $p > 2$ is equivalent to the defective LSI

$$\mu(f^2 \log f^2) \leqslant c_1\mu(|\nabla f|^2) + c_2, \quad \mu(f^2) = 1. \tag{1.4}$$

And according to [1], (1.4) is actually equivalent to the LSI in our present setting. Therefore, (1.3) is a sufficient condition for the LSI (see also [4]).

We now have two integral conditions (1.2) and (1.3), which are respectively necessary and sufficient for the LSI. The difference between them is that (1.2) is curvature free but (1.3) is not. Our first aim is to show that for a sufficient integral condition in this form, the dependence on curvature is necessary. This leads to the following result which also contains a slight improvement of (1.3).

THEOREM 1.1. *Assume that* Ric-Hess$_V \geqslant -K$ *for some* $K \geqslant 0$. *The* LSI *holds provided there exists* $\varepsilon > 0$ *such that*

$$\mu(\mathrm{e}^{(K/2+\varepsilon)\rho^2}) < \infty. \tag{1.5}$$

On the other hand, for any $K > 0$ *and* $c < \frac{1}{4}K$, *there exists an example with* Ric-Hess$_V \geqslant -K$, $\mu(\mathrm{e}^{c\rho^2}) < \infty$ *but the* LSI *does not hold.*

Let $\delta \geqslant 0$ be the smallest possible constant such that for all K, $\varepsilon > 0$, $\mu(e^{(\delta K+\varepsilon)\rho^2}) < \infty$ implies the LSI for any M and operator L with Ric-Hess$_V \geqslant -K$. It then follows from Theorem 1.1 that $\delta \in [\frac{1}{4}, \frac{1}{2}]$.

Next, let us consider the differential conditions for the LSI. Let $k(x)$ be a continuous function such that (Ric-Hess$_V$)$_x \geqslant k(x)$. By Bakry-Emery's criterion, one has $\alpha \geqslant \inf k$ and hence the LSI holds if $\inf k > 0$. As an improvement, Chen and Wang proved in [7] that the LSI holds provided o is a pole, the sectional curvatures are nonpositive and

$$\lim_{r\to\infty} \inf_{\rho(x)\geqslant r} k(x) > 0. \tag{1.6}$$

Our next result says that (1.6) is sufficient for the LSI without any other assumption.

THEOREM 1.2. *The condition* (1.6) *implies the* LSI.

But when the Ricci curvature is bounded below, the Riemannian volume of the geodesic ball grows at most radius-exponentially fast. So, in terms of (1.2) and (1.3), the LSI should essentially depend on the behavior of Hess$_V$ rather than the curvature. From this point of view, we have the following result.

Theorem 1.3. *Assume that the Ricci curvature is bounded below. The* LSI *holds provided* $\overline{\lim}_{\rho\to\infty} \mathrm{Hess}_V < 0$*, while the* LSI *does not hold if* $\underline{\lim}_{\rho\to\infty} \mathrm{Hess}_V \geqslant 0$.

Finally, we would like to mention a problem suggested by D. Stroock at the workshop on logarithmic Sobolev inequality (IHP, Paris, May 1998): how can one prove the LSI for e.g. $V = -r\rho^2$ ($r > 0$) which may be not smooth somewhere. The next two results provide a way to solve this problem by using the Sobolev inequality.

Theorem 1.4. *Assume that* W *is a locally Lipschitz continuous function such that the following Sobolev inequality holds for* $\mathrm{d}\nu = \mathrm{e}^W\,\mathrm{d}x$ *and some* $c_1, c_2 > 0$, $v > 2$*:*

$$\left\{\int f^{2v/(v-2)}\,\mathrm{d}\nu\right\}^{(v-2)/v} \leqslant c_1\int|\nabla f|^2\,\mathrm{d}\nu + c_2,\quad f\geqslant 0,\ \int f^2\,\mathrm{d}\nu = 1. \tag{1.7}$$

Next, assume that V *is a locally Lipschitz continuous function on* M *such that* $\Delta\mathrm{e}^V \geqslant \mathrm{e}^V\eta$ *in the distribution sense for some* $\eta\in C(M)$*, namely,*

$$\int \mathrm{e}^V\Delta f\,\mathrm{d}x \geqslant \int f\mathrm{e}^V\eta\,\mathrm{d}x,\quad f\geqslant 0,\ f\in C_0^\infty(M); \tag{1.8}$$

when $\partial M \neq \emptyset$*, the function* f *is assumed to satisfy the Neumann boundary condition. Let* $|\nabla V(x)|$ *denote the local Lipschitz constant of* V *at* x*. Then the* LSI *holds provided there exist* $\lambda, \varepsilon > 0$ *such that*

$$\int \exp[\lambda(|\nabla W|^2 + |\nabla V|^2 - 2\eta) + (1+\varepsilon)(W - V)]\,\mathrm{d}\mu < \infty. \tag{1.9}$$

Theorem 1.5. *Assume that the Ricci curvature is bounded below by* $-K$ *for some* $K\geqslant 0$*. Let* $W = c\rho$ *for some* $c > \sqrt{K(d-1)}$*; then* (1.7) *holds for* $v\in[d,\infty)\cap(2,\infty)$.

Corollary 1.6. *Assume that the Ricci curvature is bounded below. Let* $V = -r\rho^2$ *with* $r > 0$*, then the* LSI *holds.*

Proof. By Theorem 1.5, there exists $c > 0$ such that (1.7) holds for $W = c\rho$ and some $v > 2$. It is well known that ρ^2 is smooth except on the cut locus of o whose measure is zero. Since the Ricci curvature is bounded below, there exists $c' > 0$ such that $\Delta\rho^2 \leqslant c'(1+\rho)$ out of the cut locus. Therefore, following the argument of Yau (see [20], Appendix), we have (1.8) for $\eta = |\nabla V|^2 - c'r(1+\rho)$. Then (1.9) holds for large λ and small $\varepsilon > 0$. ∎

Professor D. Stroock has pointed out to the author that Corollary 1.6 can also be proved by showing the hypercontractivity of the corresponding semigroup. To this end, one needs to prove first the existence and some properties of the diffusion process. Refer to [16] for details.

2. INTEGRAL CONDITIONS: PROOF OF THEOREM 1.1

Let $\rho(x,y)$ be the Riemannian distance between x and y. By [19], for any $f \in C_{\mathrm{b}}(M)$, $r>1$, $t \geqslant 0$ and $x,y \in M$, one has

$$|P_t f(x)|^r \leqslant P_t|f|^r(y) \exp\left[\frac{rK\rho(x,y)^2}{2(r-1)(1-\mathrm{e}^{-2Kt})}\right]. \tag{2.1}$$

For $r>2$, let $f \geqslant 0$ be such that $\mu(f^r)=1$; we then have

$$\begin{aligned}1=\mu(P_t f^r) &\geqslant [P_t f(x)]^r \int \exp\left[-\frac{rK\rho(x,y)^2}{2(r-1)(1-\mathrm{e}^{-2Kt})}\right]\mu(\mathrm{d}y)\\ &\geqslant [P_t f(x)]^r \mu(B(o,1)) \exp\left[-\frac{rK(\rho(x)+1)^2}{2(r-1)(1-\mathrm{e}^{-2Kt})}\right].\end{aligned}$$

Therefore, there exists $c>0$ such that

$$P_t f(x) \leqslant c \exp\left[\frac{K(\rho(x)+1)^2}{2(r-1)(1-\mathrm{e}^{-2Kt})}\right], \quad t \geqslant 0,\ x \in M. \tag{2.2}$$

If (1.5) holds, then

$$\mu((P_t f)^{r+\varepsilon'}) \leqslant c\int \exp\left[\frac{(r+\varepsilon')K(\rho+1)^2}{2(r-1)(1-\mathrm{e}^{-2Kt})}\right]\mathrm{d}\mu<\infty$$

for small $\varepsilon'>0$ and large r and t. Let $\varepsilon=\frac{r/2-1}{r-1} \in (0,1)$, and let $p>2$ be such that $1/p=\varepsilon+\frac{1-\varepsilon}{r+\varepsilon'}$. Noting that P_t is contractive in $L^1(\mu)$, by Riesz-Thorin's interpolation theorem (see e.g. [9], p. 3) we obtain $\|P_t\|_{2\to p}<\infty$. This implies the LSI as we have explained in the previous section. The proof of the first assertion is then completed.

To prove the second assertion, we recall Ledoux's isoperimetric inequality for the LSI (see [15]). Define

$$\kappa_1=\inf\frac{\mu_{\mathrm{s}}(\partial A)}{\mu(A)}, \quad \kappa_2=\inf\frac{\mu_{\mathrm{s}}(\partial A)}{\mu(A)\sqrt{-\log\mu(A)}},$$

where A runs over all open smooth connected domains with $\mu(A) \leqslant 1/2$, and $\mu_{\mathrm{s}}(\partial A)$ denotes the area of ∂A induced by μ.

THEOREM 2.1. (Ledoux) *Assume that* Ric-Hess$_V$ *is bounded below. Then there exist* $c_1, c_2>0$ *such that*

$$\frac{\kappa_1^2}{4} \leqslant \lambda_1 \leqslant c_1(\kappa_1^2+\kappa_1), \quad \frac{\kappa_2^2}{c_2} \leqslant \alpha \leqslant c_2(\kappa_2^2+\kappa_2),$$

where λ_1 *denotes the spectral gap of* L.

We remark that the above result was proved by Ledoux ([15]) for compact manifolds with an argument also valid for our present case. Actually, the key point of his proof is the gradient estimate

$$\|\nabla P_t f\|_\infty \leqslant \frac{c(t_0)}{\sqrt{t}}\|f\|_\infty, \quad t \in (0,t_0).$$

This estimate remains true whenever Ric-Hess$_V$ is bounded below (see e.g. [18] for detailed estimates).

COROLLARY 2.2. *Under the assumption of Theorem* 2.1, *we have:*

(i) *If* $\lambda_1 > 0$, *then there exist* $c_1, c_2 > 0$ *such that* $\mu(\rho > r) \leqslant c_1 \mathrm{e}^{-c_2 r}$, $r \geqslant 0$.

(ii) *If* $\alpha > 0$, *then there exist* $c_1, c_2 > 0$ *such that* $\mu(\rho > r) \leqslant c_1 \mathrm{e}^{-c_2 r^2}$, $r \geqslant 0$.

In particular, for one-dimensional case, we may assume that $M = \mathbb{R}$ *or* $[0,\infty)$. *Define* $F(r) = \min\{\mu(r,\infty), 1-\mu(r,\infty)\}$. *We then have:*

(iii) $\lambda_1 > 0$ *if and only if there exists* $c > 0$ *such that* $\mathrm{e}^V \geqslant cF$.

(iv) $\alpha > 0$ *if and only if there exists* $c > 0$ *such that* $\mathrm{e}^V \geqslant cF\sqrt{-\log F}$.

Proof. We only prove (ii) and (iv) since the proofs of (i) and (iii) are similar. By Theorem 2.1, $\alpha > 0$ implies $\kappa_2 > 0$. Let $f(r) = \mu(\rho > r)$, then $\mu_{\mathrm{s}}(S_r) = -f'(r)$. Choose $r_0 > 0$ such that $f(r_0) = 1/2$, we have

$$\frac{-f'(r)}{f(r)\sqrt{-\log f(r)}} \geqslant \kappa_2 > 0, \quad r \geqslant r_0.$$

This proves (ii) immediately.

Next, for the one-dimensional case, assume that $\mathrm{e}^V \geqslant cF\sqrt{-\log F}$ for some $c > 0$. Let $A = (a,b) \subset M$ with $\mu(A) \leqslant 1/2$. Take $r_- < r_+$ be such that $\mu(r_-, r_+) > \frac{1}{2}$. If a or b is in (r_-, r_+), then $\mu_{\mathrm{s}}(\partial A) \geqslant \inf_{(r_-,r_+)} Z^{-1}\mathrm{e}^V := c_1 > 0$. But

$$\mu(a)\sqrt{-\log \mu(A)} \leqslant \sup_{(0,1/2)} r\sqrt{-\log r} = 1/\sqrt{2\mathrm{e}}.$$

We obtain

$$\frac{\mu_{\mathrm{s}}(\partial A)}{\mu(A)\sqrt{-\log \mu(A)}} \geqslant c_1\sqrt{2\mathrm{e}} > 0.$$

On the other hand, if $a, b \notin (r_-, r_+)$, without loss of generality, assume that $a, b \geqslant r_+$ (noting that $\mu(r_-, r_+) > 1/2 \geqslant \mu(A)$). We have $\mu(A) \leqslant F(a)$ and $\mu(A)\sqrt{-\log \mu(A)} \leqslant F(a)\sqrt{-\log F(a)}$ since the function $r\sqrt{-\log r}$ is increasing in $(0, 1/2)$. Then by (iv),

$$\frac{\mu_{\mathrm{s}}(\partial A)}{\mu(A)\sqrt{-\log \mu(A)}} \geqslant \frac{\mathrm{e}^{V(a)}}{ZF(a)\sqrt{-\log F(a)}} \geqslant \frac{c}{Z} > 0.$$

Therefore, $\kappa_2 > 0$ and equivalently, the LSI holds. Finally, the necessity follows directly from Theorem 2.1 since $\mu_{\mathrm{s}}(\partial A) = Z^{-1}\mathrm{e}^{V(r)}$ for $A = (r,\infty)$ or for (when $M = \mathbb{R}$) $A = (-\infty, r)$. ∎

We now turn to prove the second assertion of Theorem 1.1 by constructing some proper counterexamples. For simplicity, we consider the case when $M = [0,\infty)$ (the full line case is similar). By Corollary 2.2, for any $K > 0$ and $c \in (0, \frac{1}{4}K)$, it suffices to construct $V \in C^2[0,\infty)$ such that $V'' \leqslant K$,

$$\int_0^\infty \mathrm{e}^{V(r)+cr^2}\,\mathrm{d}r < \infty, \tag{2.3}$$

and

$$\lim_{r\to\infty} \inf_{r' \geqslant r} \frac{\mathrm{e}^{V(r')}}{\mu(r',\infty)\sqrt{-\log \mu(r',\infty)}} = 0. \tag{2.4}$$

For any $\varepsilon > 0$, choose nonnegative $h \in C^\infty[0,2]$ such that $h(0) = h(2) = 1$, $h(1) = 0$, $|h(r) - |1-r|\,| \leqslant \varepsilon$, $|h'| \leqslant 1+\varepsilon$, $h'(r) \leqslant 0$ for $r \in [0,1]$ and $h'(r) \geqslant 0$ for $r \in [1,2]$. Let $c(\varepsilon) = \|h''\|_\infty$. Next, for $m \gg \sigma \gg 1$, define

$$g(r) = \begin{cases} 1, & \text{if } r \notin \bigcup\limits_{n=2}^{\infty}(m^n, (2+\sigma)m^n); \\ h(\frac{r-m^n}{\sigma m^n})(1-m^{1-n}) + m^{1-n}, & \text{if } r \in (m^n, (1+\sigma)m^n); \\ h(\frac{r-\sigma m^n}{m^n})(1-m^{1-n}) + m^{1-n}, & \text{if } r \in [(1+\sigma)m^n, (2+\sigma)m^n),\ n \geqslant 2. \end{cases}$$

Then $g \in C^\infty[0,\infty)$ and $\lim\limits_{r\to\infty} \inf\limits_{r' \geqslant r} g(r') = 0$.

Let $\beta > 0$ to be determined, and define V and F by

$$\int_r^\infty \mathrm{e}^V = F(r) = \exp\Big[-\beta\Big(\int_0^r g\Big)^2\Big], \quad r \geqslant 0.$$

Then

$$\frac{\mathrm{e}^V}{F(r)\sqrt{-\log F(r)}} = 2\frac{\mathrm{d}}{\mathrm{d}r}\sqrt{-\log F(r)} = 2\sqrt{\beta}g(r)$$

and hence (2.4) holds.

We now estimate V''. It is easy to see that $V = \log[2\beta g(\int_0^r g)F(r)]$ and we then obtain

$$V' = \frac{g'}{g} + \frac{g}{\int_0^r g} - 2\beta g \int_0^r g, \quad V'' = \frac{g''}{g} + \frac{g'}{\int_0^r g} - \frac{{g'}^2}{g^2} - \frac{g^2}{(\int_0^r g)^2} - 2\beta g' \int_0^r g - 2\beta g^2.$$

Then $V'' \leqslant -2\beta < 0$ outside of $\bigcup\limits_{n\geqslant 2}(m^n, (2+\sigma)m^n)$. For $r \in (m^n, (2+\sigma)m^n)$, we have $g \geqslant m^{1-n}$, $\int_0^r g \geqslant m^n - (2+\sigma)m^{n-1}$, $|g''| \leqslant \frac{c(\varepsilon)}{m^{2n}}$, $-\frac{1+\varepsilon}{\sigma m^n} \leqslant g' \leqslant 0$ on $[m^n, (1+\sigma)m^n]$, $0 \leqslant g' \leqslant \frac{1+\varepsilon}{m^n}$ on $[(1+\sigma)m^n, (2+\sigma)m^n]$ and $\int_0^{(1+\sigma)m^n} g \leqslant (1 + \frac{\sigma}{2} + \sigma\varepsilon + \sigma m^{1-n})m^n$. Then

$$V'' \leqslant \frac{c(\varepsilon)}{m^{n+1}} + \frac{1+\varepsilon}{m^{2n}(1-\frac{2+\sigma}{m})} + \frac{\beta(1+\varepsilon)(2+\sigma+2\varepsilon\sigma+2\sigma m^{1-n})}{\sigma}.$$

Therefore, we have $V'' \leqslant K$ choosing β (recall that $n \geqslant 2$) to be

$$\beta = \frac{\sigma}{(1+\varepsilon)(2+\sigma+2\varepsilon\sigma+2\sigma m^{-1})}\Big[K - \frac{c(\varepsilon)}{m^3} - \frac{1+\varepsilon}{m^4(1-\frac{2+\sigma}{m})}\Big]. \tag{2.5}$$

We remark that

$$\lim_{\varepsilon\to 0}\lim_{\sigma\to\infty}\lim_{m\to\infty}\beta = K. \tag{2.6}$$

Finally, it remains to show (2.3). To this end, it suffices to prove

$$\beta \inf_{r>0} \frac{\left(\int_0^r g\right)^2}{r^2} > c \tag{2.7}$$

for some ε, m, σ. For $r \in (m^n, (1+\sigma)m^n)$, we have

$$\frac{1}{r}\int_0^r g \geqslant \frac{1}{r}\Big(m^n - (1+2\sigma)m^{n-1} + \int_{m^n}^r \Big(1 - \frac{s-m^n}{\sigma m^n}\Big)\,\mathrm{d}s\Big) - \varepsilon.$$

Noting that the right hand side is decreasing in r, we obtain

$$\begin{aligned}\frac{1}{r}\int_0^r g &\geqslant \frac{1}{(1+\sigma)m^n}\Big(m^n\Big(1 - \frac{1+2\sigma}{m}\Big) + \int_{m^n}^{(1+\sigma)m^n} \Big(1 - \frac{s-m^n}{\sigma m^n}\Big)\,\mathrm{d}s\Big) - \varepsilon \\ &= \frac{1}{1+\sigma}\Big(1 - \frac{1+2\sigma}{m} + \frac{\sigma}{2}\Big) - \varepsilon := c(m,\sigma,\varepsilon).\end{aligned} \tag{2.8}$$

On the other hand, for $r \in [(1+\sigma)m^n, (2+\sigma)m^n)$, it follows that

$$\frac{1}{r}\int_0^r g \geqslant \frac{1}{(2+\sigma)m^n}\int_0^{(1+\sigma)m^n} g \geqslant \frac{1+\sigma}{2+\sigma}c(m,\sigma,\varepsilon). \tag{2.9}$$

Now, for any $r > m^2$, let $s(r) = \sup\{s \leqslant r : s \in (m^n, (2+\sigma)m^n)$ for some $n \geqslant 2\}$. Then

$$\frac{1}{r}\int_0^r g = \frac{r - s(r) + \int_0^{s(r)} g}{r} \geqslant \frac{1}{s(r)}\int_0^{s(r)} g.$$

By combining this with (2.8) and (2.9), we obtain

$$\beta \inf_{r>0} \frac{(\int_0^r g)^2}{r^2} \geqslant \frac{\beta(1+\sigma)^2}{(2+\sigma)^2}\big[c(m,\sigma,\varepsilon)^+\big]^2$$

which goes to $\frac{1}{4}K$ by first letting $m \to \infty$, then $\sigma \to \infty$ and finally $\varepsilon \to 0$. Therefore, there exists $\varepsilon > 0$ and $m \gg \sigma \gg 1$ such that (2.7) holds. The proof is then completed. ∎

By an argument as in [7], we have the following consequence.

COROLLARY 2.3. *Assume that o is a pole and the sectional curvatures of M are nonpositive. The condition* (1.5) *in Theorem* 1.1 *can be replaced by*

$$\mu\big(\exp[(\varepsilon - \liminf_{\rho\to\infty} k/2)\rho^2]\big) < \infty. \tag{1.5$'$}$$

Proof. According to [7], under the assumption there exists $W \in C^2(M)$ such that $\|V - W\|_\infty < \infty$ and

$$\text{Ric-Hess}_W \geqslant \liminf_{\rho\to\infty} k - \frac{\varepsilon}{2} := -K_W.$$

By (1.5′) and Theorem 1.1, the LSI holds for W and hence it also holds (with a different constant) for V. ∎

3. DIFFERENTIAL CONDITIONS: PROOFS OF THEOREMS 1.2 AND 1.3

We first prove some lemmas.

LEMMA 3.1. *If either the Ricci curvature is bounded below by* $-K \leqslant 0$ *and* $\lim_{r\to\infty} \inf_{\rho\geqslant r} \mathrm{Hess}_{-V} > 0$ *or* $\lim_{r\to\infty} \inf_{\rho\geqslant r} k > 0$, *then for any* $r > 1$, *there exist* $c_1, c_2, \lambda > 0$ *such that*

$$P_t|P_1 f|^r(x) \leqslant P_{t+1}|f|^r(y) \exp\left[c_1 + c_2 \exp[-\lambda t]\rho(x,y)^2\right]$$

holds for all $t \geqslant 0$, $f \in C_{\mathrm{b}}(M)$ *and* $x, y \in M$.

Proof. Choose $\lambda > 0$ such that

$$\max\left\{\lim_{r\to\infty}\inf_{\rho\geqslant r} k,\ \lim_{r\to\infty}\inf_{\rho\geqslant r} \mathrm{Hess}_{-V}\right\} \geqslant 2\lambda. \tag{3.1}$$

For any $x, y \in M$, let (x_t, y_t) be the coupling by parallel displacement with $x_0 = x$ and $y_0 = y$, see [18] for the construction of this coupling (see also [13] for the original argument). Then (see [18]) the martingale part of $\rho(x_t, y_t)$ disappears and the remainder is the same as that for the coupling by reflection constructed in [14] and [8]. Therefore (see e.g. [6] and [17]),

$$\mathrm{d}\rho(x_t,y_t) \leqslant \left\{2\sqrt{K(d-1)} + \int_0^{\rho(x_t,y_t)} \mathrm{Hess}_V(U_s,U_s)\,\mathrm{d}s\right\}\mathrm{d}t \tag{3.2}$$

and

$$\mathrm{d}\rho(x_t,y_t) \leqslant -\left\{\int_0^{\rho(x_t,y_t)} (\text{Ric-Hess}_V)(U_s,U_s)\,\mathrm{d}s\right\}\mathrm{d}t, \tag{3.3}$$

where $U_s|[0,\rho(x_t,y_t)]$ is the unit tangent vector field along a minimal geodesic from x_t to y_t with length $\rho(x_t,y_t)$. We note that the above formulae remain true even when x_t is in the cut locus of y_t, see e.g. [6] and [17]. Therefore, by combining (3.1) with (3.2) and (3.3), we obtain

$$\mathrm{d}\rho(x_t,y_t) \leqslant [c - \lambda\rho(x_t,y_t)]\,\mathrm{d}t \tag{3.4}$$

for some $c > 0$. This implies

$$\rho(x_t,y_t) \leqslant \frac{c}{\lambda} + \rho(x,y)\exp[-\lambda t]. \tag{3.5}$$

On the other hand, by (2.1), there exists $c(r) > 0$ such that

$$|P_1 f|^r(x_t) \leqslant P_1|f|^r(y_t)\exp\left[c(r)\rho(x_t,y_t)^2\right].$$

By this and (3.5) we obtain

$$\begin{aligned}
P_t|P_1 f|^r(x) &= E|P_1 f|^r(x_t)\\
&\leqslant EP_1|f|^r(y_t)\exp\left[c(r)(c/\lambda + \exp[-\lambda t]\rho(x,y))^2\right]\\
&\leqslant P_{t+1}|f|^r(y)\exp\left[c_1 + c_2\exp[-\lambda t]\rho(x,y)^2\right]
\end{aligned}$$

for some $c_1, c_2 > 0$. ∎

LEMMA 3.2 *If* (1.6) *holds, then there exists* $\varepsilon > 0$ *such that* (1.2) *holds.*

Proof. Under the polar coordinates at $o \in M$, we have $x = (r, \xi) \in [0, \infty) \times \mathbb{S}^{d-1}$, where $r = \rho(x)$ and d is the dimension of M. Let $g(r, \xi)$ be such that $\mathrm{d}x = g(r, \xi)\,\mathrm{d}\xi\,\mathrm{d}r$. Then

$$\Delta\rho(x) = \frac{\partial}{\partial r} \log g(r, \xi)|_{r=\rho(x)}, \quad x \notin \mathrm{cut}(o). \tag{3.6}$$

If (1.6) holds, then there exist $r_0 > 1$ and $\sigma > 0$ such that $k(x) \geqslant \sigma$ for $\rho(x) \geqslant r_0$. For any $x \notin \mathrm{cut}(o)$ with $\rho(x) > r_0$, let $l(\cdot) : [0, \rho] \to M$ be the minimal geodesic from o to x. Define $T_s = \frac{\mathrm{d}l(s)}{\mathrm{d}s}$ and let $\{U_i\}_{i=1}^{d-1}$ be Jacobi fields along l such that $U_i(0) = 0$ and $\{U_i(\rho), T_\rho : 1 \leqslant i \leqslant d-1\}$ is an orthonormal basis of $T_x M$. We then have

$$\Delta\rho = \sum_{i=1}^{d-1} \int_0^\rho \big(|\nabla_T U_i|^2 - \langle R(U_i, T)T, U_i\rangle\big),$$

where R denotes the curvature tensor and the integral is taken along l. Choose $f \in C^\infty[0, \infty)$ such that $1 \geqslant f \geqslant 0$, $f(0) = 0$, $f(r) = 1$ for $r \geqslant r_0$ and $|f'| \leqslant 1$. Let $\{W_i\}_{i=1}^{d-1}$ be parallel vector fields along l such that $U_i(\rho) = W_i(\rho)$. By the index lemma, we have

$$\Delta\rho \leqslant \sum_{i=1}^{d-1} \int_0^\rho \big(|\nabla_T(fW_i)|^2 - \langle R(fW_i, T)T, fW_i\rangle\big) = (d-1)\int_0^\rho (f')^2 - \int_0^\rho f^2 \mathrm{Ric}(T, T).$$

Noting that

$$\langle \nabla V, \nabla\rho\rangle = \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}s}(f^2\langle \nabla V, T\rangle) = \int_0^\rho (f^2)'(TV) + \int_0^\rho f^2 \mathrm{Hess}_V(T, T),$$

we obtain

$$\begin{aligned} L\rho &\leqslant (d-1)\int_0^\rho (f')^2 - \int_0^\rho f^2(\mathrm{Ric}\text{-}\mathrm{Hess}_V)(T, T) + \int_0^\rho (f^2)'(TV) \\ &\leqslant (d-1)r_0 + 2r_0 \sup_{B(o, r_0)} |\nabla V| - \sigma(\rho - r_0) := c - \sigma\rho. \end{aligned}$$

Combining this with (3.6), we obtain

$$\frac{\partial}{\partial r}\big[\log g(r, \xi) + V(r, \xi)\big] \leqslant c - \sigma r, \quad r \geqslant r_0.$$

This then implies that

$$g(r, \xi)\mathrm{e}^{V(r, \xi)} \leqslant c_1 \mathrm{e}^{cr - \sigma r^2/2}$$

for some $c_1 > 0$ and all $r > 0$. Therefore,

$$\mu(\mathrm{e}^{\varepsilon\rho^2}) \leqslant Z^{-1} c_1 \int_{\mathbb{S}^{d-1}} \mathrm{d}\xi \int_0^\infty \exp[cr - (\sigma/2 - \varepsilon)r^2]\,\mathrm{d}r < \infty, \quad \varepsilon < \frac{\sigma}{2}. \quad ∎$$

Proof of Theorem 1.2. For any $f \geqslant 0$ with $\mu(f^2) = 1$ and any $p \in (2,4)$, by Lemma 3.1 there exist $c_1, c_2, \lambda > 0$ such that

$$\mu((P_{t+1}f)^p) \leqslant \int\limits_{M\times M} [P_{t+1}f(x)]^{p/2}\cdot[P_{t+1}f^2(y)]^{p/4} \exp[c_1 + c_2 \mathrm{e}^{-\lambda t}\rho(x,y)^2]\mu(\mathrm{d}x)\mu(\mathrm{d}y)$$

$$\leqslant c^{(4-p)/4}\bigg(\int (P_{t+1}f(x))^2 P_{t+1}f^2(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\bigg)^{p/4} \leqslant c^{(4-p)/p},$$

where

$$c = \int \exp\Big[\frac{4}{4-p}(c_1 + c_2\mathrm{e}^{-\lambda t}\rho(x,y)^2)\Big]\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

By Lemma 3.2, $c < \infty$ and hence $\|P_{t+1}\|_{2\to p} < \infty$ for big $t > 0$. ∎

LEMMA 3.3. *Assume that* $\mathrm{Ric} \geqslant -K$ *for some* $K \geqslant 0$. *Let* $|B(x,r)|$ *be the Riemannian volume of the geodesic ball* $B(x,r)$. *We have*

$$\frac{|B(x,r_1)|}{|B(x,r_2)|} \leqslant \frac{r_1^d}{r_2^d}\exp\Big[\sqrt{K(d-1)}(r_1 - r_2)\Big], \quad r_1 \geqslant r_2 > 0.$$

Proof. Fix $x \in M$ and consider the polar coordinates at x. For any $r > 0$, let $Q_r = \{\xi \in \mathbb{S}^{d-1} : \rho(x, \exp_x[r\xi]) = r,\ (r,\xi) := \exp[r\xi] \notin \partial M \cup \mathrm{cut}(x)\}$. Then Q_r is decreasing in r (recall that M is convex). By [5], proof of Proposition 4.1, $\frac{|B(x,r)|}{|B(r)|}$ is decreasing in r, where

$$|B(r)| = |\mathbb{S}^{d-1}| \int\limits_0^r \sinh^{d-1}\Big(\sqrt{K/(d-1)}s\Big)\,\mathrm{d}s$$

which is the volume of a geodesic ball with radius r in the d-dimensional space form with constant sectional curvature $-K/(d-1)$. The proof is then completed. ∎

Proof of Theorem 1.3. If $\lim\limits_{r\to\infty}\sup\limits_{\rho\geqslant r}\mathrm{Hess}_V < 0$, we have $V \leqslant c - \sigma\rho^2$ for some $c, \sigma > 0$. Since the Ricci curvature is bounded below, by the volume comparison theorem we have $\mu(\mathrm{e}^{\varepsilon\rho^2}) < \infty$ for $\varepsilon < \sigma$. Then the proof of the LSI is similar to that of Theorem 1.2.

On the other hand, if $\lim\limits_{r\to\infty}\inf\limits_{\rho\geqslant r}\mathrm{Hess}_V \geqslant 0$, then for any $\varepsilon > 0$, there exists $c > 0$ such that $V \geqslant -c - \frac{\varepsilon}{2}\rho^2$. Next, by Lemma 3.3 with $r_1 = \rho(x) + 1$ and $r_2 = 1/2$, we obtain $|B(x,1/2)| \geqslant c_1\exp[-c_2\rho(x)]$ for some $c_1, c_2 > 0$ and all $x \in M$. Choose $\{x_n\} \subset M$ such that $\rho(x_n) = n$. We have

$$\mu(\mathrm{e}^{\varepsilon\rho^2}) \geqslant \int \exp[-c + \varepsilon\rho^2/2]\,\mathrm{d}x \geqslant \sum_n \exp[-c + \varepsilon(n-1)^2/2]|B(x_n, 1/2)| = \infty.$$

This means that there is no $\varepsilon > 0$ such that (1.2) holds, hence the LSI does not hold. ∎

4. LSI DEDUCED FROM SOBOLEV INEQUALITIES: PROOFS OF THEOREMS 1.4 AND 1.5

LEMMA 4.1. *Let $V_i \in C^2(M)$ and denote by $P_t^{(i)}$ the semigroup of $L_i = \Delta + \nabla V_i$ $(i = 1, 2)$. If $\|\nabla(V_1 - V_2)\|_\infty := B < \infty$, then for any $f \in C_{\rm b}(M)$ and $r > 1$,*

$$|P_t^{(2)} f|^r \leqslant (P_t^{(1)} |f|^r) \exp\big[B^2 rt/4(r-1)\big].$$

Proof. For positive $f \in C_{\rm b}(M)$, let $\phi(s) = \log P_s^{(1)} (P_{t-s}^{(2)} f)^r,\ s \in [0, t]$. We have

$$\begin{aligned}\phi'(s) &= \frac{1}{P_s^{(1)}(P_{t-s}^{(2)} f)^r} \big\{ P_s^{(1)} L_1 (P_{t-s}^{(2)} f)^r - r P_s^{(1)} (P_{t-s}^{(2)} f)^{r-1} L_2 P_{t-s}^{(2)} f \big\} \\ &= \frac{r}{P_s^{(1)}(P_{t-s}^{(2)} f)^r} P_s^{(1)} (P_{t-s}^{(2)} f)^r \big\{ (r-1) |\nabla \log P_{t-s}^{(2)} f|^2 \\ &\qquad + \langle \nabla(V_1 - V_2), \nabla \log P_{t-s}^{(2)} f \rangle \big\} \\ &\geqslant -\frac{rB^2}{4(r-1)}.\end{aligned}$$

The proof is now completed by taking integral over s from 0 to t. ∎

Proof of Theorem 1.4. We first assume that V is smooth with $Z = 1$. If (1.7) holds, then (see e.g. [3], Section 4)

$$\int g^2 \log g^2 \, {\rm d}\nu \leqslant \frac{v}{2} \log\Big(c_1 \int |\nabla g|^2 \, {\rm d}\nu + c_2 \Big), \quad \int g^2 \, {\rm d}\nu = 1. \tag{4.1}$$

For any $f \in C_0^\infty(M)$ with $\mu(f^2) = 1$. By (4.1) with $g = f{\rm e}^{(V-W)/2}$, we obtain

$$\begin{aligned}\int f^2 \log f^2 \, {\rm d}\mu &\leqslant \int (W - V) f^2 \, {\rm d}\mu + \frac{v}{2} \log\Big(c_2 + c_1 \int |\nabla (f{\rm e}^{(V-W)/2})|^2 \, {\rm d}\nu \Big) \\ &\leqslant c(\lambda, \varepsilon) + \frac{2\lambda}{1+\varepsilon} \int |\nabla(f{\rm e}^{(V-W)/2})|^2 \, {\rm d}\nu + \int (W - V) f^2 \, {\rm d}\mu\end{aligned} \tag{4.2}$$

for some $c(\lambda, \varepsilon) > 0$. By Green's formula and (1.8), we have (assuming that f satisfies the Neumann boundary condition if $\partial M \neq \emptyset$)

$$\int \langle \nabla f^2, \nabla {\rm e}^V \rangle \, {\rm d}x = -\int {\rm e}^V \Delta f^2 \, {\rm d}x = -\int f^2 \Delta {\rm e}^V \, {\rm d}x.$$

We then have

$$\begin{aligned}\int |\nabla(f{\rm e}^{(V-W)/2})|^2 \, {\rm d}\nu &\leqslant 2 \int |\nabla(f{\rm e}^{V/2})|^2 \, {\rm d}x + 2 \int f^2 {\rm e}^V |\nabla {\rm e}^{-W/2}|^2 \, {\rm d}\nu \\ &= 2\mu(|\nabla f|^2) + \frac{1}{2} \int f^2 (|\nabla V|^2 + |\nabla W|^2) \, {\rm d}\mu + \int \langle \nabla f^2, \nabla {\rm e}^V \rangle \, {\rm d}x \\ &\leqslant 2\mu(|\nabla f|^2) + \frac{1}{2} \mu\big(f^2 [|\nabla V|^2 + |\nabla W|^2 - 2\eta] \big)\end{aligned}$$

by (1.8). By combining this with (4.2) and using the inequality $a \cdot b \leqslant a \log a + \mathrm{e}^b$ for $a \geqslant 0$, $b \in \mathbb{R}$, we obtain, for some $c(\lambda, \varepsilon) > 0$,

$$
\begin{aligned}
\mu(f^2 \log f^2) \leqslant & \frac{4\lambda}{1+\varepsilon}\mu(|\nabla f|^2) + c(\lambda,\varepsilon) + \frac{1}{1+\varepsilon}\mu(f^2 \log f^2) \\
& + \int \exp\left[\lambda(|\nabla W|^2 + |\nabla V|^2 - 2\eta) + (1+\varepsilon)(W - V)\right] \mathrm{d}\mu.
\end{aligned} \tag{4.3}
$$

This implies the defective LSI by (1.9) and hence the LSI holds by Aida's assertion (see [1]).

For locally Lipschitz continuous V, by Green-Wu approximation theorem ([11], Proposition 2.3), we may take $V' \in C^\infty(M)$ such that $|V - V'| < 1, |\nabla V'| \leqslant |\nabla V| + 1$ and $\Delta \mathrm{e}^{V'} \geqslant \mathrm{e}^V(\eta - 1)$. Applying (4.3) to V', we see that the LSI holds for V' and hence also holds for V. ∎

Proof of Theorem 1.5. By Green-Wu's approximation theorem ([11], Proposition 2.1), there exists $\overline{\rho} \in C^\infty(M)$ such that $|\rho - \overline{\rho}| < 1$ and $|\nabla \overline{\rho}| < 2$. For $c > \sqrt{K(d-1)}$, let $\mathrm{d}\eta = \mathrm{e}^{c\overline{\rho}}\,\mathrm{d}x$. Then it suffices to prove (1.7) for η. Let $P_t^{(1)}$ be the semigroup of $\Delta + c\nabla\overline{\rho}$. By [9], Corollary 2.4.3, (1.7) for η is equivalent to the upper bound of $P_t^{(1)}$

$$
(P_t^{(1)} f)^2 \leqslant c_3 t^{-v/2}, \quad t \in (0,1],\ \eta(f^2) = 1. \tag{4.4}
$$

Let $P_t^{(2)}$ be the semigroup of Δ and denote $r = 2^{1/3}$. By Lemma 4.1 and applying (2.1) to $P_t^{(2)}$, we obtain for $t \in (0,1]$ and $f \geqslant 0$ with $\eta(f^2) = 1$,

$$
\begin{aligned}
[P_t^{(1)} f(x)]^2 &\leqslant c_4 [P_t^{(2)} f^r(x)]^{r^2} \leqslant c_4 [P_t^{(2)} f^{r^2}(y)]^r \mathrm{e}^{c_5 \rho(x,y)^2/t} \\
&\leqslant c_6 P_t^{(1)} f^2(y) \mathrm{e}^{c_5\rho(x,y)^2/t}, \quad x, y \in M.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_6 = c_6 \int P_t^{(1)} f^2(y)\eta(\mathrm{d}y) &\geqslant [P_t^{(1)} f(x)]^2 \int \mathrm{e}^{-c_5\rho(x,y)^2/t}\eta(\mathrm{d}y) \\
&\geqslant [P_t^{(1)} f(x)]^2 \eta\big(B(x,\sqrt{t})\big)\mathrm{e}^{-c_5}, \quad x \in M.
\end{aligned}
$$

From this we see that (4.4) follows from the volume lower bound

$$
\eta(B(x,r)) \geqslant c_7 r^v, \quad r \in [0,1],\ x \in M. \tag{4.5}
$$

By Lemma 3.3, we have

$$
\frac{|B(o,1)|}{|B(x,r)|} \leqslant \frac{|B(x,\rho(x)+1)|}{|B(x,r)|} \leqslant \frac{(\rho(x)+1)^d}{r^d} \exp\Big[\sqrt{K(d-1)}(\rho(x)+1)\Big],\ r \in [0,1].
$$

Noting that $\overline{\rho} \geqslant \rho - 1$, we obtain

$$
\eta(B(x,r)) \geqslant \mathrm{e}^{c(\rho(x)-2)}|B(x,r)| \geqslant c_7 r^d
$$

for some $c_7 > 0$ and all $r \in [0,1]$. Therefore, (4.5) holds for $v \geqslant d$. ∎

5. A MORE GENERAL SETTING

The main aim of this section is to apply the argument in the proof of Theorem 1.4 to a more general setting. Let M be a separable Hausdorff space and denote by $\mathcal{B}(M)$ its Borel σ-field. Let P_t be a diffusion semigroup which is symmetric w.r.t. a σ-finite measure μ. Let L be the corresponding generator with domain $\mathcal{D}(L)$. As usual, we write $\Gamma(f,g) = \frac{1}{2}[L(fg) - fLg - gLf]$. By "diffusion property" we mean that

$$Lh(f_1,\ldots,f_n) = \sum_{i=1}^{n}(\partial_i h)Lf_i + \frac{1}{2}\sum_{i,j=1}^{n}(\partial_i\partial_j h)\Gamma(f_i,f_j)$$

holds for any $n \geqslant 1$, $h \in C_0^\infty(\mathbb{R}^n)$ and $f_i \in \mathcal{D}(L)$, $(i=1,\ldots,n)$. Assume that $\mathcal{A}$ is a subspace of $\mathcal{D}(L)$ which is dense in $L^p(\mu)$ for any $p>1$ such that

$$\mu(\Gamma(f,g)) = -\mu(fLg), \quad g,f \in \mathcal{A}. \tag{5.1}$$

Assume further that the following defective LSI holds:

$$\mu(f^2\log f^2) \leqslant c_1\mu(\Gamma(f,f)) + c_2, \quad \mu(f^2)=1,\ f \in \mathcal{A}. \tag{5.2}$$

As was shown by Aida ([1]), (5.2) is equivalent to the LSI when μ is a probability measure and there exists $t>0$ such that

$$\inf\{\mu(1_B P_t 1_A) : A,B \in \mathcal{B}(M), \mu(A),\mu(B) \geqslant \varepsilon\} > 0$$

for any $\varepsilon > 0$.

Next, let F be such that $\mu(\Gamma(f^2,\mathrm{e}^F)) = -\mu(f^2L\mathrm{e}^F)$ for all $f \in \mathcal{A}$, and $\mathcal{A}$ is dense in $L^p(\mathrm{d}\nu)$ for all $p>1$, where $\mathrm{d}\nu = \mathrm{e}^{2F}\,\mathrm{d}\mu$. Then the proof of [1], Lemma 4.1 implies that the defective LSI (5.2) holds for ν if there exists $\varepsilon > 0$ such that

$$\mu\big(\mathrm{e}^{c_1(1+\varepsilon)\Gamma(F,F)} + \mathrm{e}^{-(2+\varepsilon)F}\big) < \infty. \tag{5.3}$$

Below we present an alternative to the condition (5.3).

THEOREM 5.1. *Let μ, F and ν be as above. The defective* LSI *holds for ν if there exists $\varepsilon > 0$ such that*

$$\mu\big(\exp[-2\varepsilon F - (1+\varepsilon)c_1(\Gamma(F,F) + LF)]\big) < \infty. \tag{5.4}$$

Proof. Let $f \in \mathcal{A}$ be such that $\nu(f^2)=1$. By (5.2) with f replaced by $f\mathrm{e}^F$, we obtain

$$\nu(f^2\log f^2) \leqslant c_1\mu(\Gamma(f\mathrm{e}^F, f\mathrm{e}^F)) + c_2 - 2\nu(Ff^2). \tag{5.5}$$

Noting that (by the diffusion property and the assumption of F)

$$\Gamma(f\mathrm{e}^F, f\mathrm{e}^F) = f^2\mathrm{e}^{2F}\Gamma(F,F) + \mathrm{e}^{2F}\Gamma(f,f) + \frac{1}{2}\Gamma(f^2,\mathrm{e}^{2F}),$$
$$\mu(\Gamma(f^2,\mathrm{e}^{2F})) = -\mu(f^2L\mathrm{e}^{2F}) = -2\nu(f^2LF) - 4\nu(f^2\Gamma(F,F)),$$

the remainder of the proof is then similar to that of Theorem 1.4. ∎

Obviously, (5.3) does not recover (5.4). For instance, let $M = \mathbb{R}^d$ and $\mathrm{d}\mu = \pi^{-d/2}\mathrm{e}^{-x^2}\,\mathrm{d}x$. We have $L = \Delta - \nabla x^2$ and $c_1 = 1, c_2 = 0$. Next, let $F(x) = rx^2$ for $r \in \mathbb{R}$. Then for any r there exists $\varepsilon > 0$ such that (5.4) holds, but (5.3) holds for some $\varepsilon > 0$ if and only if $r \in (-\frac{1}{2}, \frac{1}{2})$.

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