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PROJECTIVE MODULES OVER NON-COMMUTATIVE TORI: CLASSIFICATION OF MODULES WITH CONSTANT CURVATURE CONNECTION

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Dedicated to Professor D.B. Fuchs on his 60th birthday

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Abstract. We study finitely generated projective modules over noncommutative tori. We prove that for every module E with constant curvature connection the corresponding element $[E]$ of the K-group is a generalized quadratic exponent and, conversely, for every positive generalized quadratic exponent μ in the K-group one can find a module E with constant curvature connection such that $[E] = \mu$. In physical words we give necessary and sufficient conditions for existence of 1/2 BPS states in terms of topological numbers.

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1. INTRODUCTION

In present paper we study projective modules over non-commutative tori. (We always consider finitely generated projective modules.) Our main goal is to describe all modules that admit constant curvature connections. It is well known that constant curvature connections correspond to maximally supersymmetric BPS fields ([4]); this means that we give conditions for existence of 1/2 BPS states.

The main results of the paper are formulated in the following theorems.

Theorem *Let $\mathcal{A}_\theta$ be a non-commutative torus. Then for every projective $\mathcal{A}_\theta$-module E with a constant curvature connection the corresponding element of the group $\mathrm{K}_0(\mathcal{A}_\theta)$ is a generalized quadratic exponent. Conversely, if μ is a positive generalized quadratic exponent in $\mathrm{K}_0(\mathcal{A}_\theta)$ then there exists a projective module E with constant curvature connection such that $[E] = \mu$. (Here $[E]$ stands for the*

K-*theory class of* E. *The definition of generalized quadratic exponent will be given later.)*

THEOREM *Let* $\mathcal{A}_\theta$ *be an irrational non-commutative torus. In this case projective modules over* $\mathcal{A}_\theta$ *which admit constant curvature connection are in one-to-one correspondence with positive generalized quadratic exponents in* $\mathrm{K}_0(\mathcal{A}_\theta)$.

This theorem is an immediate consequence of Theorem 1.1 and of the following very strong result by M. Rieffel (see [9]): for irrational non-commutative torus $\mathcal{A}_\theta$ the projective modules E and F are isomorphic if and only if the classes $[E], [F] \in K_0(\mathcal{A}_\theta)$ are equal.

Our main results were formulated and partially proved in [7], Appendix D. It is assumed in [7] that every linear combination of entries of the matrix θ is irrational. It is proved that in this case a projective module can be transformed into a free module by means of complete Morita equivalence iff corresponding K-theory class is a generalized quadratic exponent. This statement can be used to prove Theorem 1.1 in the conditions of Appendix D of [7]).

The paper is organized as follows. In the Introduction we remind the main notions and results we need and explain how we plan to prove the main theorem. In Section 2 we introduce the notion of generalized quadratic exponent and we study its properties. Section 3 is about integral generalized quadratic exponents and finite dimensional representations of rational non-commutative tori. The statements of Section 3 can be derived easily from well known results, but we don't know any reference containing them precisely, in the form we need. In Section 4 we present a proof of main results.

Let us remind the definition of a non-commutative torus (see [10a] for more details). Let L be the lattice $\mathbb{Z}^n$ in the vector space $V^* = \mathbb{R}^n$. Let θ be a real valued skew-symmetric bilinear form on $\mathbb{R}^n$. We will think about θ as a two-form, that is an element of $\Lambda^2 V$. The non-commutative torus $\mathcal{A}_\theta$ is the universal $\mathbb{C}^*$-algebra generated by unitary operators U_α, $\alpha \in L$ obeying relations

$$U_\alpha U_\beta = \mathrm{e}^{\pi \mathrm{i} \theta(\alpha,\beta)} U_{\alpha+\beta}. \tag{1.1}$$

Any element from $\mathcal{A}_\theta$ can be represented uniquely by a sum $a = \sum\limits_{\alpha \in L} c_\alpha U_\alpha$, where c_α are complex numbers. Assigning to every $a \in \mathcal{A}_\theta$ the coefficient c_0 in the representation above we obtain a canonical trace τ on $\mathcal{A}_\theta$.

Let $\{e_i\}$ be a basis of L. One can say that $\mathcal{A}_\theta$ is the universal $\mathbb{C}^*$-algebra generated by unitary operators $U_1, \dots, U_n$ obeying the relations

$$U_i U_j = \mathrm{e}^{2\pi \mathrm{i} \theta(e_i, e_j)} U_j U_i. \tag{1.2}$$

To check that these two definitions are equivalent one should take $U_i = U_{e_i}$.

The transformations $\delta_k U_{e_k} = U_{e_k}$, $1 \leqslant k \leqslant n$, $\delta_l U_{e_k} = 0$, $k \neq l$, $1 \leqslant k$, $l \leqslant n$ can be regarded as generators of the abelian Lie algebra L_θ of infinitesimal automorphisms on $\mathcal{A}_\theta$. We can naturally identify L_θ with V. Let us remind the definition of a connection in a $\mathcal{A}_\theta$-module following [2] (we do not need the general notion of connection from [3]). First we need the notion of a smooth part of a projective module.

Any element from $\mathcal{A}_\theta$ can be considered as a (generalized) function on the n-dimensional torus whose Fourier coefficients are c_α (see above). The space of smooth functions on $\mathbb{T}^n$ forms a subalgebra of $\mathcal{A}_\theta$. We denote it by ${\mathcal{A}_\theta}^{\text{smooth}}$ and call it the smooth part of $\mathcal{A}_\theta$. If E is a projective $\mathcal{A}_\theta$-module one can define its smooth part E^{smooth} in a similar manner (see [9]). A connection on the projective module E can be defined as follows:

An $\mathcal{A}_\theta$*-connection* on a right $\mathcal{A}_\theta$-module E is a linear map $\nabla : L_\theta \to \operatorname{End}_{\mathbb{C}} E$, satisfying the condition

$$\nabla_\delta(ea) = (\nabla_\delta e)a + e(\delta(a)),$$

where $e \in E^{\text{smooth}}$, $a \in \mathcal{A}_\theta^{\text{smooth}}$, and $\delta \in L_\theta$. The *curvature* $F_{\mu,\nu} = [\nabla_\mu, \nabla_\nu]$ of the connection ∇ is considered as a two-form on L_θ with values in $\operatorname{End}_{\mathcal{A}_\theta} E$. (Here $\operatorname{End}_{\mathcal{A}_\theta} E$ stands for the space of endomorphisms of the $\mathcal{A}_\theta^{\text{smooth}}$-module E^{smooth} and $\operatorname{End}_{\mathbb{C}} E$ denotes the space of $\mathbb{C}$-linear endomorphisms of E^{smooth}.)

We always consider hermitian modules and hermitian connections. This means that if E is a right $\mathcal{A}_\theta$-module it is equipped with an $\mathcal{A}_\theta$-valued hermitian inner product $\langle \cdot , \cdot \rangle$ (for the detailed list of properties see [1]); all connections that we will consider should be compatible with this inner product.

If E is endowed with an $\mathcal{A}_\theta$-connection, then one can define a Chern character

$$\operatorname{ch}(E) = \sum_{k=0} \frac{\widehat{\tau}(F^k)}{(2\pi \mathrm{i})^k k!} = \widehat{\tau}(\mathrm{e}^{\frac{F}{2\pi \mathrm{i}}}), \tag{1.3}$$

where F is the curvature of the connection on E, and $\widehat{\tau}$ is the canonical trace on $\widehat{A} = \operatorname{End}_{\mathcal{A}_\theta}(E)$ (we use that $\mathcal{A}_\theta$ is equipped with a canonical trace $\tau = c_0$). One can consider $\operatorname{ch}(E)$ as an element in the Grassmann algebra $\Lambda^{\cdot}(L_\theta^*) = \Lambda^{\cdot}(V^*)$. We have a lattice L in V^*. Thus we can talk about integral elements in $\Lambda^{\cdot}(V^*)$ which are just the elements of $\Lambda^{\cdot} L$. In the commutative case $\operatorname{ch}(E)$ is integral. In the non-commutative case this is wrong, but there exists an integral element $\mu(E) \in \Lambda^{\cdot}(V^*)$ related to $\operatorname{ch}(E)$ by the formula (see [5], [9])

$$\operatorname{ch}(E) = \mathrm{e}^{\iota(\theta)} \mu(E). \tag{1.4}$$

Here $\iota(\theta)$ stands for the operation of contraction with θ considered as an element of $\Lambda^2 V$. In particular, formula (1.4) means that $\mathrm{e}^{-\iota(\theta)} \operatorname{ch}(E)$ is an integral element of $\Lambda^{\cdot}(V^*)$. The group $\Lambda^{\text{even}} L$ can be naturally identified with the group $\mathrm{K}_0(\mathcal{A}_\theta)$. Moreover $\mu(E)$ is the class of the module E in the $\mathrm{K}_0(\mathcal{A}_\theta)$ group (see [5]).

Let us remind that the element $\mu \in \mathrm{K}_0(\mathcal{A}_\theta)$ is called *positive* if $(\mathrm{e}^{\iota(\theta)}\mu)_{(0)} > 0$ (the zero component is positive). A well known theorem of M. Rieffel (see [9]) says that if θ is irrational then every positive element of μ is represented by a projective module over $\mathcal{A}_\theta$.

Let E be a projective $\mathcal{A}_\theta$-module with a constant curvature connection ∇. Denote by F the curvature of ∇. Then since F is a 2-form with values in $\mathbb{C}$ we obtain that $\widehat{\tau}(F^k) = \widehat{\tau}(1)F^k$. The number $\widehat{\tau}(1)$ is called the *dimension of the module* E and we denote it by d_E. Then the formula (1.3) becomes

$$\operatorname{ch}(E) = d_E \mathrm{e}^{\frac{F}{2\pi \mathrm{i}}}. \tag{1.5}$$

We see that in this case $\operatorname{ch}(E)$ is a quadratic exponent (i.e. an expression of the form $C\mathrm{e}^a$ where C is a constant and $a \in \Lambda^2(V^+)$). It follows from (1.4) and from this fact that $\mu(E)$ is a generalized quadratic exponent (i.e. a limit of quadratic exponents). This gives a proof of the first statement of Theorem 1.1. The proof of the second statement of this theorem is based on the study of generalized quadratic exponents in Sections 2 and 3. In Section 3 we will study integral generalized quadratic exponents keeping in mind that $\mathrm{K}_0(\mathcal{A}_\theta)$ is exactly the integral lattice in $\Lambda^{\mathrm{even}}(V)$, where $V = L_\theta^*$. We will prove some auxiliary technical results saying that something is rational or integral which we will use in our construction of the module in Section 4.

In Section 4 we will construct a desired module together with constant curvature connection in four steps.

First we find explicitly the curvature F as a 2-form on L_θ.

Secondly, we construct some spaces of functions on $\mathbb{R}^p \times \mathbb{Z}^{n-2p}$ together with actions of generators of some non-commutative torus $\mathcal{A}_{\tilde\theta}$ and constant curvature connection having the curvature form F. We do not construct an $\mathcal{A}_{\tilde\theta}$-module at this step. Moreover, we even will not specify what space of functions we will take.

Thirdly, we will check that our construction in the previous step is just a particular case of Rieffel's construction in [9] where he constructs $\mathcal{A}_{\tilde\theta}$-projective modules. So we can construct the desired module $\widetilde{E}$ over $\mathcal{A}_{\tilde\theta}$ using Rieffel's construction.

Fourthly, we will see that $\tau = \theta - \widetilde{\theta}$ is a rational element of $\Lambda^2 L_\theta = \Lambda^2 V^*$. Also, we will use Rieffel's explicit calculation of $[\widetilde{E}]$ (of the class of $\widetilde{E}$ in $\mathrm{K}_0(\mathcal{A}_{\tilde\theta}) \subset \Lambda^{\mathrm{even}}(V)$) to find a simple relation between $[\widetilde{E}]$ and μ. Finally, we show that we can construct a projective module E with constant curvature connection over $\mathcal{A}_\theta$ such that $[E] = \mu$ by taking E to be a tensor product of $\widetilde{E}$ by some finite dimensional module M over $\mathcal{A}_\tau$.

2. GENERALIZED QUADRATIC EXPONENTS

In this section we introduce generalized quadratic exponents and study their properties.

Let V be a finite dimensional vector space over $\mathbb{R}$. Let V^* be the dual space. Then the space $V \oplus V^*$ has a natural symmetric bilinear product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = y_2(x_1) + y_1(x_2),$$

where $x_1, x_2 \in V$ and $y_1, y_2 \in V^*$. Consider the Clifford algebra $\mathrm{Cl}(V \oplus V^*)$. It naturally acts on the vector space $\Lambda^\cdot(V)$; we denote this action by ρ. Note that there is a natural inclusion i of $V \oplus V^*$ into $\mathrm{Cl}(V \oplus V^*)$.

DEFINITION An element $q \in \Lambda^\cdot(V)$ is called a *generalized quadratic exponent* if there exists a maximal isotropic subspace $U \subset V \oplus V^*$ such that for any $x \in U$ we have $\rho(x)q = 0$.

If the projection of U onto V^* is bijective we can represent U as the graph of a linear operator $a : V^* \to V$. The operator a is antisymmetric; it can be considered as an element of $\Lambda^2(V)$. The element q can be represented in the form $\mathrm{const} \cdot \mathrm{e}^a$, i.e. it is a quadratic exponent. The set of maximal isotropic subspaces we just considered is dense in the set of all maximal isotropic subspaces; this means that quadratic exponents are dense in the set of all generalized quadratic exponents.

In the next proposition we will describe all possible generalized quadratic exponents.

PROPOSITION *Let $q \in \Lambda^{\cdot}(V)$ be a generalized quadratic exponent. Then there exists a subspace $W \subset V$, a non-degenerate element $\widetilde{q}_1 \in \Lambda^2(V/W)$ and a non-zero element $w \in \Lambda^{\dim W} W$ such that*

$$q = w \wedge \mathrm{e}^{q_1},$$

where $q_1 \in \Lambda^2(V)$ is any preimage of $\widetilde{q}_1 \in \Lambda^2(V/W)$ under the natural projection from $\Lambda^2(V)$ onto $\Lambda^2(V/W)$.

Proof. Let U be the maximal isotropic subspace corresponding to q. It is easy to see that we can choose a basis $\{\xi_i\}$ of V and a dual basis $\{\eta_i\}$ of V^* such that U is spanned by the vectors $\eta_1 - \sum_{i=1}^{j} a_{1,i}\xi_i, \ldots, \eta_j - \sum_{i=1}^{j} a_{j,i}\xi_i, \xi_{j+1}, \ldots, \xi_{\dim W}$. Thus, q satisfies the following system of equations:

$$\begin{aligned}
&\frac{\partial q}{\partial \xi_1} - \Big(\sum_{i=1}^{j} a_{1,i}\xi_i\Big) \wedge q = 0\\
&\vdots\\
&\frac{\partial q}{\partial \xi_j} - \Big(\sum_{i=1}^{j} a_{j,i}\xi_i\Big) \wedge q = 0\\
&\xi_{j+1} \wedge q = 0\\
&\vdots\\
&\xi_{\dim W} \wedge q = 0.
\end{aligned}$$

The partial derivatives in this system are understood as left derivatives in the sense of superalgebra.

It is easy to see that any solution of this system is of the form

$$C \cdot \xi_{j+1} \wedge \cdots \wedge \xi_{\dim W} \wedge \mathrm{e}^{\sum_{k=1}^{j}\sum_{l=1}^{j} a_{k,l}\xi_k \wedge \xi_l},$$

where C is a constant. The proposition follows easily from the above formula. W is the subspace spanned by $\xi_{j+1}, \ldots, \xi_{\dim W}$ and $\widetilde{q}_1$ is the projection of $\sum_{k=1}^{j}\sum_{l=1}^{j} a_{k,l}\xi_k \wedge \xi_l$. ∎

$\Lambda^{\cdot}(V)$ is a graded vector space. If $q \in \Lambda^{\cdot}(V)$ let us denote by $q_{(i)} \in \Lambda^i V$ the projection q on $\Lambda^i V$.

COROLLARY *Let q be a generalized quadratic exponent. If $q_{(0)}$ is not zero then there is a non-degenerate element $a \in \Lambda^2 V$ and a non-zero real number C such that*

$$q = C\mathrm{e}^{a}.$$

Proof. Immediately follows from Proposition 2.1. ∎

Let $b \in \Lambda^2(V^*)$. Then b acts naturally on $\Lambda^{\cdot}(V)$. If we choose a basis $\{\xi_i\}$ in V then we can write the action of b as $\sum_{k,l} b_{k,l}\frac{\partial}{\partial\xi_k\partial\xi_l}$. Another way of thinking is to think about b as an element of $\mathrm{Cl}(V \oplus V^*)$. We have a canonical map from $\Lambda^{\cdot}(V^*)$ to $\mathrm{Cl}(V \oplus V^*)$ since V^* is an isotropic subspace in $V \oplus V^*$. Then the action of b is simply $\rho(b)$.

PROPOSITION *Let q be a generalized quadratic exponent and b any element in $\Lambda^2(V^*)$. Then $\mathrm{e}^{\rho(b)}q$ is a generalized quadratic exponent.*

Proof. We will reduce the proposition to the case where b is decomposable. Since $\rho(b) = \sum_{k,l} b_{k,l}\frac{\partial}{\partial\xi_k\partial\xi_l}$ in some basis $\{\xi_i\}$ of V and the operators $\frac{\partial}{\partial\xi_k\partial\xi_l}$ commute it is enough to prove the proposition in the case when $\rho(b) = c\frac{\partial}{\partial\xi_k\partial\xi_l}$, where c is a real number. In this case $\mathrm{e}^{\rho(b)} = 1 + \rho(b)$.

Our goal is to show that there exists a subspace $\widetilde{W} \subset V \oplus V^*$ such that for any $x \in \widetilde{W}$ we have $\rho(x)(q + \rho(b)(q)) = 0$.

Let W be a subspace of $V \oplus V^*$ such that if $x \in W$ then $\rho(x)q = 0$. We can choose a basis $\{v_1 + w_1, \ldots, v_k + w_k, v_{k+1}, \ldots, v_{\dim W}\}$ of W, where $v_i \in V^*$ and $w_i \in V$, and the vectors $\{w_i\}$ are linearly independent.

A simple calculation shows that

$$\rho(v_l)(q + \rho(b)(q)) = 0 + \rho(v_l)\rho(b)(q) = \rho(b)\rho(v_l)(q) = 0,$$

for $l > k$. Also, we can easily see that for $l < k + 1$

$$\begin{aligned}\rho(v_l+w_l)(q+\rho(b)(q)) &= \rho(v_l+w_l)\rho(b)(q) = [\rho(v_l+w_l),\rho(b)](q) + \rho(b)\rho(v_l+w_l)(q)\\ &= [\rho(w_l),\rho(b)](q) = \rho(\iota(w_l)b)q,\end{aligned}$$

where $\iota(w_l)$ is plugging the vector w_l in the 2-form b. $\iota(w_l)b$ is an element of V^*. Since $b^2 = 0$ we see that $0 = \iota(w_l)(b^2) = 2(\iota(w_l)b)b$. Thus,

$$\rho(\iota(w_l)b)q = \rho(\iota(w_l)b + (\iota(w_l)b)b)q = \rho(\iota(w_l)b)(q + \rho(q)).$$

Therefore, we see that $\rho([v_l - \iota(w_l)b] + w_l)(q + \rho(b)q) = 0$. Denote by $\widetilde{W}$ the subspace of $V \oplus V^*$ spanned by the vectors $[v_1 - \iota(w_1)b] + w_1, \ldots, [v_k - \iota(w_k)b] + w_k, v_{k+1}, \ldots, v_{\dim W}$. It is easy to check that $\widetilde{W}$ is a maximal isotropic subspace of $V \oplus V^*$ and we showed that $\rho(x)(q + \rho(b)q) = 0$ for any $x \in \widetilde{W}$. Thus, $q + \rho(b)q$ is a generalized quadratic exponent. ∎

3. INTEGRAL GENERALIZED QUADRATIC EXPONENTS

In this section we study integral generalized quadratic exponents and we prove a couple of auxiliary propositions that we will use in our construction.

Let V be a finite dimensional vector space and let L be a lattice in it. Denote by n the dimension of V. Then $V \cong \mathbb{R}^n$ and $L \cong \mathbb{Z}^n$. We denote the dual lattice to L by L^*. Obviously $L^* \subset V^*$. We call an element of $\Lambda^\cdot(V)$ *integral* if it lies in $\Lambda^\cdot L$.

Define a subspace U_μ of V^* as follows:

$$U_\mu = \{x \in V^* : \iota(x)\mu = 0\}.$$

Denote by $W_\mu \subseteq V$ the orthogonal complement to U_μ.

PROPOSITION *Let $\mu \in \Lambda^{\cdot}(V)$ be an integral generalized quadratic exponent. Then $L_\mu = L \cap W_\mu$ is a lattice in W_μ. We can identify $\Lambda^{\dim W_\mu} L_\mu$ with $\mathbb{Z}$ (the isomorphism is not canonical but it is specified up to a sign). Let $\alpha \in \Lambda^{\dim W_\mu} L_\mu$ be a volume form (an element that corresponds to 1 under the isomorphism with $\mathbb{Z}$). Then $\mu_{(\dim W_\mu)} = N\alpha$, where N is a non-zero integer.*

Proof. Let $\mu \in \Lambda^{\cdot}(V)$ be an integral generalized quadratic exponent. Let k be the largest integer such that $\mu_{(k)} \neq 0$ and for all $l > k$ we have $\mu_{(l)} = 0$. From Proposition 2.1 easily follows that $U_\mu = \{x \in V^* : \iota(x)\mu = 0\} = \{x \in V^* : \iota(x)\mu_{(k)} = 0\}$. Moreover, from Proposition 2.1 it follows that $\mu_{(k)}$ is a decomposable element of $\Lambda^{\cdot}(V)$ and that $k = \dim V - \dim U_\mu = \dim W_\mu$. Since μ is integral $\mu_{(k)}$ is also integral. Thus, the subspace U_μ is spanned by $U_\mu \cap L^*$. Therefore, $U_\mu \cap L^*$ is a lattice in U_μ. This immediately implies that $L_\mu = W_\mu \cap L$ is a lattice in W_μ since W_μ is the orthogonal complement to U_μ.

From the above discussion it is easy to see that $\Lambda^{\dim W_\mu}(W_\mu \cap L) = \Lambda^{\dim W_\mu} L_\mu \cong \mathbb{Z}$. Since by the definition α corresponds to ± 1 under such an isomorphism and $\mu_{(\dim W_\mu)} = \mu_{(k)}$ is an integral element we obtain that $\mu_{(\dim W_\mu)} = N\alpha$ for some integer N. ∎

Under the conditions in the above proposition we can easily find a complement $\widetilde{L}_\mu$ ($\widetilde{L}_\mu \cong \mathbb{Z}^{n-\dim W_\mu}$) to L_μ in L. It is not unique but we do not care about that. Let Y_μ be the subspace of V spanned by $\widetilde{L}_\mu$. Then it is obvious that $V = W_\mu \oplus Y_\mu$ and $L = L_\mu \oplus \widetilde{L}_\mu$.

The next results will be used in the construction in Section 4.1. Since they do not use any theory of non-commutative tori we state them here. But they need some explanation concerning their origin.

Let $\mu \in \Lambda^{\mathrm{even}}(V)$ be an integral generalized quadratic exponent which will be an element of K_0 representing a projective module of $\mathcal{A}_\theta$. We can think about θ as an element of $\Lambda^2 V^*$. If there exists a projective module E over $\mathcal{A}_\theta$ with constant curvature connection such that $[E] = \mu$ then by the result of G. Elliott (see [5]) $d\mathrm{e}^{\frac{F}{2\pi\mathrm{i}}} = \mathrm{e}^{\iota(\theta)}\mu$, where d is the dimension of the module. $\widetilde{\theta}$ which satisfies the conditions of the lemma below will be constructed in Section 4.1.

LEMMA *Let $\mu \in \Lambda^{\mathrm{even}}(V)$ be an integral generalized quadratic exponent. Let us assume that we fixed the isomorphism between $\Lambda^{\dim W_\mu} L_\mu$ and $\mathbb{Z}$ (see Proposition 3.1) so that $\mu_{(\dim W_\mu)} = N\alpha$ with N being a natural number (here $\alpha \in \Lambda^{\dim W_\mu} L_\mu$ corresponds to 1 in $\mathbb{Z}$). Let θ and $\widetilde{\theta}$ be elements of $\Lambda^2 V^*$ such that $\theta - \widetilde{\theta}$ is zero on $V \otimes Y_\mu$ (that is, if $X_\mu \subset V^*$ is the orthogonal complement to Y_μ then $\theta - \widetilde{\theta} \in \Lambda^2 X_\mu$). Assume that*

$$\mathrm{e}^{\iota(\theta)}\mu = c\,\mathrm{e}^{\iota(\widetilde{\theta})}\alpha, \tag{3.1}$$

where c is a real number. Then $c = N$, that is

$$\mathrm{e}^{\iota(\theta)}\mu = N\mathrm{e}^{\iota(\widetilde{\theta})}\alpha, \tag{3.2}$$

and $N(\theta - \widetilde{\theta})$ is an integral element of $\Lambda^2 X_\mu$.

Proof. Let us denote $\dim W_\mu$ by k. Then formula (3.1) implies that

$$\mu_{(k)} = (\mathrm{e}^{\iota(\theta)}\mu)_{(k)} = c(\mathrm{e}^{\iota(\widetilde{\theta})}\alpha)_{(k)} = c\alpha_{(k)}.$$

Thus, $c = N$ and we have proved formula (3.2). From formula (3.2) it easily follows that

$$\mu = N\mathrm{e}^{\iota(\widetilde{\theta}-\theta)}\alpha. \tag{3.3}$$

This means that $\mu_{k-2} = N\iota(\widetilde{\theta}-\theta)\alpha$. μ_{k-2} is an integral element. $\widetilde{\theta}-\theta$ is in $\Lambda^2 X_\mu$ which is dual to W_μ and α is a non-zero element of $\Lambda^{\dim W_\mu} W_\mu$. Thus, $N(\widetilde{\theta}-\theta)$ is an integral element of $\Lambda^2 X_\mu$. ∎

3.1. Modules over the rational non-commutative tori. Let us assume that the conditions of Lemma 3.2 are satisfied. We denote $\theta-\widetilde{\theta}$ by τ. Let us remind the definition of $\mathcal{A}_\tau$: $\mathcal{A}_\tau$ is a universal $\mathbb{C}^*$ algebra having unitary generators U_β, where $\beta \in L$, obeying the relations

$$U_{\beta_1}U_{\beta_2} = \mathrm{e}^{\pi \mathrm{i}\tau(\beta_1,\beta_2)}U_{\beta_1+\beta_2}.$$

We can reformulate this definition in a slightly different way. Let $\beta_1, \dots, \beta_n$ (where $n = \dim V$) be a basis of a free $\mathbb{Z}$ module L. $\mathcal{A}_\tau$ is a universal $\mathbb{C}^*$ algebra having unitary generators U_i, $1 \leqslant i \leqslant n$, obeying the relations

$$U_iU_j = \mathrm{e}^{2\pi\mathrm{i}\tau(\beta_i,\beta_j)}U_jU_i.$$

It is obvious that the two definitions are equivalent.

Proposition *Under the conditions of Lemma* 3.2 *there exists an N dimensional module M over $\mathcal{A}_\tau$.*

Proof. Since $N\tau$ is an integral form and we have a freedom in choosing a basis $\{\beta_i\}$ of L, we can choose it so that

$$N\tau(\beta_{2i-1},\beta_{2i}) = q_1q_2\cdots q_i,$$
$$\tau(\beta_k,\beta_l) = 0 \quad \text{unless } k = 2\mathrm{i}-1 \text{ and } l = 2i \text{ or } k = 2i \text{ and } l = 2i-1,$$

where $q_1, q_2, \dots$ are integers (see [6]) and moreover the basis $\{\beta_i\}$ respects the decomposition of L into $L_\mu \oplus \widetilde{L}_\mu$. In this basis the algebra $\mathcal{A}_\tau$ is generated by unitary generators U_i obeying the relations

$$U_{2i-1}U_{2i} = \mathrm{e}^{2\pi\mathrm{i}\frac{q_1\cdots q_i}{N}}U_{2i}U_{2i-1} \tag{3.5}$$

(all other generators commute). Note that it may happen that there exists an integer m such that if $i > m$ then all q_i are zero.

So we see that our algebra $\mathcal{A}_\tau$ is a tensor product of algebras $\mathcal{A}_{\tau i}$, where $\mathcal{A}_{\tau i}$ is generated by two unitary generators U_{2i-1} and U_{2i} obeying relations (3.5) or $\mathcal{A}_{\tau i}$ is generated by only one unitary generator (this is the case when $i > m$, in particular if $\beta_i \in \widetilde{L}_\mu$). Thus it is enough to show that we can construct finite dimensional modules M_i over $\mathcal{A}_{\tau i}$ such that $(\dim M_1)(\dim M_2)\cdots$ divides N. Indeed, in such case we can take M to be the direct sum of $M_1 \otimes M_2 \otimes \cdots$ taken $\frac{N}{(\dim M_1)(\dim M_2)\cdots}$ times.

If $\mathcal{A}_{\tau i}$ is generated by one unitary generator then it has a 1-dimensional module over it, U_i acts by 1. We choose M_i to be this module in this case.

If $\mathcal{A}_{\tau i}$ is generated by two unitary generators U_{2i-1} and U_{2i} obeying relations (3.5) then it has a module of dimension $\frac{N}{\mathrm{GCD}(N,q_1\cdots q_i)}$, where GCD stands for greatest common divisor. We choose M_i to be a module of the dimension $\frac{N}{\mathrm{GCD}(N,q_1\cdots q_i)}$.

Thus, it is enough to show that $\frac{N^m}{\mathrm{GCD}(N,q_1)\cdots\mathrm{GCD}(N,q_1\cdots q_m)}$ divides N, where m is the number of tori $\mathcal{A}_{\tau i}$ generated by two generators. Therefore it is enough to prove that $\frac{\mathrm{GCD}(N,q_1)\cdots\mathrm{GCD}(N,q_1\cdots q_m)}{N^{m-1}}$ is an integer. ∎

LEMMA
$$\frac{\mathrm{GCD}(N,q_1)\cdots\mathrm{GCD}(N,q_1\cdots q_m)}{N^{m-1}}$$
is an integer if
$$\frac{q_1^2q_2}{N},\frac{q_1^3q_2^2q_3}{N^2},\ldots,\frac{q_1^mq_2^{m-1}\cdots q_m}{N^{m-1}}$$
are integers.

Proof. Denote by $a_i=\frac{\mathrm{GCD}(N,q_1\cdots q_i)}{\mathrm{GCD}(N,q_1\cdots q_{i-1})}$ and $b_i=q_i/a_i$. Then we can write $q_1^mq_2^{m-1}\cdots q_m=(a_1^ma_2^{m-1}\cdots a_m)(b_1^mb_2^{m-1}\cdots b_m)$. We will prove by induction that $\frac{a_1^ka_2^{k-1}\cdots a_k}{N^{k-1}}$ are integers. The initial case $k=1$ is obvious. For $k>1$ we have
$$\frac{a_1^ka_2^{k-1}\cdots a_k}{N^{k-1}}=\frac{a_1^{k-1}a_2^{k-2}\cdots a_{k-1}}{N^{k-2}}\left(\frac{a_1\cdots a_k}{N}\right).$$
$\frac{a_1^{k-1}a_2^{k-2}\cdots a_{k-1}}{N^{k-2}}$ is an integer by induction hypothesis and we also know that $\frac{N}{a_1\cdots a_k}$ are relatively prime with $b_1,\cdots b_k$. But on the other hand
$$\frac{q_1^kq_2^{k-1}\cdots q_k}{N^{k-1}}=\frac{a_1^{k-1}a_2^{k-2}\cdots a_{k-1}}{N^{k-2}}\left(\frac{a_1\cdots a_k}{N}\right)\left(b_1^kb_2^{k-1}\cdots b_k\right).$$
Thus, $\frac{a_1^{k-1}a_2^{k-2}\cdots a_{k-1}}{N^{k-2}}(\frac{a_1\cdots a_k}{N})$ is an integer. ∎

To prove the proposition it is enough to show that
$$\frac{q_1^2q_2}{N},\frac{q_1^3q_2^2q_3}{N^2},\ldots,\frac{q_1^mq_2^{m-1}\cdots q_m}{N^{m-1}}$$
are all integers. Let $\{\gamma_i\}$ be a basis of V^* dual to $\{\beta_i\}$. We know that $\mu=Ne^{\iota(\tau)}\alpha$ is an integral element of $\Lambda^{\mathrm{even}}(V)$. Denote by k the dimension of W_μ as before. Then, $\alpha=\pm\beta_1\wedge\beta_2\wedge\cdots\wedge\beta_k$ and the numbers
$$\left\langle\gamma_3\wedge\cdots\wedge\gamma_k,N\iota(\tau)\alpha\right\rangle,\left\langle\gamma_5\wedge\cdots\wedge\gamma_k,N\frac{(\iota(\tau))^2}{2}\alpha\right\rangle,\ldots,\left\langle\gamma_{2m+1}\wedge\cdots\wedge\gamma_k,N\frac{(\iota(\tau))^m}{m!}\alpha\right\rangle$$
are integers. A straightforward calculation shows that
$$\left\langle\gamma_{2j+1}\wedge\cdots\wedge\gamma_k,yN\frac{(\iota(\tau))^j}{j!}\alpha\right\rangle=\pm\frac{q_1^jq_2^{j-1}\cdots q_j}{N^{j-1}}.$$
Thus all numbers $\frac{q_1^jq_2^{j-1}\cdots q_j}{N^{j-1}}$ are integers. ∎

4. PROOF OF THEOREM 1.1

First, let us show that if we have a projective $\mathcal{A}_\theta$-module with constant curvature connection then the corresponding class $\mu = [E] \in \mathrm{K}_0(\mathcal{A}_\theta)$ is a positive generalized quadratic exponent.

Indeed, we know from formula (1.4) that

$$\mu = [E] = \mathrm{e}^{-\iota(\theta)} \operatorname{ch}(E) = \mathrm{e}^{\iota(-\theta)} \operatorname{ch}(E).$$

Also, from formula (1.5) we see that

$$\operatorname{ch}(E) = d_E \mathrm{e}^{\frac{F}{2\pi \mathrm{i}}}.$$

Thus $\operatorname{ch}(E)$ is a generalized quadratic exponent since F is an element of $\Lambda^2 V$ (recall that $V = L_\theta^*$). From Proposition 2.4 it immediately follows that μ is a generalized quadratic exponent since $-\theta \in \Lambda^2(V^*)$. Therefore we have proved that $\mu = [E]$ is a generalized quadratic exponent. It is a positive element of $\mathrm{K}_0(\mathcal{A}_\theta)$ because it represents a genuine $\mathcal{A}_\theta$-module. Thus we have proved the statement of Theorem 1.1 in one direction.

This was the easy part. The hard part is to prove the second half, that is to show that if $\mu \in \mathrm{K}_0(\mathcal{A}_\theta)$ is a positive generalized quadratic exponent then there exists a projective module E with a constant curvature connection which represents the class μ, i.e., $\mu = [E]$. In the next subsection we present an explicit construction of such a module.

4.1. Construction of an $\boldsymbol{\mathcal{A}_\theta}$-module E with constant curvature connection.

In this section we will construct explicitly an $\mathcal{A}_\theta$-module E with a constant curvature connection representing $\mu \in \mathrm{K}_0(\mathcal{A}_\theta)$. Assuming that such a module exists we see that $\operatorname{ch}(E) = \mathrm{e}^{\iota(\theta)}\mu$ is a generalized quadratic exponent (follows from Proposition 2.4 and the fact that μ is a generalized quadratic exponent). Moreover, $\operatorname{ch}(E)_{(0)} > 0$ therefore from Corollary 2.3 follows that

$$\operatorname{ch}(E) = \mathrm{e}^{\iota(\theta)}\mu = d_E \mathrm{e}^{\frac{F}{2\pi \mathrm{i}}},$$

where F is the curvature form, and d_E is the dimension of the module E. Thus, reversing the previous arguments it is obvious that it is enough to construct a projective $\mathcal{A}_\theta$-module with constant curvature connection satisfying the following properties:

a) the curvature form is F;

b) the dimension of the module is d_E.

In Section 3 we defined a subspace $W_\mu \subseteq V = L_\theta^*$ associated with the generalized quadratic exponent μ. Since $\mu \in \mathrm{K}_0(\mathcal{A}_\theta)$ we see that μ is integral. Thus, $L_\mu = L \cap W_\mu$ is the integral lattice in W_μ by Proposition 3.1. As in Section 3 we denote by $k = \dim W_\mu$ and we choose a complement $\widetilde{L}_\mu$ to L_μ in L. Denote by Y_μ the span of $\widetilde{L}_\mu$ in V. It is obvious that $L_\theta^* = V = W_\mu \oplus Y_\mu$. Thus, we have a natural decomposition $L_\theta = V^* = W_\mu^* \oplus Y_\mu^*$. Note that the space $Y_\mu^* = U_\mu$ was defined in Section 3. Since $\mu_{(k)} = d_E(\mathrm{e}^{\frac{F}{2\pi \mathrm{i}}})_{(k)}$, k is an even integer, that is $k = 2p$, $p \in \mathbb{Z}$. Denote by q the rank of the free abelian group $\widetilde{L}_\mu$. We have $q = n - 2p$, where $n = \dim L_\theta$ the dimension of $\mathcal{A}_\theta$. Since F^p is non-zero it follows that $F|W_\mu^*$ (F restricted to W_μ^*) is a non-degenerate 2-form.

4.1.1. CONSTRUCTION OF OPERATORS $\boldsymbol{\nabla}_x$ FOR $x \in L_\theta$. Let $\mathcal{H}eis$ be the Heisenberg algebra generated by the operators ∇_x, for $x \in W_\mu^*$, which satisfy the relation

$$[\nabla_x, \nabla_y] = F(x, y),$$

where $x, y \in W_\mu^*$. The algebra $\mathcal{H}eis$ has a unique irreducible representation which can be realized in the space of square integrable functions on $\mathbb{R}^p$. Moreover, the action of ∇_x is given by an operator

$$(\nabla_x(f))(z) = 2\pi\mathrm{i}\langle\phi(x), z\rangle f(z) + \sum_i \psi_i(x)\frac{\partial f(z)}{\partial z_i},$$

where $\phi : W_\mu \to (\mathbb{R}^p)^*$ is some linear map, and $\psi_i : W_\mu \to \mathbb{R}$ are some linear functions on W_μ^*. In particular, we see that these operators preserve the space of Schwartz functions on $\mathbb{R}^p$.

The above construction provides us with the action of the operators ∇_x for $x \in W_\mu^*$ only. First we will extend the above construction to obtain an action of all operators ∇_x, $x \in L_\theta = W_\mu^* \oplus Y_\mu^*$. Then, we will obtain an action of some non-commutative torus $\mathcal{A}_{\tilde\theta}$ so that ∇ becomes an $\mathcal{A}_{\tilde\theta}$-connection.

We extend the space from the space of Schwartz functions on $\mathbb{R}^p$ to the space of Schwartz functions on $\mathbb{R}^p \times \widetilde{L}_\mu = \mathbb{R}^p \times \mathbb{Z}^q$. Denote it by H. If $x \in L_\theta = W_\mu^* \oplus Y_\mu^*$ we denote by x_W the projection of x on W_μ^* and by x_Y the projection of x on Y_μ^* (obviously $x = x_W + x_Y$). We define the action of ∇_x on an element $f(z, a) \in H$, where $z \in \mathbb{R}^p$ and $a \in \widetilde{L}_\mu$, as follows

$$(\nabla_x(f))(z, a) = (\nabla_{x_W}(f))(z, a) + 2\pi\mathrm{i}\langle x_Y, a\rangle f(z, a), \tag{4.1}$$

where the action of ∇_{x_W} is the same as above (only along z's). Notice that the operators ∇_x, $x \in L_\theta$ satisfy the commutation relations

$$[\nabla_x, \nabla_y] = [\nabla_{x_W}, \nabla_{y_W}] = F(x_W, y_W) = F(x, y), \quad x, y \in L_\theta,$$

since ∇_{x_Y} (recall that $(\nabla_{x_Y}(f))(z, a) = 2\pi\mathrm{i}\langle x_Y, a\rangle f(z, a)$) commutes with ∇_y for any $y \in L_\theta$.

Thus, we have constructed operators ∇_x, $x \in L_\theta$, which satisfy the desired commutation relations. Next, we will construct operators acting on the space H which generate a non-commutative torus $\mathcal{A}_{\tilde\theta}$ such that

$$\begin{aligned} &\text{(i) } \nabla \text{ is an } \mathcal{A}_{\tilde\theta}\text{-connection} \\ &\text{(ii) } \theta - \widetilde{\theta} \text{ is an element of } \Lambda^2 W_\mu^*. \end{aligned} \tag{4.2}$$

4.1.2. CONSTRUCTION OF OPERATORS SATISFYING CONDITIONS (4.2). Let us choose a basis $\beta_1, \ldots, \beta_{2p}$ of L_μ and a basis $\beta_{2p+1}, \ldots, \beta_{2p+q}$ of $\widetilde{L}_\mu$. We will construct operators V_i, $1 \leqslant i \leqslant 2p+q$ acting on H which generate a non-commutative torus $\mathcal{A}_{\tilde\theta}$ which satisfies conditions (4.2).

LEMMA *For $1 \leqslant i \leqslant 2p$ there exists an operator $\widetilde{V}_i$ acting on H such that:*

(i) $(\widetilde{V}_i(f))(z,a) = \mathrm{e}^{2\pi\mathrm{i}\chi_i(z)} f(z+y_i,a)$, *where $z \in \mathbb{R}^p$, $a \in \widetilde{L}_\mu$, for some $y_i \in \mathbb{R}^p$ and some linear function $\chi_i \in (\mathbb{R}^p)^*$;*

(ii) $[\nabla_x, \widetilde{V}_i] = 2\pi\mathrm{i}\langle x, \beta_i\rangle \widetilde{V}_i$, *for $x \in L_\theta$.*

Proof. Let us introduce an operator $W(y,\chi)$, where $y \in \mathbb{R}^p$ and $\chi \in (\mathbb{R}^p)^*$

$$(W(y,\chi)f)(z,a) = \mathrm{e}^{2\pi\mathrm{i}\chi(z)} f(z+y,a).$$

A straightforward calculation shows that $[\nabla_x, W(y,\chi)]W(y,\chi)^{-1}$ is an operator of multiplication by a real number and moreover we obtain a non-degenerate pairing between the spaces W_μ^* and $\mathbb{R}^p \oplus (\mathbb{R}^p)^*$. Thus, choosing an appropriate element $y_i \in \mathbb{R}^p$ and $\chi_i \in (\mathbb{R}^p)^*$, we can put $\widetilde{V}_i = W(y_i,\chi_i)$. ∎

We define the operators $\widetilde{V}_i$ for $1 \leqslant i \leqslant 2p$ as in the above lemma. We define the operators $\widetilde{V}_i$ for $2p+1 \leqslant i \leqslant 2p+q$ acting on H by the formula

$$(\widetilde{V}_i f)(z,a) = f(z, a-\beta_i).$$

LEMMA *For any $1 \leqslant i \leqslant n = 2p+q$, and any $x \in L_\theta$, we have*

$$[\nabla_x, \widetilde{V}_i] = 2\pi\mathrm{i}\langle x, \beta_i\rangle \widetilde{V}_i. \tag{4.3}$$

Proof. For $i \leqslant 2p$ formula (4.3) follows from Lemma 4.1. For $i > 2p$ formula (4.3) follows from an easy straightforward calculation. ∎

It is easy to check that the operators $\widetilde{V}_i$ are generators of some non-commutative torus. Moreover, these operators satisfy the first condition in (4.2) but they do not satisfy the second condition. To remedy this we will modify the operators $\widetilde{V}_i$ replacing them with operators $V_i = \mathrm{e}^{2\pi\mathrm{i}l_i(\cdot)}\widetilde{V}_i$.

If $l \in Y_\mu^*$ then the operator $\mathrm{e}^{2\pi\mathrm{i}l(\cdot)}\widetilde{V}_i$ acts on H by the formula

$$(\mathrm{e}^{2\pi\mathrm{i}l(\cdot)}\widetilde{V}_i(f))(z,a) = \mathrm{e}^{2\pi\mathrm{i}l(a)}(\widetilde{V}_i(f))(z,a).$$

Moreover, we have

$$[\nabla_x, \mathrm{e}^{2\pi\mathrm{i}l(\cdot)}\widetilde{V}_i] = 2\pi\mathrm{i}\langle x, \beta_i\rangle \mathrm{e}^{2\pi\mathrm{i}l(\cdot)}\widetilde{V}_i$$

which follows from an easy straightforward calculation (since the operator $\mathrm{e}^{2\pi\mathrm{i}l(\cdot)}$ commutes with the operators ∇_x, $x \in L_\theta$).

PROPOSITION *For $1 \leqslant i \leqslant 2p+q$ there exists a linear function $l_i \in Y_\mu^*$ on Y_μ such that if we define $V_i = \mathrm{e}^{2\pi\mathrm{i}l_i(\cdot)}\widetilde{V}_i$ then*

$$V_i V_j = \mathrm{e}^{2\pi\mathrm{i}\widetilde{\theta}_{ij}} V_j V_i, \tag{4.4}$$

and $\theta - \widetilde{\theta}$ is an element of $\Lambda^2 W_\mu^*$.

Proof. First, it is easy to see that there exists a 2-form $\sigma \in \Lambda^2 L_\theta$ such that $\widetilde{V}_i\widetilde{V}_j = \mathrm{e}^{2\pi\mathrm{i}\sigma_{ij}}\widetilde{V}_j\widetilde{V}_i$. An easy calculation shows that the operator $\mathrm{e}^{2\pi\mathrm{i}l(\cdot)}$ commutes with the operators $\widetilde{V}_i$ for $i \leqslant 2p$. If $i > 2p$ then we have

$$(\widetilde{V}_i \circ \mathrm{e}^{2\pi\mathrm{i}l(\cdot)}) = \mathrm{e}^{-2\pi\mathrm{i}l(\beta_i)}(\mathrm{e}^{2\pi\mathrm{i}l(\cdot)} \circ \widetilde{V}_i).$$

This gives us that if $i, j \leqslant 2p$ then

$$V_i V_j = e^{2\pi i \sigma_{ij}} V_j V_i; \tag{4.5}$$

if $i \leqslant 2p$ and $j > 2p$ then

$$V_i V_j = e^{2\pi i(\sigma_{ij} + l_i(\beta_j))} V_j V_i; \tag{4.6}$$

and if $i, j > 2p$ then

$$V_i V_j = e^{2\pi i(\sigma_{ij} + l_i(\beta_j) - l_j(\beta_i))} V_j V_i. \tag{4.7}$$

For $1 \leqslant i \leqslant 2p$ we define $l_i \in Y_\mu^*$ by the formula

$$l_i(\beta_j) = \theta(\beta_i, \beta_j) - \sigma_{ij}$$

on the basis $\{\beta_j\}$, $j > 2p$ of Y_μ. For $2p < i \leqslant 2p + q$ we define $l_i \in Y_\mu^*$ by the formula

$$l_i(\beta_j) = \frac{1}{2}(\theta(\beta_i, \beta_j) - \sigma_{ij})$$

on the basis $\{\beta_j\}$, $j > 2p$ of Y_μ.

Equations (4.5), (4.6), and (4.7) show that $V_i V_j = e^{2\pi i \theta(\beta_i, \beta_j)} V_j V_i$ if either i or j greater then $2p$ and $V_i V_j = e^{2\pi i \sigma_{ij}} V_j V_i$ if $i, j \leqslant 2p$. Thus we have constructed the linear functions $l_i \in Y_\mu^*$ such that the conditions (4.4) are satisfied. ∎

We define the operators V_i as in the above lemma. We easily see that the operators V_i generate a non-commutative torus $\mathcal{A}_{\tilde{\theta}}$, where $\widetilde{\theta}(\beta_i, \beta_j) = \sigma_{ij}$ if both $i, j \leqslant 2p$ and $\widetilde{\theta}(\beta_i, \beta_j) = \theta(\beta_i, \beta_j)$ otherwise.

Thus, the operators V_i satisfy the condition (4.2).

4.1.3. CONSTRUCTION OF A PROJECTIVE $\boldsymbol{\mathcal{A}_{\tilde{\theta}}}$-MODULE. Now we will identify our construction with the construction given in [9].

Let G be a central extension of the abelian group $\mathbb{R}^p \times \widetilde{L}_\mu \times (\mathbb{R}^p)^* \times (\widetilde{Y}_\mu / L_\mu^*)$ given by the natural pairing between $\mathbb{R}^p \times \widetilde{L}_\mu$ and $(\mathbb{R}^p)^* \times (\widetilde{Y_\mu / L_\mu^*})$. We see that G is a Heisenberg group and it acts naturally on H. We denote this representation by ρ. Moreover, for each V_i there exists a unique element $g_i \in G$ such that $\rho(g_i) = V_i$. One can easily recognize the construction of elementary modules over non-commutative tori in M. Rieffel's paper ([9]).

Thus, choosing an appropriate space of functions on $\mathbb{R}^p \times \widetilde{L}_\mu$ we get a projective $\mathcal{A}_{\tilde{\theta}}$-module $\widetilde{E}$ with constant curvature connection ∇ such that:

(i) the curvature of ∇ is F;
(ii) $\theta - \widetilde{\theta}$ is an element of $\Lambda^2 W_\mu^*$.

Next, we would like to find explicitly the class $[\widetilde{E}]$ in $\mathrm{K}_0(\mathcal{A}_{\tilde{\theta}})$. Note that in our construction of module $\widetilde{E}$ we canonically identified the space L_θ with the space $L_{\tilde{\theta}}$. Thus, we can think about $[\widetilde{E}]$ as an integral element of $\Lambda^{\text{even}} L_{\tilde{\theta}}^* = \Lambda^{\text{even}} V$. To find the class $[\widetilde{E}]$ we would have to do some calculations. Fortunately, they were already done by M. Rieffel in [9]. So, we will apply his results to our case.

Let us remind that in paper [9] M. Rieffel introduced a linear map $\widetilde{T} : L_{\tilde{\theta}} \to \mathbb{R}^p \times \mathbb{R}^q \times (\mathbb{R}^p)^*$. In our notation $L_{\tilde{\theta}}$ is canonically identified with $L_\theta = V = W_\mu \oplus Y_\mu$

and $\mathbb{R}^q$ with Y_μ. Thus, in our terms we have a linear map $\widetilde{T} : W_\mu \oplus Y_\mu \to \mathbb{R}^p \times Y_\mu \times (\mathbb{R}^p)^* = \mathbb{R}^p \times (\mathbb{R}^p)^* \times Y_\mu$. It is easy to see from the explicit construction of the operators V_i that $\widetilde{T}$ maps W_μ to $\mathbb{R}^p \times (\mathbb{R}^p)^*$, and Y_μ to Y_μ. Moreover the restriction of $\widetilde{T}$ on Y_μ is the identity map.

M. Rieffel found in [9] that

$$[\widetilde{E}] = d \prod_{j=1}^{p} \overline{Y}_j \wedge \overline{Y}_{j+p}, \tag{4.8}$$

where $d = \det(\widetilde{T})$ and

$$\overline{Y}_j = \begin{cases} \widetilde{T}^{-1}(\overline{e}_j) & \text{for } 1 \leqslant j \leqslant p, \\ \widetilde{T}^{-1}(e_{j-p}) & \text{for } p+1 \leqslant j \leqslant 2p; \end{cases} \tag{4.9}$$

where $\{e_j\}$ is a basis of $\mathbb{R}^p$ and $\{\overline{e}_j\}$ is the dual basis of $(\mathbb{R}^p)^*$.

Since $\widetilde{T}$ is the identity map on Y_μ we see that

$$\det(\widetilde{T}^{-1}) = \pm \frac{\prod_{j=1}^{p} \overline{Y}_j \wedge \overline{Y}_{j+p}}{\alpha},$$

where α is the volume form on W_μ (see Proposition 2.2 for the definition of α). Note, we put a $\pm$ sign because we do not want to specify precisely how to pick a volume form. The lattice L_μ specifies the volume form up to a sign. Later it will be easy to make the right choice of the sign so that everything would agree with M. Rieffel's paper ([9]). We get

$$d = \det(\widetilde{T}) = \frac{1}{\det(\widetilde{T}^{-1})} = \pm \frac{\alpha}{\prod_{j=1}^{p} \overline{Y}_j \wedge \overline{Y}_{j+p}}.$$

Thus we obtain that

$$[\widetilde{E}] = \pm\alpha. \tag{4.10}$$

4.1.4. CONSTRUCTION OF A PROJECTIVE $\boldsymbol{\mathcal{A}}_\theta$-MODULE E. From the above results and Proposition 3.1 we see that if we make the right choice of the sign (so that $[\widetilde{E}] = \alpha$) then

$$N = \frac{\mu_{(2p)}}{\alpha} \tag{4.11}$$

is a positive integer. Moreover, we have $d_{\tilde{E}} \mathrm{e}^{\frac{F}{2\pi \mathrm{i}}} = \mathrm{e}^{\iota(\theta)}\mu$ and $d\mathrm{e}^{\frac{F}{2\pi \mathrm{i}}} = \mathrm{e}^{\iota(\tilde{\theta})}\alpha$. Therefore,

$$\mathrm{e}^{\iota(\theta)}\mu = \left(\frac{d_{\tilde{E}}}{d}\right) \mathrm{e}^{\iota(\widetilde{\theta})}\alpha. \tag{4.12}$$

From equations (4.12) and (4.13) we see that $\frac{d_{\tilde{E}}}{d} = N$.

One can easily check that the conditions of Lemma 3.2 are satisfied. Therefore $N(\theta - \widetilde{\theta})$ is an integral element of $(W_\mu)^* = X_\mu$. From Proposition 3.3 follows that there exists N-dimensional module M over $\mathcal{A}_{\theta-\tilde{\theta}}$. Denote

$$E = \widetilde{E} \otimes M. \tag{4.13}$$

From Proposition 5.4 and Theorem 5.6 in M. Rieffel's paper ([9]) it follows that E is a projective module over $\mathcal{A}_\theta$ (since $\theta = \widetilde{\theta} + (\theta - \widetilde{\theta})$) with constant curvature connection with the curvature given by formula $\Omega = F \otimes \mathrm{Id}_M = F$ and $\mathrm{ch}(E) = \dim(M)\mathrm{ch}(\widetilde{E})$. Therefore we see that

$$\mathrm{ch}(E) = N\mathrm{ch}(\widetilde{E}) = N\mathrm{e}^{\iota(\widetilde{\theta})}\alpha = N\Big(\frac{d}{d_E}\Big)\mathrm{e}^{\iota(\theta)}\mu = \mathrm{e}^{\iota(\theta)}\mu.$$

Thus we have constructed a projective $\mathcal{A}_\theta$-module E with constant curvature connection such that $[E] = \mu$. This finishes the proof of Theorem 1.1 ∎

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