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THE STRUCTURE OF THE QUANTUM SEMIMARTINGALE ALGEBRAS

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Abstract. In the theory of quantum stochastic calculus one disposes of two quantum semimartingale algebras $\mathcal{S}$ and $\mathcal{S}'$. The first one is an algebra for the composition of operators and has a quantum functional calculus for analytical functions. The second one is larger and is an algebra for the operations of quantum square and angle brackets. In this article we study the algebraic and analytic properties of these algebras. This study is mainly performed through a remarkable transform of quantum processes which, surprisingly, establishes a bijection in between these two algebras. This bijection allows to define norms on these algebras that equip them with Banach algebra structures.

Keywords: *Quantum stochastic calculus, quantum semimartingales, quantum brackets, quantum Ito formula.*

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1. INTRODUCTION

The quantum stochastic calculus on Fock space, defined by Hudson and Parthasarathy ([7]) is a non-commutative extension of the usual stochastic calculus. It deals with operators on the boson Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+))$ and allows to define quantum stochastic integrals

$$\int_0^t H_s \, \mathrm{d}a_s^\varepsilon$$

of adapted operator processes $(H_t)_{t\geqslant 0}$ with respect to the three basic quantum noises $(a_t^+)_{t\geqslant 0}$, $(a_t^-)_{t\geqslant 0}$, $(a_t^\circ)_{t\geqslant 0}$ (creation, annihilation and conservation processes) and with respect to the time process $(a_t^\times)_{t\geqslant 0}$. The resulting operator $\int_0^t H_s \, \mathrm{d}a_s^\varepsilon$ is only defined on a particular subspace of Φ, the space $\mathcal{E}$ of coherent vectors. This domain constraint prevents quantum stochastic integrals from being composed. A quantum Ito formula was nevertheless established in a weak sense. It was actually a quantum Ito integration by part formula, that is, for the "composition" of two quantum stochastic integrals T_t and S_t. In the formulation in [7] all operator compositions HK were replaced by expressions of the form

$$\langle H^*\varepsilon(v), K\varepsilon(u)\rangle$$

where $\varepsilon(v), \varepsilon(u)$ are arbitrary coherent vectors. The Ito formula in [7] admitted no extension to further functional calculus. For example, it is impossible to extract from this formulation a formula for the third power of a quantum stochastic integral.

In [5], the definition of quantum stochastic integrals is extended to arbitrary domains in Φ. As a consequence, one may have bounded quantum stochastic integrals defined on the whole Φ. This allows composition and a true quantum Ito integration by part formula. In [2], a space $\mathcal{S}$ of quantum stochastic processes of the form

$$T_t = \lambda I + \sum_{\varepsilon=+,\circ,-,\times} \int_0^t H_s^\varepsilon \, \mathrm{d}a_s^\varepsilon,$$

defined on all Φ is obtained, and it is proved that $\mathcal{S}$ forms an algebra under operator composition (it is even a $*$-algebra through the adjoint mapping). This algebra $\mathcal{S}$ thus allows polynomial functional calculus. In [10] it is proved that the functional calculus on $\mathcal{S}$ can be extended to analytical function, and even to C^{2+} functions for the self-adjoint elements of $\mathcal{S}$. The algebra $\mathcal{S}$ deserves its name: the algebra of regular quantum semimartingales.

In [2] a theory of quantum square and angle brackets is also described. These quantum brackets are the non-commutative extensions of the square and angle brackets of classical stochastic calculus ([8]). The integration by part formula on $\mathcal{S}$ then takes the following familiar form:

$$S_tT_t = S_0T_0 + \int_0^t S_s \, \mathrm{d}T_s + \int_0^t \mathrm{d}S_sT_s + [S,T]_t. \tag{1.1}$$

If $(S_t)_{t\geqslant 0}$ and $(T_t)_{t\geqslant 0}$ are in $\mathcal{S}$ then so is $(S_tT_t)_{t\geqslant 0}$, but in general none of the three processes appearing in the right hand side of (1.1) belongs to $\mathcal{S}$. Indeed, they satisfy the same kind of properties as the elements of $\mathcal{S}$, but they are not in general made of bounded operators. These remarks suggest to define another space $\mathcal{S}'$, which is larger than $\mathcal{S}$, and which always contains the processes $\left(\int_0^t S_s \, \mathrm{d}T_s\right)_{t\geqslant 0}$, $\left(\int_0^t \mathrm{d}S_sT_s\right)_{t\geqslant 0}$, $([S,T]_t)_{t\geqslant 0}$ for $S, T \in \mathcal{S}$.

The space $\mathcal{S}'$ happens also to be a $*$-algebra but for the square bracket product:

$$\mathcal{S}' \times \mathcal{S}' \to \mathcal{S}', \quad (S_., T_.) \to ([S_., T_.]_t)_{t\geqslant 0}.$$

In this paper we give a deep study of $\mathcal{S}$ and $\mathcal{S}'$, their relations and their algebraic properties. We study some norms on them and prove that they are Banach algebras.

The main point is the definition of a transform $\mathcal{D}$ which maps quantum stochastic processes to quantum stochastic processes, is invertible and establishes a perfect bijection between $\mathcal{S}$ and $\mathcal{S}'$. This result is surprising for the following reasons:

— it is very simply described,
— it proves that the algebra $\mathcal{S}$ is large,
— the transform $\mathcal{D}$ has the property to make bounded operator processes which were not.

2. ELEMENTS OF QUANTUM STOCHASTIC CALCULUS

2.1. THE FOCK SPACE. The Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+))$ is the direct sum $\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^+)^{\otimes n}$ of all n-th symmetric tensor products of $L^2(\mathbb{R}^+)$ (cf. [9] for complements). A good way to understand Φ is to use Guichardet's notations ([6]). Let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$, then $\mathcal{P} = \bigcup_n \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n-elements subsets of $\mathbb{R}^+$. Identifying $\mathcal{P}_n$ with the increasing simplex $\Sigma_n = \{0 < t_1 < \cdots < t_n\}$ one can equip $\mathcal{P}_n$ with the Lebesgue measure structure. By putting the Dirac mass $\delta_\emptyset$ on $\mathcal{P}_0$, this altogether gives a structure of σ-finite measured space to $\mathcal{P}$, whose only atom is $\{\emptyset\}$. It is not difficult to see that $L^2(\mathcal{P})$ is naturally isomorphic to Φ by identifying n-variable symmetric functions on $\mathbb{R}^+$ to functions on Σ_n. In this article the Fock space Φ will always be understood as $L^2(\mathcal{P})$. Elements of $\mathcal{P}$ are denoted by small Greek letters $\sigma, \tau, \omega, \ldots$ and the corresponding volume element is denoted by $\mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\omega, \ldots$.

Let us set some notation and recall some basic results in this context. For all $\sigma \in \mathcal{P}$, we denote by $\vee\sigma$ the maximum of σ (if $\sigma \neq \emptyset$) and by σ- the set $\sigma \setminus \{\vee\sigma\}$. For all $t \in \mathbb{R}^+$ and all $\sigma \in \mathcal{P}$ let $\sigma_{t)} = \sigma \cap [0,t[$, $\sigma_{(t} = \sigma \cap]t,+\infty[$, $\sigma \cup t = \sigma \cup \{t\}$. Let $\mathcal{P}_{t)}$ (respectively $\mathcal{P}_{(t}$) be the set of $\sigma \in \mathcal{P}$ such that $\sigma \subset [0,t]$ (respectively $\sigma \subset [t,+\infty[$). The space $L^2(\mathcal{P}_{t)})$ (respectively $L^2(\mathcal{P}_{(t)})$) is identified with the subspace of $f \in L^2(\mathcal{P})$ such that $f(\sigma) = 0$ whenever $\sigma \not\subset [0,t]$ (respectively $\sigma \not\subset [t,+\infty[$). One also writes $\Phi_{t]} = L^2(\mathcal{P}_{t)})$ and $\Phi_{[t} = L^2(\mathcal{P}_{(t})$.

For all $u \in L^2(\mathbb{R}^+)$ let $\varepsilon(u)$ be the element of Φ such that $[\varepsilon(u)](\sigma) = \prod_{s\in\sigma} u(s)$ (with the empty product being equal to 1). The vectors $\varepsilon(u)$ are called *coherent vectors* on Φ. The space $\mathcal{E}$ generated by the coherent vectors is dense in Φ. The *vaccum vector* is the vector $\mathbb{1} = \varepsilon(0)$ that is $\mathbb{1}(\sigma) = \delta_\emptyset(\sigma)$. Note that for all $u \in L^2(\mathbb{R}^+)$ one has $\|\varepsilon(u)\|^2 = \mathrm{e}^{\|u\|^2}$. For $u \in L^2(\mathbb{R}^+)$ and $t \in \mathbb{R}^+$ one writes $u_{t]} = u\mathbb{1}_{[0,t]}$ and $u_{[t} = u\mathbb{1}_{[t,+\infty[}$. The mapping

$$\Phi \to \Phi_{t]} \otimes \Phi_{[t}, \quad \varepsilon(u) \mapsto \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})$$

extends to an isomorphism between Φ and $\Phi_{t]} \otimes \Phi_{[t}$. This property is the so-called *continuous tensor product structure* of Φ.

2.2. CALCULUS ON Φ. We are now going to define some useful operators on Φ. Let $t \in \mathbb{R}^+ \setminus \{0\}$. Let P_t be the orthogonal projection from Φ onto $\Phi_{t]}$. That is,

$$[P_t f](\sigma) = f(\sigma)\mathbb{1}_{\mathcal{P}_{t)}}(\sigma).$$

We define $[P_0 f](\sigma) = \delta_\emptyset(\sigma) f(\emptyset)$. The vector $P_0 f$ is often seen as a scalar, namely $f(\emptyset)$, instead of $f(\emptyset)\mathbb{1}$.

For an $f \in \Phi$ and $t \in \mathbb{R}^+$ define

$$[D_t f](\sigma) = f(\sigma \cup t)\mathbb{1}_{\mathcal{P}_{t)}}(\sigma)$$

for all $\sigma \in \mathcal{P}$. One can easily check the following (cf. [2]).

LEMMA 2.1. *For all $f \in \Phi$ one has*

$$\int_0^\infty \int_{\mathcal{P}} |[D_t f](\sigma)|^2 \,\mathrm{d}\sigma \,\mathrm{d}t = \|f\|^2 - |f(\emptyset)|^2.$$

Thus, for all $f \in \Phi$ and almost all $t \in \mathbb{R}^+$, $D_t f$ is an element of Φ.

A family $(g_t)_{t \geqslant 0}$ of elements of Φ is said to be an *Ito integrable process* if:

(i) $(t, \sigma) \mapsto g_t(\sigma)$ is measurable on $\mathbb{R}^+ \times \mathcal{P}$,
(ii) $g_t \in \Phi_{t]}$ for all t,
(iii) $\int_0^\infty \|g_t\|^2 \,\mathrm{d}t < \infty$.

If $(g_t)_{t \geqslant 0}$ is an Ito integrable process, define $\int_0^\infty g_s \,\mathrm{d}\chi_s$ by

$$\Big[\int_0^\infty g_s \,\mathrm{d}\chi_s\Big](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \\ g_{\vee\sigma}(\sigma -) & \text{otherwise.} \end{cases}$$

See [3] for the proof of the following.

LEMMA 2.2. *For all Ito integrable process $(g_t)_{t \geqslant 0}$, one has*

$$\int_{\mathcal{P}} \Big| \Big[\int_0^\infty g_s \,\mathrm{d}\chi_s\Big](\sigma)\Big|^2 \,\mathrm{d}\sigma = \int_0^\infty \|g_s\|^2 \,\mathrm{d}s < \infty.$$

Thus $\int_0^\infty g_s \,\mathrm{d}\chi_s$ belongs to Φ.

From all these definitions and lemmas, one easily deduces the following:

THEOREM 2.3. (Fock space predictable representation property) *For all $f \in \Phi$, the process $(D_t f)_{t \geqslant 0}$ is Ito integrable. One has the unique representation*

$$f = P_0 f + \int_0^\infty D_s f \,\mathrm{d}\chi_s \tag{2.1}$$

with

$$\|f\|^2 = |P_0 f|^2 + \int_0^\infty \|D_s f\|^2 \,\mathrm{d}s \tag{2.2}$$

and

$$\langle g, f\rangle = \overline{P_0 g} P_0 f + \int_0^\infty \langle D_s g, D_s f\rangle \,\mathrm{d}s \tag{2.3}$$

for all $g \in \Phi$.

We denote by $\int_a^b g_s \,\mathrm{d}\chi_s$ the Ito integral $\int_0^\infty g_s \mathbb{1}_{[a,b]}(s) \,\mathrm{d}\chi_s$.

2.3. Adaptedness. An operator H on Φ is said to be *adapted at time* t (or t-*adapted*) if H is of the form $H_t \otimes I$ in the tensor product structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$, for some operator H_t on $\Phi_{t]}$. If the operator H is defined on the coherent vector domain $\mathcal{E}$, the t-adaptedness writes as follows:

(i) $H\varepsilon(u_{t]}) \in \Phi_{t]}$,
(ii) $H\varepsilon(u) = [H\varepsilon(u_{t]})] \otimes \varepsilon(u_{[t})$

for all $u \in L^2(\mathbb{R}^+)$. An *adapted process of operators* on Φ is a family $(H_t)_{t\geqslant 0}$ of operators defined on a domain $\mathcal{D}$ such that

(i) $t \mapsto H_t f$ is measurable for all $f \in \mathcal{D}$,
(ii) H_t is t-adapted for all $t \in \mathbb{R}^+$.

2.4. Quantum stochastic integrals. Let us recall the generalized definition of quantum stochastic integrals as given in [5].

Let $\mathcal{D}$ be a domain on Φ such that $f \in \mathcal{D}$ implies $P_t f \in \mathcal{D}$ and $D_t f \in \mathcal{D}$ for almost all $t \in \mathbb{R}^+$. Such a domain $\mathcal{D}$ is called *adapted domain*.

Let $(H_t^\varepsilon)_{t\geqslant 0}$, $\varepsilon = +, -, \circ, \times$, be four adapted processes of operators on $\mathcal{D}$ satisfying

$$\int_0^t \|H_s^\circ D_s f\|^2 \,\mathrm{d}s + \int_0^t \|H_s^+ P_s f\|^2 \,\mathrm{d}s + \int_0^t \|H_s^- D_s f\| \,\mathrm{d}s + \int_0^t \|H_s^\times P_s f\| \,\mathrm{d}s < \infty \tag{2.4}$$

for all $f \in \mathcal{D}$ and all $t \in \mathbb{R}^+$.

An adapted process of operators $(T_t)_{t\geqslant 0}$ is said to be the *quantum stochastic integral process*

$$T_t = \sum_{\varepsilon=+,\circ,-,\times} \int_0^t H_s^\varepsilon \,\mathrm{d}a_s^\varepsilon, \quad t \in \mathbb{R}^+, \tag{2.5}$$

on the domain $\mathcal{D}$ if

(i) $\mathcal{D} \subset \operatorname{Dom} T_t$,
(ii) $\int_0^t \|T_s D_s f\|^2 \,\mathrm{d}s < \infty$ for all $f \in \mathcal{D}$, all $t \in \mathbb{R}^+$,

(iii) for all $t \in \mathbb{R}^+$ and all $f \in \mathcal{D}$, one has

$$(2.6)\quad \begin{aligned} T_t P_t f = \int_0^t T_s D_s f \,\mathrm{d}\chi_s + \int_0^t H_s^{\circ} D_s f \,\mathrm{d}\chi_s + \int_0^t H_s^{+} P_s f \,\mathrm{d}\chi_s \\ + \int_0^t H_s^{-} D_s f \,\mathrm{d}s + \int_0^t H_s^{\times} P_s f \,\mathrm{d}s. \end{aligned}$$

THEOREM 2.4. ([5]) *On the adapted domain $\mathcal{E}$ and under the condition* (2.4), *the equation* (2.6) *admits a unique solution $(T_t)_{t\geqslant 0}$ on $\mathcal{E}$ which is determined by the identity*

$$\langle \varepsilon(v), T_t \varepsilon(u)\rangle = \sum_{\varepsilon=+,\circ,-,\times} \int_0^t h^{\varepsilon}(s) \langle \varepsilon(v), H_s^{\varepsilon} \varepsilon(u)\rangle \,\mathrm{d}s$$

for all $u, v \in L^2(\mathbb{R}^+)$ and all $t \in \mathbb{R}^+$, and where

$$h^{\varepsilon}(s) = \begin{cases} \overline{v}(s) u(s) & \text{if } \varepsilon = \circ; \\ \overline{v}(s) & \text{if } \varepsilon = +; \\ u(s) & \text{if } \varepsilon = -; \\ 1 & \text{if } \varepsilon = \times. \end{cases}$$

THEOREM 2.5. ([5], Quantum Ito formula) *Let $T_t = \sum_{\varepsilon} \int_0^t H_s^{\varepsilon} \,\mathrm{d}a_s^{\varepsilon}$ and $S_t = \sum_{\varepsilon} \int_0^t K_s^{\varepsilon} \,\mathrm{d}a_s^{\varepsilon}$, $t \in \mathbb{R}^+$, be quantum stochastic integrals on the whole Φ. Then $(S_t T_t)_{t\geqslant 0}$ is a quantum stochastic integral process on the whole Φ and*

$$\begin{aligned} S_t T_t = \sum_{\varepsilon} \int_0^t S_s H_s^{\varepsilon} \,\mathrm{d}a_s^{\varepsilon} + \sum_{\varepsilon} \int_0^t K_s^{\varepsilon} T_s \,\mathrm{d}a_s^{\varepsilon} + \int_0^t K_s^{\circ} H_s^{\circ} \,\mathrm{d}a_s^{\circ} \\ + \int_0^t K_s^{-} H_s^{\circ} \,\mathrm{d}a_s^{-} + \int_0^t K_s^{\circ} H_s^{+} \,\mathrm{d}a_s^{+} + \int_0^t K_s^{-} H_s^{+} \,\mathrm{d}a_s^{\times}. \end{aligned}$$

3. THE ALGEBRAS OF QUANTUM SEMIMARTINGALES

3.1. THE ALGEBRA $\mathcal{S}$. Let $\mathcal{S}$ be the space of quantum stochastic integral processes $(T_t)_{t\geqslant 0}$ made of bounded operators and such that $T_t = \sum_{\varepsilon} \int_0^t H_s^{\varepsilon} \,\mathrm{d}a_s^{\varepsilon}$ on $\mathcal{E}$, with all the operators H_s^{ε} being bounded and $t \mapsto \|H_t^{\varepsilon}\| \in L^{f(\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^+)$ with $f(0) = +\infty$, $f(+) = f(-) = 2$, $f(\times) = 1$.

THEOREM 3.1. ([2]) *The stochastic integral representation of each element* $(T_t)_{t\geqslant 0}$ *of* $\mathcal{S}$ *can be extended to the whole* Φ. *The mapping* $t \mapsto \|T_t\|$ *belongs to* $L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. *The space* $\mathcal{S}$ *is a* $*$-*algebra for the adjoint mapping and the composition of operators.*

The algebra $\mathcal{S}$ admits another characterization which is expressed only in terms of $(T_t)_{t\geqslant 0}$ (and not of its coefficients H^ε_s). This characterization can be found in [2], but will not be used here.

The interesting point with $\mathcal{S}$ is that Theorem 3.1 allows to perform a polynomial functional calculus on $\mathcal{S}$ and to derive associated quantum Ito formulae ([2]). This functional calculus can even be extended to analytical functions (and to C^{2+}-functions in the case of self-adjoint elements of $\mathcal{S}$), cf. [10]. These results show that $\mathcal{S}$ really behaves like a space of quantum semimartingales.

3.2. QUANTUM BRACKETS, THE ALGEBRA $\mathcal{S}'$. For $T_t = \sum_\varepsilon \int H^\varepsilon_s \, \mathrm{d}a^\varepsilon_s$ and $S_t = \sum_\varepsilon \int_0^t K^\varepsilon_s \, \mathrm{d}a^\varepsilon_s$, elements of $\mathcal{S}$ define

$$\int_0^t S_s \, \mathrm{d}T_s = \sum_\varepsilon \int_0^t S_s H^\varepsilon_s \, \mathrm{d}a^\varepsilon_s, \qquad \int_0^t \mathrm{d}S_s T_s = \sum_\varepsilon \int_0^t K^\varepsilon_s T_s \, \mathrm{d}a^\varepsilon_s$$

$$[S,T]_t = \int_0^t K^\circ_s H^\circ_s \, \mathrm{d}a^\circ_s + \int_0^t K^-_s H^\circ_s \, \mathrm{d}a^-_s + \int_0^t K^\circ_s H^+_s \, \mathrm{d}a^+_s + \int_0^t K^-_s H^+_s \, \mathrm{d}s$$

$$\langle S,T\rangle_t = \int_0^t K^-_s H^+_s \, \mathrm{d}s.$$

The last two expressions are respectively called *quantum square bracket* and *quantum angle bracket* of S and T.

The quantum Ito formula on $\mathcal{S}$ then writes

$$S_t T_t = \int_0^t S_s \, \mathrm{d}T_s + \int_0^t \mathrm{d}S_s T_s + [S,T]_t.$$

In general, none of the processes $\left(\int_0^t S_s \, \mathrm{d}T_s\right)_{t\geqslant 0}$, $\left(\int_0^t \mathrm{d}S_s T_s\right)_{t\geqslant 0}$ and $([S,T]_t)_{t\geqslant 0}$ is in $\mathcal{S}$; only their sum is. These three process actually belong to a larger space.

Let $\mathcal{S}'$ be the space of adapted processes $(T_t)_{t\geqslant 0}$ such that $T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s \, \mathrm{d}a^\varepsilon_s$ on $\mathcal{E}$, with all the operators H^ε_s being bounded and $t \mapsto \|H^\varepsilon_t\| \in L^{f(\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^+)$ with $f(0) = +\infty$, $f(+) = f(-) = 2$, $f(\times) = 1$.

The definition of $\mathcal{S}'$ is exactly the same as the one of $\mathcal{S}$ excepted that one does not ask the operators T_t to be bounded.

Clearly $\mathcal{S}'$ contains $\mathcal{S}$. The inclusion is strict, for $(a^\circ_t)_{t\geqslant 0}$, $(a^-_t)_{t\geqslant 0}$, $(a^+_t)_{t\geqslant 0}$ belong to $\mathcal{S}'$ and not to $\mathcal{S}$.

THEOREM 3.2. ([2]) *The mapping* $(S,T)\mapsto[S,T]$ *is well-defined from* $\mathcal{S}'\times\mathcal{S}'$ *to* $\mathcal{S}'$. *The space* $\mathcal{S}'$ *is a* $*$*-algebra for the adjoint mapping and the product* $(S,T)\mapsto[S,T]$.

The mapping $(S,T)\mapsto\langle S,T\rangle$ *is well-defined from* $\mathcal{S}'\times\mathcal{S}'$ *to* $\mathcal{S}$.

The mappings $(S,T)\mapsto\int_0^{\cdot}\mathrm{d}S_sT_s$ *and* $(S,T)\mapsto\int_0^{\cdot}T_s\,\mathrm{d}S_s$ *are well-defined from* $\mathcal{S}'\times\mathcal{S}$ *to* $\mathcal{S}'$.

One has to keep in mind the different possible roles of $\mathcal{S}'$: it is a space which extends $\mathcal{S}$, but it is also an algebra for a product which is different from the one of $\mathcal{S}$. As spaces, we have $\mathcal{S}\subset\mathcal{S}'$; as algebras they are not related.

We have set all the preliminaries about these algebras, so we can start the study of their algebraic and analytic structures.

4. A REMARKABLE TRANSFORM OF QUANTUM PROCESSES

4.1. DEFINITION AND CHARACTERIZATION. Let $(T_t)_{t\geqslant0}$ be an adapted process of operators defined on $\mathcal{E}$. Assume that for all $f\in\mathcal{E}$ and all $t\in\mathbb{R}^+$ we have

$$\int_0^t\|T_sD_sf\|^2\,\mathrm{d}s<\infty. \tag{4.1}$$

Define $\big(\mathcal{D}_t(T.)\big)_{t\geqslant0}$ to be the adapted process of operators defined by the following:

$$\mathcal{D}_t(T.)P_tf=T_tP_tf-\int_0^tT_sD_sf\,\mathrm{d}\chi_s \tag{4.2}$$

for $f\in\mathcal{E}$, and $\mathcal{D}_t(T.)$ is extended t-adaptedly on $\mathcal{E}$.

LEMMA 4.1. *If* $(T_t)_{t\geqslant0}$ *satisfies* (4.1) *on* $\mathcal{E}$, *then so does* $\big(\mathcal{D}_t(T.)\big)_{t\geqslant0}$.

Proof. We have, for all $f\in\mathcal{E}$, say $f=\varepsilon(v)$,

$$\begin{aligned}\int_0^t\|\mathcal{D}_s(T.)D_sf\|^2\,\mathrm{d}s&=\int_0^t\Big\|T_sD_sf-\int_0^sT_uD_uD_sf\,\mathrm{d}\chi_u\Big\|^2\,\mathrm{d}s\\&\leqslant2\int_0^t\|T_sD_sf\|^2\,\mathrm{d}s+2\int_0^t\int_0^s\|T_uD_uD_sf\|^2\,\mathrm{d}u\,\mathrm{d}s\\&\leqslant2\int_0^t\|T_sD_sf\|^2\,\mathrm{d}s+2\int_0^t\int_0^s|v(s)|^2|v(u)|^2\|T_u\varepsilon(v_{u]})\|^2\,\mathrm{d}u\,\mathrm{d}s\\&\leqslant2\int_0^t\|T_sD_sf\|^2\,\mathrm{d}s+2\Big(\int_0^t(v(s))^2\,\mathrm{d}s\Big)\Big(\int_0^t\|T_uD_uf\|^2\,\mathrm{d}u\Big)<\infty.\end{aligned}$$

∎

PROPOSITION 4.2. *The process* $\big(\mathcal{D}_t(T.)\big)_{t\geqslant 0}$ *is the unique solution* $(X_t)_{t\geqslant 0}$ *of the equation*

$$X_t = T_t - \int_0^t X_s \,\mathrm{d}a_s^\circ \quad \textit{on } \mathcal{E}. \tag{4.3}$$

Proof. If $X_t = \mathcal{D}_t(T.)$ for all $t \in \mathbb{R}^+$, then $\int_0^t X_s \,\mathrm{d}a_s^\circ$ is well-defined on $\mathcal{E}$ by Lemma 4.1. Furthermore, we have

$$X_t P_t f = T_t P_t f - \int_0^t T_s D_s f \,\mathrm{d}\chi_s = T_t P_t f - \int_0^t (X_s - T_s) D_s f \,\mathrm{d}\chi_s - \int_0^t X_s D_s f \,\mathrm{d}\chi_s.$$

That is, the process $(Y_t)_{t\geqslant 0} = (X_t - T_t)_{t\geqslant 0}$ satisfies $Y_t P_t f = \int_0^t Y_s D_s f \,\mathrm{d}\chi_s - \int_0^t X_s D_s f \,\mathrm{d}\chi_s$ on $\mathcal{E}$.

By Theorem 2.4 and Equation (2.6) this exactly means $Y_t = -\int_0^t X_s \,\mathrm{d}a_s^\circ$.

If $(X'_t)_{t\geqslant 0}$ is another solution of (4.3), then the process $(Z_t)_{t\geqslant 0} = (X_t - X'_t)_{t\geqslant 0}$ satisfies $Z_t = -\int_0^t Z_s \,\mathrm{d}a_s^\circ$ on $\mathcal{E}$. By (2.6) this means that for all $f \in \mathcal{E}$

$$Z_t P_t f = \int_0^t Z_s D_s f \,\mathrm{d}\chi_s - \int_0^t Z_s D_s f \,\mathrm{d}\chi_s = 0. \quad \blacksquare$$

4.2. THE INVERSE TRANSFORM. Let $(T_t)_{t\geqslant 0}$ be an adapted process of operators defined on $\mathcal{E}$. Suppose that, for all $f \in \mathcal{E}$ and all $t \in \mathbb{R}^+$ we have

$$\int_0^t \|T_s D_s f\|^2 \,\mathrm{d}s < \infty.$$

That is, the same condition as (4.1). Define $\big(\mathcal{D}_t^{-1}(T.)\big)_{t\geqslant 0}$ to be the following adapted process of operators on $\mathcal{E}$:

$$\mathcal{D}_t^{-1}(T.) = T_t + \int_0^t T_s \,\mathrm{d}a_s^\circ. \tag{4.4}$$

By Theorem 2.4, one easily proves the following.

LEMMA 4.3. *If* $(T_t)_{t\geqslant 0}$ *satisfies* (4.1) *on* $\mathcal{E}$, *then so does* $\big(\mathcal{D}_t^{-1}(T.)\big)_{t\geqslant 0}$.

PROPOSITION 4.4. *$\big(\mathcal{D}_t^{-1}(T.)\big)_{t\geqslant 0}$ is the only process $(X_t)_{t\geqslant 0}$ on $\mathcal{E}$ such that*

$$X_t P_t f = T_t P_t f + \int_0^t X_s D_s f \,\mathrm{d}\chi_s \tag{4.5}$$

for all $f\in\mathcal{E}$.

Proof. We have

$$\begin{aligned}
\mathcal{D}_t^{-1}(T.)P_t f &= T_t P_t f + \int_0^t T_s \,\mathrm{d}a_s^\circ P_t f\\
&= T_t P_t f + \int_0^t \Big(\int_0^s T_u \,\mathrm{d}a_u^\circ D_s f\Big)\,\mathrm{d}\chi_s + \int_0^t T_s D_s f\,\mathrm{d}\chi_s\\
&= T_t P_t f + \int_0^t \big(\mathcal{D}_s^{-1}(T.) - T_s\big) D_s f\,\mathrm{d}\chi_s + \int_0^t T_s D_s f\,\mathrm{d}\chi_s\\
&= T_t P_t f + \int_0^t \mathcal{D}_s^{-1}(T.) D_s f\,\mathrm{d}\chi_s.
\end{aligned}$$

If X' is another process satisfying (4.5), then $(Z_t)_{t\geqslant 0} = (X_t - X'_t)_{t\geqslant 0}$ satisfies $Z_t P_t f = \int_0^t Z_s D_s f\,\mathrm{d}\chi_s$ for all $f\in\mathcal{E}$. Thus $Z_t\varepsilon(u_{t]}) = \int_0^t u(s) Z_s \varepsilon(u_{s]})\,\mathrm{d}\chi_s$ for all $u\in L^2(\mathbb{R}^+)$ and $\|Z_t\varepsilon(u_{t]})\|^2 = \int_0^t |u(s)|^2\|Z_s\varepsilon(u_{s]})\|^2\,\mathrm{d}s$. So, by Gronwall's lemma, $Z_t\varepsilon(u_{t]}) = 0$ for all $u\in L^2(\mathbb{R}^+)$. ∎

PROPOSITION 4.5. *The transforms $\mathcal{D}$ and $\mathcal{D}^{-1}$ are inverse of each other. That is, if $(T_t)_{t\geqslant 0}$ is any adapted process of operators on $\mathcal{E}$ such that $\int_0^t \|T_s D_s f\|^2\,\mathrm{d}s<\infty$ for all $f\in\mathcal{E}$ and all $t\in\mathbb{R}^+$, then*

$$\mathcal{D}_t\big(\mathcal{D}_\cdot^{-1}(T.)\big) = \mathcal{D}_t^{-1}\big(\mathcal{D}.(T.)\big) = T_t \quad \text{on } \mathcal{E}.$$

Proof. We have

$$\mathcal{D}_t^{-1}\big(\mathcal{D}.(T.)\big) = \mathcal{D}_t(T.) + \int_0^t \mathcal{D}_s(T.)\,\mathrm{d}a_s^\circ = T_t - \int_0^t \mathcal{D}_s(T.)\,\mathrm{d}a_s^\circ + \int_0^t \mathcal{D}_s(T.)\,\mathrm{d}a_s^\circ = T_t.$$

We also have

$$\begin{aligned}
\mathcal{D}_t\big(\mathcal{D}_\cdot^{-1}(T.)\big)P_t f &= \mathcal{D}_t^{-1}(T.)P_t f - \int_0^t \mathcal{D}_s^{-1}(T.)D_s f\,\mathrm{d}\chi_s\\
&= T_t P_t f + \int_0^t \mathcal{D}_s^{-1}(T.)D_s f\,\mathrm{d}\chi_s - \int_0^t \mathcal{D}_s^{-1}(T.)D_s f\,\mathrm{d}\chi_s = T_t P_t f. \quad ∎
\end{aligned}$$

4.3. THE BIJECTION. Note that $\mathcal{D}$ and $\mathcal{D}^{-1}$ are well-defined on $\mathcal{S}$ and $\mathcal{S}'$.

THEOREM 4.6. *$\mathcal{D}^{-1}(\mathcal{S}) \subset \mathcal{S}'$ and $\mathcal{D}(\mathcal{S}') \subset \mathcal{S}$. That is, $\mathcal{D}$ and $\mathcal{D}^{-1}$ realize a bijection between $\mathcal{S}$ and $\mathcal{S}'$.*

Proof. Let $(T_t)_{t\geqslant 0}$ be an element of $\mathcal{S}$. Recall that in particular $t \mapsto \|T_t\|$ belongs to L^∞_{loc}. Suppose that the integral representation of $(T_t)_{t\geqslant 0}$ is $T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s \,\mathrm{d}a^\varepsilon_s$. Then

$$\begin{aligned}\mathcal{D}^{-1}_t(T_\cdot) &= T_t + \int_0^t T_s \,\mathrm{d}a^\circ_s \\ &= \int_0^t (H^\circ_s + T_s)\,\mathrm{d}a^\circ_s + \int_0^t H^+_s \,\mathrm{d}a^+_s + \int_0^t H^-_s \,\mathrm{d}a^-_s + \int_0^t H^\times_s \,\mathrm{d}a^\times_s .\end{aligned}$$

Thus $\mathcal{D}^{-1}_t(T_\cdot) = \sum_\varepsilon \int_0^t K^\varepsilon_s \,\mathrm{d}a^\varepsilon_s$ on $\mathcal{E}$, with $t \mapsto \|K^\circ_s\| \in L^\infty_{\mathrm{loc}}$, $t \mapsto \|K^\pm_s\| \in L^2_{\mathrm{loc}}$, $t \mapsto \|K^\times_s\| \in L^1_{\mathrm{loc}}$. So $\big(\mathcal{D}^{-1}_t(T_\cdot)\big)_{t\geqslant 0}$ belongs to $\mathcal{S}'$.

Conversely, let $T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s \,\mathrm{d}a^\varepsilon_s$, $t \in \mathbb{R}^+$, be an element of $\mathcal{S}'$. By Equation (2.6) we have, for all $f \in \mathcal{E}$

$$\begin{aligned}\mathcal{D}_t(T_\cdot)P_t f &= T_t P_t f - \int_0^t T_s D_s f \,\mathrm{d}\chi_s \\ &= \int_0^t H^\circ_s D_s f \,\mathrm{d}\chi_s + \int_0^t H^+_s P_s f \,\mathrm{d}\chi_s + \int_0^t H^-_s D_s f \,\mathrm{d}s + \int_0^t H^\times_s P_s f \,\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\mathcal{D}_t(T_\cdot)P_t f\| \\ &\leqslant \Big\|\int_0^t H^\circ_s D_s f \,\mathrm{d}\chi_s\Big\| + \Big\|\int_0^t H^+_s P_s f \,\mathrm{d}\chi_s\Big\| + \Big\|\int_0^t H^-_s D_s f \,\mathrm{d}s\Big\| + \Big\|\int_0^t H^\times_s P_s f \,\mathrm{d}s\Big\| \\ &\leqslant \Big(\int_0^t \|H^\circ_s D_s f\|^2 \,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^t \|H^+_s P_s f\|^2 \,\mathrm{d}s\Big)^{1/2} \\ &\qquad + \int_0^t \|H^-_s D_s f\| \,\mathrm{d}s + \int_0^t \|H^\times_s P_s f\| \,\mathrm{d}s \qquad (4.6) \\ &\leqslant \sup_{s\leqslant t} \|H^\circ_s\| \Big(\int_0^t \|D_s f\|^2 \,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^t \|H^+_s\|^2 \,\mathrm{d}s\Big)^{1/2} \|P_t f\|\end{aligned}$$

$$+\Big(\int_0^t \|H_s^-\|^2\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t \|D_s f\|^2\,\mathrm{d}s\Big)^{1/2}+\Big(\int_0^t \|H_s^\times\|\,\mathrm{d}s\Big)\|P_t f\|$$

$$\leqslant\Big[\sup\|H_s^\circ\|+\Big(\int_0^t \|H_s^+\|^2\,\mathrm{d}s\Big)^{1/2}+\Big(\int_0^t \|H_s^-\|^2\mathrm{d}s\Big)^{1/2}+\int_0^t \|H_s^\times\|\,\mathrm{d}s\Big]\|P_t f\|.$$

Thus $\mathcal{D}_t(T.)$ is a bounded operator on $\mathcal{E}\cap\Phi_{t]}$. As $\mathcal{D}_t(T.)$ is adapted at time t, it is bounded on $\mathcal{E}$ with the same norm. It extends to a bounded operator on Φ. Furthermore, the estimate (4.6) proves that $t\mapsto\|\mathcal{D}_t(T.)\|$ belongs to L^∞_{loc}.

Finally, by Proposition 4.4 we have

$$\begin{aligned}\mathcal{D}_t(T.)&=\sum_\varepsilon\int_0^t H_s^\varepsilon\,\mathrm{d}a_s^\varepsilon-\int_0^t \mathcal{D}_s(T.)\,\mathrm{d}a_s^\circ\\&=\int_0^t (H_s^\circ-\mathcal{D}_s(T.))\,\mathrm{d}a_s^\circ+\int_0^t H_s^+\,\mathrm{d}a_s^+ +\int_0^t H_s^-\,\mathrm{d}a_s^- +\int_0^t H_s^\times\,\mathrm{d}a_s^\times.\end{aligned}$$

This integral representation, the boundedness of $\mathcal{D}_t(T.)$, the estimate on $\|\mathcal{D}_t(T.)\|$ altogether prove that $\big(\mathcal{D}_t(T.)\big)_{t\geqslant 0}$ is an element of $\mathcal{S}$.

Proposition 4.5 shows that $\mathcal{D}$ and $\mathcal{D}^{-1}$ thus realize a bijection between $\mathcal{S}$ and $\mathcal{S}'$. ∎

At this stage it is natural to wonder whether $\mathcal{D}$ is an algebra homomorphism between $\mathcal{S}'$ and $\mathcal{S}$. The following formula proves that the answer is negative.

Theorem 4.7. *If $T_t=\sum_\varepsilon\int_0^t H_s^\varepsilon\,\mathrm{d}a_s^\varepsilon$ and $S_t=\sum_\varepsilon\int_0^t K_s^\varepsilon\,\mathrm{d}a_s^\varepsilon$ are elements of $\mathcal{S}'$, then*

$$\begin{aligned}\mathcal{D}_t^{-1}\big(\mathcal{D}.(T.)\mathcal{D}.(S)\big)&=[T,S]_t+\int_0^t H_s^+\mathcal{D}_s(S.)\,\mathrm{d}a_s^+ +\int_0^t H_s^\times\mathcal{D}_s(S.)\,\mathrm{d}a_s^\times\\&\quad+\int_0^t \mathcal{D}_s(T.)K_s^-\,\mathrm{d}a_s^- +\int_0^t \mathcal{D}_s(T.)K_s^\times\,\mathrm{d}a_s^\times.\end{aligned}\tag{4.7}$$

Proof. Let $X_t=\mathcal{D}_t(T.)$ and $Y_t=\mathcal{D}_t(S.)$. We have $X_t=T_t-\int_0^t X_s\,\mathrm{d}a_s^\circ$,

$Y_t = S_t - \int_0^t Y_s \, \mathrm{d}a_s^\circ$. Thus

$$
\begin{aligned}
X_t Y_t &= \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t \mathrm{d}X_s Y_s + [X,Y]_t \\
&= \int_0^t X_s \, \mathrm{d}s_s - \int_0^t X_s Y_s \, \mathrm{d}a_s^\circ + \int_0^t \mathrm{d}T_s Y_s - \int_0^t X_s Y_s \, \mathrm{d}a_s^\circ \\
&\qquad + [T,S]_t - \Big[\int_0^{\cdot} X_s \, \mathrm{d}a_s^\circ, S. \Big]_t + \Big[T., \int_0^{\cdot} Y_s \, \mathrm{d}a_s^\circ \Big]_t + \int_0^t X_s Y_s \, \mathrm{d}a_s^\circ \\
&= [T,S]_t + \int_0^t X_s \, \mathrm{d}s_s + \int_0^t \mathrm{d}T_s Y_s - \int_0^t X_s K_s^\circ \, \mathrm{d}a_s^\circ - \int_0^t X_s K_s^+ \, \mathrm{d}a_s^+ \\
&\qquad - \int_0^t H_s^\circ Y_s \, \mathrm{d}a_s^\circ - \int_0^t H_s^- Y_s \, \mathrm{d}a_s^- - \int_0^t X_s Y_s \, \mathrm{d}a_s^\circ \\
&= [T,S]_t + \int_0^t X_s K_s^- \, \mathrm{d}a_s^- + \int_0^t X_s K_s^\times \, \mathrm{d}a_s^\times + \int_0^t H_s^+ Y_s \, \mathrm{d}a_s^+ \\
&\qquad + \int_0^t H_s^\times Y_s \, \mathrm{d}a_s^\times - \int_0^t X_s Y_s \, \mathrm{d}a_s^\circ .
\end{aligned}
$$

This means that

$$
X_t Y_t = \mathcal{D}_t \Big([T,S]. + \int_0^{\cdot} X_s K_s^- \, \mathrm{d}a_s^- + \int_0^{\cdot} X_s K_s^\times \, \mathrm{d}a_s^\times + \int_0^{\cdot} H_s^+ Y_s \, \mathrm{d}a_s^+ + \int_0^{\cdot} H_s^\times Y_s \, \mathrm{d}a_s^\times \Big). \blacksquare
$$

5. BANACH ALGEBRA STRUCTURES

5.1. A NORM ON $\mathcal{S}'$. Actually it is false to claim that $\mathcal{S}$ and $\mathcal{S}'$ admit Banach algebra structures. They only admit locally convex algebra structure. The problem comes from the fact that $\mathcal{S}$ and $\mathcal{S}'$ deal with processes indexed by $\mathbb{R}^+$ and that the norm conditions defining $\mathcal{S}$ and $\mathcal{S}'$ are only local. In order to get true Banach algebras one has to restrict to compact intervals of time. This restriction is not important for our needs.

Let A be a fixed number in $\mathbb{R}^+$. We denote by $\mathcal{S}_A$ and $\mathcal{S}'_A$ the algebras of quantum semimartingales obtained by restricting the elements of $\mathcal{S}$ and $\mathcal{S}'$, respectively, to the time interval $[0, A]$.

Let $(T_t)_{t\geqslant 0}$ be an element of $\mathcal{S}'$, with representation $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, \mathrm{d}a_s^\varepsilon$. One defines

$$\|T.\|_{\mathcal{S}'_A} = \sup_{s\leqslant A} \|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 \, \mathrm{d}s\Big)^{1/2} + \Big(\int_0^A \|H_s^-\|^2 \, \mathrm{d}s\Big)^{1/2} + \int_0^A \|H_s^\times\| \, \mathrm{d}s.$$

PROPOSITION 5.1. (i) *The mapping* $T. \mapsto \|T.\|_{\mathcal{S}'_A}$ *defines a norm on* $\mathcal{S}'_A$.

(ii) *If* $(A_n)_n$ *is an increasing sequence in* $\mathbb{R}^+$ *such that* $\lim_n A_n = +\infty$, *then the family* $(\|\cdot\|_{\mathcal{S}'_{A_n}})_{n\in\mathbb{N}}$ *is a separating family of seminorms on* $\mathcal{S}'$.

Proof. (i) The mapping $T. \mapsto \|T.\|_{\mathcal{S}'_A}$ identifies $\mathcal{S}'_A$ with

$$L^\infty\big([0,A]; B(\Phi)\big) \oplus L^2\big([0,A]; B(\Phi)\big) \oplus L^2\big([0,A]; B(\Phi)\big) \oplus L^1\big([0,A]; B(\Phi)\big)$$

as a normed vector space. The only point that needs to be developed is that $\|T.\|_{\mathcal{S}'_A} = 0$ if and only if $(T_t)_{t\in[0,A]}$ is the null process.

This is a consequence of the uniqueness theorem for quantum stochastic integrals ([1]).

(ii) It is clear that $\|\cdot\|_{\mathcal{S}'_A}$ is a seminorm on $\mathcal{S}'$ and that $\|T\|_{\mathcal{S}'_{A_n}} = 0$ if and only if $T_t \equiv 0$ for all $t \in [0, A_n]$. It is thus clear that if $\lim_n A_n = +\infty$ then the family $(\|\cdot\|_{\mathcal{S}'_{A_n}})_{n\in\mathbb{N}}$ will be separating for $\mathcal{S}'$. ∎

THEOREM 5.2. (i) *Equipped with the norm* $\|\cdot\|_{\mathcal{S}'_A}$, *the space* $\mathcal{S}'_A$ *is a Banach algebra.*

(ii) *Equipped with the family* $(\|\cdot\|_{\mathcal{S}'_n})_{n\in\mathbb{N}}$ *of seminorms, the space* $\mathcal{S}'$ *is a locally convex closed algebra.*

Proof. Let us first check that $\|\cdot\|_{\mathcal{S}'_A}$ is a $*$-algebra norm for $\mathcal{S}'_A$. If $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, \mathrm{d}a_s^\varepsilon$, $t \in [0, A]$ is an element of $\mathcal{S}'_A$, it is then clear that $\|T.^*\|_{\mathcal{S}'_A} = \|T.\|_{\mathcal{S}'_A}$. If $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon \, \mathrm{d}a_s^\varepsilon$, $t \in [0, A]$ is another element of $\mathcal{S}'_A$ then

$$[S,T]_t = \int_0^t K_s^\circ H_s^\circ \, \mathrm{d}a_s^\circ + \int_0^t K_s^- H_s^\circ \, \mathrm{d}a_s^- + \int_0^t K_s^\circ H_s^+ \, \mathrm{d}a_s^+ + \int_0^t K_s^- H_s^+ \, \mathrm{d}a_s^\times$$

and

$$\begin{aligned}\|[S,T].\|_{\mathcal{S}'_A} &= \sup_{s\geqslant A}\|K^\circ_s H^\circ_s\| + \Big[\int_0^A \|K^-_s H^\circ_s\|^2\,\mathrm{d}s\Big]^{1/2} + \Big[\int_0^A \|K^\circ_s H^+_s\|^2\,\mathrm{d}s\Big]^{1/2} \\ &\qquad + \int_0^t \|K^-_s H^+_s\|\,\mathrm{d}s \\ &\leqslant \sup_{s\leqslant A}\|K^\circ_s\| \sup_{s\leqslant A}\|H^\circ_s\| + \sup_{s\leqslant A}\|H^\circ_s\|\Big[\int_0^A \|K^-_s\|^2\,\mathrm{d}s\Big]^{1/2} \\ &\qquad + \sup_{s\geqslant A}\|K^\circ_s\|\Big[\int_0^A \|H^+_s\|^2\,\mathrm{d}s\Big]^{1/2} + \Big[\int_0^t \|K^-_s\|\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t \|H^+_s\|^2\,\mathrm{d}s\Big]^{1/2} \\ &\leqslant \|S.\|_{\mathcal{S}'_A}\|T.\|_{\mathcal{S}'_A}.\end{aligned}$$

Thus $\|\cdot\|_{\mathcal{S}'_A}$ is a $*$-algebra norm on $\mathcal{S}'_A$.

We have already noticed that, as a normed vector space, $\mathcal{S}'_A$ identifies with

$$L^\infty\big([0,A];B(\Phi)\big)\oplus L^2\big([0,A];B(\Phi)\big)\oplus L^2\big([0,A];B(\Phi)\big)\oplus L^1\big([0,A];B(\Phi)\big).$$

Thus if $T^n_t = \sum_\varepsilon \int_0^t H^{n,\varepsilon}_s\,\mathrm{d}a^\varepsilon_s$, $t\in[0,A]$ is a Cauchy sequence in $\mathcal{S}'_A$ then the coefficient $H^{n,\varepsilon}_s$ will converge to an s-adapted operator $H^\varepsilon_s \in L^{f(\varepsilon)}([0,A];B(\Phi))$, with $f(0)=+\infty$, $f(+)=f(-)=2$, $f(\times)=1$. The process $T_t=\sum_\varepsilon\int_0^t H^\varepsilon_s\,\mathrm{d}a^\varepsilon_s$ is thus an element of $\mathcal{S}'_A$, limit of $(T^n_\cdot)_{n\in\mathbb{N}}$ in $\mathcal{S}'_A$. Thus $\mathcal{S}'_A$ is a Banach algebra.

Let us prove that $\mathcal{S}'$ is closed for the family of seminorms $\|\cdot\|_{\mathcal{S}'_n}$. If $(T^n_\cdot)_n$ is a sequence in $\mathcal{S}'$ which is Cauchy in each $\mathcal{S}'_m$, $m\in\mathbb{N}$, then $(T^n_\cdot)_{n\in\mathbb{N}}$ admits a limit $(T_{m,t})_{t\in[0,m]}$ in each $\mathcal{S}'_m$. But as $\mathcal{S}'_m\subset\mathcal{S}'_{m'}$ (as Banach algebras) for $m\leqslant m'$, we clearly have $T_{m,t}=T_{m',t}$ for $m<m'$ and $t\in[0,m]$. Thus there exists a process $(T_t)_{t\geqslant 0}$ such that $T_t=T_{m,t}$ for all $t\in[0,m]$ and all $m\in\mathbb{N}$. Furthermore, if $T_t=\sum_\varepsilon\int_0^t H^\varepsilon_s\,\mathrm{d}a^\varepsilon_s$, $t\in\mathbb{R}^+$, we know that

$$\sup_{s\leqslant m}\|H^\circ_s\| + \Big[\int_0^m \|H^+_s\|^2\,\mathrm{d}s\Big]^{1/2} + \Big[\int_0^m \|H^-_s\|^2\,\mathrm{d}s\Big]^{1/2} + \int_0^m \|H^\times_s\|\,\mathrm{d}s < \infty$$

for all $m\in\mathbb{N}$. Thus $(T_t)_{t\geqslant 0}\in\mathcal{S}'$. ∎

Note the following easy result.

Proposition 5.3. *Equipped with $\|\cdot\|_{\mathcal{S}'_A}$, the space $\mathcal{S}'_A$ is a Banach algebra for the angle bracket product $(S.,T.)\mapsto\langle S.,T.\rangle$.*

5.2. Norms on $\mathcal{S}$. Now we aim to equip the algebra $\mathcal{S}$ with a norm that will give it a Banach algebra structure, in the same way as in $\mathcal{S}'$. We have seen that

$\mathcal{S}'$ admits such a norm, namely $\|\cdot\|_{\mathcal{S}'}$; we also know that $\mathcal{S}$ is a subspace (not a subalgebra!) of $\mathcal{S}'$. The first natural question one can ask is: what happens to $\|\cdot\|_{\mathcal{S}'}$ when one restricts it to $\mathcal{S}$?

Let us see, with a counter-example, that $\|\cdot\|_{\mathcal{S}'}$ is not an algebra norm for $\mathcal{S}$. (The author thanks A. Coquio from Institut Fourier, Grenoble, for providing this nice counter-example.)

For $n \in \mathbb{N}$, let Π^n be the orthogonal projection onto the n first chaoses of Φ, that is, on $\bigoplus_{k=0}^{n} L^2(\mathcal{P}_k)$.

Let $(\Pi_t^n)_{t\geqslant 0}$ be the operator martingale associated to Π^n that is,

$$\Pi_t^n \varepsilon(u) = \left[P_t \Pi^n \varepsilon(u_{t]})\right] \otimes \varepsilon(u_{[t}).$$

One can easily check that for $f \in \Phi_{t]}$ and $\sigma \in \mathcal{P}_{t]}$ we have $[\Pi_t^n f](\sigma) = \mathbb{1}_{\#\sigma\leqslant n}\, f(\sigma)$. As each Π_t^n is a norm-1 operator, the quantum stochastic integral $T_t = \int_0^t \Pi_s^n \,\mathrm{d}a_s^\circ$ is well defined, at least on $\mathcal{E}$.

One can check (cf. [4]) that for $f \in \mathcal{E}$, $T_t f$ is given by the following formula $[T_t f](\sigma) = \sum_{\substack{s\in\sigma\\ s\leqslant t}} \left[\Pi_s^n D_s D_{\sigma_{(s}} f\right](\sigma_{s)})$ where $D_\omega = D_{t_1} \cdots D_{t_n}$ if $\omega = \{t_1 < \cdots < t_n\}$.

Thus

$$[T_t f](\sigma) = \sum_{\substack{s\in\sigma\\ s\leqslant t}} \mathbb{1}_{\#\sigma_{s)}\leqslant n} \left[D_s D_{\sigma_{(s}} f\right](\sigma_{s)}) = \sum_{\substack{s\in\sigma\\ s\leqslant t}} \mathbb{1}_{\#\sigma_{s)}\leqslant n} f(\sigma) = (n+1)\wedge(\#\sigma_{t)})\; f(\sigma).$$

So $\int_{\mathcal{P}} |[T_t f](\sigma)|^2 \,\mathrm{d}\sigma \leqslant (n+1)^2 \int_{\mathcal{P}} |f(\sigma)|^2 \,\mathrm{d}\sigma$.

That is, T_t is a bounded operator (with norm $n+1$). Clearly $(T_t)_{t\geqslant 0}$ is an element of $\mathcal{S}$. We have $\|T_\cdot\|_{\mathcal{S}'_A} = \sup_{s\leqslant A} \|\Pi_s^n\| = 1$, for all A.

We have

$$T_t^2 = \int_0^t (T_s \Pi_s^n + \Pi_s^n T_s + \Pi_s^n)\,\mathrm{d}a_s^\circ$$

and $\|T_\cdot^2\|_{\mathcal{S}'_A} = \sup_{s\leqslant A} \|T_s \Pi_s^n + \Pi_s^n T_s + \Pi_s^n\|$.

Let us compute $\|T_t \Pi_t^n + \Pi_t^n T_t + \Pi_t^n\|$. We have, for $f \in \Phi_{t]}, \sigma \in \mathcal{P}_{t)}$

$$\begin{aligned}\left[(T_t \Pi_t^n + \Pi_t^n T_t + \Pi_t^n) f\right](\sigma) &= \mathbb{1}_{\#\sigma\leqslant n} [T_t f](\sigma) + [\Pi_t^n T_t f](\sigma) + \mathbb{1}_{\#\sigma\leqslant n}\, f(\sigma)\\ &= \mathbb{1}_{\#\sigma\leqslant n} \#\sigma f(\sigma) + \mathbb{1}_{\#\sigma\leqslant n}\, \#\sigma f(\sigma) + \mathbb{1}_{\#\sigma\leqslant n} f(\sigma)\\ &= (2\#\sigma + 1)\mathbb{1}_{\#\sigma\leqslant n}\, f(\sigma).\end{aligned}$$

Thus $\|T_t \Pi_t^n + \Pi_t^n T_t + \Pi_t^n\| \leqslant 2n+1$. It is even equal to $2n+1$ by taking $f \in L^2(\mathcal{P}_n)$.

This finally gives $\|T_\cdot^2\|_{\mathcal{S}'_A} = 2n+1$ and clearly $\|T_\cdot^2\|_{\mathcal{S}'_A} > \|T\|^2_{\mathcal{S}'_A}$. This shows that $\|\cdot\|_{\mathcal{S}'}$ is not an algebra norm for $\mathcal{S}$.

Anyway, it is possible to slightly modify $\|\cdot\|_{\mathcal{S}'}$ in order to produce an algebra norm for $\mathcal{S}$. This is performed through the transform $\mathcal{D}$.

Let $A \in \mathbb{R}^+$ be fixed. Let $\mathcal{S}_A$ be the restriction of $\mathcal{S}$ to processes indexed by $[0, A]$. On $\mathcal{S}_A$ define the following norm:

$$\|T_\cdot\|_{\mathcal{D}^{-1}} = \|\mathcal{D}^{-1}(T_\cdot)\|_{\mathcal{S}'_A}.$$

PROPOSITION 5.4. *$\|\cdot\|_{\mathcal{D}^{-1}}$ is a norm on $\mathcal{S}_A$ which makes it complete.*

Proof. The fact that $\|\cdot\|_{\mathcal{D}^{-1}}$ is a norm on $\mathcal{S}_A$ comes easily from the linearity and injectivity of $\mathcal{D}^{-1}$. Let us check the completeness property. If $(T^n_\cdot)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}_A, \|\cdot\|_{\mathcal{D}^{-1}})$ then $(\mathcal{D}^{-1}_\cdot(T^n_\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}'_A, \|\cdot\|_{\mathcal{S}'_A})$, thus it converges in $\mathcal{S}'_A$ to a process $(T_t)_{t\in[0,A]}$. The process $(\mathcal{D}_t(T_\cdot))_{t\in[0,A]}$ belongs to $\mathcal{S}_A$ and

$$\|T^n_\cdot - \mathcal{D}_\cdot(T_\cdot)\|_{\mathcal{S}_A} = \|\mathcal{D}^{-1}_\cdot(T^n_\cdot) - T_\cdot\|_{\mathcal{S}'_A} \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus $(\mathcal{D}_t(T_\cdot))_{t\in[0,A]}$ is the limit in $(\mathcal{S}_A, \|\cdot\|_{\mathcal{D}^{-1}})$ of $(T^n_\cdot)_{n\in\mathbb{N}}$. ∎

Unfortunately, $\|\cdot\|_{\mathcal{D}^{-1}}$ is not an algebra norm for $\mathcal{S}_A$ (this is not surprising as $\mathcal{D}$ is not an algebra morphism). Let us see a counter-example again. Let $A \in \mathbb{R}^+$ be fixed. Let $T_t = a^+_t$, $t \in [0, A]$ and $S_t = a^-_t$, $t \in [0, A]$. Let $X_t = \mathcal{D}_t(T_\cdot)$, $Y_t = \mathcal{D}_t(S_\cdot)$. Let us compute $\|X_t\|$ and $\|Y_t\|$ first. By Equation (4.2) we have $X_t P_t f = \int_0^t P_s f \,\mathrm{d}\chi_s$ and $Y_t P_t f = \int_0^t D_s f \,\mathrm{d}s$. Thus

$$\|X_t P_t f\| = \Big(\int_0^t \|P_s f\|^2 \,\mathrm{d}s\Big)^{1/2} \leqslant \sqrt{t}\|P_t f\|$$

$$\|Y_t P_t f\| = \int_0^t \|D_s f\| \,\mathrm{d}s \leqslant \sqrt{t}\Big(\int_0^t \|D_s f\|^2 \,\mathrm{d}s\Big)^{1/2} \leqslant \sqrt{t}\|P_t f\|.$$

Furthermore $X_t P_t \mathbb{1} = \int_0^t \mathbb{1} \,\mathrm{d}\chi_s = \chi_t$ thus $\|X_t P_t \mathbb{1}\| = \sqrt{t} = \sqrt{t}\|P_t \mathbb{1}\|$ and $Y_t P_t \chi_t = \int_0^t \mathbb{1} \,\mathrm{d}s = t$ thus $\|Y_t P_t \chi_t\| = t = \sqrt{t}\|P_t \chi_t\|$. This proves that $\|X_t\| = \|Y_t\| = \sqrt{t}$. Furthermore $\|X_\cdot\|_{\mathcal{D}^{-1}} = \|T\|_{\mathcal{S}'_A} = \sqrt{A}$ and $\|Y_\cdot\|_{\mathcal{D}^{-1}} = \|S\|_{\mathcal{S}'_A} = \sqrt{A}$.

By (4.7) we have $X_t Y_t = \mathcal{D}_t\Big(\int_0^\cdot Y_s \,\mathrm{d}a^+_s + \int_0^\cdot X_s \,\mathrm{d}a^-_s\Big)$ thus

$$\begin{aligned}\|X_\cdot Y_\cdot\|_{\mathcal{D}^{-1}} &= \Big\|\int_0^\cdot Y_s \,\mathrm{d}a^+_s + \int_0^\cdot X_s \,\mathrm{d}a^-_s\Big\|_{\mathcal{S}'_A} \\ &= \Big(\int_0^A \|Y_s\|^2 \,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^A \|X_s\|^2 \,\mathrm{d}s\Big)^{1/2} = 2\Big(\int_0^A s \,\mathrm{d}s\Big)^{1/2} = \sqrt{2}A.\end{aligned}$$

We do not have $\|X_\cdot Y_\cdot\|_{\mathcal{D}^{-1}} \leqslant \|X_\cdot\|_{\mathcal{D}^{-1}}\|Y\|_{\mathcal{D}^{-1}}$.

We have to define another norm on $\mathcal{S}$. Let A be fixed in $\mathbb{R}^+$. Let $(T_t)_{t\in[0,A]} \in \mathcal{S}_A$. Recall that $(T_t)_{t\in[0,A]}$ also belongs to $\mathcal{S}'_A$. Define

$$\|T.\|_{\mathcal{S}_A} = \sup_{s\leqslant A} \|T_s\| + \|T.\|_{\mathcal{S}'_A}.$$

THEOREM 5.5. *$\|\cdot\|_{\mathcal{S}_A}$ is a $*$-algebra norm on $\mathcal{S}_A$. It is equivalent to the norm $\|\cdot\|_{\mathcal{D}^{-1}}$ with:*

$$\|T.\|_{\mathcal{D}^{-1}} \leqslant \|T.\|_{\mathcal{S}_A} \leqslant 3\|T.\|_{\mathcal{D}^{-1}}, \tag{5.1}$$

the constants being optimal.

Thus $(\mathcal{S}_A, \|\cdot\|_{\mathcal{S}_A})$ is a Banach algebra.

Proof. $\|\cdot\|_{\mathcal{S}_A}$ is clearly a norm on $\mathcal{S}_A$. Let us look at its behaviour with respect to the product in $\mathcal{S}_A$. Let $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \,\mathrm{d}a_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon \,\mathrm{d}a_s^\varepsilon$ be elements of $\mathcal{S}_A$. Then

$$S_t T_t = \sum_\varepsilon \int_0^t S_s H_s^\varepsilon \,\mathrm{d}a_s^\varepsilon + \sum_\varepsilon \int_0^t K_s^\varepsilon T_s \,\mathrm{d}a_s^\varepsilon + [S.,T.]_t;$$

$$\begin{aligned}
\|S.T.\|_{\mathcal{S}_A} &= \sup_{s\leqslant A} \|S_sT_s\| + \sup_{s\leqslant A} \|S_sH_s^\circ + K_s^\circ T_s + H_s^\circ K_s^\circ\| \\
&\quad + \Big(\int_0^A \|S_sH_s^+ + K_s^+T_s + K_s^\circ H_s^+\|^2 \,\mathrm{d}s\Big)^{1/2} \\
&\quad + \Big(\int_0^A \|S_sH_s^- + K_s^-T_s + K_s^- H_s^\circ\|^2 \,\mathrm{d}s\Big)^{1/2} + \int_0^A \|S_sH_s^\times + K_s^\times T_s + K_s^- H_s^+\| \,\mathrm{d}s \\
&\leqslant \sup_{s\leqslant A} \|S_s\| \Big[\sup_{s\leqslant A} \|T_s\| + \sup_{s\leqslant A} \|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 \,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^A \|H_s^-\|^2 \,\mathrm{d}s\Big)^{1/2} \\
&\quad + \int_0^A \|H_s^\times\| \,\mathrm{d}s\Big] + \sup_{s\leqslant A} \|K_s^\circ\| \Big[\sup_{s\leqslant A} \|T_s\| + \sup_{s\leqslant A} \|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 \,\mathrm{d}s\Big)^{1/2}\Big] \\
&\quad + \Big(\int_0^A \|K_s^+\|^2 \,\mathrm{d}s\Big)^{1/2} \big(\sup_{s\leqslant A} \|T_s\|\big) \\
&\quad + \Big(\int_0^A \|K_s^-\|^2 \,\mathrm{d}s\Big)^{1/2} \Big[\sup_{s\leqslant A} \|T_s\| + \sup_{s\leqslant A} \|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 \,\mathrm{d}s\Big)^{1/2}\Big] \\
&\quad + \Big(\int_0^A \|K_s^x\| \,\mathrm{d}s\Big)\big(\sup_{s\leqslant A} \|T_s\|\big) \leqslant \|S.\|_{\mathcal{S}_A} \|T.\|_{\mathcal{S}_A}.
\end{aligned}$$

Thus $\|\cdot\|_{\mathcal{S}_A}$ is an algebra norm for $\mathcal{S}_A$. It is clearly a $*$-algebra norm.

Furthermore, the estimate (4.6) proves that for $T. \in \mathcal{S}'$

$$\sup_{s\leqslant A}\|\mathcal{D}_s(T.)\| \leqslant \|T.\|_{\mathcal{S}'_A}. \tag{5.2}$$

Let $T. \in \mathcal{S}_A$. We have

$$\begin{aligned}\|T.\|_{\mathcal{D}^{-1}} &= \|\mathcal{D}^{-1}_.(T.)\|_{\mathcal{S}'_A}\\ &= \sup_{s\leqslant A}\|H^\circ_s + T_s\| + \Big(\int_0^A \|H^+_s\|^2\,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^A \|H^-_s\|^2\,\mathrm{d}s\Big)^{1/2} + \int_0^A \|H^\times_s\|\,\mathrm{d}s\\ &\leqslant \sup_{s\leqslant A}\|T_s\| + \sup_{s\leqslant A}\|H^\circ_s\| + \Big(\int_0^A \|H^+_s\|^2\,\mathrm{d}s\Big)^{1/2} + \Big(\int_0^A \|H^-_s\|^2\,\mathrm{d}s\Big)^{1/2}\\ &\qquad + \int_0^A \|H^\times_s\|\,\mathrm{d}s = \|T\|_{\mathcal{S}_A}.\end{aligned}$$

This gives the first inequality.

Now, for $T. \in \mathcal{S}'_A$ we have

$$\begin{aligned}\|\mathcal{D}.(T)\|_{\mathcal{S}_A} &= \sup_{s\leqslant A}\|\mathcal{D}_s(T)\| + \sup_{s\leqslant A}\|H^\circ_s - \mathcal{D}_s(T.)\| + \Big(\int_0^A \|H^+_s\|^2\,\mathrm{d}s\Big)^{1/2}\\ &\qquad + \Big(\int_0^A \|H^-_s\|^2\,\mathrm{d}s\Big)^{1/2} + \int_0^A \|H^\times_s\|\,\mathrm{d}s\\ &\leqslant 2\sup_{s\leqslant A}\|\mathcal{D}_s(T)\| + \|T.\|_{\mathcal{S}'_A} \leqslant 3\|T.\|_{\mathcal{S}'}, \qquad \text{by (5.2).}\end{aligned}$$

Thus

$$\|T.\|_{\mathcal{S}_A} = \|\mathcal{D}.\circ\mathcal{D}^{-1}_.(T.)\|_{\mathcal{S}_A} \leqslant 3\|\mathcal{D}^{-1}_.(T.)\|_{\mathcal{S}'_A} = 3\|T.\|_{\mathcal{D}^{-1}}.$$

This proves the second inequality and thus the equivalence of the norms. As a consequence, $\mathcal{S}_A$ is complete for $\|\cdot\|_{\mathcal{S}_A}$ (Proposition 5.4).

Let us now check that the constants in the inequality (5.1) are optimal.

Let $T_t = a^\times_t$, $t \in \mathbb{R}^+$; this is an element of $\mathcal{S}$. We have $T_t f = tf$ and thus $\|T_t\| = t$, $\|T.\|_{\mathcal{S}_A} = 2A$. Furthermore $\mathcal{D}^{-1}_t(T.) = tI + \int_0^t s\,\mathrm{d}a^\circ_s$ and $\|\mathcal{D}^{-1}_.(T.)\|_{\mathcal{S}'_A} = 2A$.

This proves that the inequality $\|T.\|_{\mathcal{D}^{-1}} \leqslant \|T\|_{\mathcal{S}_A}$ is also optimal.

Let $T_t = a^+_t$, $t \in [0,A]$. Then $\|T.\|_{\mathcal{S}'_A} = \sqrt{A}$. Let $X_t = \mathcal{D}_t(T.)$, $t \in [0,A]$ that is $X_t P_t f = \int_0^t P_s f\,\mathrm{d}\chi_s$. We know that $\|X_t\| = \sqrt{t}$. Thus

$$\|X.\|_{\mathcal{S}'_A} = \sup_{s\leqslant A}\|-X_s\| + \Big(\int_0^A \|I\|^2\,\mathrm{d}s\Big)^{1/2} = 2\sqrt{A}$$

and finally $\|X.\|_{\mathcal{S}_A} = 3\sqrt{A}$, whereas $\|X.\|_{\mathcal{D}^{-1}} = \|\mathcal{D}_{\cdot}^{-1}(X)\|_{\mathcal{S}'_A} = \|T.\|_{\mathcal{S}'_A} = \sqrt{A}$. This proves that the inequality $\|T.\|_{\mathcal{S}_A} \leqslant 3\|T.\|_{\mathcal{D}^{-1}}$ is also optimal. ∎

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