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INVARIANT SUBSPACES OF COMPOSITION OPERATORS

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ABSTRACT. We will study the structure of invariant subspaces of a composition operator. The consequences of the lattice of one composition operator being contained in another will be discussed and some results concerning the structure of an invariant subspace shared by two composition operators will be given.

KEYWORDS: *composition operators, invariant subspaces.*

MSC (2000): 47B38.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane and denote by $\mathrm{Hol}(\mathbb{D})$ the set of holomorphic functions on $\mathbb{D}$. Define the Hardy Hilbert space $\mathbb{H}^2$ of analytic functions on $\mathbb{D}$ by

$$\mathbb{H}^2 = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{r\in(0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta < \infty \right\}.$$

For $\lambda \in \mathbb{D}$, denote by k_λ the evaluation kernel at the point λ. Hence, $\langle f, k_\lambda\rangle = f(\lambda)$ for all $f \in \mathbb{H}^2$. The evaluation kernels on $\mathbb{H}^2$ have the explicit form:

$$k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}.$$

Let φ be an analytic self-map of the unit disk. We define the composition operator induced by φ on $\mathbb{H}^2$ by

$$C_\varphi f = f \circ \varphi \quad \forall f \in \mathbb{H}^2.$$

Composition operators have been extensively studied in many papers (see [3] and [16]). The composition operator C_φ is a bounded operator on $\mathbb{H}^2$ ([15], p. 123). In fact, its norm satisfies the following estimates:

PROPOSITION *If φ is an analytic self-map of the unit disk, then*

$$\left\{\frac{1}{1-|\varphi(0)|}\right\}^{\frac{1}{p}} \leqslant \|C_\varphi\| \leqslant \left\{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}^{\frac{1}{p}}$$

on the Hardy space $\mathbb{H}^p$ for $1 \leqslant p < \infty$.

Given an analytic self-map of the unit disk, φ, denote by $\varphi^{(n)}$ the n^{th} iterate of φ under composition. A very important result in the study of the iteration of analytic self-maps of the unit disk is the Denjoy-Wolff theorem ([4] and [21]) which states the following:

THEOREM (Denjoy-Wolff Theorem) *Let φ be an analytic self-map of the unit disk other then either an elliptic disk automorphism or the identity. Then φ has a unique fixed point $a \in \overline{\mathbb{D}}$ such that $|\varphi'(a)| \leqslant 1$. Moreover, if φ is not an elliptic disk automorphism, then $\varphi^{(n)} \to a$ uniformly on compact subsets of $\mathbb{D}$. Also if $a \in \partial\mathbb{D}$, then $0 < \varphi'(a) \leqslant 1$.*

The Denjoy-Wolff theorem is used extensively in the study of composition operators. For example, note that Denjoy-Wolff theorem together with Proposition 1.1 implies that C_φ is power bounded whenever φ has the Denjoy-Wolff point in $\mathbb{D}$.

Little is known about the adjoints of composition operators. However, the following simple property which characterizes their action on evaluation kernels is very useful.

PROPOSITION *Let φ be an analytic self-map of the unit disk. Then $C_\varphi^* k_\lambda = k_{\varphi(\lambda)}$ for all $\lambda \in \mathbb{D}$.*

C. Cowen ([2]) proved that any analytic self-map of the unit disk φ with Denjoy-Wolff point a, with $\varphi'(a) \neq 0$, can be modeled after a linear fractional transformation Φ in the following way

$$\Phi \circ \sigma = \sigma \circ \varphi$$

where $\sigma : \mathbb{D} \to \Omega$ is analytic, Φ is an automorphism of Ω, and Ω is either the complex plane or a half-plane. Using this model, analytic self-maps of the unit disk can be classified as one of the following types ([3], p. 71):

(1) Plane\Dilation: $\Omega = \mathbb{C}$, $\sigma(a) = 0$, $\Phi(z) = sz$ where $0 < s < 1$.

(2) Plane\Translation: $\Omega = \mathbb{C}$, $\sigma(a) = \infty$, $\Phi(z) = z + 1$.

(3) Half-plane\Dilation: $\Omega = \{z : \Re z > 0\}$, $\sigma(a) = 0$, $\Phi(z) = sz$ where $0 < |s| < 1$.

(4) Half-plane\Translation: $\Omega = \{z : \Im z > 0\}$, $\sigma(a) = \infty$, $\Phi(z) = z \pm 1$.

The map φ is in the Plane\Dilation case if and only if φ has the Denjoy-Wolff point in $\mathbb{D}$. If φ has the Denjoy-Wolff point $a \in \partial\mathbb{D}$ and $\varphi'(a) < 1$, then φ is in the Half-plane\Dilation case. Also, by using this model it can be shown that:

PROPOSITION (i) *If φ has a fixed point $a \in \mathbb{D}$, and C_φ and C_ψ have a non-constant common eigenfunction, then C_φ and C_ψ commute.*

(ii) *Let φ be an analytic self-map of the unit disk not a disk automorphism, with the Denjoy-Wolff point $a \in \mathbb{D}$. If $\varphi'(a) \neq 0$, then $C_\varphi f = \lambda f$ has a non-zero solution if and only if $\lambda = \varphi'(a)^j$ for some non-negative integer j. Moreover, f is a non-zero solution of $C_\varphi f = \varphi'(a)^j f$ for some non-negative integer j if and only if $f(z) = c\sigma(z)^j$ where σ is the map in the model and c is a constant* ([3], p. 78).

(iii) *If C_φ is compact, then $\sigma(C_\varphi) = \{\varphi'(a)^n\}_{n=1}^{\infty} \cup \{0,1\}$* ([16], p. 94).

Proof. We present a short proof of item (1) above. Let σ be the non-constant common eigenvector of C_φ and C_ψ. Note that since σ is a non-constant eigenvector of C_φ, $\varphi'(a) \neq 0$. Let V be the fundamental domain for φ as in Theorem 2.5.3 from [3]. Then we have

$$\sigma \circ \varphi \circ \psi = \lambda\alpha\sigma = \sigma \circ \psi \circ \varphi,$$

where α and λ are eigenvalues of C_φ and C_ψ respectively corresponding to σ. Since σ is univalent on V, it follows that $\varphi \circ \psi = \psi \circ \varphi$ on V. Now since σ is non-constant, the result follows from the open mapping theorem. ∎

Note that item (i) just proved is very similar to Exercise 2.4.2 from [3].

Suppose that φ has a non-zero Denjoy-Wolff point $a \in \mathbb{D}$. Then $\psi(z) = \xi \circ \varphi \circ \xi^{-1}(z)$ has 0 as a fixed point where $\xi(z)$ is the disk automorphism given by $\xi(z) = \frac{a-z}{1-\bar{a}z}$. Similarity of C_φ to $C_\xi C_\varphi C_\xi^{-1}$ and the fact that similar operators have isomorphic invariant subspace lattices, allows us to assume without loss of generality that the interior fixed point of φ is zero when considering the lattice of invariant subspaces of C_φ.

When φ fixes zero, it can easily be shown that each subspace $\bigvee\{z^i\}_{i=0}^n$ is invariant under C_φ^*. Thus, C_φ^* has an upper-triangular matrix with respect to the basis $\{z^n\}_{n=0}^\infty$ with the diagonal elements $\{\overline{\varphi'(0)}^n\}_{n=0}^\infty$. This implies $\{\overline{\varphi'(0)}^n\}_{n=0}^\infty \subseteq \sigma_0(C_\varphi^*)$.

In this paper, we study the invariant subspaces of composition operators. Invariant subspaces of composition operators can be quite complex. For example, C_{z^n} for $n \geqslant 2$ is an isometry that is similar to $1 \oplus S$ on $\mathbb{C} \oplus z\mathbb{H}^2$, where S is a unilateral shift of infinite multiplicity. Also, in [12] it was shown that a hyperbolic composition operator is universal. An operator U is called *universal* if for every operator T, some constant multiple of T is similar to the restriction of U to one of its invariant subspaces. Another example of a universal operator is the unilateral backward shift of infinite multiplicity ([17]). This allows for example to formulate the invariant subspace problem in terms of invariant subspaces of hyperbolic composition operators ([12]). Because of this close relationship between the invariant subspaces of hyperbolic composition operators and the invariant subspace problem, the invariant subspaces of invertible composition operators have been studied in [9], [10] and [12]. In what follows, we study a broader range of composition operators, namely those in Plane\Dilation case, Half-plane\Dilation case and Half-plane\Translation case.

Throughout this paper, we work with composition operators acting on $\mathbb{H}^2$ and we assume that all the symbol maps of composition operators are analytic self-maps of the unit disk. The spectrum and the point spectrum of an operator

T are denoted by $\sigma(T)$ and $\sigma_0(T)$ respectively. Moreover, we denote the lattice of all invariant subspaces of an operator A by $\operatorname{Lat} A$.

2. THE DENJOY-WOLFF POINT IN $\partial\mathbb{D}$

We now discuss the consequences of the invariant subspaces of one composition operator being invariant under another when the Denjoy-Wolff point is in $\partial\mathbb{D}$.

THEOREM *Suppose that φ has a Denjoy-Wolff point on $\partial\mathbb{D}$ and that $\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\} \in \operatorname{Lat} C_\psi^*$ for an uncountable collection of points $\{z_\alpha\}$ satisfying $\sum_{n=0}^{\infty}(1-|\varphi^{(n)}(z_\alpha)|) < \infty$. Then $\psi = \varphi^{(m)}$ for some $m \in \mathbb{N} \cup \{0\}$.*

Proof. Since $\sum_{n=0}^{\infty}(1-|\varphi^{(n)}(z_\alpha)|) < \infty$, the Blaschke product B_α with zeros $\{\varphi^{(n)}(z_\alpha)\}$ converges. As is well-known and verified easily,

$$\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\} = (B_\alpha \mathbb{H}^2)^\perp \quad \text{for any } z_\alpha.$$

Since $\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\} \in \operatorname{Lat} C_\psi^*$, we obtain

$$k_{\psi(z_\alpha)} = C_\psi^* k_{z_\alpha} \in \bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\}.$$

Since $\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\} = (B_\alpha \mathbb{H}^2)^\perp$ and $B_\alpha \in B_\alpha \mathbb{H}^2$, we have

$$B_\alpha(\psi(z_\alpha)) = \langle B_\alpha, k_{\psi(z_\alpha)}\rangle = 0.$$

Since B_α is a Blaschke product, for each z_α, we conclude that $\psi(z_\alpha) = \varphi^{(m_\alpha)}(z_\alpha)$ for some $m_\alpha \in \mathbb{N}$. Since $\{z_\alpha\}$ is uncountable and $\mathbb{N}$ is countable, we conclude that some $m \in \mathbb{N}$ must occur uncountably many times. So $\psi(z_\beta) = \varphi^{(m)}(z_\beta)$ for an uncountable sub-collection $\{z_\beta\}$ of $\{z_\alpha\}$ and some fixed $m \in \mathbb{N}$. Since the collection $\{z_\beta\}$ has a limit point in $\mathbb{D}$ and φ and ψ are analytic, it follows that $\psi = \varphi^{(m)}$. ∎

COROLLARY *Let φ be in the Half-plane\Dilation or Half-plane\Translation case. If $\operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\psi$, then $\psi = \varphi^{(m)}$ for some $m \in \mathbb{N} \cup \{0\}$.*

Proof. If φ is in either of the above cases, then we have ([3], p. 80)

$$\sum_{n=0}^{\infty}(1-|\varphi^{(n)}(z)|) < \infty \quad \forall z \in \mathbb{D}.$$

Moreover, the assumption $\operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\psi$ implies that $\operatorname{Lat} C_\varphi^* \subseteq \operatorname{Lat} C_\psi^*$. So for each $z \in \mathbb{D}$, the subspace $\mathcal{M}_z \equiv \bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z)}\}$ is invariant under C_ψ^*. Hence the result follows from Theorem 2.1. ∎

Corollary *Let φ_1 and φ_2 be non-constant analytic self-maps of the unit disk. Suppose that there exists an uncountable collection of points $\{z_\alpha\}$ such that $\sum\limits_{n=0}^{\infty}(1-|\varphi_j^{(n)}(z_\alpha)|) < \infty$ for $j=1,2$. If $\bigvee\limits_{n=0}^{\infty}\{k_{\varphi_1^{(n)}(z)}\}$ is invariant for $C^*_{\varphi_2}$ and $\bigvee\limits_{n=0}^{\infty}\{k_{\varphi_2^{(n)}(z)}\}$ is invariant for $C^*_{\varphi_1}$ for each z_α, then $\varphi_1=\varphi_2$.*

Proof. Since for each z_α, $\sum\limits_{n=0}^{\infty}(1-|\varphi_1^{(n)}(z_\alpha)|) < \infty$ and $\bigvee\limits_{n=0}^{\infty}\{k_{\varphi_1^{(n)}(z_\alpha)}\}$ is invariant under $C^*_{\varphi_2}$, by Theorem 2.1 we conclude that $\varphi_2=\varphi_1^{(m)}$ for some $m\in\mathbb{N}\cup\{0\}$. By a similar argument, we conclude that $\varphi_1=\varphi_2^{(n)}$ for some $n\in\mathbb{N}\cup\{0\}$. Hence, $\varphi_1=\varphi_1^{(mn)}$.

We now want to show $m=n=1$. First note that since

$$\sum_{n=0}^{\infty}(1-|\varphi_1^{(n)}(z_\alpha)|) < \infty,$$

the map φ_1 must have the Denjoy-Wolff point a on $\partial\mathbb{D}$. If not, suppose $\varphi_1(a)=a\in\mathbb{D}$. Since $\{\varphi_1^{(n)}(z)\}$ accumulates at $a\in\mathbb{D}$, there exists $r\in(0,1)$ such that $r<1-|\varphi_1^{(n)}(z_\alpha)|$ for all $n\geqslant 0$. This contradicts $\sum\limits_{n=0}^{\infty}(1-|\varphi_1^{(n)}(z_\alpha)|) < \infty$.

If m or n is zero, then $\varphi_1=z=\varphi_2$. This is impossible since the identity does not satisfy the hypothesis of the corollary. Now to prove $m=n=1$, assume by way of contradiction that $mn>1$. Since $\forall z\in\mathbb{D}$ we have that $\varphi_1^{(mn)}(z)=\varphi_1(z)$, the sequence $\{\varphi_1^{(k)}(z)\}_{k=0}^{\infty}$ does not converge to the fixed point of φ_1 on $\partial\mathbb{D}$ which contradicts the Denjoy-Wolff Theorem. Hence, $m=n=1$ and so $\varphi_1=\varphi_2$. ∎

Remark In particular, this result implies: If φ and ψ are in the Half-plane\Dilation case or in the Half-plane\Translation case (not necessarily both in the same one), then $\varphi=\psi$ if and only if $\operatorname{Lat} C_\varphi=\operatorname{Lat} C_\psi$.

Theorem 2.1 will help us to conclude that the invariant subspaces of an invertible composition operator can be contained only in the lattice of a similar type of invertible composition operator.

Proposition *If $\operatorname{Lat} C_\varphi\subseteq\operatorname{Lat} C_\psi$, and φ is either a hyperbolic or parabolic disk automorphism, then ψ is also an automorphism of the same kind.*

Proof. If φ is a hyperbolic or parabolic disk automorphism, then for any $z\in\mathbb{D}$ we have that $\sum\limits_{n=0}^{\infty}(1-|\varphi^{(n)}(z)|)<\infty$. This follows because the model for a hyperbolic disk automorphism is the Half-plane\Dilation and the model for the parabolic disk automorphism is the Half-plane\Translation ([2]). Since $\operatorname{Lat} C_\varphi\subseteq\operatorname{Lat} C_\psi$, by Theorem 2.1 we conclude that $\psi=\varphi^{(m)}$ for some $m\in\mathbb{N}\cup\{0\}$. Hence, ψ will also be an automorphism. Moreover, ψ will have the same behavior under iteration as φ does. Because hyperbolic and parabolic automorphisms can be distinguished by their behavior under iteration, we conclude that ψ will be of the same type as φ. ∎

Recall that if an operator A is in the weakly closed algebra generated by an operator B, then $\operatorname{Lat} B \subseteq \operatorname{Lat} A$. Hence, the results above also indicate restrictions on the composition operators that can occur in the weakly closed algebra generated by composition operators whose symbol map is in the Half-plane\Dilation case or in the Half-plane\Translation case. This restriction is the following.

Corollary *Suppose φ is in the Half-plane\Dilation case or in the Half-plane\Translation case. If C_ψ is in the weakly closed algebra generated by C_φ, then C_ψ is in the semi-group generated by C_φ.*

Proof. Since C_ψ is in the weakly closed algebra generated by C_φ, we have $\operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\psi$. Hence, by Corollary 2.2 we conclude that $\psi = \varphi^{(n)}$ for some $n \in \mathbb{N} \cup \{0\}$. So $C_\psi = C_\varphi^n$ is in the semi-group generated by C_φ. ∎

We now discuss the relationship of the Denjoy-Wolff points of the maps φ and ψ to the common invariant subspaces for C_φ and C_ψ. We will see later on that the existence of a common fixed point for φ and ψ in $\mathbb{D}$ will imply the existence of a rich collection of common invariant subspaces for C_φ and C_ψ. The following result will work in the opposite direction, by deducing the existence of equality of the Denjoy-Wolff points for φ and ψ from assumptions about some common invariant subspaces for C_φ and C_ψ.

Theorem (i) *If φ has the Denjoy-Wolff point in $\mathbb{D}$ and $\operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\psi$ then φ and ψ have the same Denjoy-Wolff point.*

(ii) *If φ and ψ have the Denjoy-Wolff points on $\partial\mathbb{D}$, and there exists a point $\lambda \in \mathbb{D}$ such that*

$$\sum_{n=0}^{\infty}(1-|\varphi^{(n)}(\lambda)|) < \infty \quad \textit{and} \quad \bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(\lambda)}\} \in \operatorname{Lat} C_\psi^*,$$

then φ and ψ have the same Denjoy-Wolff point on $\partial\mathbb{D}$.

Proof. (i) First assume that φ has the Denjoy-Wolff point $a \in \mathbb{D}$. Then by Proposition 1.3 we have

$$C_\varphi^* k_a = k_{\varphi(a)} = k_a.$$

Since $\operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\psi$, we get

$$C_\psi^* k_a = k_{\psi(a)} \in \bigvee\{k_a\}.$$

This implies that $k_{\psi(a)} = \beta k_a$ for some $\beta \in \mathbb{C}$. Hence, $\beta = 1$ and $\psi(a) = a$.

(ii) Now suppose that φ and ψ have Denjoy-Wolff points a and b respectively on $\partial\mathbb{D}$. Since for $\lambda \in \mathbb{D}$ we have $\sum\limits_{n=0}^{\infty}(1-|\varphi^{(n)}(\lambda)|) < \infty$, we can form the Blaschke product B with zeros $\{\varphi^{(n)}(\lambda)\}_0^\infty$. It is well-known that

$$\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(\lambda)}\} = \left(B\mathbb{H}^2\right)^{\perp}.$$

Since $\bigvee\limits_{n=0}^{\infty}\{k_{\varphi^{(n)}(\lambda)}\} \in \operatorname{Lat} C_\psi^*$, we have

$$C_\psi^{m^*} k_\lambda = k_{\psi^{(m)}(\lambda)} \in (B\mathbb{H}^2)^\perp.$$

Hence, $B(\psi^{(m)}(\lambda)) = \langle B, k_{\psi^{(m)}(\lambda)}\rangle = 0$. Since the only zeros of B are $\{\varphi^{(n)}(\lambda)\}_0^\infty$, we conclude that $\psi^{(m)}(\lambda) = \varphi^{(i_m)}(\lambda)$ for some $i_m \in \mathbb{N}$.

Note that $\psi^{(n)}(\lambda)$ are all distinct. To see this, suppose that $\psi^{(m)}(\lambda) = \psi^{(m')}(\lambda)$ for some distinct m and m' in $\mathbb{N}$. Without loss of generality, assume that $m' < m$. Then $\psi^{(m-m')}(\psi^{(m')}(\lambda)) = \psi^{(m')}(\lambda)$. Thus $\psi^{(m-m')}$ fixes $\psi^{(m')}(\lambda) \in \mathbb{D}$. This contradicts the Denjoy-Wolff Theorem since then the iterates $\{\psi^{(n)}(\lambda)\}$ do not converge to $b \in \partial\mathbb{D}$. Hence, the sequence $\{\psi^{(m)}(\lambda)\}$ are all distinct. A similar argument shows that $\{\varphi^{(n)}(\lambda)\}$ are all distinct. By dropping to a subsequence if necessary, we can assume that $\{i_m\}$ are increasing. Hence, using the Denjoy-Wolff Theorem, we obtain

$$a = \lim_{i_m\to\infty} \varphi^{(i_m)}(\lambda) = \lim_{m\to\infty} \psi^{(m)}(\lambda) = b.$$

Thus the result follows. ∎

Theorem *Let φ and ψ be analytic self-maps of the unit disk. Then $C_\varphi \mathbb{H}^2 \subseteq C_\psi \mathbb{H}^2$ if and only if $C_\varphi = C_\psi C_f$ where f is some analytic self-map of the unit disk.*

Proof. If $C_\varphi = C_\psi C_f$ where f is some analytic self-map of the unit disk, then clearly $C_\varphi \mathbb{H}^2 \subseteq C_\psi \mathbb{H}^2$.

Conversely, if $C_\varphi \mathbb{H}^2 \subseteq C_\psi \mathbb{H}^2$, then $C_\varphi = C_\psi A$ for some operator A. First, assume that ψ is non-constant. Then $1 = C_\varphi 1 = C_\psi(A1)$. By the open mapping theorem, we get that $A1 = 1$ is a constant. Also, for any $n \in \mathbb{N}$ we have

$$\varphi = C_\varphi z = Az \circ \psi \quad \text{and} \quad \varphi^n = C_\varphi z^n = Az^n \circ \psi.$$

Hence, $(Az \circ \psi)^n = Az^n \circ \psi$. Thus, the open mapping theorem yields that $Az^n = (Az)^n$. Hence, A must be a composition operator ([11]).

If ψ is a constant, then C_ψ is evaluation at a point. Hence, $C_\varphi \mathbb{H}^2 \subseteq C_\psi \mathbb{H}^2$ implies that C_φ is also evaluation at a point. By taking $f = \varphi$, the result follows. ∎

Denote by $\mathcal{W}(A)$ and $\operatorname{Alg\,Lat} A$, respectively, the weak operator topology closure of the algebra generated by A and the identity and the collection of operators that leave invariant every subspace that is left invariant under A. In addition, denote by $\{A\}'$ the set of all operators that commute with A. A subspace is called hyper-invariant for an operator A if it is invariant under every operator that commutes with A. Denote by $\operatorname{Alg\,Lat}(\{A\}')$ the set of all operators that leave invariant every subspace that is invariant under $\{A\}'$. An operator A is called *reflexive* if $\operatorname{Alg\,Lat} A = \mathcal{W}(A)$.

Theorem *Suppose φ has the Denjoy-Wolff point $a \in \partial\mathbb{D}$. If $\varphi'(a) < 1$ and $z \in \mathbb{D}$, then the following hold for $A = C_\varphi^*\big| \mathcal{M}_z$, where $\mathcal{M}_z \equiv \bigvee_{n=0}^{\infty} \{k_{\varphi^{(n)}(z)}\}$:*

(i) *A is not compact;*
(ii) *$\{A\}' = \mathcal{W}(A)$;*
(iii) *A has no point spectrum;*
(iv) *$r(A) = \varphi'(a)^{-\frac{1}{2}}$;*
(v) *A is reflexive.*

Proof. It is well-known ([3], p. 284) that A is similar to a weighted shift operator with weights $w_n \equiv \frac{1-|z_n|^2}{1-|z_{n+1}|^2}$ ([3]) where $z_n \equiv \varphi^{(n)}(z)$ (note that since we are only considering a forward sequence, we do not need the assumption of analyticity on the closed unit disk). The reader can find the facts about weighted shifts used in this proposition in the article of A.L. Shields ([17], pp. 49–128). Since $\lim w_n = \varphi'(a)^{-\frac{1}{2}} \neq 0$, it follows that A is not compact and that $r(A) = \varphi'(a)^{-\frac{1}{2}}$. Since $w_n \neq 0$ for all $n \in \mathbb{N}$, A is injective and so its point spectrum is empty and also any operator that commutes with A is the limit, in the strong operator topology, of a sequence of polynomials in A. Hence, $\{A\}' = \mathcal{W}(A)$. However,

$$\emptyset \neq \{z : |z|, \varphi'(a)^{-\frac{1}{2}}\} \subseteq \sigma_0(PC_\varphi P) \subseteq \{z : |z| \leqslant \varphi'(a)^{-\frac{1}{2}}\}$$

where $\sigma_0(B)$ is the point spectrum of B. Since A is injective and its adjoint has non-empty point spectrum, it follows that it is reflexive ([17], p. 104). ∎

3. GENERALIZATION TO THE HARDY SPACE OF VECTOR VALUED FUNCTIONS

Let H be a Hilbert space. A function F taking values in H is said to be *measurable* if $\langle F(\mathrm{e}^{\mathrm{i}\theta}), h\rangle$ is a measurable function for every $h \in H$. Then $\mathbb{L}^2(H)$ is defined by (page 52 from [7]):

$$\mathbb{L}^2(H) = \left\{F \text{ measurable} : \|F\|_2 \equiv \left(\frac{1}{2\pi}\int_0^{2\pi} \|F(\mathrm{e}^{\mathrm{i}\theta})\|^{\frac{1}{2}}\, \mathrm{d}\theta\right)^2 < \infty\right\}.$$

Further reference concerning vector-valued functions can be found in [19], p. 183. The Hardy space $\mathbb{H}^2(H)$ of functions taking values in H is defined to be the subspace of $L^2(H)$ consisting of functions which have vanishing negative Fourier coefficients ([7], p. 55).

Let $\{e_n\}$ be a basis for H. Given $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$ consider the linear functional $\Phi_{n,\lambda}$ on $\mathbb{H}^2(H)$ defined by

$$\Phi_{n,\lambda}(F) = \langle F(\lambda), e_n\rangle \quad \forall F \in \mathbb{H}^2(H)$$

where we have identified functions in $\mathbb{H}^2(H)$ with their extension to the interior of the disk.

Note that throughout this section we denote the inner product on $\mathbb{H}^2(H)$ by $(\cdot\,,\cdot)$ and the inner product on H by $\langle\cdot\,,\cdot\rangle$. Since $\mathbb{H}^2(H)$ is a Hilbert space, there exist $K_{n,\lambda} \in \mathbb{H}^2(H)$ such that

$$\Phi_{n,\lambda}(F) = \langle F(\lambda), e_n\rangle = (F, K_{n,\lambda}).$$

LEMMA $\bigvee\limits_{n\in\mathbb{N},\lambda\in\mathbb{D}} \{K_{n,\lambda}\} = \mathbb{H}^2(H)$.

Proof. Suppose that $F \in \mathbb{H}^2(H)$ is orthogonal to $\bigvee\limits_{n\in\mathbb{N},\,\lambda\in\mathbb{D}} \{K_{n,\lambda}\}$. Then

$$0 = (F, K_{n,\lambda}) = \langle F(\lambda), e_n\rangle \quad \forall \lambda \in \mathbb{D},\, n \in \mathbb{N}.$$

Hence, for any $\lambda \in \mathbb{D}$, we have that $F(\lambda)$ is orthogonal to every e_n. The result now follows since $\{e_n\}$ form a basis for H. ∎

Remark Consider the basis $\{z^m e_n\}_{m,n=0}^{\infty}$ for $\mathbb{H}^2(H)$. Then

$$(z^m e_k, K_{n,\lambda}) = \langle \lambda^m e_k(\lambda), e_n\rangle = \begin{cases} 0 & \text{if } n \neq k,\\ \lambda^m & \text{if } n = k.\end{cases}$$

Lemma *For any $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$, we have $K_{n,\lambda} = k_\lambda e_n$, where k_λ is the evaluation kernel at λ in $\mathbb{H}^2$.*

Proof. Let $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} K_{n,\lambda} &= \sum_{m,k} (K_{n,\lambda}, z^m e_k) z^m e_k = \sum_m \overline{\lambda}^m z^m e_n \quad \text{(by Remark 3.2)}\\ &= e_n \sum_m (\overline{\lambda} z)^m = e_n \frac{1}{1-\overline{\lambda} z} = k_\lambda e_n. \quad ∎ \end{aligned}$$

Given an analytic self-map of the unit disk φ, the ampliation $\widehat{C}_\varphi$ acts on $\mathbb{H}^2(H)$ by

$$\widehat{C}_\varphi F = F \circ \varphi \quad \forall F \in \mathbb{H}^2(H).$$

The action of $\widehat{C}_\varphi^*$ on evaluation kernels is described by the following.

Lemma *Let φ be an analytic self-map of the unit disk. Then $\widehat{C}_\varphi^* k_\lambda e_n = k_{\varphi(\lambda)} e_n$ for all $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$.*

Proof. Let $F \in \mathbb{H}^2(H)$. Then for any $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$ we have that

$$(F, \widehat{C}_\varphi^* K_{n,\lambda}) = (F \circ \varphi, K_{n,\lambda}) = \langle F \circ \varphi(\lambda), e_n\rangle = (F, K_{n,\varphi(\lambda)}).$$

Since $F \in \mathbb{H}^2(H)$ was arbitrary, we conclude that $\widehat{C}_\varphi^* K_{n,\lambda} = K_{n,\varphi(\lambda)}$. So the result follows from Lemma 3.4. ∎

We now prove a theorem similar to Theorem 2.1 for the ampliation of composition operators on $\mathbb{H}^2(H)$.

Theorem *Consider the space $\mathbb{H}^2(H)$. Suppose φ has the Denjoy-Wolff point on $\partial\mathbb{D}$ and $\bigvee_{n=0}^{\infty} \{k_{\varphi^{(n)}(z_\alpha)}\} e_m \subseteq \operatorname{Lat} \widehat{C}_\psi^*$ for an uncountable collection of points $\{z_\alpha\}$ and every $m \geqslant 1$. If $\sum_{n=0}^{\infty} (1 - |\varphi^{(n)}(z_\alpha)|) < \infty$ for all z_α, then $\psi = \varphi^{(m)}$ for some $m \in \mathbb{N}$.*

Proof. Since $\sum_{n=0}^{\infty} (1 - |\varphi^{(n)}(z_\alpha)|) < \infty$ for each z_α, we can form the Blaschke product B_α with zeros $\{\varphi^{(n)}(z_\alpha)\}$. Denote $\varphi^{(n)}(z_\alpha)$ by $z_{\alpha,n}$ and $k_{\varphi^{(n)}(z_\alpha)}$ by $k_{\alpha,n}$.

For each e_m, we have

$$(B_\alpha \mathbb{H}^2)^\perp e_m = \bigvee_{n=0}^{\infty} \{k_{\alpha,n}\} e_m.$$

Now since $\bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\}e_m \subseteq \operatorname{Lat}\widehat{C}_\psi^*$, we have that

$$k_{\psi(z_\alpha)}e_m = \widehat{C}_\psi^* k_{z_\alpha}e_m \in \bigvee_{n=0}^{\infty}\{k_{\varphi^{(n)}(z_\alpha)}\} = (B_\alpha\mathbb{H}^2)^\perp e_m.$$

So $B_\alpha(\psi(z_\alpha)) = (B_\alpha, k_{\psi(z_\alpha)}) = 0$. Hence, we have $\psi(z_\alpha) = \varphi^{(n_\alpha)}(z_\alpha)$ for some $n_\alpha \in \mathbb{N}$. Since $\{z_\alpha\}$ is uncountable and $\mathbb{N}$ is countable, we conclude that some $n \in \mathbb{N}$ must occur uncountably many times. Now an argument as in the proof of Theorem 2.1 yields that $\psi = \varphi^{(n)}$ for some $n \in \mathbb{N}$. ∎

Using the above theorem, we can easily prove analogous results to Corollary 2.2, Corollary 2.3, Proposition 2.5, and Corollary 2.6 for the ampliation of composition operators on $\mathbb{H}^2(H)$. The following is a generalization of Theorem 2.8.

Theorem *Let φ and ψ be analytic self-maps of the unit disk. Then $\widehat{C}_\varphi\mathbb{H}^2(H) \subseteq \widehat{C}_\psi\mathbb{H}^2(H)$ if and only if $\varphi = f \circ \psi$ for some analytic self-map of the unit disk f.*

Proof. Let $\{e_n\}$ be a basis for H. If $\varphi = f \circ \psi$ where f is some analytic self-map of the unit disk, then $\widehat{C}_\varphi = \widehat{C}_\psi\widehat{C}_f$. So the result follows.

Conversely, since $\{e_i\}$ is an orthonormal basis, $\widehat{C}_\varphi\mathbb{H}^2 e_n \subseteq \mathbb{H}^2 e_n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. Since $\widehat{C}_\varphi\mathbb{H}^2(H) \subseteq \widehat{C}_\psi\mathbb{H}^2(H)$, for any $g \in \mathbb{H}^2$ we have

$$\widehat{C}_\varphi g e_n = g \circ \varphi e_n \in [\widehat{C}_\psi\mathbb{H}^2(H)] \cap \mathbb{H}^2 e_n.$$

So $g \circ \varphi e_n = \widehat{C}_\varphi g e_n = (h \circ \psi)e_n$ for some $h \in \mathbb{H}^2$. Now it follows from Theorem 2.8 that $\varphi = f \circ \psi$ where f is some analytic self map of the unit disk. ∎

4. The Denjoy-Wolff point in $\mathbb{D}$

We now turn our attention to invariant subspaces of composition operators whose symbol map has the Denjoy-Wolff point in $\mathbb{D}$. An easy example of such an operator is $C_{\alpha z}$ where $0 < |\alpha| < 1$. This operator is compact and normal, and an application of the spectral theorem implies that all invariant subpaces of $C_{\alpha z}$ are closed spans of some powers of z. We will also study invariant subspaces common to several composition operators whose symbol maps have Denjoy-Wolff points in $\mathbb{D}$.

Definition A collection of operators is said to be *simultaneously triangularizable* if there exists a maximal chain of subspaces each of which is invariant under all operators in the collection.

Simultaneous triangularization is studied in detail in [14]. If the inducing maps of a collection of composition operators have the same fixed point in $\mathbb{D}$, then that collection of composition operators has a rich collection of common invariant subspaces.

PROPOSITION *Let $\{\varphi_\alpha\}_{\alpha\in I}$ be a collection of analytic self-maps of the unit disk that have a common fixed point in $\mathbb{D}$. Then $\{C_{\varphi_\alpha}\}_{\alpha\in I}$ is simultaneously triangularizable.*

Proof. By performing a similarity we can assume that $\varphi_\alpha(0) = 0$ for all $\alpha \in I$. Then each C_{φ_α} leaves invariant $z^n\mathbb{H}^2$ for all $n \in \mathbb{N}$. Since $\{z^n\mathbb{H}^2\}_{n=0}^{\infty}$ is a maximal chain, the result follows from the definition. ∎

We start by investigating 2-dimensional invariant subspaces of composition operators.

LEMMA *Let $f \in \mathbb{H}^2$ be a non-constant function. Suppose that $\varphi(a) = a \in \mathbb{D}$ and let $\bigvee\{1, f\} \in \operatorname{Lat} C_\varphi$. Then $f - f(a)$ is an eigenvector of C_φ.*

Proof. Let $g = f - f(a)$. Note that g is non-constant. Since $\mathcal{M} = \bigvee\{1, f\} \in \operatorname{Lat} C_\varphi$, we have that $g \in \mathcal{M}$ and

$$C_\varphi g = \alpha + \beta g \quad \text{for some } \alpha, \beta \in \mathbb{C}.$$

Since $g(a) = 0$, we have

$$g(\varphi(a)) = \alpha + \beta g(a), \quad g(a) = \alpha + \beta g(a), \quad \alpha = 0.$$

So $f - f(a)$ is an eigenfunction of C_φ. ∎

Note that if σ is an eigenvector of C_φ, then $\bigvee\{1, \sigma\} \in \operatorname{Lat} C_\varphi$. For example, let $\varphi(z) = \frac{z}{4.5z+10}$. Then $\sigma(z) = \frac{2z}{z+2}$ and $\bigvee\{1, \sigma\} \in \operatorname{Lat} C_\varphi$.

THEOREM *Suppose that $\bigvee\{1, f\}$ is a common 2-dimensional invariant subspace for C_φ and C_ψ and that φ and ψ have fixed points in $\mathbb{D}$. Then C_φ commutes with C_ψ if and only if φ and ψ have the same fixed point.*

Proof. Since φ and ψ have fixed points in $\mathbb{D}$, if they commute, then by an iteration argument it easily follows that they have the same fixed point.

Now assume that φ and ψ have the same fixed point $a \in \mathbb{D}$. It follows from Lemma 4.3 that $f - f(a)$ is a non-constant common eigenvector for C_φ and C_ψ. Hence, C_φ and C_ψ commute by Proposition 1.4. ∎

THEOREM *Suppose that $\bigvee\{1, f\} \in \operatorname{Lat} C_\varphi$ is 2-dimensional and that $\varphi(0) = 0$. If f is univalent, then $\varphi(z) = z^m h$ for some $m \in \mathbb{N}$ where h is a non-vanishing analytic function.*

Proof. We first prove the following:

CLAIM: Let ψ be an analytic self-map of the unit disk such that $\mathcal{M} = \bigvee\{1, g\}$ is a 2-dimensional invariant subspace for C_ψ for some $g \in \mathbb{H}^2$. If $\psi(0) = \psi(b) = 0$ where $b \in \mathbb{D}$ is non-zero, then $g(b) = g(0)$.

Let $\widehat{g} = g - g(0)$. Then by Lemma 4.3 we obtain that $\widehat{g}$ is an eigenvector of C_ψ. So we have

$$C_\psi \widehat{g} = \beta\widehat{g}.$$

Hence, evaluating at b we get

$$\widehat{g}(\psi(b)) = \widehat{g}(0) = \beta\widehat{g}(b).$$

Since $\widehat{g}(0) = 0$, we have that either $\widehat{g}(b) = 0$ or $\beta = 0$. However, β cannot be zero, since then $C_\psi \widehat{g} = 0$, a contradiction to C_ψ being $1-1$. Hence, we must have $g(b) - g(0) = \widehat{g}(b) = 0$. This proves the claim.

Since $\varphi \in \mathbb{H}^2$, we can write $\varphi = Bh$ where B is a Blaschke product that vanishes at all the zeros of φ, say $\{a_n\}$, and h is non-vanishing. Since $\bigvee\{1, f\} \in \operatorname{Lat} C_\varphi$ and $\varphi(a_n) = 0 = \varphi(0)$, by the above claim it follows that $f(a_n) = f(0)$. Because f is univalent, we conclude that $a_n = 0$ for every $n \in \mathbb{N}$. Hence, $\varphi(z) = z^m h$ for some $m \in \mathbb{N}$. ∎

COROLLARY *Let $\varphi(0) = 0$ and φ have zeros that accumulate at some point $a \in \partial\mathbb{D}$. Then there does not exist a 2-dimensional invariant subspace of the form $\bigvee\{1, f\}$ for C_φ where f is analytic at a.*

Proof. Let $\{a_n\}$ be the non-zero zeros of φ that accumulate at the point $a \in \partial\mathbb{D}$. By way of contradiction, suppose there exist such subspace $\bigvee\{1, f\}$ that is invariant under C_φ. Then since $\varphi(a_n) = 0 = \varphi(0)$, by the claim in the proof of Theorem 4.5 we have $f(a_n) = f(0)$ for every $n \in \mathbb{N}$. Since the set $\{a_n\}$ accumulates at $a \in \partial\mathbb{D}$ and f is analytic at a, it follows that $f \equiv f(0)$ is a constant. However, this contradicts the assumption that $\bigvee\{1, f\}$ is 2-dimensional. ∎

PROPOSITION *Let φ be an analytic self-map of the unit disk with a fixed point $a \in \mathbb{D}$. If $\varphi'(a) = 0$, then C_φ has no non-trivial finite dimensional invariant subspace other than the constants.*

Proof. Without loss of generality, assume that $\varphi(0) = 0$ and let $\mathcal{M}$ be a non-trivial finite dimensional invariant subspace for C_φ other than the constants. First, we show that $\mathcal{M}$ must contain the constants. Since $\mathcal{M}$ is finite dimensional, $C_\varphi|\mathcal{M}$ has an eigenvector. Note that since $\varphi(0) = 0$, the point spectrum of C_φ is contained in $\{1\} \cup \{\varphi'(0)^n\}_{n=1}^{\infty}$ and that the eigenvector corresponding to eigenvalue 1 is the constant 1 ([3], p. 78). However, since $\varphi'(0) = 0$, C_φ has no non-constant eigenvector. Since the eigenvector corresponding to eigenvalue 1 is the constant 1, the only eigenvector of C_φ which is 1 is in $\mathcal{M}$. Hence, constants are contained in $\mathcal{M}$. Now since $\varphi(0) = 0$, $\mathbb{C}$ is reducing for C_φ, $\mathcal{N} = \mathcal{M} \cap \mathbb{C}^\perp$ is a finite dimensional invariant subspace for C_φ. So, $C_\varphi|\mathcal{N}$ has an eigenvalue other than 1. This yields a contradiction since the point spectrum of C_φ is $\{1\}$. ∎

Suppose C_φ is not normal and that φ has the Denjoy-Wolff point in $\mathbb{D}$. It would be interesting to find whether or not C_φ has any finite dimensional invariant subspaces that do not contain the constants or to show that all such subspaces must have a basis consisting of eigenvectors of C_φ.

In [12] there is an example of a 2-dimensional common invariant subspace for elliptic and hyperbolic composition operators that does not contain the constants.

THEOREM *Suppose φ is not an elliptic disk automorphism. If $\varphi(a) = a \in \mathbb{D}$ and a 2-dimensional invariant subspace $\mathcal{M}$ of C_φ^* contains a kernel function k_b, then $a = b$ or $\varphi(b) = a$.*

Proof. Since $\varphi(a) = a \in \mathbb{D}$, by Proposition 1.1 we have that C_φ is power bounded. The sequence $\{C_\varphi^{n} k_b\}_{n=0}^{\infty}$ is a bounded sequence that converges uniformly on compact subsets to k_a. Hence, $k_a \in \mathcal{M}$. If $b \neq a$, then k_a and k_b form a basis for $\mathcal{M}$. It follows that $k_{\varphi(b)}$ must be k_a. Otherwise, the 2-dimensional subspace $\mathcal{M}$ would contain the three linearly independent vectors k_a, k_b, and $k_{\varphi(b)}$. ∎

Note that there are examples of composition operators satisfying the conditions of Theorem 4.8. For example, let $\varphi(z) = \frac{z}{5}(z^3 + z^2 + z + 1)$. Then $\bigvee\{k_0, z\} \in \operatorname{Lat} C_\varphi^*$ is a 2-dimensional invariant subspace for C_φ^* where $C_\varphi^* z = \frac{1}{5}z$.

The following result implies that if every linear manifold invariant under C_φ^* is also invariant under C_ψ^*, then $\psi = \varphi^{(n)}$ for some $n \in \mathbb{N}$.

THEOREM *Let φ and ψ be analytic self-maps of the unit disk. Suppose that the linear span of $\{k_{\varphi^{(n)}(z_\alpha)}\}$ is invariant under C_ψ^* for each z_α where $\{z_\alpha\} \subseteq \mathbb{D}$ is an uncountable collection of points. Then $\psi = \varphi^{(m)}$ for some $m \in \mathbb{N}$.*

Proof. By assumption

$$k_{\psi(z_\alpha)} = C_\psi^* k_{z_\alpha} \in \operatorname{span}\{k_{\varphi^{(n)}(z_\alpha)}\}.$$

So we have that

$$k_{\psi(z_\alpha)} = \sum_{i=1}^{N} \gamma_i k_{\varphi^{(n_i)}(z_\alpha)}.$$

Since an evaluation kernel cannot be written as a non-trivial linear combination of other evaluation kernels, we get that $k_{\psi(z_\alpha)} = k_{\varphi^{(j_\alpha)}(z_\alpha)}$ for some $j_\alpha \in \mathbb{N}$. Since $\{z_\alpha\} \subseteq \mathbb{D}$ is uncountable and $\mathbb{N}$ is countable, there exist $m \in \mathbb{N}$ such that $\psi(z_\beta) = \varphi^{(m)}(z_\beta)$ for some fixed $m \in \mathbb{N}$ and an uncountable sub-collection of points $\{z_\beta\} \subseteq \{z_\alpha\}$. Since ψ and φ are analytic we conclude that $\psi = \varphi^{(m)}$. ∎

THEOREM *Let φ be an analytic self-map of unit disk with fixed point $a \in \mathbb{D}$ that is not an elliptic disk automorphism. If $\mathcal{M}$ is an invariant subspace for C_φ, then either $\mathcal{M}$ contains the constants or every function in $\mathcal{M}$ vanishes at $a \in \mathbb{D}$.*

Proof. Suppose that $\mathcal{M}$ does not contain the constants. Let $f \in \mathcal{M}$. Since $\varphi(a) = a \in \mathbb{D}$, it follows that C_φ is power bounded by Proposition 1.1. So $\{C_\varphi^n f\}$ is a bounded sequence. Moreover, since $\{C_\varphi^n f\}$ converges to $f(a)$ uniformly on compact subsets of $\mathbb{D}$, we conclude that $\{C_\varphi^n f\}$ converges to $f(a)$ weakly. Since $\mathcal{M}$ is a subspace, we get that $f(a) \in \mathcal{M}$. Since $\mathcal{M}$ does not contain the constants, we must have $f(a) = 0$. The result follows since f was arbitrary. ∎

COROLLARY *Let φ be an analytic self-map of the unit disk that is not an elliptic disk automorphism. Assume $\varphi(0) = 0$ and $\varphi = Bg$ where B is a Blaschke product and g is a non-vanishing analytic function. If $\mathcal{M}$ is an invariant subspace for C_φ that does not contain the constants, then $C_\varphi \mathcal{M} \subseteq B\mathbb{H}^2$.*

Proof. Since $\varphi(0) = 0$ and $\mathcal{M}$ does not include the constants, by Theorem 4.10 we must have that $f(0) = 0$ for any $f \in \mathcal{M}$. Let $\{a_i\}$ be the zeros of φ. Then

$$(C_\varphi f)(a_i) = f(\varphi(a_i)) = f(0) = 0.$$

So for any $f \in \mathcal{M}$ we have $C_\varphi f \in B\mathbb{H}^2$. ∎

Using Theorem 4.10 we can also conclude that if φ is not an elliptic disk automorphism and $\varphi(a) = a \in \mathbb{D}$, then for any outer function g, $1 \in \bigvee_{n=0}^{\infty} \{C_\varphi^n g\}$. This happens because if not, then for any $f \in \bigvee_{n=0}^{\infty} \{C_\varphi^n g\}$ we would have $f(a) = 0$. In particular, $g(a) = 0$ which is a contradiction to g being outer.

THEOREM *Suppose that φ and ψ have fixed points a and b respectively in $\mathbb{D}$ and that they are not elliptic disk automorphisms. Let $\mathcal{M}$ be a common invariant subspace for C_φ and C_ψ not containing the constants. Then*

$$\mathcal{M} \subseteq (z - \gamma(b))\mathbb{H}^2 \cap (z - \gamma(a))\mathbb{H}^2,$$

where γ is any map in the semi-group generated by φ and ψ. In particular, if $\mathcal{M}$ is non-zero, then $\{\varphi^{(i)}(\gamma(a))\}$, $\{\varphi^{(i)}(\gamma(b))\}$, $\{\psi^{(i)}(\gamma(a))\}$, and $\{\psi^{(i)}(\gamma(b))\}$ must be finite.

Proof. Since $\varphi(a) = a$ and $\psi(b) = b$ are in $\mathbb{D}$ and $\mathcal{M}$ does not contain the constants, by Theorem 4.10 we conclude that $\{k_a, k_b\} \subseteq \mathcal{M}^\perp$. Let γ be any map in the semi-group generated by φ and ψ. Since $C_\varphi C_\psi = C_{\psi\circ\varphi}$ for any pair of composition operators and $\mathcal{M}^\perp$ is invariant under C_φ^* and C_ψ^*, it will be invariant under C_γ^*. Hence, Proposition 1.3 will yield that

$$\bigvee\{k_{\gamma^{(n)}(b)}, k_{\gamma^{(n)}(a)}\} \subseteq \mathcal{M}^\perp.$$

So for every function $f \in \mathcal{M}$ and every $n \geqslant 0$ we have

$$f(\gamma^{(n)}(b)) = \langle f, k_{\gamma^{(n)}(b)}\rangle = 0,$$
$$f(\gamma^{(n)}(a)) = \langle f, k_{\gamma^{(n)}(a)}\rangle = 0.$$

Hence the result follows if we let $n = 1$.

We now prove that if $\mathcal{M}$ is non-zero, then $\{\psi^{(n)}(\gamma(a))\}$ is finite. By way of contradiction, assume that $\{\psi^{(n)}(\gamma(a))\}$ is infinite. Let $f \in \mathcal{M}$. By the Denjoy-Wolff Theorem, $\{\psi^{(n)}(\gamma(a))\}$ accumulates at the point b. Since every function in $\mathcal{M}$ vanishes at $\{\psi^{(n)}(\gamma(a))\}$ by the preceding argument, it follows that f vanishes on a set with an accumulation point $b \in \mathbb{D}$. So f must be zero. A similar argument can be given for the other cases. ∎

Note that if φ and ψ have fixed points in $\mathbb{D}$, their composition may not necessary have a fixed point in $\mathbb{D}$. For instance, let $\varphi(z) = \frac{-z-1/2}{1+z/2}$ and $\psi(z) = -z$, where both have fixed points in $\mathbb{D}$. However, $\varphi \circ \psi(z) = \frac{z+1/2}{1-z/2}$ does not have a fixed point in $\mathbb{D}$.

COROLLARY *Suppose that C_φ is compact and ψ has a fixed point in $\mathbb{D}$ and is not an elliptic disk automorphism. If $\mathcal{M}$ is a common invariant subspace for C_φ and C_ψ that does not contain the constants, then every function in $\mathcal{M}$ vanishes at $\gamma(\beta)$ where γ is any map in the semi-group generated by φ and ψ and β is the Denjoy-Wolff point of any map in the semi-group generated by φ and ψ.*

Proof. Let γ and β be as above and let σ be the map with fixed point β. Note that for any pair of composition operators C_τ and C_ξ we have $C_\tau C_\xi = C_{\xi\circ\tau}$.

CLAIM: $\beta \in \mathbb{D}$.

If the map φ is needed to generate the map σ, then by the above observation C_σ is compact and so σ has a fixed point $\beta \in \mathbb{D}$. If the map φ is not used in generating σ then $\sigma = \psi^{(n)}$ for some $n \in \mathbb{N}$. Hence, σ has the same fixed point as ψ in $\mathbb{D}$. This proves the claim.

We now consider two cases.

Case 1. The map φ is used to generate γ.

Then by the earlier observation, we have that C_γ is compact. Since $\mathcal{M}$ is invariant under C_φ and C_ψ, it is invariant under C_γ and C_σ. Hence, by using Theorem 4.10, it follows that $k_\beta \in \mathcal{M}^\perp$. Since $\mathcal{M}^\perp$ is invariant under C_γ^* we have $C_\gamma^* k_\beta = k_{\gamma(\beta)} \in \mathcal{M}^\perp$. So every function in $\mathcal{M}$ vanishes at $\gamma(\beta)$.

Case 2. The map φ is not used to generate γ.

Then $\gamma = \psi^{(n)}$ for some $n \in \mathbb{N}$. Since $\mathcal{M}$ is invariant under C_ψ and C_σ and $\sigma(\beta) = \beta \in \mathbb{D}$, by Theorem 4.10 we have that $k_\beta \in \mathcal{M}^\perp$. Hence, we have $C_\gamma^* k_\beta = k_{\gamma(\beta)} \in \mathcal{M}^\perp$. So every function in $\mathcal{M}$ vanishes at $\gamma(\beta)$. ∎

PROPOSITION *Suppose that φ and ψ have fixed points in $\mathbb{D}$ and are not elliptic disk automorphisms. Let $\mathcal{M}$ be a common invariant subspace for C_φ and C_ψ that does not contain the constants. If $\mathcal{M}$ contains a function that vanishes at most at one point, then φ and ψ have the same fixed point.*

Proof. Let a and b be the Denjoy-Wolff points of φ and ψ in $\mathbb{D}$ respectively. Let $f \in \mathcal{M}$ be a function that vanishes at most at one point. Since φ and ψ have fixed points in $\mathbb{D}$ and $\mathcal{M}$ does not contain the constants, Theorem 4.10 yields that every function in $\mathcal{M}$ vanishes at a and b. So $f(a) = 0 = f(b)$. The result follows from the assumption on f. ∎

If f is $1-1$, then f satisfies the assumptions of the previous Proposition 4.14. A condition weaker than f being $1-1$ is f being $1-1$ at a point.

DEFINITION Let $\lambda \in \mathbb{D}$. A function f is said to be $1-1$ *at* λ if whenever $f(\lambda) = f(z)$ for some $z \in \mathbb{D}$, then $z = \lambda$.

THEOREM *Let φ and ψ be analytic self-maps of the unit disk that are not elliptic disk automorphisms with fixed points a and b, respectively, in $\mathbb{D}$. Suppose $\mathcal{M}$ is a common non-zero invariant subspace for C_φ and C_ψ that does not contain the constants. If φ is $1-1$ at a, then $a = b$.*

Proof. First assume that φ is $1-1$ at a. Since $\mathcal{M}$ is a non-zero invariant subspace for C_φ and C_ψ that does not contain the constants, by Theorem 4.12 it follows that $\{\varphi^{(m)}(b)\}$ is finite. So we have that $\varphi^{(m)}(b) = \varphi^{(n)}(b)$ for some distinct $m, n \in \mathbb{N}$. Without loss of generality assume that $m < n$. So $\varphi^{(m)}(b)$ is fixed by $\varphi^{(n-m)}$. Since $\varphi^{(n-m)}$ fixes $a \in \mathbb{D}$ and it cannot have two distinct fixed points in $\mathbb{D}$, it follows that $\varphi^{(m)}(b) = a = \varphi(a)$. Since φ is univalent at a, we get $\varphi^{(m-1)}(b) = a$. By repeated applications of this argument, we get that $a = b$. ∎

THEOREM *Suppose that C_φ and C_ψ have a common invariant subspace $\mathcal{M}$ that does not contain the constants. If φ, not an elliptic disk automorphism, has the Denjoy-Wolff point $a \in \mathbb{D}$ and ψ has the Denjoy-Wolff point on $\partial\mathbb{D}$, then $\mathcal{M} \subseteq B\mathbb{H}^2$ where B is the Blaschke product formed with zeros $\{\psi^{(n)}(a)\}$.*

Proof. Since φ has the Denjoy-Wolff point $a \in \mathbb{D}$ and $\mathcal{M}$ does not contain the constants, by Theorem 4.10 we conclude that $k_a \in \mathcal{M}^{\perp}$. Since $\mathcal{M}^{\perp} \in \operatorname{Lat} C_{\psi}^{*}$ we have that

$$\bigvee_{n=0}^{\infty}\{C_{\psi}^{*^{n}} k_a\} = \bigvee_{n=0}^{\infty}\{k_{\psi^{(n)}(a)}\} \subseteq \mathcal{M}^{\perp}.$$

Hence for any $f \in \mathcal{M}$ we have

$$f(\psi^{(n)}(a)) = \langle f, k_{\psi^{(n)}(a)}\rangle = 0.$$

This shows that $f = Bg$ for some $g \in \mathbb{H}^2$ where

$$B = \prod_{n=0}^{\infty} \frac{\psi^{(n)}(a)}{|\psi^{(n)}(a)|}\frac{\psi^{(n)}(a) - z}{1 - \overline{\psi^{(n)}(a)}z}.$$

So we have that $\mathcal{M} \subseteq B\mathbb{H}^2$. ∎

Theorem *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that* $\operatorname{Lat} C_{\varphi} \subseteq \operatorname{Lat} C_{\psi}$ *and that $\varphi(a) = a \in \mathbb{D}$. Then*

$$\{z : \varphi(z) = a\} \subseteq \{z : \psi(z) = a\}.$$

In particular, if φ fixes zero, then ψ vanishes at all the zeros of φ.

Proof. Suppose that $\varphi(\lambda) = a$ for some $\lambda \in \mathbb{D}$. Since $\operatorname{Lat} C_{\varphi} \subseteq \operatorname{Lat} C_{\psi}$ and $\varphi(a) = a \in \mathbb{D}$, Theorem 2.7 yields that $\psi(a) = a$. Moreover, by Proposition 1.3 we have that

$$\bigvee\{k_a, k_{\lambda}\} \in \operatorname{Lat} C_{\varphi}^{*} \subseteq \operatorname{Lat} C_{\psi}^{*}.$$

So we have

$$C_{\psi}^{*} k_{\lambda} = k_{\psi(\lambda)} = \alpha k_a + \beta k_{\lambda} \quad \text{for some } \alpha, \beta \in \mathbb{C}.$$

Since an evaluation kernel can not be written as a non-trivial linear combination of other evaluation kernels, it follows that $\psi(\lambda) = a$ or $\psi(\lambda) = \lambda$. If $\psi(\lambda) = a$ then we are done. If $\psi(\lambda) = \lambda$ then $\lambda = a$ since ψ can have only one fixed point in $\mathbb{D}$. Hence the result follows. ∎

The inner-outer decomposition for functions in $\mathbb{H}^2$, together with the above theorem, implies that if $\varphi(0) = 0$ and $\operatorname{Lat} C_{\varphi} \subseteq \operatorname{Lat} C_{\psi}$, then the Blaschke product that appears in φ as part of its inner-outer factorization also appears as a part of the Blaschke product of decomposition of ψ.

Theorem *Let φ and ψ be Blaschke products such that $\varphi(0) = 0$. If* $\operatorname{Lat} C_{\varphi} = \operatorname{Lat} C_{\psi}$, *then $\varphi = \mathrm{e}^{\mathrm{i}\theta}\psi$ for some real number θ.*

Proof. If $\operatorname{Lat} C_{\varphi} = \operatorname{Lat} C_{\psi}$, then by Theorem 2.7 we have $\psi(0) = 0$ since $\varphi(0) = 0$. Moreover, by Theorem 4.18, we have $\varphi = \psi g$ and $\psi = \varphi h$ for some $g, h \in \mathbb{H}^2$. Since φ and ψ are Blaschke products, g and h must also be Blaschke products. Hence, $\varphi = \psi g = \varphi h g$. It follows that $gh = 1$. Since g and h are Blaschke products, it follows that they must be constants. Hence, $\varphi = \mathrm{e}^{\mathrm{i}\theta}\psi$ for some real number θ. ∎

PROPOSITION *Let φ be a non-constant analytic self-map of the unit disk that is not an elliptic disk automorphism. Then for all but countably many $z \in \mathbb{D}$, the collection $\{\varphi^{(n)}(z)\}$ is infinite.*

Proof. Let a be the Denjoy-Wolff point of φ. If $a \in \partial\mathbb{D}$, then $\varphi^{(n)}(z) \neq a$ for any $n \in \mathbb{N}$ and any $z \in \mathbb{D}$. So the result follows immediately.

Now assume that $a \in \mathbb{D}$ and that $\mathcal{A} = \{z_\alpha\}$ is an uncountable collection of points each of which has finite orbit under φ. Then for each z_α there exists $n_\alpha \in \mathbb{N}$ such that $\varphi^{(n_\alpha)}(z_\alpha) = a$. Since $\mathcal{A}$ is uncountable and $\mathbb{N}$ is countable, there exists $m \in \mathbb{N}$ such that $\varphi^{(m)}(z_\beta) = a$ where $\{z_\beta\}$ is an uncountable sub-collection of $\{z_\alpha\}$. Since φ is analytic, we conclude that $\varphi^{(m)} \equiv a$ is a constant. By an application of the open mapping theorem, we conclude that φ is a constant. This yields a contradiction. ∎

H. Heidler in [6] characterizes algebraic composition operators on a variety of spaces. The following is an alternative way of showing the non-existence of non-trivial algebraic composition operators on $\mathbb{H}^2$, by using their invariant subspaces. Note that if $\varphi \equiv a$ is constant, then C_φ is algebraic. For example, the polynomial $p(z) = z^2 - z$ is satisfied by C_a.

COROLLARY *Let φ be a non-constant analytic self-map of the unit disk that is not an elliptic disk automorphism. Then C_φ is not algebraic.*

Proof. If C_φ is algebraic, then so is C_φ^*. Hence, all cyclic subspaces of C_φ^* are finite dimensional. However, for any $z \in \mathbb{D}$

$$\mathcal{M}_z \equiv \bigvee_{n=0}^{\infty} \{C_\varphi^{n^*} k_z\} = \bigvee_{n=0}^{\infty} \{k_{\varphi^{(n)}(z)}\} \in \operatorname{Lat} C_\varphi^*.$$

Since $\mathcal{M}_z$ is finite dimensional and $\{k_{\varphi^{(n)}(z)}\}$ are linearly independent for distinct $\varphi^{(n)}(z)$, it follows that for each $z \in \mathbb{D}$, the orbit $\{\varphi^{(n)}(z)\}$ is finite. However, this contradicts Proposition 4.20. Hence, the result follows. ∎

Recall that an operator is called *reductive* if every invariant subspace for the operator is reducing. The following determines the reductive composition operators.

PROPOSITION *A composition operator is reductive if and only if $\varphi(z) = \alpha z$ for some constant α where $|\alpha| \leqslant 1$.*

Proof. First assume that C_φ is reductive. Since $C_\varphi 1 = 1$, it follows that $\mathbb{C}$ is reducing. This implies that $\varphi(0) = 0$ and so

$$z^2\mathbb{H}^2 \in \operatorname{Lat} C_\varphi \subseteq \operatorname{Lat} C_\varphi^*.$$

It follows that

$$\bigvee\{1, z\} = (z^2\mathbb{H}^2)^\perp \in \operatorname{Lat} C_\varphi.$$

So we have

$$C_\varphi z = \varphi = \alpha z + \beta \quad \text{for some } \alpha, \beta \in \mathbb{C}.$$

Since $\varphi(0) = 0$, we conclude that $\beta = 0$ and so $\varphi(z) = \alpha z$ where $|\alpha| \leqslant 1$.

Conversely, assume that $\varphi(z) = \alpha z$. If $|\alpha| < 1$, then $C_{\alpha z}$ is a compact normal operator and hence every invariant subspace of $C_{\alpha z}$ contains a spanning set of eigenvectors. Thus, by the work of Wermer ([20]), it follows that $C_{\alpha z}$ is reductive. If $\varphi(z) = \mathrm{e}^{\mathrm{i}\theta} z$ for some real number θ, then $C_{\mathrm{e}^{\mathrm{i}\theta} z}$ is a unitary operator whose eigenvectors form a basis, namely $\{z^n\}_0^\infty$. So, there exists a sequence of polynomials $p_n(z)$ such that $p_n(C_{\mathrm{e}^{\mathrm{i}\theta} z})$ converge strongly to $C^*_{\mathrm{e}^{\mathrm{i}\theta} z}$ ([20]). Hence $C_{\mathrm{e}^{\mathrm{i}\theta} z}$ is reductive. ∎

PROPOSITION *Let φ be an analytic self-map of the unit disk with the Denjoy-Wolff point $a \in \mathbb{D}$ such that $0 \neq \varphi'(a)$. Then the commutant of C_φ is reflexive.*

Proof. Without loss of generality, assume that $\varphi(0) = 0$. First assume that $0 < |\varphi'(0)| < 1$. Then C^*_φ is upper triangular with respect to the basis $\{z^n\}$. Hence, C^*_φ has eigenvectors $f_n \in \mathcal{M}_n \equiv \bigvee_{i=0}^{n} \{z^i\}$ with the corresponding eigenvalues $\overline{\varphi'(0)}^{\,n}$. Since $0 < |\varphi'(0)| < 1$, $\{\varphi'(0)^n\}$ are all distinct and so the eigenvectors f_n corresponding to distinct eigenvalues are linearly independent. So $\bigvee_{n=0}^{\infty} \{f_n\} = \mathbb{H}^2$. Hence, C^*_φ has a spanning collection of eigenvectors. So C^*_φ and also C_φ are hyper-reflexive ([5]).

If $|\varphi'(0)| = 1$, then C_φ is normal and so its commutant is reflexive ([22]). ∎

REMARK A characterization of compact composition operators on $\mathbb{H}^2$ is the following ([3]):

C_φ is compact if and only if whenever $\{f_n\}$ is bounded and f_n converges uniformly on compact subsets to 0, then $C_\varphi f_n$ converges to 0 in the norm.

Suppose C_φ is compact and $\varphi(a) = a$. Then C_φ is power bounded by Proposition 1.1 and so for any $f \in \mathbb{H}^2$, $\{C^n_\varphi f\}$ is a bounded sequence that converges to $f(a)$ uniformly on compact subsets of $\mathbb{D}$. Since C_φ is compact, the above characterization implies that $\{C^n_\varphi f\}$ converges to $f(a)$ in norm.

Let $\mathcal{M}_0 \subseteq \mathcal{M}_1$ be subspaces of $\mathbb{H}^2$. If $f \in \mathcal{M}_1$, then denote by $\widehat{f}$ the equivalence class of f in $\mathcal{M}_1/\mathcal{M}_0$. In the case of Hilbert spaces we can identify the quotient $\mathcal{M}_1/\mathcal{M}_0$ with the orthogonal complement of $\mathcal{M}_0$ in $\mathcal{M}_1$. Also, denote by $C_\varphi|\mathcal{M}_1/\mathcal{M}_0$ the compression of C_φ to the orthogonal complement of $\mathcal{M}_0$ in $\mathcal{M}_1$.

THEOREM *Let C_φ be power compact. Suppose C_φ has a continuous chain of invariant subspaces $\{\mathcal{M}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ such that $\mathcal{M}_0$ does not include the constants. Then $f(a) = 0$ for any $f \in \mathcal{M}_1$, where a is the Denjoy-Wolff point of φ.*

Proof. Let C^N_φ be compact for some $N \in \mathbb{N}$ and suppose there exists $f \in \mathcal{M}_1$ such that $f(a) \neq 0$. Since $C^{nN}_\varphi f \to f(a)$ in norm by the above remark, we have $f(a) \in \mathcal{M}_1$. Hence, $1 \in \mathcal{M}_1$. Since $1 \notin \mathcal{M}_0$, it follows that $C_{\varphi^{(N)}}|\mathcal{M}_1/\mathcal{M}_0$ has eigenvalue 1 corresponding to eigenfunction $\widehat{1} \in \mathcal{M}_1/\mathcal{M}_0$.

On the other hand, since $\{\mathcal{M}_\alpha\}$ is a continuous chain of invariant subspaces, $C_{\varphi^{(N)}}\big|\mathcal{M}_1/\mathcal{M}_0$ is a compact operator with a maximal continuous chain of invariant subspaces. Hence, $C_{\varphi^{(N)}}\big|\mathcal{M}_1/\mathcal{M}_0$ is quasi-nilpotent. This yields a contradiction to $1 \in \sigma(C_{\varphi^{(N)}}\big|\mathcal{M}_1/\mathcal{M}_0)$. ∎

PROPOSITION *Let C_φ^N be power compact and σ a non-constant eigenvector of C_φ^N. If $\bigvee_{n=0}^{\infty}\{\sigma^n\}$ has finite codimension, then there does not exist a continuous chain $\{\mathcal{M}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ of invariant subspaces for C_φ such that $\mathcal{M}_1 = \mathbb{H}^2$.*

Proof. Since σ is a non-constant eigenfunction of C_φ^N, we have that $C_\varphi^N\sigma = \varphi'(a)^{Nm}\sigma$ for some $m \in \mathbb{N}$. Note that since C_φ is $1-1$, $\varphi'(a) \not= 0$. By way of contradiction, suppose that there exists such a continuous chain of invariant subspaces for C_φ.

CLAIM: $\sigma^n \in \mathcal{M}_0$ for all $n \in \mathbb{N}$.

Suppose $\sigma^j \notin \mathcal{M}_0$ for some $j \in \mathbb{N}$. Then $C_\varphi^N\big|(\mathbb{H}^2/\mathcal{M}_0)$ has a non-zero eigenvalue, $\varphi'(a)^j$, corresponding to $\widehat{\sigma}^j \in \mathbb{H}^2/\mathcal{M}_0$. However, since $C_\varphi^N\big|(\mathbb{H}^2/\mathcal{M}_0)$ is compact and $\{\mathcal{M}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ is a continuous maximal chain of invariant subspaces, $C_\varphi^N\big|(\mathbb{H}^2/\mathcal{M}_0)$ is quasi-nilpotent which contradicts $C_\varphi^N\big|(\mathbb{H}^2/\mathcal{M}_0)$ having a non-zero eigenvalue. This proves the claim.

By using the claim, we obtain that $\bigvee_{n=0}^{\infty}\{\sigma^n\} \subseteq \mathcal{M}_0$. Since $\bigvee_{n=0}^{\infty}\{\sigma^n\}$ has finite co-dimension, it follows that $\mathcal{M}_0^\perp$ has finite dimension.

However, since $\{\mathcal{M}_\alpha^\perp\}_{0\leqslant\alpha\leqslant 1}$ form a non-trivial increasing continuous chain of invariant subspaces for C_φ^*, it is not possible for $\mathcal{M}_0^\perp$ to have finite dimension. Hence, the result follows. ∎

THEOREM *Suppose C_φ is power compact and $0 < |\varphi'(a)| < 1$, where a is the Denjoy-Wolff point of φ. Then there does not exist a non-trivial continuous chain of invariant subspaces for C_φ starting from 0.*

Proof. Let C_φ^N be compact for some $N \in \mathbb{N}$. Since C_φ^N is compact, $\varphi^{(N)}$ has no finite angular derivative at any point in $\partial\mathbb{D}$ ([3], p. 132). Also, since C_φ^N is compact, $\varphi^{(N)}$ is neither the identity nor an elliptic disk automorphism since these operators are invertible. Hence, by Theorem 1.2 and Julia-Caratheodory Theorem ([3], p. 51) it follows that $\varphi^{(N)}$ and so φ must have the Denjoy-Wolff point in $\mathbb{D}$. Without loss of generality, assume that $\varphi(0) = 0$. By considering orthogonal components, we will prove the equivalent formulation of the theorem for C_φ^*. By way of contradiction, suppose that $\{\mathcal{M}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ is a continuous chain of invariant subspaces for C_φ^* such that $\mathcal{M}_1 = \mathbb{H}^2$. Denote by f_n the eigenvector satisfying

$$C^*_{\varphi^{(N)}} f_n = \overline{\varphi'(0)}^{\,nN} f_n \qquad \text{where } f_n \in \bigvee_{i=0}^{n}\{z^i\}.$$

Note that $\bigvee_{n=0}^{\infty}\{f_n\} = \mathbb{H}^2$ since $0 < |\varphi'(a)| < 1$.

Since C_φ^N is compact and $C^*_{\varphi^{(N)}}|(\mathcal{M}_1/\mathcal{M}_0)$ has a maximal chain of invariant subspaces, $C^*_{\varphi^{(N)}}|(\mathcal{M}_1/\mathcal{M}_0)$ is quasi-nilpotent.

CLAIM: Every eigenvector f_n is in $\mathcal{M}_0$.

Assume that some $f_m \notin \mathcal{M}_0$ for some $m \in \mathbb{N}$. Then since $\mathcal{M}_1 = \mathbb{H}^2$, $\widehat{f_m} \in \mathcal{M}_1/\mathcal{M}_0$ is an eigenvector of $C^*_{\varphi^{(N)}}|(\mathcal{M}_1/\mathcal{M}_0)$ that corresponds to a non-zero eigenvalue $\overline{\varphi'(0)}^{mN}$. This contradicts the fact that $C^*_{\varphi^{(N)}}|(\mathcal{M}_1/\mathcal{M}_0)$ is quasi-nilpotent. Hence, $f_n \in \mathcal{M}_0$ for every $n \in \mathbb{N}$.

So $\mathbb{H}^2 = \bigvee_{n=0}^{\infty} \{f_n\} \subseteq \mathcal{M}_0$. Hence, $\{\mathcal{M}_\alpha\}_{0 \leqslant \alpha \leqslant 1}$ is a trivial chain. This yields a contradiction. ∎

THEOREM *Suppose C_φ is power compact and $\varphi'(a) \neq 0$, where a is the Denjoy-Wolff point of φ. Then there does not exist a non-trivial continuous chain of invariant subspaces for C_φ starting from $\mathbb{C}$.*

Proof. Let C_φ^N be compact for some $N \in \mathbb{N}$. Then $\varphi^{(N)}$ and so φ must have the Denjoy-Wolff point in $\mathbb{D}$. Without loss of generality, assume that $\varphi(0) = 0$. We will prove the equivalent of this theorem for C_φ^*. By way of contradiction, suppose that $\{\mathcal{M}_\alpha\}_{0 \leqslant \alpha \leqslant 1}$ is a non-trivial continuous chain of invariant subspaces for C_φ^* such that $\mathcal{M}_1 = z\mathbb{H}^2$. For each $n \geqslant 0$, denote by f_n the eigenvector satisfying

$$C^*_{\varphi^{(N)}} f_n = \overline{\varphi'(0)}^{\,nN} f_n \qquad \text{where } f_n \in \bigvee_{i=0}^{n} \{z^i\}.$$

Note that if $|\varphi'(a)| = 1$, then C_φ would be an elliptic disk automorphism and so cannot be compact. Hence, we have $0 < |\varphi'(a)| < 1$ and so $\bigvee_{n=0}^{\infty} \{f_n\} = \mathbb{H}^2$. Since $f_0 = 1$ is a constant, $\mathbb{C}$ is reducing for C_φ.

CLAIM: $f_n(0) = 0$ for all $n \geqslant 1$.

Let $n \geqslant 1$. Then

$$\overline{\varphi'(0)}^{nN} f_n(0) = \overline{\varphi'(0)}^{nN} \langle f_n, 1\rangle = \langle C_\varphi^{N^*} f_n, 1\rangle = \langle f_n, C_\varphi^N 1\rangle = \langle f_n, 1\rangle = f_n(0).$$

Hence, $f_n(0) = 0$ or $\overline{\varphi'(0)}^{nN} = 1$. Since $0 < |\varphi'(0)| < 1$, $\varphi'(0)^m \neq 1$ for any $m \in \mathbb{N}$. Hence, we have $f_n(0) = 0$ for all $n \geqslant 1$.

Thus, $f_n \in z\mathbb{H}^2$ for all $n \in \mathbb{N}$ and so $\bigvee_{n=1}^{\infty} \{f_n\} = z\mathbb{H}^2$. An argument similar to that of Theorem 4.27 will yield that $f_n \in \mathcal{M}_0$ for all $n > 0$. So $z\mathbb{H}^2 = \bigvee_{n=1}^{\infty} \{f_n\} \subseteq \mathcal{M}_0$. Hence, $\{\mathcal{M}_\alpha\}_{0 \leqslant \alpha \leqslant 1}$ is a trivial chain. This yields a contradiction. ∎

COROLLARY *Suppose C_φ is power compact and $\varphi'(a) \neq 0$, where a is the Denjoy-Wolff point of φ. If $\{\mathcal{M}_\alpha\}_{0 \leqslant \alpha \leqslant 1}$ is a non-trivial continuous chain of invariant subspaces for C_φ, where $\mathcal{M}_1 = \mathbb{H}^2$, then $\bigvee_{n=0}^{\infty} \{\sigma^n\} \subseteq \mathcal{M}_0$, where σ is a non-constant eigenvector of C_φ corresponding to the eigenvalue $\varphi'(a)$.*

Proof. Suppose that $\{\mathcal{M}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ is such a collection of invariant subspaces for C_φ. Assume C_φ^N is compact for some $N \in \mathbb{N}$. Then $C^*_{\varphi^{(N)}}|(\mathcal{M}_1/\mathcal{M}_0)$ is quasinilpotent. Arguing as in Proposition 4.26 will yield that every σ^n must be in $\mathcal{M}_0$. Hence, $\bigvee_{n=0}^{\infty} \{\sigma^n\} \subseteq \mathcal{M}_0$. ∎

If $\varphi(z) = \alpha z$, where $|\alpha| < 1$, then 1 and z are both common eigenfunctions of C_φ and C_ψ with eigenvalues

$$C_\varphi 1 = 1, \quad C_\varphi^* 1 = 1, \quad C_\varphi^* z = \overline{\alpha} z, \quad C_\varphi z = \alpha z.$$

In general, however, not much is known about the characteristics of common eigenfunctions of C_φ and C_ψ. The following proposition proves the lack of these common eigenfunctions under some assumptions.

PROPOSITION *If $\{\varphi^{(n)}(0)\}$ has a limit point in $\mathbb{D}$, then C_φ and C_φ^* do not have a common eigenfunction.*

Proof. Suppose that

$$C_\varphi f = \lambda f, \tag{4.1}$$

$$C_\varphi^* f = \gamma f, \tag{4.2}$$

for some constants $\lambda, \gamma \in \mathbb{C}$ where $\|f\| = 1$. Note that $f \neq 1$ since otherwise $\{\varphi^{(n)}(0)\}$ would not have a limit point in $\mathbb{D}$. Then

$$\lambda = \langle \lambda f, f\rangle = \langle C_\varphi f, f\rangle = \langle f, C_\varphi^* f\rangle = \overline{\gamma}\langle f, f\rangle = \overline{\gamma}.$$

This yields that

$$\overline{\lambda} f(0) = \langle C_\varphi^* f, 1\rangle = \langle f, C_\varphi 1\rangle = \langle f, 1\rangle = f(0).$$

So $f(0) = 0$ or $\lambda = 1$. If $\lambda = 1$, then $C_\varphi f = f$. By repeated applications of C_φ and using the Denjoy-Wolff Theorem, we conclude that f is a constant, which is a contradiction.

Hence we must have that $f(0) = 0$. Then by repeated applications of C_φ to equation 4.1 and evaluating at 0 we obtain that

$$f(\varphi^{(n)}(0)) = C_\varphi^n f(0) = \lambda^n f(0) = 0.$$

So f vanishes on $\{\varphi^{(n)}(0)\}_{n=0}^{\infty}$. Since $\{\varphi^{(n)}(0)\}_{n=0}^{\infty}$ has a limit point in $\mathbb{D}$, f must be zero, which contradicts that f is an eigenvector. ∎

PROPOSITION *Let φ be an analytic self-map of the unit disk. Then φ is univalent if and only if $\overline{C_\varphi \mathbb{H}^2}$ contains a univalent function.*

Proof. Suppose that $\overline{C_\varphi \mathbb{H}^2}$ contains a univalent function f and that $\varphi(z_1) = \varphi(z_2)$ for some $z_1, z_2 \in \mathbb{D}$. Let $\{g_n\} \subseteq \mathbb{H}^2$ be such that $C_\varphi g_n \to f$ in $\mathbb{H}^2$ norm. Thus,

$$(C_\varphi g_n)(z_1) = g_n(\varphi(z_1)) = g_n(\varphi(z_2)) = (C_\varphi g_n)(z_2).$$

Since $C_\varphi g_n \to f$, we conclude that $f(z_1) = f(z_2)$. Hence $z_1 = z_2$ since f is univalent.

Conversely, if φ is univalent, then clearly $\varphi = C_\varphi z \in \overline{C_\varphi \mathbb{H}^2}$. ∎

REMARK The following result can also be proven analogously;
If φ fixes a point $a \in \mathbb{D}$ and $\overline{C_\varphi \mathcal{M}} = \mathcal{M}$ for any invariant subspace $\mathcal{M}$ that does not contain the constants, then φ is univalent.

PROPOSITION *If φ is an analytic self-map of the unit disk such that $\varphi(0) = 0$, then no polynomial can be cyclic for C_φ^*.*

Proof. If $\varphi(0) = 0$, then each subspace $z^n \mathbb{H}^2$ is invariant for C_φ and so $\mathcal{M}_n = \bigvee_{i=0}^{n} \{z^i\}$ is invariant for C_φ^*. Hence, $\mathcal{L} \equiv \bigcup_{n=1}^{\infty} \mathcal{M}_n$ consists of non-cyclic vectors for C_φ^*. Thus no polynomial can be cyclic for C_φ^*. ∎

Given two operators A and B, denote by $[A, B]$ the commutator $AB - BA$ of A and B. Moreover, two operators A and B are said to *quasi-commute* if each commutes with $[A, B]$.

LEMMA *Let φ and ψ be analytic self-maps of the unit disk. Suppose that φ fixes a point in $\mathbb{D}$. If f is a non-constant eigenfunction of C_φ and $f \in \ker[C_\varphi, C_\psi]$, then C_φ commutes with C_ψ.*

Proof. Suppose $C_\varphi f = \lambda f$ for some $\lambda \in \mathbb{D}$. Since $f \in \ker[C_\varphi, C_\psi]$, we have

$$[C_\varphi, C_\psi] f = (C_\varphi C_\psi - C_\psi C_\varphi) f = 0.$$

By using $C_\varphi f = \lambda f$, this yields

$$C_\varphi (f \circ \psi) = \lambda (f \circ \psi).$$

Since f is an eigenfunction of C_φ corresponding to λ, and the eigenspaces of C_φ corresponding to each eigenvalue are 1-dimensional, we obtain that $f \circ \psi = \gamma f$ for some $\gamma \in \mathbb{C}$. So f is a non-constant common eigenvector of C_φ and C_ψ. Since φ has the Denjoy-Wolff point in $\mathbb{D}$, Proposition 1.4 yields that C_φ and C_ψ commute. ∎

THEOREM *Let φ and ψ be non-constant analytic self-maps of $\mathbb{D}$. Suppose that C_φ commutes with $[C_\varphi, C_\psi]$ and that φ fixes a point in $\mathbb{D}$. If C_φ has a non-constant eigenvector in $\mathbb{H}^2$, then C_φ commutes with C_ψ.*

Proof. First we prove that φ and ψ have the same fixed point. Without loss of generality, assume that $\varphi(0) = 0$. Since C_φ commutes with $[C_\varphi, C_\psi]$, we have

$$C_\varphi (C_\varphi C_\psi - C_\psi C_\varphi) = (C_\varphi C_\psi - C_\psi C_\varphi) C_\varphi.$$

Taking the adjoint of the above equation and evaluating at 1 while using $\varphi(0) = 0$ we get

$$\begin{aligned} &(C_\varphi C_\psi - C_\psi C_\varphi)^* C_\varphi^* 1 = C_\varphi^* (C_\varphi C_\psi - C_\psi C_\varphi)^* 1, \\ &k_{\psi(0)} - k_{\varphi(\psi(0))} = k_{\varphi(\psi(0))} - k_{\varphi^{(2)}(\psi(0))}, \\ &2k_{\varphi(\psi(0))} = k_{\varphi^{(2)}(\psi(0))} + k_{\psi(0)}. \end{aligned}$$

Hence, $\varphi(\psi(0)) = \varphi^{(2)}(\psi(0)) = \psi(0)$. This implies that $\psi(0)$ is a fixed point of φ. Since φ cannot have two distinct fixed points in $\mathbb{D}$, it follows that $\psi(0) = 0$.

Suppose that $C_\varphi\sigma = \lambda\sigma$ where $\sigma \in \mathbb{H}^2$ is non-constant. Since C_φ commutes with $[C_\varphi, C_\psi]$ and the eigenspaces of C_φ corresponding to each eigenvalue are one dimensional by Proposition 1.4, we get that

$$[C_\varphi, C_\psi]\sigma = \lambda'\sigma \quad \text{for some } \lambda' \in \mathbb{C}.$$

However, since C_φ commutes with $[C_\varphi, C_\psi]$, we conclude ([8] and [18]) that $[C_\varphi, C_\psi]$ is quasi-nilpotent. Hence, $[C_\varphi, C_\psi]\sigma = 0$. The result now follows from Lemma 4.34. ∎

COROLLARY *Suppose that C_φ and C_ψ quasi-commute and that φ fixes a point in $\mathbb{D}$. If C_φ has a non-constant eigenvector in $\mathbb{H}^2$, then C_φ nad C_ψ commute.*

Proof. This follows immediately from Theorem 4.35. ∎

An operator T is called *unicellular* if the lattice of its invariant subspaces is totally ordered.

THEOREM *If φ has the Denjoy-Wolff point in $\mathbb{D}$, then C_φ is not unicellular. If there exists $\lambda \in \mathbb{D}$ such that $\{\varphi^{(n)}(\lambda)\}$ are the zeros of a Blaschke product, then C_φ is not unicellular. In particular, if φ is in the Half-Plane\Dilation or Half-Plane\Translation case, then C_φ is not unicellular.*

Proof. Suppose that C_φ is unicellular and that $\varphi(a) = a \in \mathbb{D}$. Note that since C_φ leaves the constants invariant, every invariant subspace of C_φ must contain the constants. If $\varphi(a) = a \in \mathbb{D}$, then $\mathbb{C}$ and $(z-a)\mathbb{H}^2$ are invariant under C_φ. This yields a contradiction to C_φ being unicellular.

Now assume that C_φ is unicellular and that there exists λ such that $\{\varphi^{(n)}(\lambda)\}$ are the zeros of a Blaschke product. Then $\mathcal{M} = \bigvee_{n=0}^{\infty} \{k_{\varphi^{(n)}(\lambda)}\}$ is invariant under C_φ^* and so $\mathcal{M}^\perp$ is invariant under C_φ and is a non-trivial invariant subspace. Since C_φ is unicellular, $1 \in \mathcal{M}^\perp$. So $0 = \langle k_\lambda, 1\rangle$. This yields a contradiction. ∎

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