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OVERCONVERGENCE AND CYCLIC VECTORS IN BERGMAN SPACES

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ABSTRACT. In this paper we develop an overconvergence result in the context of polynomial approximation in the mean, primarily for certain weighted area measures. We then explore applications of this result pertaining to the existence and variety of cyclic vectors for the shift on Bergman spaces.

KEYWORDS: *Overconvergence, polynomial approximation, cyclic vectors, shift operator, Bergman spaces.*

MSC (2000): Primary 30E10, 30D55, 46E15; Secondary 47B20, 47B28.

1. INTRODUCTION

Let μ be a finite, positive Borel measure with compact support K in the complex plane $\mathbb{C}$ and, for $1 \leqslant t < \infty$, let $P^t(\mathrm{d}\mu)$ denote the closure of the polynomials in $L^t(\mathrm{d}\mu)$. Choose a point from each of the bounded components of $\mathbb{C} \setminus K$, let E be the set of these points and define Λ_{E} to be the collection of all linear combinations of functions of the form $f(z) = \frac{1}{z-a}$, where $a \in E$. One can learn much about (and in many cases completely determine) $P^t(\mathrm{d}\mu)$ by discovering which functions q in Λ_{E} are also in $P^t(\mathrm{d}\mu)$. If $q \in P^t(\mathrm{d}\mu) \cap \Lambda_E$, then none of the poles of q are in $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ (the collection of analytic bounded point evaluations for $P^t(\mathrm{d}\mu)$); see Lemma 2.1. Indeed, if $\{p_n\}$ is a sequence of polynomials such that $p_n \to q$ in $L^t(\mathrm{d}\mu)$ (as $n \to \infty$), then $p_n \to q$ uniformly on compact subsets of $\mathrm{abpe}(P^t(\mathrm{d}\mu))$. Therefore, $\mathrm{abpe}(P^t(\mathrm{d}\mu)) \setminus K$ can be thought of as a set of overconvergence (see [13] and [7] for related work). In many applications, though, we do not need the full strength of this last result that $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ affords, but instead need an answer to the question:

Question 1.1. Which open subsets W of $\mathbb{C}$ have the property: If $q \in P^t(\mathrm{d}\mu) \cap \Lambda_E$, then there exists a sequence $\{p_n\}$ of polynomials such that $p_n \to q$ in $L^t(\mathrm{d}\mu)$ and $p_n \to q$ uniformly on compact subsets of W?

In this paper we establish some results (see Theorems 3.1 and 3.2) that provide a strategy for answering Question 1.1 in a variety of contexts. As a consequence, we gain some rather general qualitative information concerning what it means for q to be in $P^t(\mathrm{d}\mu) \cap \Lambda_E$ (see, e.g., Theorem 3.8). We end the paper with some consequences and remarks concerning cyclic vectors for the shift on Bergman spaces.

2. PRELIMINARIES

With μ and $P^t(\mathrm{d}\mu)$ as before, a point z in $\mathbb{C}$ is called a *bounded point evaluation* for $P^t(\mathrm{d}\mu)$ if there is a positive constant c such that $|p(z)| \leqslant c\|p\|_{L^t(\mathrm{d}\mu)}$ for all polynomials p; the collection of all such points is denoted $\mathrm{bpe}(P^t(\mathrm{d}\mu))$. If $z \in \mathbb{C}$ and there are positive constants M and r such that $|p(w)| \leqslant M\|p\|_{L^t(\mathrm{d}\mu)}$ whenever p is a polynomial and $|w - z| < r$, then we call z an *analytic bounded point evaluation* for $P^t(\mathrm{d}\mu)$; the set of all points z of this sort is denoted $\mathrm{abpe}(P^t(\mathrm{d}\mu))$. Notice that $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ is an open subset of $\mathrm{bpe}(P^t(\mathrm{d}\mu))$ and, by the Maximum Modulus Theorem, each component of $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ is simply connected. If $z \in \mathrm{bpe}(P^t(\mathrm{d}\mu))$, then, by the Hahn-Banach and Riesz Representation Theorems, there exists k_z in $L^s(\mathrm{d}\mu)$ ($\frac{1}{s} + \frac{1}{t} = 1$) such that $p(z) = \int p(\zeta)k_z(\zeta)\,\mathrm{d}\mu(\zeta)$ for each polynomial p. For f in $P^t(\mathrm{d}\mu)$, define $\widehat{f}$ on $\mathrm{bpe}(P^t(\mathrm{d}\mu))$ by $\widehat{f}(z) = \int f(\zeta)k_z(\zeta)\,\mathrm{d}\mu(\zeta)$. Observe that $\widehat{f} = f$ a.e. μ on $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ and $z \mapsto \widehat{f}(z)$ is analytic on $\mathrm{abpe}(P^t(\mathrm{d}\mu))$. J. Thomson has given a direct sum decomposition of $P^t(\mathrm{d}\mu)$ that involves the components of $\mathrm{abpe}(P^t(\mathrm{d}\mu))$ (see [18], Theorem 5.8). Throughout this paper, unless otherwise specified, we let G be a bounded, simply connected region in $\mathbb{C}$. And, for $1 \leqslant t \leqslant \infty$, we let $H^t(G)$ and $L^t_a(G)$ be the corresponding Hardy and Bergman spaces for G; one may consult [3] and [8] as references for these spaces. We now state and prove a rather well-known result concerning point evaluations; we include a proof since it gives the opportunity to review some useful techniques.

Lemma 2.1. *Let μ be a finite, positive Borel measure with compact support K and suppose $\lambda \in \mathbb{C} \setminus K$. Then the following are equivalent:*

(i) $\lambda \in \mathrm{abpe}(P^t(\mathrm{d}\mu))$;
(ii) $\lambda \in \mathrm{bpe}(P^t(\mathrm{d}\mu))$;
(iii) $f(z) := \frac{1}{z-\lambda} \notin P^t(\mathrm{d}\mu)$.

Proof. Clearly, (i) implies (ii). Let us now assume that $\lambda \in \mathrm{bpe}(P^t(\mathrm{d}\mu))$. So, as earlier noted, there exists k_λ in $L^s(\mathrm{d}\mu)$ ($\frac{1}{s} + \frac{1}{t} = 1$) such that $p(\lambda) = \int p(\zeta)k_\lambda(\zeta)\,\mathrm{d}\mu(\zeta)$ for every polynomial p. If $f(z) = \frac{1}{z-\lambda}$ were the limit in $L^t(\mathrm{d}\mu)$ of a sequence of polynomials $\{p_n\}$, then we would have

$$0 = \int (z - \lambda)p_n(z)k_\lambda(z)\,\mathrm{d}\mu(z) \to \int k_\lambda(z)\,\mathrm{d}\mu(z) = 1$$

as $n \to \infty$; an obvious contradiction. Therefore, (ii) implies (iii). Lastly, assume that $f(z) = \frac{1}{z-\lambda} \notin P^t(\mathrm{d}\mu)$. So, by the Hahn-Banach and Riesz Representation Theorems, there exists g in $L^s(\mathrm{d}\mu)$ ($\frac{1}{s} + \frac{1}{t} = 1$) such that $g \perp P^t(\mathrm{d}\mu)$ and yet $\int \frac{g(\zeta)}{\zeta-\lambda}\,\mathrm{d}\mu(\zeta) \neq 0$. Let V be the component of $\mathbb{C} \setminus K$ that contains λ. Then, the Cauchy transform $\widehat{g}(z) := \int \frac{g(\zeta)}{\zeta-z}\,\mathrm{d}\mu(\zeta)$, which is defined and analytic in V, is not identically zero there. So, there exists $r > 0$ such that $\{z : |z-\lambda| \leqslant r\} \subseteq V$ and $|\widehat{g}(z)| \geqslant \varepsilon > 0$ whenever $|z-\lambda| = r$. Observe that if p is any polynomial and $z \in V$, then $p(z)\widehat{g}(z) = \int \frac{p(\zeta)g(\zeta)}{\zeta-z}\,\mathrm{d}\mu(\zeta)$. So, if $|z-\lambda| = r$, then $|p(z)| \leqslant c\|p\|_{L^t(\mathrm{d}\mu)}$, where c depends only on ε, $\|g\|_{L^s(\mathrm{d}\mu)}$ and $\mathrm{dist}(K, \{z : |z-\lambda| = r\})$. From the Maximum Modulus Theorem, it now follows that $\lambda \in \mathrm{abpe}(P^t(\mathrm{d}\mu))$. ∎

3. AN OVERCONVERGENCE RESULT

We begin this section with a well-known result concerning analytic bounded point evaluations. It is a straightforward consequence of Lemma 2.6 in [12].

Theorem 3.1. *Let μ be a finite, positive Borel measure with compact support in $\mathbb{C}$ and let K be a compact subset of* $\mathrm{abpe}(P^t(\mathrm{d}\mu))$. *Then*

$$\mathrm{abpe}(P^t(\mathrm{d}\mu|(\mathbb{C} \setminus K))) = \mathrm{abpe}(P^t(\mathrm{d}\mu)).$$

Without the assumption that K is a compact subset of $\mathrm{abpe}(P^t(\mathrm{d}\mu))$, the conclusion of Theorem 3.2 would fail even for some relatively small sets K; see [4]. Nevertheless, given λ in $\mathrm{abpe}(P^t(\mathrm{d}\mu))$, there are (sometimes quite large) Borel subsets E of support(μ) that are contained in no compact subset of $\mathrm{abpe}(P^t(\mu))$ such that $\lambda \in \mathrm{abpe}(P^t(\mathrm{d}\mu|(\mathbb{C} \setminus E)))$; in the terminology of J. Thomson ([18]), $\mu|E$ (for such E) does not play a vital role in the "sequence of barriers around λ". Our next result is the first step in a strategy for discovering such nonessential sets E. Before we get to this result, we need some more terminology. As is standard in the literature, we call γ a *cross-cut* of a region Ω if γ is a Jordan arc ($\gamma : [0,1] \to \mathbb{C}$) such that $\gamma(0)$ and $\gamma(1)$ are in $\partial\Omega$ and $\gamma((0,1)) \subset \Omega$; we let γ denote both the Jordan arc and its *trace* $\gamma([0,1])$. If, in addition, γ is continuously differentiable and $\gamma'(t) \neq 0$ for $0 \leqslant t \leqslant 1$, then we call γ a *smooth cross-cut* of Ω. Our next result involves certain cross-cuts and subregions of $\mathbb{D} := \{z : |z| < 1\}$ which we now describe (and illustrate — see figure 1). Let γ be a smooth cross-cut of $\mathbb{D}$ with endpoints a and b (neither of which are equal to 1). Then $\mathbb{D} \setminus \gamma$ is the disjoint union of two Jordan regions A and B. We assume that γ is chosen so that $0 \in A$ and $1 \in \partial B$. Let γ_{o} be a smooth cross-cut of B with endpoints a and b and with the property: There are positive constants ε and M such that

$$\mathrm{dist}(z, \gamma) \geqslant \varepsilon |z-a|^M |z-b|^M$$

whenever $z \in \gamma_{\mathrm{o}}$.

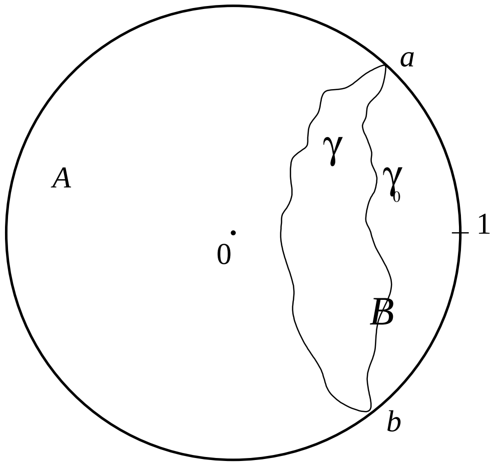

Figure 1.

THEOREM 3.2. *With γ, γ_o, A and B as described above, let μ be a finite, positive Borel measure with support contained in $\overline{A} \cup (\partial\mathbb{D})$. Let σ denote arclength measure on γ_o and let $\nu = \mu|\overline{A} + \sigma$. If $0 \in \operatorname{abpe}(P^t(\mathrm{d}\mu))$, then $0 \in \operatorname{abpe}(P^t(\mathrm{d}\nu))$.*

Proof. By [9], Exercise 2 of Chapter III and a similar argument involving an appropriately chosen outer function, we may assume that $\mu|\partial\mathbb{D} = h\mathrm{d}m$, where $0 \leqslant h \leqslant 1$ and m denotes normalized Lebesgue measure on $\partial\mathbb{D}$. Furthermore, by Theorem 3.1, we may assume that $0 \notin \operatorname{support}(\mu)$. Therefore, $f(z) := \frac{1}{z} \in L^t(\mathrm{d}\mu)$ and yet $f \notin P^t(\mathrm{d}\mu)$. Applying the Hahn-Banach and Riesz Representation Theorems, there exists g in $L^s(\mathrm{d}\mu)$ ($\frac{1}{s} + \frac{1}{t} = 1$) such that $g \perp P^t(\mathrm{d}\mu)$ and yet $\int \frac{1}{\zeta} g(\zeta)\,\mathrm{d}\mu(\zeta) \neq 0$. As before, we let $\widehat{g}$ denote the Cauchy transform of g — that is, $\widehat{g}(z) = \int \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\mu(\zeta)$ — which is defined and analytic off support(μ). Notice that ∂B consists of two Jordan arcs: γ and an arc of $\partial\mathbb{D}$ (with endpoints a and b) which we call α. Let W denote the Jordan region whose boundary consists of γ_o along with α. Now, by our choice of γ_o, there exist $\varepsilon > 0$ and a positive integer N such that

$$\operatorname{dist}(z, \gamma) \geqslant \varepsilon |z - a|^N |z - b|^N$$

for all z in γ_o. Observe that $g_o(z) := \varepsilon(z - a)^N (z - b)^N g(z)$ satisfies:

(i) $g_o \in L^s(\mathrm{d}\mu)$;
(ii) $g_o \perp P^t(\mathrm{d}\mu)$;
(iii) $\int \frac{1}{\zeta} g_o(\zeta)\,\mathrm{d}\mu(\zeta) \neq 0$; and
(iv) $\widehat{g_o}|\gamma_o \in L^\infty(\mathrm{d}\sigma)$.

So, by replacing g with g_o if necessary, we may assume that $\widehat{g}|\gamma_o \in L^\infty(\mathrm{d}\sigma)$.

Claim. $\widehat{g}|W \in H^s(W)$.

By [11], Corollaire, $\widehat{g}|W \in H^p(W)$ for $0 < p < 1$. Since $\widehat{g} \in L^\infty(\mathrm{d}\sigma)$, our claim will be established if we show that the nontangential boundary values of $\widehat{g}|W$

on α are in $L^s(\mathrm{d}m)$. Now if ξ is in the relative interior of α, then, for $0 < r < 1$ sufficiently near 1, $r\xi \in W$. Since $\widehat{g}|(\mathbb{C} \setminus \overline{\mathbb{D}}) \equiv 0$, we have (for such r) that

$$\widehat{g}(r\xi) = \widehat{g}(r\xi) - \widehat{g}\Big(\frac{\xi}{r}\Big) = \int g(\zeta)\left(\frac{1}{\zeta - r\xi} - \frac{1}{\zeta - \frac{\xi}{r}}\right) \mathrm{d}\mu(\zeta)$$

$$= \int\limits_{\overline{\mathrm{A}}\cap\mathbb{D}} g(\zeta)\left(\frac{1}{\zeta - r\xi} - \frac{1}{\zeta - \frac{\xi}{r}}\right) \mathrm{d}\mu(\zeta) + \int\limits_{\partial\mathbb{D}} g(\zeta)P_{r\xi}(\zeta)\overline{\zeta}h(\zeta)\,\mathrm{d}m(\zeta),$$

where $P_{r\xi}$ denotes the Poisson kernel on $\partial\mathbb{D}$ for evaluation at $r\xi$. Therefore, by Fatou's Theorem, $\widehat{g}(r\xi) \to g(\xi)\overline{\xi}h(\xi)$ as $r \to 1^-$. Since $0 \leqslant h \leqslant 1$, it follows that $g(\xi)\overline{\xi}h(\xi) \in L^s(\mathrm{d}m)$ and so our claim holds. Let $\widetilde{g}$ denote the boundary values of $\widehat{g}$ on ∂W; so $\widetilde{g} = \widehat{g}$ on γ_{o} and $\widetilde{g}(\xi) = g(\xi)\overline{\xi}h(\xi)$ for ξ in α. So if β is a Jordan curve that traverses ∂W once in the counterclockwise direction, then, by our claim and Cauchy's Theorem,

$$\widehat{g}(z) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\beta} \frac{\widetilde{g}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

if $z \in W$, and

$$0 = \frac{1}{2\pi\mathrm{i}} \int\limits_{\beta} \frac{\widetilde{g}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

if $z \in \mathbb{C} \setminus \overline{W}$; we may assume that (for $0 \leqslant t \leqslant \frac{1}{2}$) $\beta(t) = \gamma_{\mathrm{o}}(1-2t)$. Therefore, if we let $\tau = {\gamma_{\mathrm{o}}}^{-1}$ and define κ a.e. ν by: $\kappa = g$ on $\overline{A}$, and $\kappa(\zeta) = \frac{1}{2\pi\mathrm{i}}\widehat{g}(\zeta)\frac{|\tau'(\zeta)|}{\tau'(\zeta)}$ for ζ in γ_{o}, then we have: $\kappa \in L^s(\mathrm{d}\nu)$, $\widehat{\kappa}|(\mathbb{C} \setminus \overline{\mathbb{D}}) \equiv 0$ and so $\kappa \perp P^t(\mathrm{d}\nu))$ and yet, $\widehat{\kappa}(0) = \widehat{g}(0) \neq 0$. By the proof of Lemma 2.1, we can now assert that $0 \in \operatorname{abpe}(P^t(\mathrm{d}\nu))$. ∎

The first of the next two lemmas has its counterpart in the literature (cf. [16]); let us review our notation. If G is any bounded region in $\mathbb{C}$, then $H^\infty(G)$ denotes the collection of bounded analytic functions on G, $\mathcal{N}(G)$ denotes the Nevanlinna class of G and $L^1_a(G)$ denotes the collection of functions f that are analytic in G and that satisfy $\int_G |f|\,\mathrm{d}m_2 < \infty$, where m_2 is area measure on $\mathbb{C}$. For $0 < r < 1$, we let $W_r := \{z : |z - r| < 1 - r\}$.

Lemma 3.3. *If $f \in L^1_a(\mathbb{D})$, then $f|W_r \in \mathcal{N}(W_r)$ for $0 < r < 1$.*

Proof. For z in $\mathbb{D}$ and $0 < \rho < 1 - r$, let $\Delta(z) = \{w : |w - z| < 1 - |z|\}$ and let $\Gamma_\rho = \{z : |z - r| = \rho\}$. If $z \in \mathbb{D}$ and $f \in L^1_a(\mathbb{D})$, then

$$|f(z)| \leqslant \frac{1}{\pi(1-|z|)^2} \int\limits_{\Delta(z)} |f|\,\mathrm{d}m_2 \leqslant \frac{1}{\pi(1-|z|)^2} \int\limits_{\mathbb{D}} |f|\,\mathrm{d}m_2.$$

Evidently, $\sup\Big\{\int_{\Gamma_\rho} \log^+|f(z)|\,|\mathrm{d}z| : 0 < \rho < 1 - r\Big\} < \infty$ and so $f|W_r \in \mathcal{N}(W_r)$. ∎

LEMMA 3.4. *If* $0<\rho<1$ *and* $0\not\equiv f\in L^1_a(W_\rho)$, *then, for each* r $(\rho<r<1)$, *there exists* $f_r\not\equiv 0$ *in* $H^\infty(\mathbb{D})$ *such that* $|f_r(z)|\leqslant|f(z)|$ *for all* z *in* W_r.

Proof (Sketch). Choose r $(\rho<r<1)$ and then select s and t such that $\rho<t<s<r$. By Lemma 3.3, $f|W_t\in\mathcal{N}(W_t)$ and so, if $\{z_n\}$ is an enumeration of the zeros of f in $W_s(\subseteq W_t)$, then $\sum\limits_n((1-t)-|z_n-t|)<\infty$. Now, from elementary geometric considerations, there is a positive costant M such that $1-|z|\leqslant M((1-t)-|t-z|)$ for all z in W_s. Therefore, $\sum\limits_n(1-|z_n|)<\infty$ and hence $\{z_n\}$is a Blaschke sequence; let B be the associated Blaschke product and let $f_{\mathrm{o}}=\frac{f}{B}$.

Let T be the Möbius transformation from $H^{\wedge+}:=\{\zeta:\Im(\zeta)>0\}$ onto $\mathbb{D}$ given by $T(\zeta)=\frac{\zeta-\mathrm{i}}{\zeta+\mathrm{i}}$ and let φ be the Möbius transformation given by $\varphi(z)=T\big(T^{-1}(z)-\frac{\mathrm{i}s}{1-s}\big)$ $(=\frac{(2-s)z-s}{sz+(2-3s)})$; let $\psi=\varphi^{-1}$. Notice that φ maps W_s onto $\mathbb{D}$ and $\varphi(1)=1$. Now $(f_{\mathrm{o}}\circ\psi)|\mathbb{D}\in\mathcal{N}(\mathbb{D})$ and $f_{\mathrm{o}}\circ\psi$ has no zeros in $\mathbb{D}$. Therefore, $(f_{\mathrm{o}}\circ\psi)|\mathbb{D}=(F_hS_\mu)/S_\nu$, where F_h is an outer function and both S_μ and S_ν are singular inner functions. Let h^* be defined a.e. m (normalized Lebesgue measure on $\partial\mathbb{D}$) by $h^*(\zeta)=\min(h(\zeta),1)$ and let F_{h^*} be the corresponding outer function. For w in $\mathbb{D}$, we let $P_w(\cdot)$ denote the Poisson kernel on $\partial\mathbb{D}$ for evaluation at w. Carrying $P_w(\cdot)$ to H^+ by composition with T, one can find a constant $c>1$ such that $P_{\varphi(z)}(\zeta)\leqslant cP_z(\zeta)$ whenever $z\in W_r$ and $|\zeta|=1$. So, there is a natural number n such that

$$|F^n_{h^*}(z)|\leqslant|F_{h^*}(\varphi(z))|\leqslant|F_h(\varphi(z))|\quad\text{ and }\quad|S^n_\mu(z)|\leqslant|S_\mu(\varphi(z))|$$

whenever $z\in W_r$. Evidently, $f_r:=BF^n_{h^*}S^n_\mu$ satisfies our conclusion. ∎

We are now in a position to give the consequence of (Theorem 3.2) that we earlier described as an analogue of Theorem 3.1; once again we set the stage. Let Ω be a subregion of $\mathbb{D}$ such that the relative interior of $(\partial\Omega)\cap(\partial\mathbb{D})$ is nonempty. Let Γ be a smooth cross-cut of Ω with endpoints a and b in the relative interior of $(\partial\Omega)\cap(\partial\mathbb{D})$ such that $0\notin\Gamma$, Γ approaches $\partial\mathbb{D}$ nontangentially at a and at b, and $(\partial\Omega)\cap\mathbb{D}$ has no accumulation point in Γ. Now $\mathbb{D}\setminus\Gamma$ is the disjoint unoin of two Jordan regions — let W be the one that does not contain 0, let $\alpha=\overline{W}\cap(\partial\mathbb{D})$ and let $G=\Omega\setminus\overline{W}$.

THEOREM 3.5. *With* Ω *and* G *as described above, choose* f *in* $L^1_a(\Omega)$ $(f\not\equiv 0)$ *and define* μ *and* η *by:* $\mathrm{d}\mu=|f|\,\mathrm{d}m_2|\Omega$ *and* $\eta=\mu|G$. *If* $0\in\operatorname{abpe}(P^t(\mathrm{d}\mu))$, *then* $0\in\operatorname{abpe}(P^t(\mathrm{d}\eta))$.

Proof. By our hypothesis, we may construct two other smooth cross-cuts γ and γ_{o} of Ω, each with endpoints a and b, and having the properties:

(i) both γ and γ_{o} have nontangential approach to $\partial\mathbb{D}$ at a and at b;
(ii) $\gamma\cap\overline{W}=\{a,b\}=\gamma_{\mathrm{o}}\cap\overline{W}$;
(iii) any two of Γ, γ and γ_{o} form a positive angle at a and at b,
(iv) γ_{o} is a cross-cut of the Jordan region V whose boundary is $\gamma\cup\Gamma$, $V\subset\Omega$ and $0\notin\overline{V}$; see figure 2.

Let E be the Jordan region whose boundary is $\alpha\cup\gamma$ and let $\mu_{\mathrm{o}}=\mu|(\Omega\setminus\overline{E})+\widehat{(\mu|E)}$, where $\widehat{(\mu|E)}$ denotes the sweep of $\mu|E$ to ∂E. Then, by our hypothesis and the definition of the sweep (see [8], Chapter V, Section 9), $0 \in \mathrm{abpe}(P^t\,\mathrm{d}\mu_{\mathrm{o}})$. So, we can apply Theorem 3.2 to get that $0 \in \mathrm{abpe}(P^t(\mathrm{d}\nu))$, where $\nu = \mu|(\Omega \setminus \overline{E}) + \mu_{\mathrm{o}}|\gamma + \sigma$ and σ denotes arclength measure on γ_{o}. For z in G, let $r_z = \mathrm{dist}(z, \partial G)$ and let $\Delta_z = \{\zeta : |\zeta - z| < r_z\}$. If $z \in G$ and p is a polynomial, then $p(z)f(z) = \frac{1}{\pi r_z^2} \int_{\Delta_z} pf\,\mathrm{d}m_2$ and so there is a positive constant c (independent of p and z) such that $|p(z)|^t|f(z)|^t \leqslant \frac{c}{\pi^t r_z^{2t}} \int_G |p|^t\,\mathrm{d}\mu$. Moreover, by our construction, there exists $\varepsilon > 0$ such that $r_z \geqslant \varepsilon|z-a|\,|z-b|$ whenever $z \in (\gamma \cup \gamma_{\mathrm{o}}) \setminus \{a, b\}$. Therefore, by Lemma 3.4, we can find g in $H^\infty(\mathbb{D})$ ($g \not\equiv 0$) such that

$$|p(z)|^t|g(z)|^t \leqslant \int_G |p|^t\,\mathrm{d}\mu \tag{3.1}$$

whenever $z \in (\gamma \cup \gamma_{\mathrm{o}}) \setminus \{a, b\}$ and p is a polynomial.

Claim. $0 \in \mathrm{abpe}(P^t(|g|^t\,\mathrm{d}\nu))$.

To establish this claim, first observe that we may assume that $g(0) \neq 0$ and so there exists δ, $0 < \delta < 1$, and $\lambda > 0$ such that $|g(\zeta)| \geqslant \lambda$ whenever $|\zeta| < \delta$. Since $0 \in \mathrm{abpe}(P^t(\mathrm{d}\nu))$, there exists r, $0 < r < 1$ and $M > 0$ such that $|p(\zeta)|^t \leqslant M\|p\|^t_{L^t(\mathrm{d}\nu)}$ whenever $|\zeta| < r$ and p is a polynomial; we may assume that $\delta \leqslant r$. Therefore, $|p(\zeta)|^t \leqslant \frac{M}{\lambda^t}\|pg\|^t_{L^t(\mathrm{d}\nu)}$ whenever $|\zeta| < \delta$ and p is a polynomial; evidently our claim holds. Notice that, for any polynomial p,

$$\int_{\Omega\setminus\overline{E}} |p|^t|g|^t\,\mathrm{d}\mu \leqslant \|g\|^t_\infty \int_{\Omega\setminus\overline{E}} |p|^t\,\mathrm{d}\mu \quad \text{and, by (3.1),}$$

$$\int_{\gamma\cup\gamma_{\mathrm{o}}} |p|^t|g|^t\,\mathrm{d}\nu \leqslant \nu(\gamma \cup \gamma_{\mathrm{o}}) \int_G |p|^t\,\mathrm{d}\mu.$$

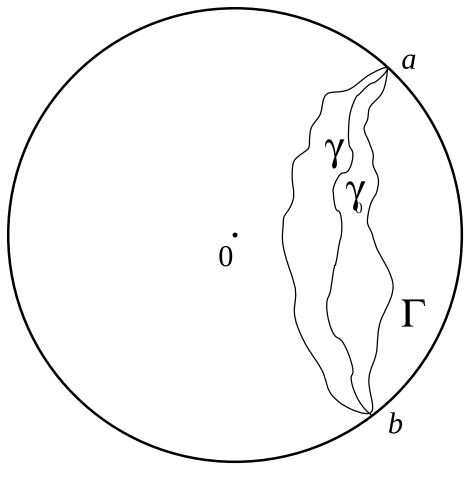

Figure 2.

Since $\Omega\setminus\overline{E} \subseteq G$, we can now apply our claim to get that $0 \in \mathrm{abpe}(P^t(\mathrm{d}\eta))$. ∎

Remark 3.6. The role of $\mathbb{D}$ in Theorem 3.5 can be assumed by any Jordan region U. Indeed, if $\Omega \subseteq U$, $f \in L^1_a(\Omega)$ and φ is a conformal mapping from $\mathbb{D}$ ont U, then, under a change of variables, $|f|\,\mathrm{d}m_2|\Omega$ corresponds to $|f\circ\varphi|\,|\varphi'|^2\,\mathrm{d}m_2|\varphi^{-1}(\Omega)$ and clearly $(f\circ\varphi)(\varphi')^2 \in L^1_a(\varphi^{-1}(\Omega))$. Using this and the fact that φ^{-1} extends to a homeomorphism between $\overline{U}$ and $\overline{\mathbb{D}}$ that is uniformly approximable by polynomials on $\overline{U}$, the result of Theorem 3.5 carries over (with $\varphi(0)$ in place of 0).

In some of the subsequent applications of Theorem 3.5 we restrict our attention to regions called crescents and do so primarily to minimize the technical details.

Definition 3.7. Let B and U be Jordan regions such that $B \subseteq U$ and $\overline{B}\cap(\partial U)$ is a single point. Then the region $\Omega := U\setminus\overline{B}$ is called a *crescent* and the point $\overline{B}\cap(\partial U)$ is called the *multiple boundary point* (mbp) of Ω.

If $\Omega = U\setminus\overline{B}$ is a crescent, μ is a finite positive Borel measure with support in $\overline{\Omega}$ and $b\in B$, then $R(\overline{\Omega}) \subseteq P^t(\mathrm{d}\mu)$ ($R(\overline{\Omega})$ is the uniform closure in $C(\overline{\Omega})$ of the rational functions with poles off $\overline{\Omega}$) if and only if $z\mapsto \frac{1}{z-b} \in P^t(\mathrm{d}\mu)$. Our next result addresses this point and, in part, justifies the title of this paper.

Theorem 3.8. *Let $\Omega = U\setminus\overline{B}$ be a crescent with mbp equal to λ, let $\mathrm{d}\mu = |f|\,\mathrm{d}m_2|\Omega$, where $f\in L^1_a(\Omega)$, and select b in B. Then the following are equivalent:*
(i) $R(\overline{\Omega}) \subseteq P^t(\mathrm{d}\mu)$;
(ii) $z\mapsto \frac{1}{z-b} \in P^t(\mathrm{d}\mu)$;
(iii) *there is a sequence $\{p_n\}$ of polynomials such that p_n converges to $\frac{1}{z-b}$ in $L^t(\mathrm{d}\mu)$ and uniformly on $\overline{\Omega}\setminus V$ (as $n\to\infty$) for any neighborhood V of λ.*

Proof. As we indicated just before the statement of Theorem 3.9, the equivalence of (i) and (ii) is well-known; we nevertheless outline the argument. Clearly (i) implies (ii). Conversely, if $\frac{1}{z-b}\in P^t(\mathrm{d}\mu)$, then, since $\frac{1}{z-b}\in L^\infty(\mathrm{d}\mu)$, we have: $\frac{1}{(z-b)^n}\in P^t(\mathrm{d}\mu)$ for $n = 1,2,3,\ldots$. From Runge's Theorem it now follows that $R(\overline{\Omega})\subseteq P^t(\mathrm{d}\mu)$. Trivially, (iii) implies (ii); what remains to be shown is the converse of this. Now, by Remark 3.6, we may assume that $U=\mathbb{D}$, and indeed that $b=0$ and that $\lambda=1$. Our proof involves a thickening of Ω; as before, if $r<1$, then we let $W_r = \{z : |z-r| < 1-r\}$. Let g be a homeomorphism from $\{z : |z|\leqslant 2\}$ onto $\{z : |z|\leqslant 2\}$ such that $g(\overline{\mathbb{D}}) = g(\overline{\mathbb{D}})$, $g(0)=0$, $g(1)=1$ and $g(B) = W_{\frac14}$; therefore, $g(\Omega) = \mathbb{D}\setminus\overline{W}_{\frac14}$. Let $E = W_{\frac15}\setminus\overline{W}_{\frac13}$, let $F = W_{-\frac12}\setminus\overline{W}_{\frac18}$ and (for $n=1,2,3,\ldots$) let $E_n = E\cap\{z : |z-1| > \frac1n\}$ and let $F_n = F\cap\{z : |z-1| > \frac1n\}$. Let $\Omega_n = \Omega\cup g^{-1}(E_n\cup F_n)$, let $I_n = \Omega_n\setminus g^{-1}(\overline{E}_n)$ and let $J_n = \Omega\setminus g^{-1}(\overline{E}_n\cup\overline{F}_n)$; notice that Ω_n, I_n and J_n are crescents and $0\notin\overline{\Omega}_n$. Define μ_n with support in $\overline{\Omega}_n$ by $\mu_n = \mu + m_2|g^{-1}(E_n\cup F_n)$. If $\frac1z\notin P^t(\mathrm{d}\mu_n)$, then, by [1], Theorem 3.12, $\mathbb{D}\subseteq \operatorname{abpe}(P^t(\mathrm{d}\mu_n))$. So, by Theorem 3.1, $\mathbb{D}\subseteq\operatorname{abpe}(P^t(\mathrm{d}\mu_n|I_n))$. We now apply Theorem 3.5 to get that $0\in\operatorname{abpe}(P^t(\mathrm{d}\mu_n|J_n))$. It follows that $0\in\operatorname{abpe}(P^t(\mathrm{d}\mu))$ and hence, by Lemma 2.1, $\frac1z\notin P^t(\mathrm{d}\mu)$. But this contradicts our assumption in (ii). So, we conclude that $\frac1z\in P^t(\mathrm{d}\mu_n)$. Hence, we can find a polynomial p_n such that $\|\frac1z - p_n\|_{L^t(\mathrm{d}\mu_n)} < \frac1n$. Evidently, $\{p_n\}$ converges to $\frac1z$ in $L^t(\mathrm{d}\mu)$. Furthermore, for any $r>0$, there exists n such that $\overline{\Omega}\setminus\{z : |z-1|<r\}$ is a compact subset of

Ω_n ($\subseteq \Omega_m$ for $m = n, n+1, n+2, \ldots$). So, from our definition of μ_n, we also have that $\{p_n\}$ converges to $\frac{1}{z}$ uniformly on $\overline{\Omega} \setminus V$ for any neighborhood V of 1. This completes our proof. ∎

4. CYCLIC VECTORS

If T is a bounded linear transformation on a Banach space $\mathcal{B}$, $x \in \mathcal{B}$ and the linear span of $\{T^n(x) : n = 0, 1, 2, \ldots\}$ is dense in $\mathcal{B}$, then x is called a *cyclic vector* for T and T is said to be *cyclic* (on $\mathcal{B}$). If G is a bounded region in $\mathbb{C}$ and $1 \leqslant t < \infty$, then the shift M_z on $L^t_a(G)$, defined by $M_z(f) = zf$, is a bounded linear transformation. If, in addition, G is simply connected, then M_z on $L^t_a(G)$ might be cyclic; which, in this setting, would mean that there exists f in $L^t_a(G)$ such that $\{pf : p \text{ is a polynomial}\}$ is dense in $L^t_a(G)$. For which G is M_z on $L^t_a(G)$ cyclic? This question has been addressed in the contexts of the Bergman space $L^t_a(G)$ and the Hardy space $H^t(G)$ (see [3], [2] and [17]), though only for a very limited collection of regions G. In a somewhat different direction, it was shown in [3] (via a change of variables argument suggested by P. Bourdon) that if M_z on $H^2(G)$ is cyclic, then M_z on $L^2_a(G)$ is also cyclic. This change of variables argument, however, does not guarantee a bounded cyclic vector for M_z on $L^2_a(G)$. In Theorem 4.11 we improve upon this result and show that if M_z on $H^t(G)$ is cyclic, then M_z on $L^t_a(G)$ has a bounded cyclic vector. Prior to any of this it was shown (see [6], Corollary 3.4) that if 1 is a cyclic vector for M_z on $L^2_a(G)$, then 1 is a cyclic vector for M_z on $H^2(G)$. Could it in fact be true that if M_z on $L^2_a(G)$ is cyclic, then M_z on $H^2(G)$ is cyclic; completing an equivalence? This question is hard to answer (it remains open) in large part because, if f is a cyclic vector for M_z on $L^2_a(G)$, then $f \circ \varphi$ (φ is a conformal mapping from $\mathbb{D}$ onto G) could have a variety of forms. This is in contrast with M_z on $H^2(G)$, where f is a cyclic vector only if $f \circ \varphi$ is an outer function. So, it would be helpful to know that if M_z on $L^2_a(G)$ is cyclic, then there is a cyclic vector f such that $f \circ \varphi$ is an outer function. Using Theorem 3.8, we take a step in this direction; first, though, we need to lay some groundwork. For the remainder of this paper, unless otherwise specified, we let G be a bounded, simply connected region in $\mathbb{C}$. We also let $\mathcal{P}$ denote the collection of polynomials. Our next result follows immediately from the fact that $\mathcal{P} \subset L^\infty(G)$.

Proposition 4.1. *A function f in $L^t_a(G)$ is a cyclic vector for M_z on $L^t_a(G)$ if and only if $fH^\infty(G)$ is dense in $L^t_a(G)$ and*

$$\inf_{p \in \mathcal{P}} \int_G |p - g|^t |f|^t \, \mathrm{d}m_2 = 0$$

for each g in $H^\infty(G)$.

Definition 4.2. A compact subset K of $\partial\mathbb{D}$ is called a *Carleson set* if $m(K) = 0$ and $\int_{\partial\mathbb{D}} \log(\operatorname{dist}(z, K)) \, \mathrm{d}m(z) > -\infty$.

The following theorem is a consequence of [5], Propositions 1 and 2 and Theorem.

THEOREM 4.3. *A function f is a bounded cyclic vector for M_z on $L^t_a(\mathbb{D})$ if and only if $f = FS_\nu$, where F is a bounded outer function and S_ν is a singular inner function such that $\nu(C) = 0$ for every Carleson set C.*

For an outline of the proof of the following result, one may consult [16].

THEOREM 4.4. (L.I. Hedberg) *If G is a bounded region in $\mathbb{C}$ such that ∂G consists of finitely many continua, then $H^\infty(G)$ is dense in $L^t_a(G)$.*

LEMMA 4.5. *Let E and G be bounded, simply connected regions and let φ be a conformal mapping from E onto G; let $\psi = \varphi^{-1}$. Then $(\psi')^{\frac{2}{t}} H^\infty(G)$ is dense in $L^t_a(G)$.*

Proof. Since ψ' is never zero, $(\psi')^{\frac{2}{t}}$ is defined and analytic in G. Moreover, $\int_G \left|(\psi')^{\frac{2}{t}}\right|^t \mathrm{d}m_2 = \int_E |\psi' \circ \varphi|^2 |\varphi'|^2 \,\mathrm{d}m_2 = m_2(E) < \infty$. Therefore, $(\psi')^{\frac{2}{t}} H^\infty(G) \subseteq L^t_a(G)$. Now, for any h in $H^\infty(G)$,

$$\inf_{g\in H^\infty(G)} \int_G \left|h - g(\psi')^{\frac{2}{t}}\right|^t \mathrm{d}m_2 = \inf_{g\in H^\infty(G)} \int_E \left|(h\circ\varphi)(\varphi')^{\frac{2}{t}} - (g\circ\varphi)\right|^t \mathrm{d}m_2 = 0,$$

since, by Theorem 4.4, $H^\infty(E)$ is dense in $L^t_a(E)$. By Theorem 4.4 (once again), our proof is now complete. ∎

Using Lemma 4.5 and a straightforward change of variables argument, we have:

PROPOSITION 4.6. *Suppose $f \in H^\infty(G)$ and let φ be a conformal mapping from $\mathbb{D}$ onto G. Then $fH^\infty(G)$ is dense in $L^t_a(G)$ if and only if $f\circ\varphi$ is a bounded cyclic vector for M_z on $L^t_a(\mathbb{D})$.*

Our next result is a consequence of work found in [5].

THEOREM 4.7. *The following are equivalent:*

(i) *M_z on $L^t_a(G)$ has a Nevanlinna class cyclic vector;*

(ii) *M_z on $L^t_a(G)$ has a bounded cyclic vector;*

(iii) *there exists f in $H^\infty(G)$ such that $\inf_{p\in\mathcal{P}} \int_G |p-h|^t |f|^t \,\mathrm{d}m_2 = 0$ for all h in $H^\infty(G)$, where $f\circ\varphi = FS_\nu$ (φ is a conformal mapping from $\mathbb{D}$ onto G), F is a bounded outer function and S_ν is a singular inner function with the property that $\nu(C) = 0$ for every Carleson set C.*

REMARK 4.8. If $\Omega = U \setminus \overline{B}$ is a crescent and $b \in B$, then we may replace the general h (in $H^\infty(\Omega)$) that appears in Theorem 4.7 (iii) by just one function, namely $h(z) = \frac{1}{z-b}$.

Our next result follows from rather standard measure-theoretic methods, though a detailed proof of it would almost certainly land us in a technical quagmire. For this reason, we give only the briefest sketch of a proof.

THEOREM 4.9. *Let W be Jordan subregion of $\mathbb{D}$ and let $X = (\partial W) \cap (\partial \mathbb{D})$.*

(i) *If S_ν is a singular inner function and $\nu(X) = 0$, then there is an outer function F in $H^\infty(\mathbb{D})$ such that $|F(z)| \leqslant |S_\nu(z)|$ for all z in W.*

(ii) *If F is an outer function and $m(X) = 0$, then there is a singular inner function S_ν, where $\nu(C) = 0$ for every Carleson set C, such that $|S_\nu(z)| \leqslant |F(z)|$ for all z in W.*

Proof. (Sketch) For z in $\mathbb{D}$ we let $\zeta \mapsto P_z(\zeta)$ denote the Poisson kernel on $\partial\mathbb{D}$ for evaluation at z. The central ingredient in the proofs of either (i) or (ii) is the following observation.

Observation. If K is a compact subset of $(\partial\mathbb{D}) \setminus X$, then there are bounds on $|P_z(\mathrm{e}^{\mathrm{i}\theta})|$ and $\left|\frac{\partial P_z(\mathrm{e}^{\mathrm{i}\theta})}{\partial\theta}\right|$ that are independent of $\mathrm{e}^{\mathrm{i}\theta}$ in K and z in W.

The proof of part (i) reduces to showing that there exists h in $L^1(\mathrm{d}m)$ $(h \geqslant 0)$ such that $\int_{\partial\mathbb{D}} P_z(\zeta)\,\mathrm{d}\nu(\zeta) \leqslant \int_{\partial\mathbb{D}} P_z(\zeta)h(\zeta)\,\mathrm{d}m(\zeta)$ whenever $z \in W$. Using our observation and basic measure-theoretic methods, one can construct such an h systematically over the sets E_1 and $E_{k+1} \setminus E_k$ (for $k = 1, 2, 3, \ldots$), where $E_k := \{\zeta : \operatorname{dist}(\zeta, X) > 2^{-k}\}$; h can be chosen to be zero on X. The proof of (ii) requires an additional ingredient. We first construct a strictly increasing continuous but singular function f on $[0,1]$ whose modulus of continuity $\omega_f(\delta)$ is $\mathrm{O}\big(\delta \log\big(\frac{1}{\delta}\big)\big)$ and let μ be the measure whose cumulative distribution function is f (see [14]). For $0 < c \leqslant 1$ and $0 \leqslant d \leqslant 1-c$, define $\mu_{c,d}$ on $[d, 1-c]$ by $\mu_{c,d}(B) = \mu\big(\frac{1}{c}(B-d)\big)$ and let $\mathcal{F}$ be the collection of measures of the form $\mu_{c,d}$ carried to $\partial\mathbb{D}$ under the mapping $s \mapsto \mathrm{e}^{2\pi\mathrm{i}s}$. If $\eta \in \mathcal{F}$, then $\eta(C) = 0$ for every Carleson set C. Furthermore, $\mathcal{F}$ is weak-star dense in the collection of measures that are absolutely continuous with respect to m. So we can now proceed, as we did in part (i), to piece-together a singular measure ν (using the collection $\mathcal{F}$) for which a reverse inequality (to that of part (i)) holds; ν satisfies $\nu(C) = 0$ for every Carleson set C. ∎

THEOREM 4.10. *Let $\Omega = U \setminus \overline{B}$ be a crescent and let φ be a conformal mapping from $\mathbb{D}$ onto Ω. If M_z on $L^t_a(\Omega)$ has a Nevanlinna class cyclic vector, then there are cyclic vectors f_1 and f_2 such that $f_1 \circ \varphi$ is an outer function in $H^\infty(\mathbb{D})$ and $f_2 \circ \varphi = S_\eta$ is a singular inner function such that $\eta(C) = 0$ for every Carleson set C.*

Proof. Mapping by an appropriate choice of Möbius transformation, we may assume that the mbp of Ω is 1 and that $0 \in B$. Now if M_z on $L^t_a(\Omega)$ has a Nevanlinna class cyclic vector, then, by Theorem 4.7, there is a cyclic vector f in $H^\infty(\Omega)$ such that $f \circ \varphi = FS_\nu$, where F is an outer function in $H^\infty(\mathbb{D})$ and S_ν is a singular inner function such that $\nu(C) = 0$ for every Carleson C. Moreover, $\inf_{p\in\mathcal{P}} \int_\Omega \big|p - \frac{1}{z}\big|^t |f|^t\,\mathrm{d}m_2 = 0$. So, by Theorem 3.8, there is a sequence $\{p_n\}$ of polynomials such that p_n converges to $\frac{1}{z}$ in $L^t(|f|^t\,\mathrm{d}m_2|\Omega)$ and uniformly on $\overline{\Omega} \setminus V$ for any neighborhood V of 1 (as $n \to \infty$). Now, since Ω is a crescent, φ extends continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. In fact, there are distinct points a and b in $\partial\mathbb{D}$ such that $\varphi(a) = \varphi(b) = 1$ (the mbp of Ω) and φ maps $\overline{\mathbb{D}} \setminus \{a, b\}$ univalently onto $\overline{\Omega} \setminus \{1\}$; we may assume that $a = -1$ and $b = 1$. If $\{r_k\}$ is a sequence of real numbers such that $0 < r_1 < r_2 < \cdots < r_k < r_{k+1} \to 1$ as $k \to \infty$, then we let $W(\{r_k\}) = \{z : |z| < r_1\} \cup \bigcup_{k=2}^{\infty} \{re^{i\theta} : 0 < r < r_k \text{ and } \operatorname{dist}(\theta, \{0, \pi\}) < \frac{\pi}{2^k}\}$. Notice that $W(\{r_k\})$ is a Jordan subregion of $\mathbb{D}$ and $(\partial W(\{r_k\})) \cap (\partial \mathbb{D}) = \{-1, 1\}$. Recalling that p_n converges to $\frac{1}{z}$ uniformly on $\overline{\Omega} \setminus V$ for any neighborhood V of 1, we can apply a Hastings-type argument (see [10], Lemma) to find a sequence $\{r_k\}$ that converges to 1 quickly enough so that $W := W(\{r_k\})$ satisfies:

$$\lim_{n\to\infty} \int\limits_{\Omega\setminus\varphi(W)} \left| p_n - \frac{1}{z} \right|^t \mathrm{d}m_2 = 0. \tag{4.1}$$

Now, by Theorem 4.9, there is a bounded outer function F_{o} and there is a singular inner function S_μ, where $\mu(C) = 0$ for every Carleson set C, such that $|F_{\mathrm{o}}(z)| \leqslant |S_\nu(z)|$ and $|S_\mu(z)| \leqslant |F(z)|$ for all z in W. Define f_1 and f_2 on Ω by $f_1 \circ \varphi = F F_{\mathrm{o}}$ and $f_2 \circ \varphi = S_\eta$, where $\eta = \nu + \mu$. By (4.1), Propositions 4.1 and 4.6, and Theorem 4.3, f_1 and f_2 are cyclic vectors for M_z on $L_a^t(\Omega)$. ∎

The next two results contribute to the theme of this paper and yet their proofs stand alone and do not involve the notion of overconvergence. The first of these represents a considerable improvement upon [3], Added in proof.

THEOREM 4.11. *If M_z on $H^t(G)$ is cyclic, then M_z on $L_a^t(G)$ is cyclic and has a bounded cyclic vector f such that $f \circ \varphi$ is an outer function; φ is a conformal mapping from $\mathbb{D}$ onto G.*

Proof. Since M_z on $H^t(G)$ is cyclic, there is a bounded outer function F_{o} ($F_{\mathrm{o}} \not\equiv 0$) such that

$$\inf_{p\in\mathcal{P}} \int\limits_{\partial\mathbb{D}} |\widetilde{p\circ\varphi} - \widetilde{h}|^t \cdot |\widetilde{F_{\mathrm{o}}}|^t \,\mathrm{d}m = 0$$

for any h in $H^\infty(\mathbb{D})$, where $\mathcal{P}$ denotes the collection of polynomials and $\widetilde{h}$ etc. denotes the nontangential boundary values of h on $\partial\mathbb{D}$. Now $|\varphi'|^2 \,\mathrm{d}m_2$ represents a finite, positive Borel measure on $\mathbb{D}$ and so the sweep of this measure to $\partial\mathbb{D}$ (we let μ denote this sweep) satisfies: $\mu << m$; see [8], Chapter V, Section 9. Hence, there is a bounded outer function F_1 ($F_1 \not\equiv 0$) such that $|F_1|^t \,\mathrm{d}\mu \leqslant \mathrm{d}m$; let $F = F_{\mathrm{o}} \cdot F_1$ and let $f = F \circ \varphi^{-1}$. Then, for any h in $H^\infty(\mathbb{D})$, we have:

$$\begin{aligned}\inf_{p\in\mathcal{P}} \int\limits_{G} |p - h\circ\varphi^{-1}|^t \cdot |f|^t \,\mathrm{d}m_2 &= \inf_{p\in\mathcal{P}} \int\limits_{\mathbb{D}} |p\circ\varphi - h|^t \cdot |F|^t \cdot |\varphi'|^2 \,\mathrm{d}m_2 \\ &\leqslant \inf_{p\in\mathcal{P}} \int\limits_{\partial\mathbb{D}} |\widetilde{p\circ\varphi} - \widetilde{h}|^t \cdot |\widetilde{F}|^t \,\mathrm{d}\mu \leqslant \inf_{p\in\mathcal{P}} \int\limits_{\partial\mathbb{D}} |\widetilde{p\circ\varphi} - \widetilde{h}|^t \cdot |\widetilde{F_{\mathrm{o}}}|^t \,\mathrm{d}m = 0.\end{aligned}$$

Since $\{f \cdot (h\circ\varphi^{-1}) : h \in H^\infty(\mathbb{D})\}$ is dense in $L_a^t(G)$, we conclude that f is a cyclic vector for M_z on $L_a^t(G)$. ∎

As we mentioned earlier, it is part of the literature that 1 is a cyclic vector for M_z on $H^2(G)$ if 1 is a cyclic vector for M_z on $L^2_a(G)$; see [6], Corollary 3.4. With this in mind, it is natural to conjecture:

If M_z on $L^2_a(G)$ is cyclic and has a cyclic vector f such that $f \circ \varphi$ is an outer function (φ is a conformal mapping from $\mathbb{D}$ onto G), then M_z on $H^2(G)$ is cyclic.

This, of course, would give us the converse to Theorem 4.11 for $t = 2$. The following proposition establishes an ever so slightly weaker form of this conjecture.

Proposition 4.12. *If M_z on $L^2_a(G)$ is cyclic and has a cyclic vector f such that $F := f \circ \varphi$ is an outer function and $\log |\widetilde{F}| \in L \log L(m)$, then M_z on $H^2(G)$ is cyclic.*

Proof. By our assumption that $\log |\widetilde{F}| \in L \log L(m)$, we can find a bounded outer function F_{o} ($F_{\mathrm{o}} \not\equiv 0$) such that $|\widetilde{F_{\mathrm{o}}}(\mathrm{e}^{\mathrm{i}\theta})| \leqslant \inf_{0 \leqslant r < 1} |F(r\mathrm{e}^{\mathrm{i}\theta})|$ a.e. m; see [19], Chapter IV, Theorem 5.3. Choose h in $H^\infty(G)$ and p in $\mathcal{P}$. Let g be a primitive of h and let q be the primitive of p such that $q(\varphi(0)) = g(\varphi(0))$. Then, for $0 < s < 1$,

$$\int_{\partial\mathbb{D}} |(q \circ \varphi)(s\zeta) - (g \circ \varphi)(s\zeta)|^2 \cdot |\widetilde{F}_{\mathrm{o}}(\zeta)|^2 \,\mathrm{d}m(\zeta)$$

$$= \int_{\partial\mathbb{D}} \left| \int_0^s (p \circ \varphi)(r\zeta) \cdot \varphi'(r\zeta) - (h \circ \varphi)(r\zeta) \cdot \varphi'(r\zeta) \,\mathrm{d}r \right|^2 \cdot |\widetilde{F}_{\mathrm{o}}(\zeta)|^2 \,\mathrm{d}m(\zeta)$$

$$\leqslant \int_{\partial\mathbb{D}} \int_0^1 |(p \circ \varphi)(r\zeta) - (h \circ \varphi)(r\zeta)|^2 \cdot |\varphi'(r\zeta)|^2 \cdot |F(r\zeta)|^2 \,\mathrm{d}r \,\mathrm{d}m(\zeta)$$

$$\leqslant C \int_{\mathbb{D}} |p \circ \varphi - h \circ \varphi|^2 \cdot |F|^2 \cdot |\varphi'|^2 \,\mathrm{d}m_2 = C \int_G |p - h|^2 \cdot |f|^2 \,\mathrm{d}m_2,$$

where C is a constant independent of s, p and h. Therefore,

$$\int_{\partial\mathbb{D}} |(\widetilde{q \circ \varphi}) - \widetilde{g \circ \varphi}|^2 \cdot |\widetilde{F}_{\mathrm{o}}|^2 \,\mathrm{d}m \leqslant C \int_G |p - h|^2 \cdot |f|^2 \,\mathrm{d}m_2.$$

Since $\{g : g' \in H^\infty(G)\}$ is dense in $H^2(G)$ and f is a cyclic vector for M_z on $L^2_a(G)$, it follows that $f_{\mathrm{o}} := F_{\mathrm{o}} \circ \varphi^{-1}$ is a cyclic vector for M_z on $H^2(G)$. ∎

Remark 4.13. There is another approach to the proof of our conjecture that is worth mentioning. If F, $g \in H^\infty(\mathbb{D})$, $g(0) = 0$ and F is an outer function, then, by Green's Theorem (see [9], Chapter VI, Section 3),

$$\int_{\partial\mathbb{D}} |\widetilde{g}\widetilde{F}|^2 \,\mathrm{d}m = \frac{2}{\pi} \int_{\mathbb{D}} |g'F + gF'|^2 \cdot \log\left(\frac{1}{|z|}\right) \mathrm{d}m_2(z).$$

Therefore,

$$\Big\{ \int\limits_{\partial\mathbb{D}} |\widetilde{g}|^2 |\widetilde{F}|^2 \, \mathrm{d}m \Big\}^{\frac{1}{2}}$$
$$\leqslant \sqrt{\frac{2}{\pi}} \bigg(\Big\{ \int\limits_{\mathbb{D}} |g'F|^2 \cdot \log\Big(\frac{1}{|z|}\Big) \, \mathrm{d}m_2(z) \Big\}^{\frac{1}{2}} + \Big\{ \int\limits_{\mathbb{D}} |gF'|^2 \cdot \log\Big(\frac{1}{|z|}\Big) \, \mathrm{d}m_2(z) \Big\}^{\frac{1}{2}} \bigg).$$

Choosing g to be $p \circ \varphi - h$, as in the proof of Proposition 4.12, we see that our objective is reached if we can find an outer function F_{o} ($F_{\mathrm{o}} \not\equiv 0$) with the properties that (for all such g):

(i) $|F_{\mathrm{o}}| \leqslant |F|$; and

(ii) $\int\limits_{\mathbb{D}} |gF_{\mathrm{o}}'|^2 \cdot \log\Big(\frac{1}{|z|}\Big) \, \mathrm{d}m_2(z) \leqslant \int\limits_{\mathbb{D}} |g'F|^2 \, \mathrm{d}m_2(z)$.

Despite the merits of this Green's Theorem approach, Proposition 4.12 is still about the best that the authors have been able to do in this direction; the validity of our conjecture remains an open question.

REMARK 4.14. Assuming that M_z on $L_a^t(G)$ is cyclic, we have no guarantee that there is a Nevanlinna class cyclic vector; this remains an open question even for crescents. However, there is an obervation that gets us tantalizingly close to the converse of Theorem 4.11 (in its full generality). Let Ω be a Jordan subregion of $\mathbb{D}$ such that

$$\int\limits_{\partial\Omega} \log(1 - |z|) \, \mathrm{d}\omega(z, \Omega, z_{\mathrm{o}}) > -\infty,$$

where $\omega(\cdot, \Omega, z_{\mathrm{o}})$ denotes harmonic measure on $\partial\Omega$ for evaluation at some point z_{o} in Ω; of course, $(\partial\mathbb{D}) \cap (\partial\Omega)$ might still be quite large under this restriction. Let φ be a conformal mapping from $\mathbb{D}$ onto G and let $W = \varphi(\Omega)$. If $0 \not\equiv f \in L_a^t(G)$, then one can sharpen Lemma 3.4 and argue (using r_z) as in the proof of Theorem 3.5 to produce a bounded outer function F_{o} ($F_{\mathrm{o}} \not\equiv 0$) such that $f_{\mathrm{o}} := F_{\mathrm{o}} \circ \varphi^{-1}$ satisfies:

$$\int\limits_{\partial W} |h(\zeta)|^t |f_{\mathrm{o}}(\zeta)|^t \, \mathrm{d}\omega(\zeta, W, w_{\mathrm{o}}) \leqslant \int\limits_{G} |h|^t |f|^t \, \mathrm{d}m_2$$

for all h in $H^\infty(G)$. So, if f is a cyclic vector for M_z on $L_a^t(G)$, then f_{o} is a cyclic vector for M_z on $H^t(W)$. This observation suggests the following strategy for the proof of the converse of Theorem 4.11. Show that if M_z on $L_a^t(G)$ is cyclic, then one can slightly thicken G (over certain of its boundary points) to a bounded, simply connected region E so that the relationship of G to E is like that of W to G as described above and such that M_z on $L_a^t(E)$ is cyclic. This is nothing more than establishing overconvergence, yet with a mild requirement on the amount of overconvergence. Whether or not this requirement can be satisfied seems to call for some delicate estimates that remain beyond the authors' grasps.

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