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$(\mathcal{U}+\mathcal{K})$-ORBITS, A BLOCK TRIDIAGONAL DECOMPOSITION TECHNIQUE AND A MODEL WITH MULTIPLY CONNECTED SPECTRUM

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ABSTRACT. Two operators on a separable Hilbert space are $(\mathcal{U}+\mathcal{K})$-*equivalent* ($A \cong_{\mathcal{U}+\mathcal{K}} B$) if $A = R^{-1}BR$, where R is invertible and $R = U+K$, U unitary, K compact. The $(\mathcal{U}+\mathcal{K})$-*orbit* of A is defined as $(\mathcal{U}+\mathcal{K})(A) = \{B \in \mathcal{B}(\mathcal{H}) : A \cong_{\mathcal{U}+\mathcal{K}} B\}$. This orbit lies between the unitary and the similarity orbit. In addition, two $(\mathcal{U}+\mathcal{K})$-equivalent operators are compalent.

In this article we develop a block tridiagonal decomposition technique that allows us to show that an operator is in the $(\mathcal{U}+\mathcal{K})$-orbit of another operator in some cases where the similarity of the two operators is apparent. We construct an essentially normal operator (model) with multiply connected (non-essential) spectrum and describe the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of this model.

KEYWORDS: *$(\mathcal{U}+\mathcal{K})$-orbit, essentially normal, model, multiply connected domain, block tridiagonal decomposition.*

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INTRODUCTION AND PRELIMINARIES

For $\mathcal{H}$ a complex separable infinite-dimensional Hilbert space $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on $\mathcal{H}$ equipped with the usual operator norm. Whenever we speak of closures of subsets of $\mathcal{B}(\mathcal{H})$, we will have the norm topology in mind.

$\mathcal{K}(\mathcal{H})$ denotes the closed ideal of all compact operators on $\mathcal{H}$. The Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted by $\mathcal{A}(\mathcal{H})$. The canonical quotient map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$ is denoted by π and $\sigma_{\mathrm{e}}(T) = \sigma(\pi(T))$ denotes the essential spectrum of T.

The Fredholm index of a (semi-)Fredholm operator T is denoted by $\mathrm{ind}(T)$. The Fredholm domain $\{z \in \mathbb{C} : T - zI \text{ is Fredholm}\}$ of an arbitrary operator T is denoted by $\rho_{\mathrm{F}}(T)$. Recall that $\rho_{\mathrm{F}}(X) = \mathbb{C} \setminus \sigma_{\mathrm{e}}(X)$.

In this article, the notation $\mathrm{cl}(Q)$ will be used for the closure of Q when Q is a subset of a topological space. When Q is a subset of $\mathbb{C}$, Q^* will denote $\{\overline{z} : z \in Q\}$. The symbol $\overline{Q}$, where Q is a set, will be avoided here.

Many equivalence relations exist on $\mathcal{B}(\mathcal{H})$. In the present article we are interested in $(\mathcal{U}+\mathcal{K})$-equivalence and the related notion of $(\mathcal{U}+\mathcal{K})$-orbit which were first introduced by Herrero in [8].

For a Hilbert space $\mathcal{H}$, we set

$$\begin{aligned}\mathcal{U}+\mathcal{K} = (\mathcal{U}+\mathcal{K})(\mathcal{H}) = \{R \in \mathcal{B}(\mathcal{H}) : &R \text{ is invertible in } \mathcal{B}(\mathcal{H}) \text{ and}\\ &R \text{ is of the form unitary plus compact}\}.\end{aligned}$$

Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be $(\mathcal{U}+\mathcal{K})$-*equivalent* ($A \cong_{\mathcal{U}+\mathcal{K}} B$) if $A = R^{-1}BR$, for some $R \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$. Note that this defines an equivalence relation on $\mathcal{B}(\mathcal{H})$. We define the $(\mathcal{U}+\mathcal{K})$-*orbit* of an operator T as

$$(\mathcal{U}+\mathcal{K})(A) = \{B \in \mathcal{B}(\mathcal{H}) : A \cong_{\mathcal{U}+\mathcal{K}} B\}.$$

Clearly, we have

$$\mathcal{U}(T) \subseteq (\mathcal{U}+\mathcal{K})(T) \subseteq \mathcal{S}(T),$$

where $\mathcal{U}(T)$ is the unitary orbit of T and $\mathcal{S}(T)$ is the similarity orbit.

One can also check that whenever two operators are $(\mathcal{U}+\mathcal{K})$-equivalent, they are compalent.

The $(\mathcal{U}+\mathcal{K})$-orbit of an operator need not be closed. As is the case with other orbits, one can find out more about the closures of $(\mathcal{U}+\mathcal{K})$-orbits than about the orbits themselves. We will write $A \to_{\mathcal{U}+\mathcal{K}} B$ when $B \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(A)$. Note that this defines a transitive relation.

The concept of compalence comes up often in investigation of $(\mathcal{U}+\mathcal{K})$-orbits. Since compalence is best understood for essentially normal operators, it comes as no surprise that descriptions of closures of $(\mathcal{U}+\mathcal{K})$-orbits of specific operators are only available for some classes of essentially normal operators. (See [6], [7], [12], [10], [1].)

Some basic properties of the relation $\to_{\mathcal{U}+\mathcal{K}}$ are as follows:

PROPOSITION 0.1. *Let A be essentially normal. If $A \to_{\mathcal{U}+\mathcal{K}} B$, we have*
(i) *$\sigma(A) \subseteq \sigma(B)$;*
(ii) *$A \sim B$ (and hence B is essentially normal);*
(iii) *$\sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B)$;*
(iv) *$\mathrm{ind}(A - z) = \mathrm{ind}(B - z)$ for $z \in \rho_{\mathrm{F}}(A) = \rho_{\mathrm{F}}(B)$;*
(v) *$\mathrm{nul}(A - z) \leqslant \mathrm{nul}(B - z)$ for $z \in \rho_{\mathrm{F}}(A) = \rho_{\mathrm{F}}(B)$;*
(vi) *$\mathrm{nul}(A - z)^* \leqslant \mathrm{nul}(B - z)^*$ for $z \in \rho_{\mathrm{F}}(A) = \rho_{\mathrm{F}}(B)$.*

The proof of (i), (ii) and (iii) is elementary (see [4]) and the properties (iv) and (v) follow from Theorem 1.13 in [9].

The idea of using a model — a specific operator with certain spectral properties — as a first step towards the investigation of the closures of $(\mathcal{U}+\mathcal{K})$-orbits of a whole class of essentially normal operators sharing the same spectral properties is due to Marcoux ([12]). In a previous article ([6]) by Guinand and himself, the following is shown:

THEOREM 0.2. ([6]) *Let S be the forward unilateral shift. An operator T is in* $\mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$ *if and only if it satisfies the following conditions:*

(i) *T is essentially normal;*
(ii) $\sigma(T)=\sigma(S)=\{z\in\mathbb{C}:|z|\leqslant 1\}$*;*
(iii) $\sigma_{\mathrm{e}}(T)=\sigma_{\mathrm{e}}(S)=\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$*;*
(iv) $\mathrm{ind}(T-z)=\mathrm{ind}(S-z)=-1$ *for* $|z|<1$.

Descriptions of closures of orbits of Hilbert space operators are usually in terms of spectral properties, one may therefore suspect that any operator T that has the same spectral properties as S will have $\mathrm{cl}(\mathcal{U}+\mathcal{K})(T)=\mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$. This is indeed the case, as was shown in [12].

There, an operator $T\in\mathcal{B}(\mathcal{H})$ is called *shift-like* if

(a) T is essentially normal;
(b) $\sigma(T)=\mathbb{D}=\{z\in\mathbb{C}:|z|\leqslant 1\}$;
(c) $\sigma_{\mathrm{e}}(T)=\mathbb{T}$;
(d) $\mathrm{ind}(T-\lambda)=-1$ for all $\lambda\in\{z\in\mathbb{C}:|z|<1\}$;
(e) $\mathrm{nul}(T-\lambda)=0$ for all $\lambda\in\{z\in\mathbb{C}:|z|<1\}$.

With this definition one can prove a theorem which is in a certain sense complementary to the above theorem. One can then use the transitivity of the relation $\rightarrow_{\mathcal{U}+\mathcal{K}}$ to obtain a description of the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the whole class of shift-like operators.

THEOREM 0.3. ([12]) *Suppose that T is shift-like. Then* $S\in\mathrm{cl}(\mathcal{U}+\mathcal{K})(T)$.

COROLLARY 0.4. ([12]) *Therefore, since the relation $\rightarrow_{\mathcal{U}+\mathcal{K}}$ is transitive, we have* $\mathrm{cl}(\mathcal{U}+\mathcal{K})(T)=\mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$.

Summing up we see that describing the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of one particular operator S was the first step towards finding this description for the whole class of essentially normal operators with the same spectral properties.

In [4], the present author constructs model operators and describes the closures of the $(\mathcal{U}+\mathcal{K})$-orbits of these models for various spectral pictures. The investigation includes operators with different indices, operators with disconnected spectra, operators with enlarged essential spectra, and operators with isolated spectral points.

The above information on $(\mathcal{U}+\mathcal{K})$-orbits is merely the necessary minimum that is needed to present the main results of this article. For an up-to-date survey of this area the reader is referred to [13].

Having defined some basic concepts, we can now outline the subject of the present article. In the remainder of this preliminary section we shall develop some holomorphic functional calculus techniques that can be applied to the investigation of closures of $(\mathcal{U}+\mathcal{K})$-orbits.

In the main part of the article we shall construct another model and describe the closure of its $(\mathcal{U}+\mathcal{K})$-orbit. We shall be interested in the case where the non-essential spectrum $\sigma(X)\setminus\sigma_{\mathrm{e}}(X)$ is a multiply connected domain. Note that so far the only existing description of the $(\mathcal{U}+\mathcal{K})$-orbit of an operator with multiply connected non-essential spectrum is due to Guinand and Marcoux, who describe the closures of $(\mathcal{U}+\mathcal{K})$-orbits of weighted shifts (including bilateral shifts), in [7]. That result provides a model whose non-essential spectrum is an annulus. The model whose closure of $(\mathcal{U}+\mathcal{K})$-orbit will be described here allows for non-essential spectra of more general shape, including multiply connected domains with more than one hole.

A block tridiagonal decomposition technique is developed in Section 2 which may be of use in the investigation of the closures of $(\mathcal{U}+\mathcal{K})$-orbits of other operators. (See Section 3 of [4] for another application of this technique.)

Let us now introduce several auxiliary lemmas that deal with the relationship between functional calculus, $(\mathcal{U}+\mathcal{K})$-orbits and spectral properties. Suppose that A and B are essentially normal operators and φ is a function holomorphic on (a neighbourhood of) $\sigma(A)$. We shall show here that $A \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(B))$ implies that $\varphi(A) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\varphi(B)))$.

LEMMA 0.5. *Suppose $A \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(B)$ and let φ be a holomorphic function on (a neighbourhood of) $\sigma(A)$. Then $\varphi(A) \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\varphi(B))$.*

Proof. Suppose $R_n^{-1}BR_n \to A$, $R_n \in (\mathcal{U}+\mathcal{K})$. Then $p(R_n^{-1}BR_n) = R_n^{-1}p(B)R_n$ whenever p is a rational function with poles outside $\sigma(A)$ and hence, by passing to the limit, we see that $\varphi(R_n^{-1}BR_n) = R_n^{-1}\varphi(B)R_n$. We are using Runge's theorem here to approximate φ uniformly on $\sigma(A)$, see [14], Theorem 13.6.

Consequently, $R_n^{-1}\varphi(B)R_n = \varphi(R_n^{-1}BR_n) \to \varphi(A)$ by the continuity of the functional calculus. This shows that $\varphi(A) \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\varphi(B))$. ∎

One can also easily show that holomorphic functions preserve compalence of essentially normal operators and that the existence of eigenvalues is preserved:

LEMMA 0.6. *Let $A \sim B$, A, B essentially normal. Let φ be a holomorphic function on (a neighbourhood of) $\sigma(A) \cup \sigma(B)$. Then $\varphi(A)$ and $\varphi(B)$ are essentially normal and $\varphi(A) \sim \varphi(B)$.*

LEMMA 0.7. *Suppose that z_0 is an eigenvalue of A and φ is holomorphic on (a neighbourhood of) $\sigma(A)$. Then $\varphi(z_0)$ is an eigenvalue of $\varphi(A)$.*

The auxiliary results that we have proved here will be used at the end of Section 3. To see an easy application right now, we shall generalize the description of the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of shift-like operators from [12]. (Recall that shift-like operators are operators with the same spectral properties as the unilateral shift.) Note that a stronger result (with a more involved proof) was shown in [11].

THEOREM 0.8. *Suppose Ω is a simply connected analytic Cauchy domain and A is an essentially normal operator on a separable Hilbert space H with the following spectral properties:*
(a) $\sigma(A) = \mathrm{cl}(\Omega)$;
(b) $\sigma_{\mathrm{e}}(A) = \partial\Omega$;
(c) $\mathrm{ind}(A-z) = -1$, $z \in \Omega$:
(d) $\mathrm{nul}(A-z) = 0$, $z \in \Omega$;
Then the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the operator A is

$$\begin{aligned}\mathrm{cl}((\mathcal{U}+\mathcal{K})(A)) = \{T \in \mathcal{B}(H) :\ & \\ &\text{(i) } T \textit{ is essentially normal},\\ &\text{(ii) } \sigma(T) = \mathrm{cl}(\Omega),\\ &\text{(iii) } \sigma_{\mathrm{e}}(T) = \partial\Omega,\\ &\text{(iv) } \mathrm{ind}(T-\lambda) = -1 \textit{ for all } \lambda \in \Omega\}.\end{aligned}$$

Proof. Let φ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\varphi|\mathbb{D}$ is a conformal map of $\mathbb{D}$ onto Ω. Let A, T be as in the Theorem. First we will use [12], Theorem 2.5, to show that $\varphi^{-1}(T) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\varphi^{-1}(A)))$. Applying Lemma 0.6 to operators A, T and to the map φ^{-1}, we see that $\varphi^{-1}(A), \varphi^{-1}(T)$ are essentially normal and the index properties of $\varphi^{-1}(A), \varphi^{-1}(T)$ are as needed. Similarly, Lemma 0.7 shows that $\mathrm{nul}(\varphi^{-1}(A)-z) = 0$ for $|z| < 1$. Hence by [12], Theorem 2.5, $\varphi^{-1}(T) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\varphi^{-1}(A)))$ and so by Lemma 0.5, $T \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(A))$. ∎

Before we proceed, let us recall some more notation. For two operators A, B we shall write $A \cong_\varepsilon B$ if there exists a unitary operator U such that $\|A - U^*BU\| < \varepsilon$. Note that we do not require that $A - U^*BU$ be compact.

For $X \subseteq \mathbb{C}$, $\varepsilon > 0$, we set $X_\varepsilon = \{x \in \mathbb{C} : \mathrm{dist}(x, X) < \varepsilon\}$.

1. A MODEL WITH MULTIPLY CONNECTED (NON-ESSENTIAL) SPECTRUM

As in [4], the basic building block of our model will be a generalization of the Hardy space H^2. Recall that a nonempty bounded open subset Ω of the complex plane $\mathbb{C}$ is a *Cauchy domain* if Ω has finitely many components, the closures of any two of which are disjoint, and the boundary $\partial\Omega$ of Ω is composed of a finite positive number of closed rectifiable Jordan curves, no two of which intersect. A Cauchy domain with an analytic boundary will be called an *analytic Cauchy domain*.

Let us briefly recall the way the Hardy spaces $H^2(\Omega, \mu)$ were constructed in [4], to which the reader is referred for more details. Although the space $H^2(\Omega, \mu)$ is a generalization of the space H^2, it is still a rather special case because of the conditions we impose here on Ω. For more on generalized Hardy spaces, see [5].

Let Ω be a simply connected analytic Cauchy domain. Then there exists a $\rho > 1$ and an invertible holomorphic function φ from $\{z : |z| < \rho\}$ to $\mathbb{C}$ such that $\varphi|\mathbb{D}$ is a conformal map of $\mathbb{D}$ onto Ω. Let us fix a φ with these properties.

We start by defining the set $H^2(\Omega)$ of holomorphic functions on Ω :

$$H^2(\Omega) = \{f \circ \varphi^{-1} : f \in H^2(\mathbb{D})\}.$$

For g holomorphic on Ω, we shall denote

$$\widehat{g}(z) = \lim_{r \to 1-} g(\varphi(r \cdot \varphi^{-1}(z))), \quad z \in \partial\Omega.$$

For $g \in H^2(\Omega)$, $\widehat{g}(z)$ exists almost everywhere on $\partial\Omega$ with respect to the arc length measure λ and $\widehat{g} \in L^2(\partial\Omega, \lambda)$.

Next, we let μ be a measure on $\partial\Omega$ equivalent to λ (i.e. μ and λ are assumed to be absolutely continuous with respect to each other). For g, $h \in H^2(\Omega)$, define an inner product

$$\langle g, h \rangle = \int_{\partial\Omega} g(z) \cdot \overline{h(z)} \, \mathrm{d}\mu.$$

$H^2(\Omega)$ with this inner product becomes a Hilbert space; we shall denote it $H^2(\Omega, \mu)$. This space inherits many of its properties from $H^2(\mathbb{D})$. Some of the properties that we shall need here follow.

LEMMA 1.1. *Let Ω be a simply connected analytic Cauchy domain.*

(i) *$g \in H^2(\Omega)$ if and only if g is holomorphic on Ω, $\widehat{g}(z)$ exists almost everywhere on $\partial\Omega$ and $\widehat{g} \in L^2(\partial\Omega, \lambda)$;*

(ii) *Let $g \in H^2(\Omega)$. For $r \in [0,1)$, let*

$$g_r(z) = g(\varphi(r \cdot \varphi^{-1}(z))), \quad z \in \Omega.$$

Then $g_r \in \mathcal{C}(\mathrm{cl}(\Omega)) \cap H^2(\Omega)$ and $g_r \to g$ as $r \to 1-$ in any $H^2(\Omega,\mu)$. Note that g_r can also be viewed as a function which is holomorphic on an open set that includes $\mathrm{cl}(\Omega)$.

We define $\widehat{H}^2(\Omega,\mu) = \{\widehat{g} : g \in H^2(\Omega,\mu)\}$. There is a one-to-one correspondence $g \mapsto \widehat{g}$ between these two sets. As is the custom for $H^2(\mathbb{D})$, we shall identify these two sets whenever it is convenient.

We can now define a multiplication operator $M(\Omega,\mu)$ on $H^2(\Omega,\mu)$ by

$$M(\Omega,\mu)(g)(z) = z \cdot g(z), \quad z \in \Omega.$$

Then $M(\Omega,\mu)$ is an essentially normal operator and the spectral properties of $M(\Omega,\mu)$ are as follows:

(i) $\sigma(M(\Omega,\mu)) = \mathrm{cl}(\Omega)$;

(ii) $\sigma_{\mathrm{e}}(M(\Omega,\mu)) = \partial\Omega$;

(iii) $\mathrm{ind}(M(\Omega,\mu) - z) = -1$, $z \in \Omega$;

(iv) $\min\mathrm{ind}(M(\Omega,\mu) - z) = 0$, $z \in \Omega$.

(See [9], Sections 3.2 and 4.1.3.)

In [4] we used the operator $M(\Omega,\mu)$ by itself as a model for the class of operators sharing its spectral properties. We also used it as a building block for more involved models. Let us recall the description of $\mathrm{cl}(\mathcal{U}+\mathcal{K})(M(\Omega,\mu))$ and some of the auxiliary results about $M(\Omega,\mu)$ which will also be needed here.

LEMMA 1.2. *For $z_0 \in \Omega$ there is a function $\psi \in H^2(\Omega,\mu)$ such that ψ has a simple zero at z_0, $\psi(z) \neq 0$ for $z \neq z_0$, $|\psi(z)| = 1$ almost everywhere on $\partial\Omega$.*

LEMMA 1.3. *Let z_0 be in Ω. Then $M(\Omega,\mu)$ is unitarily equivalent to an operator of the form*

$$\begin{pmatrix} z_0 & 0 \\ Q & M(\Omega,\mu) \end{pmatrix},$$

for some $Q \in \mathcal{B}(\mathbb{C}, H^2(\Omega,\mu))$.

LEMMA 1.4. *Let C be an operator of the form*

$$C = \begin{pmatrix} F_{\mathrm{d}} & 0 \\ T & M(\Omega,\mu) \end{pmatrix},$$

where F_{d} is a diagonal matrix. Then the following statements are equivalent:

(i) *$C \cong_{\mathcal{U}+\mathcal{K}} M(\Omega,\mu)$;*

(ii) *C is similar to $M(\Omega,\mu)$;*

(iii) *the diagonal entries $\{z_1, z_2, \ldots, z_n\}$ of F_{d} are distinct, they lie in Ω, and C has no eigenvalues;*

(iv) *the diagonal entries $\{z_1, z_2, \ldots, z_n\}$ of F_{d} are distinct and lie in Ω, and, for $1 \leqslant i \leqslant n$, the i-th column t_i of T is not in*

$$\mathrm{ran}(M(\Omega,\mu) - z_i I) = \{f \in H^2(\Omega,\mu) : f(z_i) = 0\}.$$

LEMMA 1.5. *Let $z_0 \in \Omega$. Then there exists an orthonormal basis $\{e_0, e_1, \ldots\}$ of $H^2(\Omega,\mu)$ such that the matrix of the operator $M(\Omega,\mu)$ with respect to this basis is the Toeplitz matrix*

$$\begin{pmatrix} z_0 & 0 & & & \\ z_1 & z_0 & 0 & & \\ z_2 & z_1 & z_0 & 0 & \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

THEOREM 1.6. *Let Ω be a simply connected analytic Cauchy domain. The closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the operator $M(\Omega,\mu)$ is*

$$\begin{aligned} \mathrm{cl}((\mathcal{U}+\mathcal{K})(M(\Omega,\mu))) = \{T \in \mathcal{B}(H^2(\Omega,\mu)) :& \\ &\text{(i) } T \textit{ is essentially normal},\\ &\text{(ii) } \sigma(T) = \mathrm{cl}(\Omega),\\ &\text{(iii) } \sigma_{\mathrm{e}}(T) = \partial\Omega,\\ &\text{(iv) } \mathrm{ind}(T-\lambda) = -1 \textit{ for all } \lambda \in \Omega\}. \end{aligned}$$

In this article we shall not be interested in Hardy spaces over domains other than simply connected analytic Cauchy domains. Nevertheless, note that when a domain $\Omega = \bigcup_{i=1}^{n} \Omega_n$ consists of n simply connected analytic Cauchy components and μ_i are measures on $\partial\Omega_i$ equivalent to the corresponding arc length measures, we can regard any element $f = (f_1, f_2, \ldots, f_n)$ of $\bigoplus_{i=1}^{n} H(\Omega_i, \mu_i)$ as a holomorphic function on Ω by setting $f(z) = f_i(z)$, where i is such that $z \in \Omega_i$. We shall adopt this point of view when it is convenient, mostly to simplify notation. With this in mind, note that the operator $M = \bigoplus_{i=1}^{n} M(\Omega_i, \mu_i)$ can also be defined by the formula $M(f)(z) = z \cdot f(z)$, $z \in \Omega$.

We are now ready to proceed to construct the model with which we shall be concerned in the rest of this article. Let us consider a connected analytic Cauchy domain which is not simply connected. Assume that $\Omega = \Omega_1 \setminus \mathrm{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \bigcup_{1}^{n} \Omega_{2,i}$, $\mathrm{cl}(\Omega_2) \subseteq \Omega_1$. We want to construct an essentially normal operator M with the following spectral properties:

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_{\mathrm{e}}(M) = \partial\Omega$;
(iii) $\mathrm{ind}(M - z) = -1$, $z \in \Omega$;
(iv) $\min\mathrm{ind}(M - z) = 0$, $z \in \Omega$.

This operator will then serve as a model for the class of operators sharing the same spectral properties.

One possible construction of the model would consist of constructing a Hardy space $H^2(\Omega)$ and using a multiplication operator on this space. This would require a different, more general definition of the Hardy space than the one used at the beginning of this section.

While a model could be constructed in this way, we would run into difficulties if we attempted to describe the closure of its $(\mathcal{U}+\mathcal{K})$-orbit using the same techniques as in [4]. In particular, the Lemmas 1.2 and 1.5 would no longer hold in this setting. We will therefore construct our model in a different manner. The operator $M(\Omega_1, \mu)$, where Ω_1 is as above, will be one of its building blocks. When we investigate the model we construct here, we shall be able to make use of our investigation of the properties of $M(\Omega_1, \mu)$ in [4].

Let now μ be a measure on $\partial(\Omega_1)$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega_{2,i}^*)$; all of these measures are assumed to be equivalent to the respective arc length measures. Let $A = M(\Omega_1, \mu)$ and let $B = \bigoplus_{i=1}^{n} M(\Omega_{2,i}^*, \mu_i)$. We are already familiar with the spectral properties of A and B. As a first step in constructing our model, let us consider the (essentially normal) operator $M_0 = A \oplus B^*$. The spectral properties of M_0 are as follows:

(i) $\sigma(M_0) = \mathrm{cl}(\Omega_1)$;
(ii) $\sigma_{\mathrm{e}}(M_0) = \partial\Omega$;
(iii) $\mathrm{nul}(M_0 - z) = 0$, $z \in \Omega$;
(iv) $\mathrm{nul}(M_0^* - \overline{z}) = 1$, $z \in \Omega$;
(v) $\mathrm{ind}(M_0 - z) = -1$, $z \in \Omega$;
(vi) $\mathrm{nul}(M_0 - z) = 1$, $z \in \Omega_2$;
(vii) $\mathrm{nul}(M_0^* - \overline{z}) = 1$, $z \in \Omega_2$;
(viii) $\mathrm{ind}(M_0 - z) = 0$, $z \in \Omega_2$.

We see that M_0 has some of the properties we require of M: the properties (ii), (iii), (iv), (v) and (viii) are as required. We shall now construct M as compact perturbation of M_0. This will allow us to change the spectrum of our operator (to exclude Ω_2) without disturbing the already correct essential spectrum and index properties.

The following lemma shows how this can be accomplished. In fact, the lemma is more general than necessary for the construction of the model. The additional information will be useful when we investigate the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the model.

LEMMA 1.7. *Let $\Omega, \Omega_1, \Omega_{2,1}, \Omega_{2,2}, \ldots, \Omega_{2,n}, \mu, \mu_1, \mu_2, \ldots, \mu_n$ be as above. Let 1_{Ω_1} be the constant function equal to 1 on Ω_1. Let A, B be as above. For $b \in \bigoplus_{i=1}^{n} H^2(\Omega_{2,i}^*, \mu_i)$, let C_b be an operator from $\bigoplus_{i=1}^{n} H^2(\Omega_{2,i}^*, \mu_i)$ into $H^2(\Omega_1, \mu)$ defined by $C_b g = (1_{\Omega_1} \otimes b^*)(g) = \langle g, b\rangle \cdot 1_{\Omega_1}$, $g \in \bigoplus_{i=1}^{n} H^2(\Omega_{2,i}, \mu_i)$. Next, define an operator M_b on $H^2(\Omega_1, \mu) \oplus (\bigoplus_{i=1}^{n} H^2(\Omega_{2,i}^*, \mu_i))$ by*

$$M_b = \begin{pmatrix} A & C_b \\ 0 & B^* \end{pmatrix}.$$

(a) *If $b(z) \neq 0$ for $z \in \Omega_2$, we have*
(i) $\sigma(M_b) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_{\mathrm{e}}(M_b) = \partial\Omega$;
(iii) $\mathrm{nul}(M_b - z) = 0$, $z \in \Omega$;
(iv) $\mathrm{nul}(M_b^* - \overline{z}) = 1$, $z \in \Omega$;

(v) $\operatorname{ind}(M_b - z) = -1$, $z \in \Omega$.

(b) *If* $b(z_0) = 0$ *for some* $z \in \Omega_2$, *then* z_0 *is an eigenvalue of* M_b.

With this lemma in hand, we can finish the construction of the model by letting $b(z) = 1$ for $z \in \Omega_2$ and letting $M = M_b$.

Proof. (a) Since the essential spectrum and index properties of M_b are already known (M_b being a compact perturbation of M_0), it suffices to show that $\operatorname{nul}(M_b - z) = 0$ for $z \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$ and $\operatorname{nul}(M_b - z) = 0$ for $z \in \Omega_2$.

Let $z \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$. We want to show that $M_b - z$ does not have any eigenvalues. Suppose that

$$\begin{pmatrix} A - z & C_b \\ 0 & B^* - z \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $f \in H^2(\Omega_1, \mu)$, $g \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$, i.e.

$$(A - z)f + C_b g = 0, \quad (B^* - z)g = 0.$$

But $z \notin \sigma(B^*)$, so $g = 0$ and hence $(A - z)f = 0$. Since $\operatorname{nul}(A - z) = 0$, we must have $f = 0$. We have shown that $\operatorname{nul}(M_b - z) = 0$.

Suppose next that $z \in \Omega_2$ and again

$$\begin{pmatrix} A - z & C_b \\ 0 & B^* - z \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $f \in H^2(\Omega_1, \mu)$, $g \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$, i.e.

$$(A - z)f + C_b g = 0, \quad (B^* - z)g = 0.$$

Assume that $g \neq 0$. Then $g \perp \operatorname{ran}(B - \overline{z})$. Since $\operatorname{codim} \operatorname{ran}(B - \overline{z}) = 1$, we see that $\operatorname{ran}(B - \overline{z}) = \{g\}^\perp$. But $b(\overline{z}) \neq 0$, which means that $b \notin \operatorname{ran}(B - \overline{z})$ and so $\langle g, b \rangle \neq 0$. From above, we know that $(A - z)f = -C_b g = -\langle g, b \rangle \cdot 1_{\Omega_1}$, hence $(A - z)f$ is a non-zero multiple of 1_{Ω_1}, i.e. a non-zero constant function on Ω_1. This is a contradiction, as $[(A - z)f](z) = 0$.

Hence we must have $g = 0$. This implies $(A - z)f = 0$ and so, as above, $f = 0$. We see that $\operatorname{nul}(M - z) = 0$ in this case too.

(b) Suppose that $b(z_0) = 0$ for some $z_0 \in \Omega_2$. We can then choose $g_0 \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$ such that $(B^* - z_0)g_0 = 0$, $g_0 \neq 0$. We now have $g_0 \perp \operatorname{ran}(B - \overline{z}_0)$ and $b \in \operatorname{ran}(B - \overline{z}_0)$, hence $g_0 \perp b$. Consequently, $(M - z_0)\begin{pmatrix} 0 \\ g_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $z_0 \in \sigma(M)$. ∎

2. BLOCK TRIDIAGONAL DECOMPOSITION TECHNIQUE

The following two auxiliary results will allow us to show that an operator is in the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of another operator in some situations where at first only the similarity of the two operators is apparent. Since even the statements are rather technical, the reader may prefer to first have a quick glance at Lemma 2.3 and its proof.

LEMMA 2.1. (a) *Suppose that $L, N, R_0, R_1, \ldots, R_n$ are operators on a Hilbert space $\mathcal{H}$ with the following properties:*

(i) $LR_k - R_k N = 0$, $k = 0, 1, \ldots, n$.

(ii) *$\mathcal{H}$ can be decomposed as $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$, where each subspace $\mathcal{H}_i$ is finite-dimensional and the operators $L, N, R_0, R_1, \ldots, R_n$ have a block tri-diagonal form with respect to this decomposition, i.e. $R_{ij}^{(k)} = L_{ij} = N_{ij} = 0$, if $|i-j| > 1$, $i \geqslant 0$, $j \geqslant 0$, where $R_{ij}^{(k)}$ is the (i,j)-entry of the operator matrix of R_k with respect to the above decomposition and L_{ij}, N_{ij} are the (i,j)-entries of L, N, respectively.*

(iii) $\|R_k - R_{k+1}\| < \varepsilon$ *for* $k = 0, 1, \ldots, n-1$.

Construct an operator R whose matrix is $\{R_{ij}\}_{i,j=0}^{\infty}$, where

$$R_{i,j} = \begin{cases} R_{i,j}^{(n-i-j)}, & n-i-j \geqslant 0, \\ R_{i,j}^{(0)}, & n-i-j < 0. \end{cases}$$

Then $\|LR - RN\| \leqslant 15\varepsilon(\|N\| + \|L\|)$

(b) *Suppose that $R_0, R_1, \ldots, R_n, S_0, S_1, \ldots, S_n$ are operators on a Hilbert space $\mathcal{H}$ with the following properties:*

(i) $R_0 = S_0 = I$, $S_k = R_k^{-1}$, $k = 1, 2, \ldots, n$.

(ii) *$\mathcal{H}$ can be decomposed as $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$, where each subspace $\mathcal{H}_i$ is finite-dimensional and the operators S_k and R_k, $k = 1, 2, \ldots, n$ have a block tri-diagonal form with respect to this decomposition.*

(iii) $\|R_k - R_{k+1}\| < \varepsilon$ *and* $\|S_k - S_{k+1}\| < \varepsilon$ *for* $k = 0, 1, \ldots, n-1$.

Suppose that R is constructed as above and construct an operator S whose matrix is $\{S_{ij}\}_{i,j=0}^{\infty}$, where

$$S_{i,j} = \begin{cases} S_{i,j}^{(n-i-j)}, & n-i-j \geqslant 0, \\ S_{i,j}^{(0)}, & n-i-j < 0. \end{cases}$$

Let $m = \max(\|R_0\|, \|R_1\|, \ldots, \|R_n\|, \|S_0\|, \|S_1\|, \ldots, \|S_n\|)$. *Then* $\|RS - I\| \leqslant 30\varepsilon m$ *and* $\|SR - I\| \leqslant 30\varepsilon m$.

Note that the operators R and S constructed here also have block tri-diagonal matrices with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. This is what the entries of the matrix of R look like when n is even:

$$\begin{array}{cccccccccc}
R^{(n)}_{0,0} & R^{(n-1)}_{0,1} & & & & & & & & \\
R^{(n-1)}_{1,0} & R^{(n-2)}_{1,1} & R^{(n-3)}_{1,2} & & & & & & & \\
 & R^{(n-3)}_{2,1} & R^{(n-4)}_{2,2} & R^{(n-5)}_{2,3} & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & \\
 & & & R^{(3)}_{\frac{n}{2}-1,\frac{n}{2}-2} & R^{(2)}_{\frac{n}{2}-1,\frac{n}{2}-1} & R^{(1)}_{\frac{n}{2}-1,\frac{n}{2}} & & & & \\
 & & & & R^{(1)}_{\frac{n}{2},\frac{n}{2}-1} & R^{(0)}_{\frac{n}{2},\frac{n}{2}} & R^{(0)}_{\frac{n}{2},\frac{n}{2}+1} & & & \\
 & & & & & R^{(0)}_{\frac{n}{2}+1,\frac{n}{2}} & R^{(0)}_{\frac{n}{2}+1,\frac{n}{2}+1} & R^{(0)}_{\frac{n}{2}+1,\frac{n}{2}+2} & & \\
 & & & & & & \ddots & \ddots & \ddots &
\end{array}$$

For an odd n, the picture is similar. The entries which are left blank equal zero.

Proof. (a) Denote $P = LR - RN$. We will investigate the entries P_{ij} of the matrix of P with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. As the matrices of L, N and R are block tri-diagonal, we see that $P_{ij} = 0$ for $|i-j| > 2$.

To simplify notation, we shall set

$$R_k = R_0, \quad R^{(k)}_{ij} = R^{(0)}_{ij} \qquad \text{for } k < 0,\ i, j \geqslant 0.$$

With this convention, for any $n \geqslant 0$, we have $R_{ij} = R^{(n-i-j)}_{ij}$, $i, j \geqslant 0$ and we still have $LR_k - R_kN = 0$ for $k \leqslant n$.

For $|i-j| \leqslant 2$, we have

$$\begin{aligned}
\|P_{ij}\| &= \|(LR-RN)_{ij}\| \\
&= \|(LR-RN)_{ij} - (LR_{n-j-i} - R_{n-j-i}N)_{ij}\| \\
&= \left\| \sum_{l=0}^{\infty} L_{il}R_{lj} - L_{il}R^{(n-i-j)}_{lj} - R_{il}N_{lj} + R^{(n-i-j)}_{il}N_{lj} \right\| \\
&= \left\| \sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} L_{il}R^{(n-l-j)}_{lj} - L_{il}R^{(n-i-j)}_{lj} - R^{(n-l-i)}_{il}N_{lj} + R^{(n-i-j)}_{il}N_{lj} \right\|.
\end{aligned}$$

The restriction of the summation range is possible because of the block tri-diagonal form of L, N and R. Note that, as $|l-i| \leqslant 1$, $|l-j| \leqslant 1$, we have

$$|(n-i-j)-(n-l-j)| \leqslant 1, \quad |(n-i-j)-(n-l-i)| \leqslant 1.$$

Hence

$$\|P_{ij}\| \leqslant 3(\|L\| \cdot \varepsilon + \|N\| \cdot \varepsilon) = 3 \cdot \varepsilon(\|N\| + \|L\|).$$

Now

$$P = \sum_{r=-2}^{2} P_r,$$

where

$$(P_r)_{ij} = \begin{cases} P_{ij} & \text{if } j = i + r, \\ 0 & \text{otherwise.} \end{cases}$$

From the above estimate, we see that $\|P_r\| \leqslant 3 \cdot \varepsilon(\|N\| + \|L\|)$ and hence

$$\|P\| \leqslant 15\varepsilon(\|N\| + \|L\|).$$

(b) The proof is similar to that of (a). Since the whole situation is symmetric, it suffices to show that $\|RS - I\| \leqslant 30\varepsilon m$. Denote $Q = RS - I$. We will investigate the entries Q_{ij} of the matrix of Q with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. We can again see that $Q_{ij} = 0$ for $|i - j| > 2$.

To simplify notation, we shall set

$$R_k = R_0, \quad R_{ij}^{(k)} = R_{ij}^{(0)} \qquad \text{for } k < 0,\ i, j \geqslant 0,$$

$$S_k = S_0, \quad S_{ij}^{(k)} = S_{ij}^{(0)} \qquad \text{for } k < 0,\ i, j \geqslant 0.$$

With this convention, we have $R_{ij} = R_{ij}^{(n-i-j)}$ and $S_{ij} = S_{ij}^{(n-i-j)}$, $i, j \geqslant 0$ and we have $R_k S_k - I = 0$ for $k \leqslant n$.

For $|i - j| \leqslant 2$, we have

$$\begin{aligned}
\|Q_{ij}\| &= \|(RS - I)_{ij}\| = \|(RS - I)_{ij} - (R_{n-j-i} S_{n-j-i} - I)_{ij}\| \\
&= \Big\| \sum_{l=0}^{\infty} R_{il} S_{lj} - R_{il}^{(n-i-j)} S_{lj}^{(n-i-j)} \Big\| \\
&= \Big\| \sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} R_{il}^{(n-l-i)} S_{lj}^{(n-l-j)} - R_{il}^{(n-i-j)} S_{lj}^{(n-i-j)} \Big\| \\
&= \Big\| \sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} R_{il}^{(n-l-i)} (S_{lj}^{(n-l-j)} - S_{lj}^{(n-i-j)}) \\
&\qquad + (R_{il}^{(n-l-i)} - R_{il}^{(n-i-j)}) S_{lj}^{(n-i-j)} \Big\| \\
&\leqslant 3(m\varepsilon + \varepsilon m) = 6\varepsilon m.
\end{aligned}$$

Using the fact that $Q_{ij} = 0$ for $|i - j| > 2$, we can estimate

$$\|Q\| \leqslant 30\varepsilon m. \quad ∎$$

COROLLARY 2.2. (a) *Suppose that $L, N, R_0, \ldots, R_n$ are operators satisfying* (i), (iii) *in Lemma* 2.1 (a). *Suppose that K is a compact operator and let $Q = R_n K$. Then there exists an operator R such that*

(i) *$R - R_0$ has finite rank;*
(ii) *$\|RK - Q\| < \varepsilon(\|K\| + 6 \max\{\|R_0\|, \|R_1\|, \ldots, \|R_n\|\})$;*
(iii) $\|LR - RN\| \leqslant 15\varepsilon(\|L\| + \|N\|)$.

(b) *Suppose that we have in addition operators $S_0, S_1, \ldots, S_n$ satisfying* (i), (iii) *in Lemma* 2.1 (b). *Then we can also construct an operator S such that, in addition to the properties* (i), (ii) *and* (iii) *in part* (a) *of this lemma, we have*

$$\|RS - I\| \leqslant 30\varepsilon \max(\|R_0\|, \|R_1\|, \ldots, \|R_n\|, \|S_0\|, \|S_1\|, \ldots, \|S_n\|),$$

$$\|SR - I\| \leqslant 30\varepsilon \max(\|R_0\|, \|R_1\|, \ldots, \|R_n\|, \|S_0\|, \|S_1\|, \ldots, \|S_n\|).$$

Proof. (a) Since K is compact, we can fix a finite-dimensional $\mathcal{H}_0 \subset \mathcal{H}$ such that $\|P_{\mathcal{H}_0}K - K\| < \varepsilon$. Next, fix a basis $e_1, e_2, \ldots$ of $\mathcal{H}$. Set

$$\mathcal{K}_1 = \operatorname{span}\{\mathcal{H}_0, L\mathcal{H}_0, L^*\mathcal{H}_0, N\mathcal{H}_0, N^*\mathcal{H}_0, R_0\mathcal{H}_0, R_0^*\mathcal{H}_0, \ldots, e_1\},$$
$$\mathcal{H}_1 = \mathcal{K}_1 \ominus \mathcal{H}_0,$$

and continue in this manner: with $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_i, \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_i$ constructed, we set

$$\mathcal{K}_{i+1} = \operatorname{span}\{\mathcal{K}_i, L\mathcal{K}_i, L^*\mathcal{K}_i, N\mathcal{K}_i, N^*\mathcal{K}_i, R_0\mathcal{K}_i, R_0^*\mathcal{K}_i, \ldots, e_{i+1}\},$$
$$\mathcal{H}_{i+1} = \mathcal{K}_{i+1} \ominus \mathcal{H}_i.$$

Then the conditions (i), (ii), (iii) of Lemma 2.1 (a) are satisfied. We can therefore construct R such that $\|LR - RN\| \leqslant 15\varepsilon(\|N\| + \|L\|)$, $R - R_0$ is finite dimensional and

$$\begin{aligned}\|RK - Q\| &= \|(R - R_n)K\| \\ &\leqslant \|(R - R_n)P_{\mathcal{H}_0}K\| + \|(R - R_n)(P_{\mathcal{H}_0}K - K)\| \\ &\leqslant \varepsilon \cdot \|K\| + 6\max\{\|R_0\|, \|R_1\|, \ldots, \|R_n\|\} \cdot \varepsilon.\end{aligned}$$

To see why the last inequality holds, consider the (i, j) entry of the operator matrix of $(R - R_n)$:

$$\|(R - R_n)_{ij}\| = \|R_{ij}^{(n-i-j)} - R_{ij}^{(n)}\| \leqslant 2 \cdot \max\{\|R_0\|, \|R_1\|, \ldots, \|R_n\|\},$$

and since $(R - R_n)$ is block tri-diagonal, we have

$$\|R - R_n\| \leqslant 6 \cdot \max\{\|R_0\|, \|R_1\|, \ldots, \|R_n\|\}.$$

(b) It suffices to alter the construction of the spaces $\mathcal{H}_k$ so that the resulting decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$ makes the matrices of $S_0, S_1, \ldots, S_n$ also block tridiagonal. Then we can finish the proof by applying Lemma 2.1 (b). ∎

We are now ready to use these results to make the first step towards the investigation of the model M constructed in the previous section.

LEMMA 2.3. *Assume that $\Omega = \Omega_1 \setminus \operatorname{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \bigcup_1^n \Omega_{2,i}$, $\operatorname{cl}(\Omega_2) \subseteq \Omega_1$. Let now μ be a measure on $\partial(\Omega_1)$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega_{2,i}^*)$; all of these measures are assumed to be equivalent to the respective arc length measures. Let $A = M(\Omega_1, \mu)$ and let $B = \bigoplus_{i=1}^n M(\Omega_{2,i}^*, \mu_i)$. Let 1_{Ω_1} be the constant function equal to 1 on Ω_1 and let $1_{\Omega_2^*}$ be the constant function equal to 1 on Ω_2^*. Let C be an operator from $\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i)$ into $H^2(\Omega_1, \mu)$ defined by $C = 1_{\Omega_1} \otimes 1_{\Omega_2^*}^*$. Define an operator M on $H^2(\Omega_1, \mu) \oplus \Big(\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i)\Big)$ by*

$$M = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}.$$

Next, let $D = 1_{\Omega_1} \otimes d^*$, *where* $d \in \mathcal{C}(\mathrm{cl}(\Omega_2^*)) \cap H^2(\Omega_2^*)$ *with* $d(z) \neq 0$ *for* $z \in \mathrm{cl}(\Omega_2^*)$ *and set*

$$X = \begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}.$$

Then we have

$$X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M).$$

Proof. One can easily see that M and X are similar. Indeed, since $d \in \mathcal{C}(\mathrm{cl}(\Omega_2^*))$ with $d(z) \neq 0$ for $z \in \mathrm{cl}(\Omega)$, the operator H on $H^2(\Omega_2^*, \mu)$ defined by

$$(Hf)(z) = d(z)f(z), \quad z \in \Omega_2^*$$

is invertible. We also have $HB = BH$ and hence $H^*B^* = B^*H^*$. The operator

$$\begin{pmatrix} I & 0 \\ 0 & (H^{-1})^* \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & H^* \end{pmatrix} = \begin{pmatrix} A & CH^* \\ 0 & (H^{-1})^*B^*H^* \end{pmatrix} = \begin{pmatrix} A & CH^* \\ 0 & B^* \end{pmatrix}$$

is then similar to M. But for all $g \in H^2(\Omega_2^*, \mu)$

$$CH^*g = \langle H^*g, 1_{\Omega_2^*}\rangle 1_{\Omega_1} = \langle g, H1_{\Omega_2^*}\rangle 1_{\Omega_1} = \langle g, d\rangle 1_{\Omega_1} = Dg,$$

i.e. $CH^* = D$ and X is similar to M.

Now we will show that $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$. Let $m = \max(\|d\|_{\sup}, \|d^{-1}\|_{\sup})$. Fix $\varepsilon > 0$ so that $30\varepsilon m < \frac{1}{2}$. Let γ be a holomorphic logarithm of d on Ω_2^* (see [14], Theorem 13.11), i.e. $\mathrm{e}^{\gamma} = d$. Fix a large n so that $m \cdot \|\mathrm{e}^{\gamma/n} - 1\|_{\sup} < \varepsilon$. Then we have

$$\|\mathrm{e}^{k\cdot\gamma/n} - \mathrm{e}^{(k-1)\cdot\gamma/n}\|_{\sup} < \varepsilon, \quad k = -n, -n+1, \ldots, n.$$

Let R_k be the multiplication by $\mathrm{e}^{k\cdot\gamma/n}$ on $H^2(\Omega_2^*)$, $k = 0, 1, \ldots, n$ and let S_k be the multiplication by $\mathrm{e}^{-k\cdot\gamma/n}$.

We have $R_n = H$, $\|R_k - R_{k+1}\| < \varepsilon$, $S_n = H^{-1}$, $\|S_k - S_{k+1}\| < \varepsilon$ for $k = 0, 1, \ldots, n-1$ and $R_kH = HR_k, S_k = R_k^{-1}$, $k = 0, 1, \ldots, n$. If we let both L and N equal B, we see that the conditions (i), (iii) in Lemma 2.1 (a) as well as the conditions (i), (iii) in Lemma 2.1 (b) are satisfied. Moreover, we have $R_nC^* = HC^* = (CH^*)^* = D^*$. We can now apply Corollary 2.2 to see that there exist operators R, S such that

(i) $R - R_0 = R - I$ is finite-dimensional, and

(ii) we have

$$\begin{aligned} \|CR^* - D\| = \|RC^* - D^*\| &< \varepsilon(\|C\| + 6\max\{\|R_0\|, \|R_1\|, \ldots, \|R_n\|\}) \\ &\leqslant \varepsilon(\|C\| + 6m), \end{aligned}$$

(iii) $\|RB - BR\| \leqslant 30\varepsilon\|H\| \leqslant 30\varepsilon m$,

(iv) $\|RS - I\| \leqslant 30\varepsilon m < \frac{1}{2}$ and $\|SR - I\| \leqslant 30\varepsilon m < \frac{1}{2}$.

Note that (i) says that R is of the form unitary plus compact. The condition (iv) implies that SR and RS are both invertible and $\|(SR)^{-1}\| < 2$, $\|(RS)^{-1}\| < 2$. Therefore R is invertible and $\|R^{-1}\| \leqslant 2\|S\| \leqslant 6m$, and we have

$$\|(R^*)^{-1}B^*R^* - B^*\| = \|RBR^{-1} - B\| = \|(RB - BR)R^{-1}\| \leqslant 180\varepsilon m^2.$$

Summing up, we get

$$\left\|\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & (R^{-1})^* \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix}\right\|$$
$$= \left\|\begin{pmatrix} A-A & D-CR^* \\ 0 & B^*-(R^*)^{-1}B^*R^* \end{pmatrix}\right\| \leqslant \varepsilon(\|C\|+6m)+180\varepsilon m^2.$$

Since now $\begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \in (\mathcal{U}+\mathcal{K})(H^2(\Omega_1,\mu)\oplus H^2(\Omega_2^*,\mu))$ and since ε can be chosen to be arbitrarily small, we see that

$$\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})\left(\begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}\right). \quad ∎$$

3. THE CLOSURE OF THE $(\mathcal{U}+\mathcal{K})$-ORBIT OF M

We shall now continue the investigation of the $(\mathcal{U}+\mathcal{K})$-orbit of our model. First we want to know which operators of the form $\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}$ are in $\mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$. The following is an easy corollary of Lemma 2.3.

LEMMA 3.1. *Let* $M=\begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}$ *be as constructed above. Let* $D=1_{\Omega_1}\otimes d^*$, *where* $d\in H^2(\Omega_2^*)$ *and set*

$$X=\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}.$$

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), *where*

(i) $d(z)\neq 0$ *for* $z\in\Omega_2^*$*;*
(ii) $\sigma(X)\cap\Omega_2=\emptyset$*;*
(iii) $X\in\mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Lemma 1.7.

Assume that d satisfies (i). Lemma 1.1 allows us to find a $d'\in\mathcal{C}(\mathrm{cl}(\Omega_2^*))$ such that $d'(z)\neq 0$ for $z\in\mathrm{cl}(\Omega_2^*)$ and $\|d-d'\|_{H^2(\Omega_2^*)}$ can be made arbitrarily small. (This construction is to be done on each component of Ω_2^* separately.)

By Lemma 2.3, we have

$$\begin{pmatrix} A & 1_{\Omega_1}\otimes d'^* \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M),$$

and hence $X\in\mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$. ∎

LEMMA 3.2. *Let* $k\in\mathbb{N}$. *Denote by* $\mathcal{F}_k$ *the subspace of* $H^2(\Omega_1)$ *spanned by* $1,z,z^2,\dots,z^k$. *Let* X *be an operator of the form*

$$X=\begin{pmatrix} A & F \\ 0 & B^* \end{pmatrix},$$

where F is a finite rank operator with $\operatorname{ran} F \subseteq \mathcal{F}_k$. *Then X is $(\mathcal{U}+\mathcal{K})$-equivalent to an operator of the form*

$$\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix},$$

where $D = 1_{\Omega_1} \otimes d^*$, for some $d \in H^2(\Omega_2^*)$. This $(\mathcal{U}+\mathcal{K})$-equivalence is of the form

$$X = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix},$$

where Z is a finite-rank operator.

Moreover, d can be calculated as $d = \mathcal{G}(F)$, where $\mathcal{G}$ is the linear map from

$$\{Y \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1)) : \operatorname{ran} Y \subseteq \mathcal{F}_k \text{ for some } k\}$$

into $H^\infty(\Omega_2^*)$ such that

$$\mathcal{G}(p \otimes g^*) = \theta(p)g,$$

where p is a polynomial on Ω_1, $g \in H^2(\Omega_2^*)$ and θ is a bounded map from $H^2(\Omega_1)$ into $H^\infty(\Omega_2^*)$ defined by

$$\theta(f)(z) = \overline{f(\overline{z})}, \quad z \in \Omega_2^*.$$

Proof. Observe that

$$\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} A & F \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & F + ZB^* - AZ \\ 0 & B^* \end{pmatrix}.$$

We will find a compact Z such that $G = F + ZB^* - AZ$ satisfies $\operatorname{ran} G \subseteq \mathcal{F}_{k-1}$ (for $k > 1$). The lemma will then follow by induction. (Note that the expression $ZB^* - AZ$ is linear in the variable Z.)

Let g_0 be such that $F - z^n \otimes g_0^* \in \mathcal{F}_{k-1}$ and let $Z = z^{n-1} \otimes g_0^*$. Then

$$\begin{aligned}(F + ZB^* - AZ)g &= Fg + \langle B^* g, g_0\rangle z^{n-1} - \langle g, g_0\rangle A z^{n-1} \\ &= (Fg - \langle g, g_0\rangle z^n) + \langle g, z \cdot g_0\rangle z^{n-1} \in \mathcal{F}_{k-1}.\end{aligned}$$

The fact that $d = \mathcal{G}(F)$ can be verified by an easy calculation. The boundedness of θ follows from the Cauchy theorem. ∎

LEMMA 3.3. *Suppose $f \in H^2(\Omega_1)$, $g \in H^2(\Omega_2^*)$. Define $g_0 \in H^2(\Omega_2^*)$ by $g_0 = \theta(f)g$, where θ is the map introduced in Lemma* 3.2. *Then for every $\varepsilon > 0$ there exists a finite-rank operator $X : H^2(\Omega_2^*) \to H^2(\Omega_1)$ for which*

$$\|f \otimes g^* - XB^* + AX - 1_{\Omega_1} \otimes g_0^*\| < \varepsilon.$$

In particular, we have

$$\begin{pmatrix} A & f \otimes g^* \\ 0 & B^* \end{pmatrix} \in \operatorname{cl}(\mathcal{U}+\mathcal{K}) \left(\begin{pmatrix} A & 1_{\Omega_1} \otimes g_0^* \\ 0 & B^* \end{pmatrix} \right),$$

$$\begin{pmatrix} A & 1_{\Omega_1} \otimes g_0^* \\ 0 & B^* \end{pmatrix} \in \operatorname{cl}(\mathcal{U}+\mathcal{K}) \left(\begin{pmatrix} A & f \otimes g^* \\ 0 & B^* \end{pmatrix} \right).$$

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of polynomials such that $f_k \to f$ in $H^2(\Omega_1)$. Then by Lemma 3.2, there exist a finite-rank $Z_k : H^2(\Omega_2^*) \to H^2(\Omega_1)$ and $g_k \in H^2(\Omega_2^*)$ such that

$$f_n \otimes g^* - Z_k B^* + AZ_k = 1_{\Omega_1} \otimes g_k^*,$$

for $k = 1, 2, \ldots$.

Recall that $g_k = \theta(f_k) \cdot g$. We have $f_k \to f$ in $H^2(\Omega_1)$, hence $\theta(f_k) \to \theta(f)$ in $H^\infty(\Omega_2^*)$ and $g_k = \theta(f_k) \cdot g \to g_0 = \theta(f) \cdot g$ in $H^2(\Omega_2^*)$.

With $\varepsilon > 0$ given, choose k such that $\|g_k - g_0\| < \varepsilon/2$, $\|f_k - f\| < \varepsilon/2\|g\|$ and set $X = Z_k$. Then we have

$$\begin{aligned}
&\|f \otimes g^* - XB^* + AX - 1_{\Omega_1} \otimes g_0^*\| \\
&= \|(f_k \otimes g^* - Z_k B^* + AZ_k - 1_{\Omega_1} \otimes g_0^*) + (f - f_k) \otimes g^* + 1_{\Omega_1} \otimes (g - g_0)^*\| \\
&= \|0 + (f - f_k) \otimes g^* + 1_{\Omega_1} \otimes (g - g_0)^*\| < \varepsilon.
\end{aligned}$$

The second statement follows easily using $(\mathcal{U}+\mathcal{K})$-similarities of the form $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$. ∎

COROLLARY 3.4. *Let $W \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1))$ be finite-rank, say $W = \sum_{i=1}^{k} f_i \otimes g_i^*$. Set $g_0 = \sum_{i=1}^{k} \theta(f_i) \cdot g_i$. Then for every $\varepsilon > 0$ there exists a finite-dimensional operator $X : H^2(\Omega_2^*) \to H^2(\Omega_1)$ for which*

$$\|W - XB^* + AX - 1_{\Omega_1} \otimes g_0^*\| < \varepsilon.$$

Proof. This follows from the linearity of the expression $XB^* - AX$ in X. ∎

LEMMA 3.5. *Let $W \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1))$ be compact. Then there exists a $g \in H^2(\Omega_2^*)$ such that*

$$\begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K}) \left(\begin{pmatrix} A & 1_{\Omega_1} \otimes g^* \\ 0 & B^* \end{pmatrix} \right)$$
$$\begin{pmatrix} A & 1_{\Omega_1} \otimes g^* \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K}) \left(\begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix} \right).$$

Proof. Choose $\varepsilon > 0$. Choose $z_0 \in \Omega_1$ arbitrarily. Let $\psi \in H^2(\Omega_1)$ be the function constructed in Lemma 1.2. Let $\{e_k\}_{k=1}^{\infty}$, $e_n = \psi^n \cdot e_0$ be the basis of $H^2(\Omega_1)$ constructed in Lemma 1.5. Denote $r = \max\{|\psi(z)| : z \in \mathrm{cl}(\Omega_2)\}$ and note that, by the maximum modulus principle, we have $r < 1$.

Write W as

$$W = \sum_{i=0}^{\infty} e_i \otimes h_i^*$$

and let

$$W_k = \sum_{i=0}^{k} e_i \otimes h_i^*, \quad k = 0, 1, \ldots.$$

Then by Corollary 3.4, if we define $g_k \in H^2(\Omega_2^*)$ by $g_k = \sum_{i=0}^{k} \theta(e_i) \cdot h_i$, one can find a finite dimensional $X_k : H^2(\Omega_2^*) \to H^2(\Omega_1)$ such that

$$\|W_k - X_k B^* + AX_k - 1_{\Omega_1} \otimes g_k^*\| < \varepsilon.$$

Observe that

$$\|\theta(e_i)\|_{H^\infty(\Omega_2^*)} \leqslant M\|\psi^i|\Omega_2\|_\infty = Mr^i, \quad M = \max_{\mathrm{cl}(\Omega_2^*)}\{e_0\}$$

and hence

$$\left\|\sum_{i=k}^{\infty} \theta(e_i) \cdot h_i\right\| \leqslant \sum_{i=k}^{\infty} Mr^i\|W\| = M\|W\|\frac{r^k}{1-r}.$$

Therefore the sequence g_k has a limit in $H^2(\Omega_2^*)$. We shall call it g.

We can now choose k_0 such that $\|W - W_{k_0}\| < \varepsilon$ and $M\frac{r^{k_0+1}}{1-r} < 1$. Then we have $\|h_i\| < \|W - W_{k_0}\| < \varepsilon$, $i > k_0$ and hence

$$\|g - g_{k_0}\| \leqslant \sum_{i=k_0+1}^{\infty} Mr^i\varepsilon = \varepsilon M\frac{r^{k_0+1}}{1-r} < \varepsilon.$$

We have

$$\begin{aligned}
&\|W - X_{k_0}B^* + AX_{n_0} - 1_{\Omega_1} \otimes g^*\| \\
&\leqslant \|W_{k_0} - X_{k_0}B^* + AX_{n_0} - 1_{\Omega_1} \otimes g_{k_0}^*\| + \|W - W_{k_0}\| + \|g - g_{k_0}\| \\
&\leqslant \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\end{aligned}$$

Both statements of the lemma now follow easily using $(\mathcal{U} + \mathcal{K})$-similarities of the form $\begin{pmatrix} I & X_{k_0} \\ 0 & I \end{pmatrix}$. ∎

Lemmas 3.5 and 3.1 together give

COROLLARY 3.6. *Let* $X = \begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix}$, *$W$ compact. Suppose that* $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$. *Then* $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

This is as far as we are able to get with the investigation of the original model. We shall now restrict the class of models we are investigating. This will also restrict the class of spectral pictures we study. We will subsequently use functional calculus to get back to the original class of spectral pictures.

THEOREM 3.7. *Assume that in the model* $M = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}$ *constructed above we have* $\Omega_1 = \mathbb{D}$ *and* μ *is the arc length measure. In other words, A is unitarily equivalent to the forward unilateral shift. Let X be an essentially normal operator such that*

(i) $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$;
(ii) $\sigma_{\mathrm{e}}(X) = \sigma_{\mathrm{e}}(M) = \partial\Omega_1 \cup \partial\Omega_2$;
(iii) $\mathrm{ind}(X - \lambda) = \mathrm{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \mathrm{cl}(\Omega_2)$.

Then $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

Proof. By the Brown–Douglas–Fillmore theorem ([3]), if X satisfies the above conditions, then there exists a unitary U and a compact L so that, setting $K = ULU^*$, we have $X = U^*MU + L = U^*(M+K)U$. Thus it suffices to show that $M + K \in \mathrm{cl}((\mathcal{U}+\mathcal{K})M)$. Let us therefore assume without loss of generality that $X = M + K$, where K is compact.

Fix $\varepsilon > 0$. Let $\{1, z, z^2, \ldots\}$ be the canonical basis of $H^2(\Omega_1, \mu) = H^2(\mathbb{D})$. For each $i = 1, \ldots n$, use Lemma 1.5 to construct a basis of $H^2(\Omega_{2,i}, \mu_i)$ with respect to which the matrix of $M(\Omega_{2,i}, \mu_i)$ is a Toeplitz matrix. Denote the k-th element of this basis as e'_{ik}.

We shall now construct a basis $\{e_k\}_{k=1}^{\infty}$ of $H^2(\Omega_1, \mu) \oplus \Big(\bigoplus_{i=1}^{n} H^2(\Omega^*_{2,i}, \mu_i)\Big)$ as follows:

$$e_k = z^l, \qquad \text{for } k = l(n+1)+1,\ l = 0, 1, \ldots,$$
$$e_k = e'_{j,l}, \qquad \text{for } k = l(n+1)+j,\ l = 0, 1, \ldots,\ j = 1, 2 \ldots, n.$$

Let P_k be the orthogonal projection onto $\mathrm{span}\{e_i\}_{i=0}^{k}$, $k = 0, 1, \ldots$. Since K is compact, the sequence $\{M + P_k K P_k\}_{k=1}^{\infty}$ converges to $M+K$ in the norm. Find k_0 such that $\|P_{k_0} K P_{k_0} - K\| < \varepsilon$ and $\sigma(M + P_{k_0} K P_{k_0} - K) \subseteq (\sigma(X))_\varepsilon$. Denote $K_0 = P_{k_0} K P_{k_0} - K$ and notice that $X_1 = M + K_0$ is of the form

$$\begin{pmatrix} A & C_1 & 0 \\ & F_1 & C_3 \\ & & B^* \end{pmatrix}.$$

Notice that the entries of the matrix of C_1 are zeros except for the bottom row, i.e. $\mathrm{ran}\, C_1 \subseteq \mathrm{span}\{1_{\Omega_1}\}$, if we consider C_1 as an operator from $\mathbb{C}^{2k_0}$ into $H^2(\mathbb{D})$. We want to show that X_1 is close to $(\mathcal{U}+\mathcal{K})(M)$.

Let F_2 be a perturbation of F_1 such that

- $\|F_2 - F_1\| < \varepsilon$;
- $\sigma(F_2) \subseteq \Omega_1$;
- the eigenvalues of F_2 are simple.

Let

$$X_2 = \begin{pmatrix} A & C_1 & 0 \\ & F_2 & C_3 \\ & & B^* \end{pmatrix}.$$

Then we have $\|X_2 - X\| < 2\varepsilon$ and $\sigma(X_2) \subseteq \mathrm{cl}(\Omega_1)$.

The fact that F_2 has simple eigenvalues allows us now to use Lemma 1.4 to see that an arbitrarily small perturbation of C_1 to C'_1 will cause $\begin{pmatrix} A & C'_1 \\ & F_2 \end{pmatrix}$ to have no eigenvalues and to be $(\mathcal{U}+\mathcal{K})$-equivalent to A. Moreover, one can do this so that $\mathrm{ran}(C_1 - C'_1) \subseteq \mathrm{span}\{1_{\Omega_1}\}$. Choose such a perturbation small enough so that in addition to this we have $\|C_1 - C'_1\|$ and the spectrum of

$$X_3 = \begin{pmatrix} A & C'_1 & 0 \\ & F_2 & C_3 \\ & & B^* \end{pmatrix}$$

lies in $(\sigma(X))_\varepsilon \cap \mathrm{cl}(\Omega_1)$.

We now have the following situation:

(i) $\|X_3 - X\| < 3\varepsilon$;

(ii) $\sigma(X_3) \subseteq \mathrm{cl}(\Omega_1)$;

(iii) $\begin{pmatrix} A & C_1' \\ & F_2 \end{pmatrix}$ has no eigenvalues and is $(\mathcal{U}+\mathcal{K})$-equivalent to A;

(iv) X_3 may have eigenvalues in Ω_2. These are not more than ε away from $\partial\Omega_2$. There may be infinitely many of these. If this is the case, any cluster point of the set of eigenvalues will be in $\partial\Omega_2$ (because $\sigma_{\mathrm{e}}(X_3) = \partial(\Omega_1) \cup \partial(\Omega_2)$).

In the next step, we want to find X_4 close to X_3 with the same properties except that there will be only finitely many eigenvalues in Ω_2.

Notice that condition (iii) implies that X_3 is $(\mathcal{U}+\mathcal{K})$-equivalent to an operator of the form $Y = \begin{pmatrix} A & L \\ & B^* \end{pmatrix}$, L compact, which of course has the same spectral properties as X_3. We want to show that there is an X_4 of the form

$$X_4 = \begin{pmatrix} A & C_1' & 0 \\ & F_2 & C_3' \\ & & B^* \end{pmatrix}$$

such that $\|X_3 - X_4\| < \varepsilon$, and X_4 has the desired spectral properties. This will easily follow once we prove:

CLAIM. *Let $Y = \begin{pmatrix} A & F \\ & B^* \end{pmatrix}$, F finite-dimensional. Suppose that Y has the spectral properties described above for X_3. Let $\eta > 0$. Then there exists a F' such that $Y' = \begin{pmatrix} A & F' \\ & B^* \end{pmatrix}$ has only finitely many eigenvalues in Ω_2 and $\|F - F'\| < \eta$,* $\operatorname{ran} F' \subseteq \operatorname{ran} F$.

Proof of the Claim. Suppose $F = \sum_{i=1}^{n_0} f_i \otimes g_i^*$, $g_i \in H^2(\Omega_2^*)$, $f_i \in H^2(\Omega_1)$. Using Runge's theorem and the definition of $H^2(\Omega_2^*)$, we can find polynomials g_i' such that for $F' = \sum_{i=1}^{n_0} f_i \otimes {g_i'}^*$ we have $\|F - F'\| < \varepsilon$ and moreover $\|F - F'\|$ is small enough so that $\Omega_2 \setminus \sigma(Y') \neq \emptyset$. From Lemmas 1.7 and 3.3, we know that the eigenvalues of Y' inside Ω_2 correspond to the zeros of $k = \sum_{i=1}^{n_0} \theta(f_i) \cdot g_i'$. This is (can be extended to) a holomorphic function on Ω_1. If k had infinitely many zeros in Ω_2, it would be a constant equal to zero, causing $\Omega_2 \setminus \sigma(Y') = \emptyset$, contradiction. This proves the claim, we can now resume the proof of Proposition 3.7. ∎

We now have

$$X_4 = \begin{pmatrix} A & C_1' & 0 \\ & F_2 & C_3' \\ & & B^* \end{pmatrix}$$

with respect to $\mathcal{H} = H^2(\Omega_1) \oplus \mathbb{C}^{k_0} \oplus H^2(\Omega_2^*)$. We know that $\operatorname{ran} C_1' \subseteq \operatorname{span}\{1_{\Omega_1}\}$, $\|X_4 - X\| < 4\varepsilon$, $\sigma(X_4) \subseteq \mathrm{cl}(\Omega_2)$, $\begin{pmatrix} A & C_1' \\ & F_2 \end{pmatrix}$ has no eigenvalues and is $(\mathcal{U}+\mathcal{K})$-equivalent to A. X_4 may have eigenvalues in Ω_2. These are not further than ε away from $\partial\Omega_2$ and there are only finitely many of them. Suppose these eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Denote $H_0 = \operatorname{span}\{H(\lambda_1; X_4), H(\lambda_2; X_4), \ldots, H(\lambda_m; X_4)\}$. If we knew that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$, i.e. H_0 is a subspace of the underlying space of F_2, we could move the eigenvalues λ_i away from Ω_2 by perturbing F_2. As we shall see next, it is true that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$ can be achieved by altering the decomposition of $\mathcal{H}$.

For each $i = 1, 2, \ldots, m$, let n_i be such that $H(\lambda_i; X_4) \subseteq \ker(X_4 - \lambda_i)^{n_i}$. Note that for any λ and k, X_4^k is of the form

$$\begin{pmatrix} A^k & C_1'' & C_2'' \\ & F_2^k & C_3'' \\ & & B^{*k} \end{pmatrix}.$$

One can verify by induction that

$$\operatorname{ran}\begin{pmatrix} C_1'' & C_2'' \end{pmatrix} \subseteq \operatorname{span}\{1, (x-\lambda), \ldots, (x-\lambda)^{k-1}\}.$$

If now $\begin{pmatrix} f \\ h \\ g \end{pmatrix}$ is in $\ker(X_4 - \lambda_i)^k$, we have

$$\begin{aligned}(X_4 - \lambda_i)^k \begin{pmatrix} f \\ h \\ g \end{pmatrix} &= \begin{pmatrix} A^k - \lambda_i & C_1'' & C_2'' \\ & F_2^k - \lambda_i & C_3'' \\ & & B^{*k} - \lambda_i \end{pmatrix} \begin{pmatrix} f \\ h \\ g \end{pmatrix} \\ &= \begin{pmatrix} (A^k - \lambda_i) f + C_1'' h + C_2'' g \\ (F_2^k - \lambda_i) h + C_3'' g \\ (B^{*k} - \lambda_i) g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Now $(A^k - \lambda_i) f$ is linearly independent of $\operatorname{ran}\begin{pmatrix} C_1'' & C_2'' \end{pmatrix}$, forcing $(A^k - \lambda_i) f = 0$ and hence $f = 0$. This implies that $H_0 \perp H^2(\Omega_1)$.

Notice also that the $H^2(\Omega_2^*)$ component of any vector in $\ker(X_4 - \lambda_i)^k$ (denoted here by g) is in $\ker(B^* - \lambda_i)^k$.

We use Lemma 1.3 n_1 times to find vectors $\{f_1, f_2, \ldots, f_{n_1}\}$ such that the matrix of B^* is

$$\begin{pmatrix} \lambda_1 & \cdots & & & \\ & \lambda_1 & \cdots & & \\ & & \ddots & & \\ & & & \lambda_1 & \cdots \\ & & & & B^{*\prime} \end{pmatrix}$$

with respect to the decomposition $\operatorname{span}\{f_1, f_2, \ldots, f_{n_1}\} \oplus \operatorname{span}\{f_1, f_2, \ldots, f_{n_1}\}^\perp$, where $B^{*\prime}$ is a unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_1)^{n_1} = \operatorname{span}\{f_1, f_2, \ldots, f_{n_1}\}$. We can now continue in this manner until we can write B^* as

$$\begin{pmatrix} \lambda_1 & \cdots & & & & & & \\ & \lambda_1 & \cdots & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_1 & \cdots & & & \\ & & & & \lambda_2 & \cdots & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_m & \cdots \\ & & & & & & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition $\operatorname{span}\{f_1, f_2, \dots, f_{n_0}\} \oplus \operatorname{span}\{f_1, f_2, \dots, f_{n_0}\}^\perp$, where $n_0 = \sum\limits_{i=1}^{m} n_i$ and $B^{*\prime\prime}$ is another unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_i)^{n_i} \subseteq \operatorname{span}\{f_1, f_2, \dots, f_{n_0}\}$ for $i = 1, 2, \dots, m$ and consequently we have

$$H_0 \perp (H^2(\Omega_2^*) \ominus \operatorname{span}\{f_1, f_2, \dots, f_{n_0}\}).$$

We can now rewrite X_4 as

$$X_4 = \begin{pmatrix} A & C_1' & & \\ & F_2 & C_{31}' & C_{32}' \\ & & B_1 & B_2 \\ & & & B^{*\prime\prime} \end{pmatrix} = \begin{pmatrix} A & C_1' & 0 \\ & F_3 & B_2' \\ & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition

$$\begin{aligned} \mathcal{H} &= H^2(\Omega_1) \oplus \mathbb{C}^{k_0} \oplus \operatorname{span}\{f_1, f_2, \dots, f_{n_0}\} \oplus H^2(\Omega_2^*)' \\ &\cong \mathcal{H} = H^2(\Omega_1) \oplus \mathbb{C}^{k_0+n_0} \oplus H^2(\Omega_2^*)' \end{aligned}$$

where $H^2(\Omega_2^*)'$ is an isometrically isomorphic copy of $H^2(\Omega_2^*)$. We have now $H_1 \subseteq \mathbb{C}^{k_0} \oplus \operatorname{span}\{f_1, f_2, \dots, f_{n_0}\} \cong \mathbb{C}^{k_0+n_0}$. We shall denote $B^{*\prime\prime}$ as B^* from now on.

For $i = 1, 2, \dots, m$, let I_i be the identity operator on

$$\operatorname{span}_{j=1}^{i} H(\lambda_j; X_4) \ominus \operatorname{span}_{j=1}^{i-1} H(\lambda_j; X_4)$$

and let $\lambda_i' \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$ be chosen so that $|\lambda_i - \lambda_i'| < \varepsilon$. Set

$$X_5 = X_4 + \sum_{i=1}^{m} (\lambda_i' - \lambda_i) I_i.$$

Then X_5 is of the form

$$X_5 = \begin{pmatrix} A & C_1' & 0 \\ & F_4 & B_2' \\ & & B^* \end{pmatrix},$$

we have $\sigma(X_5) = \operatorname{cl}(\Omega_1) \setminus \Omega_2$, $\|X_5 - X\| < 5\varepsilon$ and λ_i', $i = 1, 2, \dots, m$, are the only points with $\operatorname{nul}(X_5 - \lambda_i) > 0$.

Next, we decompose the finite dimensional operator F_4 as

$$F_4 = \begin{pmatrix} F_5 & D_1 \\ & F_6 \end{pmatrix},$$

where $\sigma(F_5) \subseteq \operatorname{cl}(\Omega_2)$ and $\sigma(F_6) \cap \operatorname{cl}(\Omega_2) = \emptyset$. We have

$$X_5 = \begin{pmatrix} A & D_2 & D_3 & \\ & F_5 & D_1 & D_4 \\ & & F_6 & D_5 \\ & & & B^* \end{pmatrix}.$$

Since $\sigma(F_6) \cap \sigma(B^*) = \emptyset$, Corollary 2.5 of [6] allows us to find a Z such that

$$\begin{pmatrix} I & -Z \\ & I \end{pmatrix} \begin{pmatrix} F_6 & D_5 \\ & B^* \end{pmatrix} \begin{pmatrix} I & Z \\ & I \end{pmatrix} = \begin{pmatrix} F_6 & 0 \\ & B^* \end{pmatrix}.$$

The operator X_5 is now $(\mathcal{U}+\mathcal{K})$-equivalent to

$$X_6 = \begin{pmatrix} I &&& \\ & I && \\ && I & -Z \\ &&& I \end{pmatrix} X_5 \begin{pmatrix} I &&& \\ & I && \\ && I & Z \\ &&& I \end{pmatrix} = \begin{pmatrix} A & D_2 & D_3' & D_6' \\ & F_5 & D_1' & D_4' \\ && F_6 & 0 \\ &&& B^* \end{pmatrix}.$$

Let $M_1 = \left\| \begin{pmatrix} I & -Z \\ & I \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} I & Z \\ & I \end{pmatrix} \right\|$. Recall that the only eigenvectors of X_5 were of the form $\begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}$ and hence the eigenvectors of X_6 will in fact be the same:

$$\begin{pmatrix} I &&& \\ & I && \\ && I & -Z \\ &&& I \end{pmatrix} \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}.$$

Next, consider Lemma 1.4 and perturb D_3', D_1', F_6 so that

$$\begin{pmatrix} A & D_2 & E_1 \\ & F_5 & E_2 \\ && F_7 \end{pmatrix} \cong_{\mathcal{U}+\mathcal{K}} A, \qquad \left\| \begin{pmatrix} D_3' \\ D_1' \\ F_6 \end{pmatrix} - \begin{pmatrix} E_1 \\ E_2 \\ F_7 \end{pmatrix} \right\| < \frac{\varepsilon}{M_1},$$

and F_7 has simple eigenvalues in $\Omega_1 \setminus \mathrm{cl}(\Omega_2)$.

Then

$$X_7 = \begin{pmatrix} A & D_2 & E_1 & D_6' \\ & F_5 & E_2 & D_4' \\ && F_7 & 0 \\ &&& B^* \end{pmatrix}$$

is $(\mathcal{U}+\mathcal{K})$-equivalent to an operator of the form

$$\begin{pmatrix} A & K_7 \\ & B^* \end{pmatrix},$$

where K_7 is compact.

Let us check if X_7 has any eigenvalues in Ω_2. Suppose $\lambda \in \Omega_2$ and

$$(X_7 - \lambda) \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} A-\lambda & D_2 & E_1 & D_6' \\ & F_5 - \lambda & E_2 & D_4' \\ && F_7 - \lambda & 0 \\ &&& B^* - \lambda \end{pmatrix} \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have $(F_7 - \lambda)h = 0$ and hence $h = 0$. But then

$$(X_7 - \lambda) \begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = (X_6 - \lambda) \begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is a contradiction — X_6 does not have any such eigenvalues.

So X_7 has no eigenvalues in Ω_2 and by Corollary 3.6, we have $X_7 \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$. Hence

$$\mathrm{dist}(X_6, (\mathcal{U}+\mathcal{K})(M)) < \frac{\varepsilon}{M_1},$$

which implies

$$\operatorname{dist}(X_5, (\mathcal{U}+\mathcal{K})(M)) < \varepsilon,$$

and hence

$$\operatorname{dist}(X, (\mathcal{U}+\mathcal{K})(M)) < 6\varepsilon.$$

This last statement holds for any $\varepsilon > 0$, so finally $X \in \operatorname{cl}(\mathcal{U}+\mathcal{K})(M)$. ∎

Finally, we can use functional calculus to obtain models for the original more general class of spectral pictures.

COROLLARY 3.8. *Let $\Omega = \Omega_1 \setminus \operatorname{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \bigcup\limits_1^n \Omega_{2,i}$, $\operatorname{cl}(\Omega_2) \subseteq \Omega_1$. Let φ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\varphi|\mathbb{D}$ is a conformal map of $\mathbb{D}$ onto Ω_1. (This is the map which was used to construct $H^2(\Omega_1)$.) Then we have $\varphi^{-1}(\Omega) = \mathbb{D} \setminus \operatorname{cl}(\Omega_2')$, where Ω_2' is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2' = \bigcup\limits_1^n \Omega_{2,i}'$, $\operatorname{cl}(\Omega_2') \subseteq \mathbb{D}$. Let now μ be a measure on $\partial(\mathbb{D})$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial({\Omega_{2,i}'}^*)$ equivalent to the arc length measure; all these measures are assumed to be equivalent to the respective arc length measures. Let $M' = \begin{pmatrix} M(\mathbb{D},\mu) & C \\ 0 & M({\Omega_{2,i}'}^*) \end{pmatrix}$ be the model constructed above. Then $M = \varphi(M')$ has the following spectral properties:*

(i) $\sigma(M) = \operatorname{cl}(\Omega)$;
(ii) $\sigma_{\mathrm{e}}(M) = \partial\Omega$;
(iii) $\operatorname{nul}(M - z) = 0$, $z \in \Omega$;
(iv) $\operatorname{nul}(M^* - \overline{z}) = 1$, $z \in \Omega$;
(v) $\operatorname{ind}(M - z) = -1$, $z \in \Omega$.

Let X be an essentially normal operator such that:

(i) $\sigma(X) = \sigma(M) = \operatorname{cl}(\Omega_1) \setminus \Omega_2$;
(ii) $\sigma_{\mathrm{e}}(X) = \sigma_{\mathrm{e}}(M) = \partial\Omega_1 \cup \partial\Omega_2$;
(iii) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$.

Then $X \in \operatorname{cl}(\mathcal{U}+\mathcal{K})(M)$.

Proof. The proof is very similar to that of Theorem 0.8. Lemmas 0.6 and 0.7 allow us to verify that M has the spectral properties described in the statement. Note that M' is the type of operator for which Proposition 3.7 provides conditions that are sufficient for an operator to lie in $\operatorname{cl}(\mathcal{U}+\mathcal{K})(M')$. Using Lemmas 0.6 and 0.7 again, we see that $\varphi^{-1}(X)$ satisfies the conditions of Proposition 3.7 and hence $\varphi^{-1}(X) \in \operatorname{cl}(\mathcal{U}+\mathcal{K})(M')$. Now $X \in \operatorname{cl}(\mathcal{U}+\mathcal{K})(\varphi(M')) = \operatorname{cl}(\mathcal{U}+\mathcal{K})(M)$ by Lemma 0.5. ∎

THEOREM 3.9. *Let $M = \varphi(M')$ be as in Corollary 3.8. Then $X \in \operatorname{cl}(\mathcal{U}+\mathcal{K})(M)$ if and only if:*

(i) $\sigma(X) = \sigma(M) = \operatorname{cl}(\Omega_1) \setminus \Omega_2$ *or* $\sigma(X) = \operatorname{cl}(\Omega_1)$;
(ii) $\sigma_{\mathrm{e}}(X) = \sigma_{\mathrm{e}}(M) = \partial\Omega_1 \cup \partial\Omega_2$;
(iii) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$;
(iv) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = 0$ *for* $\lambda \in \Omega_2$.

Note that if we know that $X = M + K$, where K is compact, the only condition which is not satisfied automatically is condition (i).

Proof. The necessity of these conditions is easily verified.

Suppose that X satisfies the conditions of the theorem. We can use Proposition 4.4 of [2] to find an operator X_0 such that $\|X - X_0\|$ is arbitrarily small, and X_0 satisfies the conditions of Corollary 3.8. Then $X_0 \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$ and consequently $X \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$. ∎

4. FURTHER COMMENTS

In this chapter we shall state two open questions related to the results of this paper.

The model investigated in Corollary 3.8 only allows for the index to be equal to -1 in the interior of the spectrum. A model with index equal to 1 can be dealt with easily using the adjoint. One would however like to know the following.

QUESTION 4.1. Suppose M is the model investigated in Corollary 3.8. Let $i > 1$. What is $\mathrm{cl}(\mathcal{U}+\mathcal{K})\Big(\bigoplus\limits_{k=1}^{i} M\Big)$?

Finally, whenever the closure of the $(\mathcal{U}+\mathcal{K})$-orbit is described for a model with a certain spectral picture, one may wish to go beyond the model and investigate the whole class of operators sharing the same spectral picture.

QUESTION 4.2. Suppose M is the model investigated here. Call an operator X M-like if it has the same spectral picture (including nullity) as M. Is it true that for any M-like operator, we have $M \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(X)$?

If the answer to this question is affirmative, transitivity of the relation $\rightarrow_{\mathcal{U}+\mathcal{K}}$ implies that $\mathrm{cl}(\mathcal{U}+\mathcal{K})(X) = \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$ whenever X is an M-like operator. (Compare [12].)

See also the final section of [4] for more open questions concerning $(\mathcal{U}+\mathcal{K})$-orbits of essentially normal operators.

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