J. OPERATOR THEORY
47(2001), 131–143

THE DENSITY OF HYPERCYCLIC OPERATORS ON A HILBERT SPACE

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Communicated by Norberto Salinas

ABSTRACT. On a separable infinite dimensional complex Hilbert space, we show that the set of hypercyclic operators is dense in the strong operator topology, and moreover the linear span of hypercyclic operators is dense in the operator norm topology. Both results continue to hold if we restrict to only those hypercyclic operators with an infinite dimensional closed hypercyclic subspace. Our works make connections with the classical result on the nondenseness of cyclic operators in the operator norm topology, as well as the recent developments on hypercyclic subspaces.

KEYWORDS: *Separable Hilbert space, hypercyclic operators, cyclic operators, closed hypercyclic subspaces, strong operator topology, norm topology, operator algebra.*

MSC (2000): Primary 46A32, 47A16; Secondary 47L05.

1. INTRODUCTION

Throughout this paper, we use H to denote a separable infinite dimensional Hilbert space over the complex field, and use $B(H)$ to denote the algebra of all bounded linear operators $T : H \to H$. For an operator T in $B(H)$ and a vector f in H, we define the *orbit* to be the set $\operatorname{orb}(T, f) = \{f, Tf, T^2 f, T^3 f, \ldots\}$. If the orbit $\operatorname{orb}(T, f)$ is dense in H for some vector f in H, then the operator T is called a *hypercyclic operator*, and in that case the vector f is said to be a *hypercyclic vector* of T. An example of a hypercyclic operator on a Hilbert space was first given by Rolewicz ([24]) who showed that if B is the unilateral backward shift on H then the operator λB is hypercyclic for any scalar λ with $|\lambda| > 1$. In fact, Rolewicz proved this result in the setting of Banach spaces c_0 and ℓ^p for finite $p \geqslant 1$.

The existence of hypercyclic operators on H naturally leads to the problem of determining how large a set the hypercyclic operators form in the operator algebra $B(H)$. In the present paper, we study this problem in terms of density in a topology of $B(H)$. Along this line, two classical results are helpful. To explain them, we need the following definition: An operator T in $B(H)$ is called a *cyclic operator* if there exists a vector f in H such that the linear span of the orbit $\mathrm{orb}(T, f)$ is dense in H, and in that case the vector f is called a *cyclic vector* of T. It is well-known, as documented in Halmos' Problem Book ([18], p. 88), that the cyclic operators are not dense in the operator norm topology of $B(H)$. On the other hand, the noncyclic operators are dense, due to Fillmore, Stampfli, and Williams ([13]). These two results give us a hint on the situation of hypercyclic operators. From the definitions, it is easy to see that the norm of a hypercyclic operator is strictly larger than one, and that every nonzero scalar multiple of a hypercyclic operator is cyclic. Hence if the hypercyclic operators were dense in the complement of the closed unit ball of $B(H)$, then the cyclic operators would be dense in $B(H)$. To summarize our discussion, we provide the following fact.

FACT 1.1. *The nonhypercyclic operators are dense in $B(H)$, but the hypercyclic operators are not dense in the complement of the closed unit ball of $B(H)$.*

It follows from this fact that the closure of hypercyclic operators is a special class of operators in $B(H)$. This class of operators was completely characterized by Herrero ([19]) in terms of Weyl spectra, normal eigenvalues, and Fredholm indices.

On the other hand, it is interesting to investigate whether the hypercyclic operators are dense in other topologies that are weaker than the operator norm topology in $B(H)$. Among the natural topologies that $B(H)$ carries, the strongest one next to the operator norm topology is the strong operator topology. These two topologies are the only ones that we consider in the present paper. To distinguish the two, we use the convention that when a topological term is used for $B(H)$ it always refers to the operator norm topology, otherwise we specify the strong operator topology, by adding in most cases the prefix "SOT" in front of the term.

In Section 2 a major result states that the hypercyclic operators in $B(H)$ are SOT-dense. At the first glance, the result may not seem to be possible, because every hypercyclic operator must have norm larger than 1. Nevertheless the zero operator is the SOT-limit of a sequence of hypercyclic operators. For instance, if we take the unilateral backward shift $B : H \to H$ and define operators T_n for all $n \geqslant 1$ by

$$T_n = \frac{n+1}{n} B^n,$$

then it is clear that $T_n \to 0$ in the strong operator topology, as $n \to \infty$. Furthermore, we can prove that each T_n is a hypercyclic operator, by using Rolewicz' result ([24]) that the operator

$$\sqrt[n]{\frac{n+1}{n}}\, B$$

is hypercyclic, along with Ansari's result ([1]) that if $A : H \to H$ is a hypercyclic operator then A^n is also hypercyclic. Alternatively, we can repeat Rolewicz' argument on each T_n, or use the hypercyclic criterion that we discuss in Section 2.

Instead of proving directly in Section 2 that the hypercyclic operators are SOT-dense, we prove a better result which states that the set of those hypercyclic operators with a hypercyclic Hilbert subspace is SOT-dense.

Definition 1.2. A *hypercyclic Hilbert subspace* of an operator T in $B(H)$ is an infinite dimensional closed subspace of H consisting, except for the zero vector, entirely of hypercyclic vectors of T.

The interests in the linear structure of hypercyclic vectors was originated by Beauzamy ([3], [4], [5]) who constructed an operator T in $B(H)$ with a dense invariant linear manifold consisting, except for the zero vector, of hypercyclic vectors of T. Manifolds of this kind were also studied extensively by Godefroy and Shapiro ([16]) for operators possessing some major properties of the unilateral backward shift. Then Herrero ([19]) and independently Bourdon ([10]) proved that any hypercyclic operator on H has such a dense invariant linear manifold of hypercyclic vectors. In fact, Bourdon's proof works for a complex locally convex space. Recently, Bès ([8]) gave a proof for a real locally convex space.

The construction of a hypercyclic Hilbert subspace was originated by Bernal-González and Montes-Rodríguez ([7]). Then Montes-Rodríguez ([23]) provided a sufficient condition for a bounded linear operator on a separable infinite dimensional Banach space to have an infinite dimensional closed subspace consisting, except for the zero vector, of hypercyclic vectors. This sufficient condition gives us a method to construct in Section 2 an SOT-dense set of hypercyclic operators with a hypercyclic Hilbert subspace.

Though the set of hypercyclic operators can only be SOT-dense but not dense in $B(H)$, we show in Section 3 that its linear span is indeed dense. To prove that, we require the recent results of Salas ([25]) and León-Saavedra and Montes-Rodríguez ([22]) on the hypercyclicity of the operators in form of the identity plus a unilateral weighted backward shift.

For more information on the recent developments on hypercyclicity, one may refer to the survey article by K.G. Grosse-Erdmann ([17]).

2. STRONG OPERATOR TOPOLOGY

In this section, we study the SOT-density of hypercyclic and cyclic operators in $B(H)$, for a separable infinite dimensional complex Hilbert space H. To begin, we extract the following definition from [21], Theorem A.

Definition 2.1. Let $B_{\rm hy}(H)$ denote the set of all operators T in $B(H)$ such that T has a sequence of positive integers $\{n_k\}$ satisfying the following three Axioms:

(1) There is a dense subset D_1 of H such that $\|T^{n_k}f\| \to 0$ for every vector f in D_1.

(2) There is a dense subset D_2 of H, and a mapping $A : D_2 \to D_2$ such that $TA =$ the identity map on D_2 and $\|A^{n_k}f\| \to 0$ for every vector f in D_2.

(3) There is an infinite dimensional closed subspace H_0 of H such that $\|T^{n_k}f\| \to 0$ for every vector f in H_0.

Every operator in $B_{\rm hy}(H)$ is hypercyclic and moreover it has a hypercyclic Hilbert subspace, as proved by Montes–Rodríguez ([23]). In fact Montes-Rodríguez proved the result not only for a Hilbert space but for a Banach space. Then Chan ([11]) gave a simple proof for the Hilbert space version of this result. Certainly not every hypercyclic operator on H has a hypercyclic Hilbert subspace, and in fact León-Saavedra and Montes-Rodríguez ([22]) showed that Axiom 3 is essential for the existence of a hypercyclic Hilbert subspace.

Axioms 1 and 2 combined is called the *hypercyclicity criterion,* which is a sufficient condition, first shown by Kitai ([20]), for the operator T to be hypercyclic. In fact, Kitai's work takes place in a Banach space setting. Then Gethner and Shapiro ([15]) rediscovered the criterion in a Fréchet space setting, by using an argument totally different from Kitai's. Axiom 3 was added by Montes-Rodríguez ([23]) in order to show that T has a hypercyclic Hilbert subspace.

Before we state the results of this section, we remark that the finite rank operators are SOT-dense in $B(H)$. To see that, we fix a countable orthonormal basis of H, and let P_n denote the orthogonal projection onto the linear span of first n members of the orthonormal basis. Then for any operator S in $B(H)$, the sequence P_nSP_n converges to S in the strong operator topology, as $n \to \infty$. Furthermore, it is easy to construct a sequence of nonzero finite rank operators that converges to the zero operator in the operator norm topology. Hence the nonzero finite rank operators are SOT-dense in $B(H)$. With this observation, we prove the main theorem of this section.

THEOREM 2.2. *The set* $B_{\mathrm{hy}}(H)$ *is* SOT-*dense in* $B(H)$.

Proof. We are to prove that every SOT-open set U contains an operator T in $B_{\mathrm{hy}}(H)$. Since U contains a nonzero finite rank operator S, we can assume that U is the SOT-basic open set given by

$$U \ = \{T \in B(H) : \|(T-S)f_\gamma\| < \varepsilon \text{ for all } \gamma = 1, 2, \dots, n\},$$

for some nonzero vectors $f_1, f_2, \dots, f_n$ in H and some positive ε, and then construct an operator T in U that is also in $B_{\mathrm{hy}}(H)$.

We begin by letting the dimension of the range of S be $k \geqslant 1$, and let $\{h(1), h(2), \dots, h(k)\}$ be an orthonormal basis of the range of S. It follows that a vector f is in $\ker S$ if and only if

$$\langle f, S^*h(i)\rangle = \langle Sf, h(i)\rangle = 0, \qquad \text{for all } i = 1, \dots, k.$$

Thus $\operatorname{span}\{S^*h(i) : 1 \leqslant i \leqslant k\} = \ker S^\perp$ is a finite dimensional subspace, and so is the subspace M defined by

$$M = \ker S^\perp + \operatorname{span}\{h(1), \dots, h(k)\}.$$

This subspace M coincides with $\operatorname{span}\{h(1), \dots, h(k)\}$ if and only if the subspace $\ker S^\perp$ is contained in $\operatorname{span}\{h(1), \dots, h(k)\}$. Otherwise there exist orthonormal vectors $g(1), g(2), \dots, g(m)$ such that the vectors $h(1), \dots, h(k)$, $g(1), \dots, g(m)$ form an orthonormal basis of M. We continue the proof only for the case when $m \geqslant 1$, but the same proof works for the case when the vectors $g(i)$ do not exist and consequently we take $m = 0$ whenever m appears in an algebraic expression.

Thus an orthonormal basis of the orthogonal complement $M^\perp$ of M is a countably infinite set, which we divide into three different sequences to facilitate the definition of our operator T. In other words, we let

$$\{e(i), a(i), b(i) : i \geqslant 1\}$$

be an orthonormal basis of $M^\perp$ so that the set E defined by

$$\begin{aligned}E = \{h(1), \dots, h(k)\} \cup \{g(1), \dots, g(m)\} \cup \{e(1), e(2), e(3), \dots\}\\ \cup \{a(1), a(2), a(3), \dots\} \cup \{b(1), b(2), b(3), \dots\}\end{aligned}$$

is an orthonormal basis of H. In addition, we may assume that there exists a large enough positive integer p so that if $P : H \to H$ denotes the orthogonal projection onto the closed linear span of the set

$$\{e(i) : i \geqslant p+1\} \cup \{a(i) : i \geqslant 1\}$$

and if C denotes the positive constant given by

$$C = \max\{\|f_\gamma\| : \gamma = 1, 2, \dots, n\},$$

then

$$\|Pf_\gamma\| < \min\left(\frac{\varepsilon^2}{9Ck\|S\|^2 + 3k\varepsilon\|S\|}, \frac{\varepsilon}{6\sqrt{2}}\right), \quad \text{for all } \gamma = 1, 2, \dots, n. \tag{2.1}$$

In terms of the orthonormal basis E, we define a linear mapping $T : H \to H$ by the following equations:

$$Tg(i) = Sg(i), \qquad \text{if } 1 \leqslant i \leqslant m; \tag{2.2}$$

$$Th(i) = Sh(i) + \frac{\varepsilon}{3C}e(p+i), \qquad \text{if } 1 \leqslant i \leqslant k; \tag{2.3}$$

$$Te(i) = 0, \qquad \text{if } 1 \leqslant i \leqslant p. \tag{2.4}$$

Note that Equation (2.3) defines T on the range of S, and hence we can continue to give the following definition in Equation (2.5):

$$Te(p+i) = \frac{-3C}{\varepsilon}TSh(i), \qquad \text{if } 1 \leqslant i \leqslant k; \tag{2.5}$$

$$Te(p+k+i) = g(i), \qquad \text{if } 1 \leqslant i \leqslant m; \tag{2.6}$$

$$Te(p+k+m+i) = h(i), \qquad \text{if } 1 \leqslant i \leqslant k; \tag{2.7}$$

$$Te(p+2k+m+i) = 2e(i), \qquad \text{if } i \geqslant 1; \tag{2.8}$$

$$Ta(1) = 0. \tag{2.9}$$

$$Ta(2i+1) = 2a(i), \qquad \text{if } i \geqslant 1; \tag{2.10}$$

$$Ta(2i) = b(i), \qquad \text{if } i \geqslant 1; \tag{2.11}$$

$$Tb(i) = 0, \qquad \text{if } i \geqslant 1. \tag{2.12}$$

Equations (2.2) to (2.12) define a bounded linear operator T in $B(H)$. In the case that the vectors $g(i)$ do not exist, we do not have Equations (2.2) and (2.6), and so $m = 0$ in Equations (2.7) and (2.8) as we have commented before.

We now proceed to show that $T \in U$. We begin with the following estimate: If $\rho_1, \rho_2, \ldots, \rho_k$ are scalars and if $h = \rho_1 h(1) + \cdots + \rho_k h(k)$, then

$$\begin{aligned}
\|Th\| &\leqslant |\rho_1|\,\|Th(1)\| + \cdots + |\rho_k|\,\|Th(k)\| \\
&\leqslant \big(|\rho_1|^2 + \cdots + |\rho_k|^2\big)^{1/2}\big(\|Th(1)\|^2 + \cdots + \|Th(k)\|^2\big)^{1/2} \\
&= \|h\| \left(\sum_{i=1}^{k} \|Th(i)\|^2 \right)^{1/2} \\
&\leqslant \|h\| \left(\sum_{i=1}^{k} (\|S\| + \frac{\varepsilon}{3C})^2 \right)^{1/2}, \quad \text{by Equation (2.3)} \\
&= \|h\| \sqrt{k} \left(\|S\| + \frac{\varepsilon}{3C} \right).
\end{aligned} \tag{2.13}$$

This allows us to estimate $\|(T-S)f_\gamma\|$ for a fixed γ with $1 \leqslant \gamma \leqslant n$. For that purpose, we first observe that $M^\perp$ is contained in $\ker S$ and so

$$Se(i) = Sa(i) = Sb(i) = 0 \quad \text{for all } i \geqslant 1.$$

This observation, along with Equation (2.4), implies that $(T-S)e(i) = 0$ whenever $1 \leqslant i \leqslant p$. In addition, Equation (2.2) gives $(T-S)g(i) = 0$ whenever $1 \leqslant i \leqslant m$. Hence if we write the vector f_γ as

$$f_\gamma = \sum_{i=1}^{m} \sigma_i g(i) + \sum_{i=1}^{k} \rho_i h(i) + \sum_{i=1}^{\infty} c_i e(i) + \sum_{i=1}^{\infty} \alpha_i a(i) + \sum_{i=1}^{\infty} \beta_i b(i)$$

such that the three infinite sums $\sum |c_i|^2, \sum |\alpha_i|^2$, and $\sum |\beta_i|^2$ are all finite, then we derive the following inequality:

$$\|(T-S)f_\gamma\| \leqslant \left\| \sum_{i=1}^{k} \rho_i (T-S) h(i) \right\| + \left\| \sum_{i=p+1}^{\infty} c_i Te(i) + \sum_{i=1}^{\infty} \alpha_i Ta(i) + \sum_{i=1}^{\infty} \beta_i Tb(i) \right\|.$$

Then we use Equations (2.3), and (2.5) through (2.12), to continue our estimation.

$$\begin{aligned}
\|(T-S)f_\gamma\| \leqslant & \left\| \sum_{i=1}^{k} \frac{\varepsilon}{3C} \rho_i e(p+i) \right\| + \left\| \sum_{i=p+1}^{p+k} c_i \frac{3C}{\varepsilon} TSh(i-p) \right\| + \\
& + \sqrt{\sum_{i=p+k+1}^{\infty} 2^2 |c_i|^2} + \sqrt{\sum_{i=1}^{\infty} 2^2 |\alpha_i|^2}.
\end{aligned} \tag{2.14}$$

For the second summand on the right-hand side of Inequality (2.14), we observe that each $Sh(i-p)$ is in the range of S, which is spanned by $h(1), \ldots, h(k)$. Hence we can use Inequality (2.13) to deduce that

$$\begin{aligned}
&\Big\| \sum_{i=p+1}^{p+k} c_i \frac{3C}{\varepsilon} TSh(i-p) \Big\| \\
&\qquad \leqslant \frac{3C}{\varepsilon}\sqrt{k}\Big(\|S\| + \frac{\varepsilon}{3C}\Big) \sum_{i=p+1}^{p+k} |c_i|\,\|Sh(i-p)\| \\
&\qquad \leqslant \frac{3C}{\varepsilon}\sqrt{k}\Big(\|S\| + \frac{\varepsilon}{3C}\Big)\|S\| \Big(\sum_{i=p+1}^{p+k} |c_i|^2\Big)^{1/2} \Big(\sum_{i=p+1}^{p+k} 1^2\Big)^{1/2} \\
&\qquad < \frac{3C}{\varepsilon} k\Big(\|S\| + \frac{\varepsilon}{3C}\Big)\|S\| \frac{\varepsilon^2}{9Ck\|S\|^2 + 3k\varepsilon\|S\|}, \qquad \text{by Inequality (2.1)} \\
&\qquad = \frac{\varepsilon}{3}.
\end{aligned}$$

This, along with Inequalities (2.14) and (2.1), implies that

$$\|(T-S)f_\gamma\| < \frac{\varepsilon}{3C}\|f_\gamma\| + \frac{\varepsilon}{3} + 2\sqrt{2}\|Pf_\gamma\| \; < \frac{\varepsilon}{3C}C + \frac{\varepsilon}{3} + 2\sqrt{2}\frac{\varepsilon}{6\sqrt{2}} = \varepsilon,$$

which means that $T \in U$.

We now turn our attention to showing that T is in $B_{\mathrm{hy}}(H)$. We first remark that if H_0 is the closed linear span of $\{b(i) : i \geqslant 1\}$, then $Tf = 0$ for all vectors f in H_0. It remains to show that T satisfies the hypercyclicity criterion; that is, Axioms 1 and 2 in the definition of $B_{\mathrm{hy}}(H)$. For Axiom 1, we note that $\operatorname{span} E$ is dense in H and hence it suffices to show that $T^j \to 0$ pointwise on E, as $j \to \infty$. To do that we first notice that if $1 \leqslant i \leqslant k$, then we use Equations (2.3) and (2.5) to derive that

$$T^2 h(i) = T(Th(i)) = TSh(i) + \frac{\varepsilon}{3C} Te(p+i) = 0. \tag{2.15}$$

Since the vectors $h(i)$ span the range of S, we have that

$$T^2 S = 0. \tag{2.16}$$

Then it follows from Equation (2.2) that $T^3 g(i) = T^2 S g(i) = 0$ for all $i = 1, \dots, m$.

We now consider the basis elements $e(i)$ in E. First $Te(i) = 0$ if $1 \leqslant i \leqslant p$, by Equation (2.4). Then we deduce from Equations (2.5) and (2.6) that

$$T^2 e(p+i) = \frac{-3C}{\varepsilon} T^2 Sh(i) = 0, \qquad \text{if } 1 \leqslant i \leqslant k.$$

If $1 \leqslant i \leqslant m$, then we observe by Equations (2.6), (2.2) and (2.16) that

$$T^4 e(p+k+i) = T^2(Tg(i)) = T^2 Sg(i) = 0.$$

Furthermore, if $1 \leqslant i \leqslant k$, then we use Equations (2.7) and (2.15) to derive that

$$T^3 e(p+k+m+i) = T^2 h(i) = 0.$$

For a fixed integer i with $i > p+2k+m$, we write $i = q(p+2k+m)+r$ for some positive integer q and some nonnegative integer $r < p+2k+m$. It follows from Equations (2.8) and (2.7) that

$$T^j e(i) = 2T^{j-1}e(i-(p+2k+m)) = \cdots = \begin{cases} 2^q T^{j-q} e(r) & \text{if } r \neq 0 \\ 2^{q-1} T^{j-q} h(k) & \text{if } r = 0, \end{cases}$$

which is 0 whenever $j - q \geqslant 4$. Finally, we note that $Ta(1) = 0$ by our definition, and that $T^2 a(2) = Tb(1) = 0$. In general, we can use induction to show that for any positive integer i, there exists a large enough positive integer j such that $T^j a(i) = 0$. Hence $T^j f \to 0$ for all vectors f in $\operatorname{span} E$, when $j \to \infty$.

To finish the whole proof, we need to show that T satisfies Axiom 2. For that, we define a linear operator $A : H \to H$ in terms of the orthonormal basis E by

$$\begin{aligned} Ag(i) &= e(p+k+i), && \text{if } 1 \leqslant i \leqslant m, \\ Ah(i) &= e(p+k+m+i), && \text{if } 1 \leqslant i \leqslant k, \\ Ae(i) &= \tfrac{1}{2} e(p+2k+m+i), && \text{if } i \geqslant 1, \\ Aa(i) &= \tfrac{1}{2} a(2i+1), && \text{if } i \geqslant 1, \\ Ab(i) &= a(2i), && \text{if } i \geqslant 1. \end{aligned}$$

This definition gives an operator A in $B(H)$ satisfying $TA = I$ and $A^j f \to 0$ for all vectors f in $\operatorname{span} E$. ∎

The previous proof shows a slightly stronger statement than the theorem, because of some special properties that the operator T in the proof has. Actually the proof shows that those operators T in $B_{\mathrm{hy}}(H)$ having all following four additional properties are SOT-dense in $B(H)$: First, $\ker T = \operatorname{span}\{b(i) : i \geqslant 1\}$ is an infinite dimensional closed subspace of H. Second, we can take the sequence of positive integers $\{n_k\}$ of T in the definition of $B_{\mathrm{hy}}(H)$ to be the entire sequence of positive integers. Third, the sets D_1 and D_2 for T in the definition of $B_{\mathrm{hy}}(H)$ are the same set, namely the linear span of an orthonormal basis of H. Lastly T has a right inverse A satisfying Axiom 2 in the definition of $B_{\mathrm{hy}}(H)$. Despite these refinements of the theorem, we point out the following direct consequence of the theorem.

COROLLARY 2.3. *The set of all hypercyclic operators on H is* SOT-*dense in $B(H)$.*

Since every hypercyclic operator is cyclic, we derive the following result from Corollary 2.3.

COROLLARY 2.4. *The set of all cyclic operators on H is* SOT-*dense in $B(H)$.*

The cyclic operators are not dense in $B(H)$, as we have mentioned in the Introduction. In particular, a co-rank argument in [18], p. 88 shows that if $V = S \oplus S$, where S is the unilateral forward shift, then every operator T satisfying $\|T - V\| < 1$ cannot be cyclic. Nevertheless, Corollary 2.4 shows that the cyclic operators are dense in the topology of $B(H)$ that is one step weaker than the operator norm topology.

3. NORM TOPOLOGY

In this section, we investigate how large the set $B_{\rm hy}(H)$ is in terms of the operator norm topology. Though $B_{\rm hy}(H)$ is not dense, we can prove, in Theorem 3.3 below, that the linear span of $B_{\rm hy}(H)$ is dense. This result does not follow from Theorem 2.2, because the linear span of an SOT-dense set in $B(H)$ is not necessarily dense. For instance, the set of all finite rank operators is an SOT-dense linear manifold in $B(H)$, but it fails to be dense in $B(H)$. Before we prove Theorem 3.3, we need the following lemma that helps us identify some operators in the linear span of $B_{\rm hy}(H)$.

LEMMA 3.1. *If S is an orthogonal projection with rank one, then S is the sum of two operators in $B_{\rm hy}(H)$.*

Proof. Let $\{e_n : n \geqslant 0\}$ be an orthonormal basis of H such that $Se_0 = e_0$ and $Se_n = 0$ for all $n \geqslant 1$. Then we define an operator T_1 in $B(H)$ by

$$T_1 e_n = \begin{cases} \frac{1}{2}e_0 & \text{if } n = 0, \\ 2e_{(n-2)/2} & \text{if } n \text{ is nonzero even}, \\ 0 & \text{if } n \text{ is odd}; \end{cases}$$

and define an operator T_2 in $B(H)$ by

$$T_2 e_n = \begin{cases} \frac{1}{2}e_0 & \text{if } n = 0, \\ -2e_{(n-2)/2} & \text{if } n \text{ is nonzero even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Since $S = T_1 + T_2$, it remains to show that T_1 and T_2 are in $B_{\rm hy}(H)$. For that, we define A_1 and A_2 in $B(H)$ by

$$A_1 e_n = \frac{1}{2}e_{2n+2} \quad \text{and} \quad A_2 e_n = -\frac{1}{2}e_{2n+2} \quad \text{for all } n \geqslant 0.$$

Then $T_1A_1 = T_2A_2 =$ the identity map on H. Furthermore, for all $i = 1, 2$ and for all vectors f in the linear span of the orthonormal basis $\{e_0, e_1, e_2, \ldots\}$, we can check that $T_i^j f \to 0$ and $A_i^j f \to 0$ when $j \to \infty$.

If we take H_0 to be the infinite dimensional closed subspace spanned by $\{e_n : n \text{ is odd}\}$, then for all vectors f in H_0 we have that $T_1 f = T_2 f = 0$. ∎

We now turn our attention to approximating the identity operator I, by beginning with the following definition: A bounded linear operator $B : H \to H$ is called a *unilateral weighted backward shift* if there exist an orthonormal basis $\{e_n : n \geqslant 0\}$ of H and a bounded positive weight sequence $\{w_n : n \geqslant 1\}$ such that $Be_0 = 0$ and $Be_n = w_n e_{n-1}$ for all $n \geqslant 1$. For every unilateral weighted backward shift B on H, Salas ([25]) proved that the operator $I + B$ is hypercyclic. Then F. León-Saavedra and A. Montes-Rodríguez ([21], Theorem 4.1) further proved that if B is a compact unilateral weighted backward shift, then the operator $I + B$ is in $B_{\rm hy}(H)$. Consequently, the sequence

$$\Big\{ I + \frac{1}{n}B : n \geqslant 1 \Big\}$$

is a sequence of operators in $B_{\rm hy}(H)$ converging to I in the operator norm, and so we have the following statement.

LEMMA 3.2. *The identity map $I : H \to H$ is the limit of a sequence of operators in $B_{\mathrm{hy}}(H)$.*

In Lemmas 3.1 and 3.2, we considered two special operators and showed that they are in the closed linear span of $B_{\mathrm{hy}}(H)$. As it turns out, they are two important cases in the proof of the following general statement.

THEOREM 3.3. *The linear span of $B_{\mathrm{hy}}(H)$ is dense in $B(H)$.*

Proof. We need to show that if S is an operator in $B(H)$, then there exists a sequence $\{T_n : n \geqslant 1\}$ of operators such that each T_n is a finite linear combination of operators in $B_{\mathrm{hy}}(H)$ and $T_n \to S$ in operator norm. In fact we can assume that S is self-adjoint, because every operator S can be decomposed as a sum $S = \operatorname{Re} S + \mathrm{i}\operatorname{Im} S$, where $\operatorname{Re} S = (S + S^*)/2$ and $\operatorname{Im} S = -\mathrm{i}(S - S^*)/2$ are two self-adjoint operators. Furthermore we can apply the spectral theorem ([12], p. 272–273) to find L^∞ and L^2 spaces of a σ-finite measure, and also a function ψ in L^∞ such that S is unitarily equivalent to the multiplication operator $M_\psi : L^2 \to L^2$ defined by $M_\psi f = \psi f$. Since unitary equivalence preserves all three Axioms in the definition of $B_{\mathrm{hy}}(H)$, it suffices for us to continue our argument only for the case where $S = M_\psi$ and $H = L^2$.

Since the function ψ is bounded almost everywhere, there is a sequence of simple functions (ψ_n) such that $\psi_n \to \psi$ in L^∞; see, for example, [14], p. 45. Hence

$$\|M_{\psi_n} - M_\psi\| = \|\psi_n - \psi\|_\infty \to 0;$$

see, for example, [12], p. 265. Since each ψ_n is a finite linear combination of measurable characteristic functions, we need only to finish the proof under a further assumption that the operator S is the multiplication operator M_χ on L^2, where χ is a measurable characteristic function.

With the function χ, the Hilbert space L^2 can be written as the orthogonal sum of two closed subspaces, namely χL^2 and $(1-\chi)L^2$, correponding to which our argument continues in three separate cases.

In the first case we assume that χL^2 is a finite dimensional subspace, spanned by orthonormal vectors $f_1, \ldots, f_N$ in χL^2. For each f_i, we let P_i be the orthogonal projection onto $\operatorname{span}\{f_i\}$, and so we can write M_χ as $M_\chi = P_1 + P_2 + \cdots + P_N$. Since each P_i is the sum of two operators in $B_{\mathrm{hy}}(L^2)$ by Lemma 3.1, the operator M_χ is the sum of $2N$ operators in $B_{\mathrm{hy}}(L^2)$.

Secondly we discuss the situation where χL^2 and $(1-\chi)L^2$ are two infinite dimensional closed subspaces. We begin by writing the operator M_χ as

$$M_\chi = \frac{1}{2} I + \frac{1}{2}(M_\chi - M_{1-\chi}),$$

where I is the identity operator on L^2. Thus by using Lemma 3.2, we need only to show that $M_\chi - M_{1-\chi}$ is the limit of a sequence of operators in $B_{\mathrm{hy}}(H)$. Note that M_χ and $M_{1-\chi}$ are two identity operators respectively on χL^2 and $(1-\chi)L^2$, both of which are isomorphic to our separable infinite dimensional Hilbert space H. Hence we can use Lemma 3.2 again to find a sequence of operators (W_n) in $B_{\mathrm{hy}}(\chi L^2)$ and a sequence of operators (T_n) in $B_{\mathrm{hy}}((1-\chi)L^2)$ such that

$$W_n \to M_\chi \text{ in } B(\chi L^2), \quad \text{and} \quad T_n \to M_{1-\chi} \text{ in } B((1-\chi)L^2),$$

and furthermore each W_n is unitarily equivalent to T_n for each n. Hence each direct sum operator $W_n \bigoplus(-T_n)$ is on $\chi L^2 \bigoplus(1-\chi)L^2 = L^2$ and they together satisfy

$$W_n \oplus (-T_n) \to M_\chi - M_{(1-\chi)} \text{ in } L^2.$$

Moreover one can use the unitary equivalence of W_n and T_n to check that the same sequence of positive integers $\{n_k\}$ in the definition of $B_{\mathrm{hy}}(L^2)$ can be used for both W_n and $-T_n$, and hence for the direct sum operator $W_n \oplus (-T_n)$ also. Thus each operator $W_n \oplus (-T_n)$ is in $B_{\mathrm{hy}}(L^2)$.

Lastly we must discuss the case in which χL^2 is infinite dimensional and $(1-\chi)L^2$ is finite dimensional. We begin by letting G be the measurable set such that $\chi = 1$ on G and $\chi = 0$ off G. Then G can be written as $G = G_1 \cup G_2$, where G_1 and G_2 are disjoint measurable subsets of G with positive measures. This is because if G could not be written as such a union, then every measurable function on G would be a scalar multiple of χ, and so χL^2 would be one dimensional. Repeating this argument, we see that G can be written as a countable union $\bigcup\{G_n : n \geqslant 0\}$ of disjoint measurable subsets G_n, each of which has positive measure. Consequently if we let χ_1 be the characteristic function for the union of G_n for all odd integers n, and let χ_2 be the characteristic function for the union of G_n for all even integers n, then $\chi = \chi_1 + \chi_2$ and both $\chi_1 L^2$ and $\chi_2 L^2$ are infinite dimensional. Thus $(1-\chi_1)L^2 = (1-\chi)L^2 + \chi_2 L^2$ is infinite dimensional, and so is $(1-\chi_2)L^2$. Hence we can write the multiplication operator M_χ as the sum of two multiplication operators M_{χ_1} and M_{χ_2}, on each of which we can apply the argument in the second case. This finishes the whole proof. ∎

Since every operator in $B_{\mathrm{hy}}(H)$ is necessarily hypercyclic, we derive the following statement directly from Theorem 3.3.

COROLLARY 3.4. *The linear span of all hypercyclic operators is dense in* $B(H)$.

Since every hypercyclic operator is necessarily cyclic, we deduce the following statement from Corollary 3.4.

COROLLARY 3.5. *The linear span of all cyclic operators is dense in* $B(H)$.

This corollary shows that the structure in $B(H)$ of the cyclic operators is similar to the structure in H of the cyclic vectors of a particular cyclic operator T, as it is proved in [18], p. 285 that the linear span of the cyclic vectors of T is dense in H.

To conclude this paper, we remark that the techniques that we use in proving Theorems 2.2 and 3.3 rely very heavily on the assumption that the underlying space H is a Hilbert space. Thus it is very natural to raise the following question.

QUESTION 3.6. Do Theorems 2.2 and 3.3 hold for a separable infinite dimensional Banach space?

When Rolewicz ([24]) exhibited the first example of a hypercyclic operator on a Banach space, he raised the question whether every separable infinite dimensional Banach space X admits a hypercyclic operator in $B(X)$. Recently Ansari ([2]) and independently Bernal-González ([6]) provided an affirmative answer for Rolewicz' question. Both of them used Salas' result ([25]) that the perturbation of the identity map on ℓ^2 by a unilateral weighted backward shift is hypercyclic. This result produced a class of operators on X, which are the only operators on X known to be hypercyclic. However, this class of operators does not seem to help answer our question.

Based on Ansari's argument, Bonet and Peris ([9]) recently showed that every separable infinite dimensional Fréchet space admits a continuous hypercyclic operator. Though the continuous operators on a Fréchet space do not carry the operator norm topology, we can rephrase the above question as follows: What results analogous to Theorems 2.2 and 3.3 hold for a Fréchet space?

Acknowledgements. The author thanks Juan Bès for providing several valuable references.

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Received June 15, 1999; revised November 1, 1999.