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STABLE APPROXIMATE UNITARY EQUIVALENCE OF HOMOMORPHISMS

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Abstract. Let A be a separable unital nuclear C^*-algebra and let B be a unital C^*-algebra. Suppose that A satisfies the Universal Coefficient Theorem and $\alpha, \beta : A \to B$ are homomorphisms. We show that α and β are stably approximately unitarily equivalent if they induce the same element in $\mathrm{KK}(A, B)$ and either A or B is simple. In particular, an automorphism α on A is stably approximately inner if $[\alpha] = [\mathrm{id}_E]$ in $\mathrm{KK}(A, A)$. If B is simple and A is "K-theoretically locally finite" then α and β are stably approximately unitarily equivalent if and only if they induce the same element in $\mathrm{KL}(A, B)$. In the case that A and B are separable purely infinite simple C^*-algebras and A is nuclear and satisfies the UCT, then φ and ψ are approximately unitarily equivalent if and only if $[\alpha] = [\psi]$ in $\mathrm{KL}(A, B)$.

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0. INTRODUCTION

We study homomorphisms from a C^*-algebra A to another C^*-algebra B. Consider two homomorphisms $\varphi, \psi : A \to B$. The question that we consider here is when these two homomorphisms are equivalent (in some suitable sense). The theory of C^*-algebras is often regarded as noncommutative topology. Continuous maps between topological spaces are of fundamental importance in topology. As noncommutative counterparts of continuous maps, homomorphisms are of fundamental importance in the theory of C^*-algebras. Classification of homomorphisms is of great interest.

Let $A = B$ and $\varphi \in \mathrm{Aut}(A)$ be an automorphism of A. A long-studied problem is when φ is (approximately) inner, i.e., when φ is (approximately) unitarily equivalent to the identity. The study of automorphisms is very closely related to group representations, crossed products, dynamical systems and the classification of type III factors. Results in early study state that, when the spectrum (the usual spectrum, Borchers', or Connes' spectrum) of the automorphism is small, the automorphism is inner (or approximately inner) (see [59]). With the recent developments in theory of C^*-algebras, K-theory seems more popular and indeed a more important invariant (at least for separable C^*-algebras). Therefore it is quite appropriate to ask: is the automorphism φ (stably) approximately inner, if $[\varphi] = [\mathrm{id}]$ in $\mathrm{KK}(A, A)$?

If $A = C(X)$, where X is a metric space and B is the Calkin algebra, the BDF-theory classifies the monomorphisms from A to B ([8] and [9]). It shows that two monomorphisms $\varphi, \psi : A \to B$ are unitarily equivalent if and only if $[\varphi] = [\psi]$ in $\mathrm{KK}(A, B)$ ($= \mathrm{KK}^1(A, \mathcal{K})$). The BDF-theory is motivated by classification of essentially normal operators on a separable Hilbert space. However it has profound impact on many other areas of mathematics. The Calkin algebra is certainly a very unique C^*-algebra and quite different from other interesting and more widely used C^*-algebras. But we note that the Calkin algebra is a unital simple C^*-algebra of real rank zero. It is not only natural but also important to replace it by more common separable (simple) C^*-algebras.

We began to study homomorphisms in 1992 (see [41] and [43]). The results in [41] and [43] led to the discovery that many simple C^*-algebras of real rank zero have the property that every normal element with vanishing index can be approximated by normal elements with finite spectra. It also leads to the solution of a long standing open problem in linear algebra and operator theory: almost commuting self-adjoint matrices can be approximated by commuting self-adjoint matrices (see [45] and [25]). It turns out that these results have applications to the classification theory of nuclear C^*-algebras. Several papers about the homomorphisms appeared later ([23], [48], [49], [54] and [12]). Dădârlat ([12]) shows that, in the case that $A = C(X)$, φ and ψ are stably approximately unitarily equivalent if and only if $[\varphi] = [\psi]$ in $\mathrm{KL}(A, B)$. These results play important roles in the classification theory of nuclear C^*-algebras (see also [14]). More recently, we have shown that the above holds for any unital C^*-algebra of continuous trace A ([52]). With the explosion of the theory of classification of C^*-algebras, it is increasingly important to handle homomorphisms from a general (nuclear) simple C^*-algebra to another simple C^*-algebra. The main results of this paper classify these homomorphisms (up to stable approximate unitary equivalence). In order to use K-theory freely, however, we will assume that A is nuclear and satisfies the Universal Coefficient Theorem (UCT) (see 2.1). It is known, for example, that direct limits of type I C^*-algebras satisfy the UCT. It seems that all nuclear C^*-algebras of interest satisfy the UCT. The main result of the paper is, assuming that A is a separable unital simple nuclear C^*-algebra which satisfies the Universal Coefficient Theorem (UCT), two homomorphisms φ and ψ from A to B are stably approximately unitarily equivalent if $[\varphi] = [\psi]$ in $\mathrm{KK}(A, B)$ (see 4.3). If A is the closure of $\bigcup_n A_n$, where each A_n is nuclear and $\mathrm{K}_i(A_n)$ is finitely generated, and B is a simple unital C^*-algebra then φ and ψ are stably approximately unitarily equivalent if and only if $[\varphi] = [\psi]$ in $\mathrm{KL}(A, B)$. The condition on A is certainly satisfied by all separable C^*-algebras which are direct limits of type I. In the case that both A and B are also assumed to be separable and purely infinite simple C^*-algebras, then the main result implies that φ and ψ are in fact approximately unitarily equivalent if (and only if) $[\varphi] = [\psi]$ in $\mathrm{KL}(A, B)$.

The strategy that we used in this paper is very different from that of [43] (and [23], [48], [12] and [52]). We will briefly describe our approach. To avoid certain ambiguity, we will only consider monomorphisms. Therefore we assume that there is at least one monomorphism from A to B. For convenience, we will assume that B is a unital C^*-algebra and A is a unital C^*-subalgebra of B.

Given a monomorphism $\varphi : A \to B$ we construct a "mapping torus" for φ. Put

$$M_\varphi(A,B) = \{x \in C([0,1],B) : x(0) \in A, x(1) = \varphi(x(0))\}.$$

This gives an essential extension

$$0 \to \mathrm{S}B \to M_\varphi(A,B) \to A \to 0,$$

where $\mathrm{S}B = C_0((0,1),B)$. We let $\tau_\varphi : A \to M(\mathrm{S}B)/\mathrm{S}B$ be the extension determined by φ in the above. If we find a unitary $U \in M(\mathrm{S}B)$ such that

$$\pi(U)^*\tau_\varphi\pi(U) = \tau_\psi,$$

where $\pi : M(\mathrm{S}B) \to M(\mathrm{S}B)/\mathrm{S}B$ is the quotient map, then, for any $a \in A$, let $x \in C([0,1],B)$ with $x(0) = a$ and $x(1) = \varphi(a)$, and let $y \in C([0,1],B)$ with $y(0) = a$ and $y(1) = \psi(a)$; we have

$$U^*xU - y \in \mathrm{S}B.$$

This implies that, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there is t (close to 1) such that

$$\|U^*(t)\varphi(a)U(t) - \psi(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$. We might not be able to find such U in general. However, suppose that $[\varphi] = [\psi]$ in $\mathrm{KK}(A,B)$, then $[\tau_\varphi] = [\tau_\psi]$ in $\mathrm{Ext}(A,\mathrm{S}B)$. Thus, we obtain a unitary $U \in M(\mathrm{S}B \otimes \mathcal{K})$ such that

$$\pi(U)^*\mathrm{diag}(\varphi,\tau)\pi(U) = \mathrm{diag}(\psi,\tau)$$

for some trivial essential extension $\tau : A \to M(\mathrm{S}B \otimes \mathcal{K})/\mathrm{S}B \otimes \mathcal{K}$. We will show that, if B is simple, τ can be chosen to be τ_∞, where τ_∞ is the trivial extension given by the diagonal map $h_\infty : a \mapsto \mathrm{diag}(a, a, \ldots, a, \ldots)$, for $a \in A$.

From the above, we will have, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$,

$$\|U^*(t)\mathrm{diag}(f(t), a, \ldots)U(t) - \mathrm{diag}(g(t), a, \ldots)\| < \varepsilon$$

and

$$U^*(t)\mathrm{diag}(f(t), a, \ldots)U(t) - \mathrm{diag}(g(t), a, \ldots) \in B \otimes \mathcal{K}$$

for all $a \in \mathcal{F}$, $f, g \in C([0,1],B)$ with $f(0) = a$, $f(1) = \varphi(a)$, $g(0) = a$, $g(1) = \psi(a)$ and for some t close to 1. But $\mathrm{diag}(a, \ldots)$ is in $M(\mathrm{S}B \otimes \mathcal{K})$. So the above inequality does not seem useful. However, if in the above inequality there are only finitely many a's in both diagonals and $U(t) \in B \otimes M_n$, then it will imply that φ and ψ are stably approximately unitarily equivalent. Our next effort is to cut the "tail" of τ_∞.

Let $E = D + \mathcal{B} \otimes \mathcal{K}$, where $D = \{(a, a, \ldots,) \in M(\mathrm{S}B \otimes \mathcal{K}) : a \in A\}$. Note that $D \cong A$. Then E is an essential extension of A by $B \otimes \mathcal{K}$. Define $\Gamma_t(x) = U(t)^*xU(t)$ for $x \in E$. Then Γ_t is an automorphism on E. Let $\pi_0 : E \to E/B \otimes \mathcal{K}\,(\cong D \cong A)$. Then Γ_t induces the identity map on $E/B \otimes \mathcal{K}$. We will show that Γ_t is approximately inner. Once we have done that, for any $\varepsilon > 0$ and any finite subset $\mathcal{G} \subset E$, we obtain a unitary $W_t \in E$ such that

$$\|W_t^* x W_t - U(t)^* x U(t)\| < \frac{\varepsilon}{4}$$

for all $x \in \mathcal{G}$. In particular, if $\mathcal{G}$ is large enough, for t close to 1, we have

$$\|W_t^* \mathrm{diag}(\varphi(a), h_\infty(a)) W_t - \mathrm{diag}(\psi(a), h_\infty(a))\| < \frac{\varepsilon}{2}$$

for all $a \in \mathcal{F}$.

Since $W_t \in E$, we may assume that $W_t = W_1 \oplus W_2$, where

$$W_1 = \Big(\sum_{i=1}^n e_{ii}\Big) W \Big(\sum_{i=1}^n e_{ii}\Big) \quad \text{and} \quad W_2 = \Big(1 - \sum_{i=1}^n e_{ii}\Big) \mathrm{diag}(u_1, u_1, \ldots) \Big(1 - \sum_{i=1}^n e_{ii}\Big)$$

for some unitary $u_1 \in e_{11} E e_{11}$, with an error of no more than $\varepsilon/2$ (and $n \geqslant 1$). Therefore we can cut the "tail", i.e., we have

$$\|W_1^* \mathrm{diag}(\varphi(a), h_n(a)) W_1 - \mathrm{diag}(\psi(a), h_n(a))\| < \varepsilon,$$

where $h_n : a \mapsto \mathrm{diag}(a, a, \ldots, a)$ (from A to $M_{n-1}(M(\mathrm{S}B))$), for all $a \in \mathcal{F}$.

This shows that φ and ψ are stably unitarily equivalent.

While the approximate unitary equivalence is stronger than stable approximate unitary equivalence, the latter is more available and therefore very useful, as demonstrated in the classification theory (see [21], [23], [22], for example). Here is a rather simple example which shows that stable approximate unitary equivalence is a much more accessible relation than approximate unitary equivalence. Let $A = C([0,1])$ and $\varphi(f)(t) = f(1-t)$ with $f \in [0,1]$. Of course, $[\varphi] = [\mathrm{id}_A]$ in $\mathrm{KK}(A, A)$. By considering the traces on A, one sees immediately that φ is not approximately unitarily equivalent to id_A. However, φ and id_A are stably approximately unitarily equivalent. When either A is simple or the target algebra B is simple, the stably approximately unitarily equivalence becomes much more rigorous. As we mentioned earlier, classification of homomorphisms is very important in the theory of C^*-algebras. The results in this paper have many significant applications. Because of the length of this paper, these applications will not be included here. In a subsequent paper, we will give applications of the main results of this paper to the theory of classification of nuclear C^*-algebras. For example, we will show, using the main results of this paper, that a separable unital nuclear simple C^*-algebras of TAF with the right K-theory is isomorphic to the UHF-algebra with rational K_0-group (see [53]) and certain class of quasidiagonal nuclear simple C^*-algebras can be classified with their K-theoretical data (in particular, the algebras are not assumed to be direct limits of (sub-)homogeneous C^*-algebras). Since our main results do not assume that C^*-algebras have real rank zero, they can also be applied to C^*-algebras having real rank other than zero. Furthermore, in another subsequent paper, we will show that the results in Sections 4 and 5 can be used to classify C^*-algebra extensions of a separable nuclear C^*-algebra by a simple C^*-algebra (up to unitary equivalence). Applications to automorphism groups on nuclear simple C^*-algebras will also appear.

The paper is organized as follows. Section 1 shows that the trivial extension $\tau_\infty : A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ given by the diagonal map $d(a) = \mathrm{diag}(a, \ldots)$, for $a \in A$, is absorbing if A or B is simple, where A is a nuclear C^*-subalgebra of

B. We also point out that the condition that one of the C^*-algebras is simple is crucial and can not be removed in general. In fact, we show that, in the case that $A = B$, τ_∞ is absorbing if and only if A is simple. Section 2 gives certain K-theory computation which is needed in Section 3. Section 3 shows that certain automorphisms are approximately inner. In Section 4 we prove the main result. In Section 5, we show that a version of the main result also holds for approximately multiplicative morphisms.

Throughout the paper, we will denote by $M(A)$ the multiplier algebra of a C^*-algebra A and by $\mathrm{S}A$ the suspension of A, $C_0((0,1), A)$.

1. CERTAIN ABSORBING EXTENSIONS

1.1. Definition. Let A and B be C^*-algebras and let $\varphi : A \to B$ be a contractive completely positive linear map. The map φ is called *factorable* if it factors through M_n for some n, i.e., if there exist contractive completely positive linear maps $L_1 : A \to M_n$ and $L_2 : M_n \to B$ such that $\varphi = L_2 \circ L_1$. We say that φ is *nuclear* if it is the pointwise limit of factorable maps.

It is known ([10]) that every contractive completely positive linear map from A to B is nuclear if A or B is nuclear.

1.2. Lemma. ([37]) *Let A be a simple C^*-algebra and let B be a separable C^*-subalgebra of A. For every nuclear map $\varphi : B \to A$, every compact subset $S \subset B$ and every $\varepsilon > 0$ there are $a_1, a_2, \dots, a_n$ in A such that $\Big\| \sum\limits_{i=1}^{n} a_i^* a_i \Big\| \leqslant 1$ and*

$$\Big\| \varphi(b) - \sum a_i^* b a_i \Big\| < \varepsilon \quad \textit{for all } b \in S.$$

1.3. Lemma. *Let A be a simple C^*-algebra and let B be a separable C^*-subalgebra of A. For every nuclear map $\varphi : B \to \mathrm{S}A$, every compact subset $S \subset B$, every $\varepsilon > 0$ and every integer $N > 0$, there are an integer n and $a_1, a_2, \dots, a_m$ in $M_{N+n}(\mathrm{S}A)$ with $\Big\| \sum\limits_{i=1}^{m} a_i^* a_i \Big\| \leqslant 1$ such that*

$$\Big\| \varphi(b) - \sum a_i^* d_n(b) a_i \Big\| < \varepsilon \quad \textit{for all } b \in S,$$

where $d_n : B \to M_{N+n}(C([0,1], B))$ is defined by $d_n(b) = \mathrm{diag}(0, 0, \dots, 0, b, b, \dots, b)$, 0 repeats N times and d repeats n times, b is identified with the constant function in $C([0,1], B)$, and $\mathrm{S}A$ is identified with the first corner of $M_{N+n}(\mathrm{S}A)$.

Proof. It suffices to consider the case when S is finite. Let $\varepsilon > 0$. There are $0 < t_1 < t_2 < \cdots < t_n < 1$ such that

$$\|\varphi(b)(t) - \varphi(b)(t')\| < \frac{\varepsilon}{4}$$

for $|t-t'| < \max\{t_{i+1}-t_i : i=1,\dots,n-1\}$ and $\|\varphi(b)(t)\| < \varepsilon/3$ if $0 < t \leqslant t_1$ or $t_n < t < 1$ for all $b \in S$. By applying 1.2, since $M_{N+n}(A)$ is simple, there are $b_i^{(j)} \in M_{N+n}(A)$, $i = 1, 2, \dots, m(j)$, $j = 1, 2, \dots, n$ such that $\left\|\sum_i (b_i^{(j)})^* b_i^{(j)}\right\| \leqslant 1$,

$$\left\|\varphi(b)(t_j) - \sum_i (b_i^{(j)})^*(e_{N+j,1} b e_{1,N+j}) b_i^{(j)}\right\| < \frac{\varepsilon}{4},$$

for all $b \in S$, where we identify A with $e_{1,1}M_n(A)e_{1,1}$, and $\{e_{i,k}\}$ is a matrix unit for M_{N+n}. Without loss of generality, we may assume that $b_i^{(j)} = e_{N+j,N+j} b_i^{(j)} e_{11}$ and $m(j) = m$ (by adding zeros if necesary). So $(b_i^{(j)})^* b_{i'}^k = 0$ and $(b_i^{(j)})^* d_n(b) b_{i'}^{(k)} = 0$ if $j \neq k$ for all $b \in S$. Define

$$a_i(t) = [(t_{k+1}-t)/(t_{k+1}-t_k)]^{1/2} b_i^{(k)} + [(t-t_k)/(t_{k+1}-t_k)]^{1/2} b_i^{(k+1)}$$

for $t \in [t_k, t_{k+1}]$, $a_i(t) = (t/t_1)^{1/2} b_i^{(1)}$ for $t \in [0, t_1]$ and $a_i(t) = [(1-t)/(1-t_n)]^{1/2} b_i^{(n)}$ for $t \in [t_n, 1]$, $i = 1, 2, \dots, m$. Then $a_i \in C_0((0,1), A)$. It is also easy to check that

$$\left\|\sum_{i=1}^m a_i^* a_i\right\| \leqslant 1.$$

Furthermore, for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}
&\left\|\varphi(b)(t) - \sum_{i=1}^m a_i^*(t) d_n(b) a_i(t)\right\| \\
&\qquad \leqslant [(t_{k+1}-t)/(t_{k+1}-t_k)]\left\|\varphi(b)(t) - \sum_{i=1}^m (b_i^{(k)})^* b b_i^{(k)}\right\| \\
&\qquad\qquad + [(t-t_k)/(t_{k+1}-t_k)]\left\|\varphi(b)(t) - \sum_{i=1}^m (b_i^{(k+1)})^* b b_i^{(k+1)}\right\| \\
&\qquad < \frac{\varepsilon}{4} + [(t_{k+1}-t)/(t_{k+1}-t_k)]\left\|\varphi(b)(t_k) - \sum_{i=1}^m (b_i^{(k)})^* b b_i^{(k)}\right\| \\
&\qquad\qquad + \frac{\varepsilon}{4} + [(t-t_k)/(t_{k+1}-t_k)]\left\|\varphi(b)(t_{k+1}) - \sum_{i=1}^m (b_i^{(k+1)})^* b b_i^{(k+1)}\right\| \\
&\qquad < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

and for $t \in [0, t_1]$

$$\begin{aligned}
&\left\|\varphi(b)(t) - \sum_{i=1}^m a_i^*(t) d_n(b) a_i(t)\right\| \\
&\qquad \leqslant \|\varphi(b)(t)\| + [1 - t/t_1]\|\varphi(b)(t_1)\| + (t/t_1)\left\|\varphi(b)(t_1) - \sum_{i=1}^m (b_i^{(1)})^* b b_i^{(1)}\right\| \\
&\qquad < \frac{\varepsilon}{4} + [1 - t/t_1]\frac{\varepsilon}{4} + (t/t_1)\frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

for all $b \in S$. Similarly, we also have

$$\Big\|\varphi(b)(t) - \sum_{i=1}^{m} a_i^*(t)d_n(b)a_i(t)\Big\| < \varepsilon$$

for $t \in [t_n, 1]$ and $b \in S$. Thus

$$\Big\|\varphi(b) - \sum_{i=1}^{m} a_i^* d_n(b)a_i\Big\| < \varepsilon$$

for all $s \in S$. ∎

1.4. Let A be a unital C^*-subalgebra of a unital C^*-algebra B and let $d : A \to M(B \otimes \mathcal{K})$ be defined by $d(a) = \operatorname{diag}(a, a, \dots,)$ for $a \in A$. By identifying a with the constant function $a(t) = a$, for $t \in [0,1]$, we may also regard d as a homomorphism from A to $C([0,1], M(B \otimes \mathcal{K}))$. Note that $C([0,1], B) \subset M(\mathrm{S}B)$ and $M(\mathrm{S}B \otimes \mathcal{K}) = C^{\mathrm{b}}((0,1), M(B \otimes \mathcal{K})_\sigma)$, the C^*-algebra of bounded continuous maps from $(0,1)$ (with the usual topology on $(0,1)$) to $M(B \otimes \mathcal{K})$ (with the strict topology of $M(B \otimes \mathcal{K})$) (see [2]). So we may also regard d as a homomorphism from A to $M(\mathrm{S}B \otimes \mathcal{K})$. These conventions will be used repeatedly without warning.

1.5. Lemma. *Let A be a unital nuclear separable C^*-subalgebra of a unital simple C^*-algebra B ($1_A = 1_B$) and let $d : A \to M(\mathrm{S}B \otimes \mathcal{K})$ be defined by $d(a) = \operatorname{diag}(a, a, \dots, a, \dots)$, where $a \in C([0,1], A)$ is the constant function with $a(t) = a$ for each $t \in [0,1]$. Then for every unital contractive completely positive linear map $\varphi : A \to M(\mathrm{S}B \otimes \mathcal{K})$ there exists a sequence $\{v_n\}$ of isometries in $M(\mathrm{S}B \otimes \mathcal{K})$ such that*

$$\varphi(a) - v_n^* d(a) v_n \in \mathrm{S}B \otimes \mathcal{K} \quad \textit{and} \quad \|\varphi(a) - v_n^* d(a) v_n\| \to 0$$

for all $a \in A$.

Proof. As in the proof of Theorem 5 in [36] it suffices to show that there is a sequence $\{s_n\}$ of contractions satisfying the above two conditions. Let $C = C^*(\varphi(A))$, the C^*-subalgebra generated by $\varphi(A)$. So C is separable. Note that $\mathrm{S}B \otimes \mathcal{K}$ is σ-unital. It is well known (see [1] and the proof of 3.12.15 of [59]) that there is a countable approximate unit $\{e_n\}$ for $\mathrm{S}B \otimes \mathcal{K}$ which is quasi-central for C. Since A is separable, we can choose a compact subset $S \subset A$ such that the linear span of S is dense in A. Set $f_n = (e_n - e_{n-1})^{1/2}$. Let $\varepsilon > 0$. Passing to a subsequence if necessary, we may assume that

$$\|\varphi(a) f_n - f_n \varphi(a)\| < 2^{-n}\varepsilon \quad \text{for all } , a \in S.$$

Define the maps $\psi_n : A \to \mathrm{S}B \otimes \mathcal{K}$ by $\psi_n(a) = f_n \varphi(a) f_n$. Note that the infinite sum $\sum_{n=1}^{\infty} \psi_n(a)$ converges in the strict topology for every $a \in A$. Thus $\psi : a \to \sum_{n=1}^{\infty} \psi_n(a)$ defines a completely positive linear map from A to $M(\mathrm{S}B \otimes \mathcal{K})$. Note also that $\sum_{n=1}^{\infty} f_n^2 = 1$ and $\sum_{n=1}^{\infty} \varphi f_n^2 = \varphi$ converge (in the strict topology). We have

$$\|\varphi(a) - \psi(a)\| \leqslant \sum_{n=1}^{\infty} \|\varphi(a) f_n - f_n \varphi(a)\| \, \|f_n\| < \varepsilon.$$

Moreover $\varphi(a)f_n^2 - f_n\varphi(a)f_n \in \mathrm{S}B\otimes\mathcal{K}$ for all n and $a\in S$. So it follows that $\varphi(a)-\psi(a)\in \mathrm{S}B\otimes\mathcal{K}$.

Denote by $\{e_{ij}\}$ a matrix unit for $\mathcal{K}$.

CLAIM. for any $\varepsilon>0$ there is a sequence of $\{x_n\}$ of elements in $\mathrm{S}B\otimes\mathcal{K}$ and a sequence of integers $\{l(n)\}$ such that:

(i) $\|\psi_n(a)-x_n^*d(a)x_n\|<2^{-n}\varepsilon$, for $a\in S$;
(ii) $\|x_n^*d(a)x_m\|=0$ for all $a\in A$ and $n\neq m$;
(iii) $\|x_n^*e_m\|<2^{-n}$ for all $m\leqslant n$;
(iv) $x_n\in\Big(\sum_{i=l(n-1)+1}^{l(n)}e_{ii}\Big)\mathrm{S}B\otimes\mathcal{K}\Big(\sum_{i=l(n-1)+1}^{l(n)}e_{ii}\Big)$ and $\|x_n\|\leqslant 1$.

We construct $\{x_n\}$ inductively. Assume that we have elements $x_1,x_2,\dots,x_n$ satisfying (i)–(iv).

There is $k(n)$ ($\geqslant n$) such that

$$\|(1-e_{k(n)k(n)})f_m\|<\frac{2^{-2n-2}\varepsilon}{1+\sup\{\|s\|:s\in S\}}$$

for $m=1,2,\dots,n$. Let $\psi_n'(a)=e_{k(n)k(n)}\psi_n(a)e_{k(n)k(n)}$ for all $a\in A$. Note that

$$\|\psi_n'(a)-\psi_n(a)\|<2^{-n-1}\varepsilon$$

for all $a\in S$. By Lemma 1.3, there are an integer $j(n)\geqslant k(n)+l(n)+1$ and $a_1(n),a_2(n),\dots,a_{m(n)}(n)\in\Big(\sum_{i=1}^{j(n)}e_{ii}\Big)\mathrm{S}B\otimes\mathcal{K}\Big(\sum_{i=1}^{j(n)}e_{ii}\Big)$ such that $\Big\|\sum a_i^*(n)a_i(n)\Big\|\leqslant 1$ and

$$\Big\|\psi_n'(a)-\sum_i a_i^*(n)d_n(a)a_i(n)\Big\|<2^{-n-1}\varepsilon$$

for all $a\in S$, where $d_n(a)=\mathrm{diag}(0,\dots,0,a,a,\dots,a)$ is diagonal in $\Big(\sum_{i=l(n)+1}^{j(n)}e_{ii}\Big)\mathrm{S}B\otimes\mathcal{K}\Big(\sum_{i=l(n)+1}^{j(n)}e_{ii}\Big)$ (there are $l(n)$ zeros and $j(n)-l(n)-1$ a's). We may assume that $a_i(n)=\Big(\sum_{i=l(n)+1}^{j(n)}e_{ii}\Big)a_i(n)$, $i=1,2,\dots,m(n)$. Therefore $\sum_i a_i^*(n)d(a)a_i(n)=\sum_i a_i^*(n)d_n(a)a_i(n)$. Set $x_{n+1}=\sum_{i=1}^{m(n)}E_{i1}a_i(n)$, where $E_{i1}=1_{C_n}\otimes e_{i1}$ and 1_{C_n} is the identity of $\Big(\sum_{i=l(n)+1}^{j(n)}e_{ii}\Big)(\mathrm{S}B\otimes\mathcal{K})\Big(\sum_{i=l(n)+1}^{j(n)}e_{ii}\Big)^{\sim}$. Note that $\|x_n\|\leqslant 1$ and $x_{n+1}\in\Big(\sum_{i=l(n)+1}^{l(n+1)}e_{ii}\Big)\mathrm{S}B\otimes\mathcal{K}\Big(\sum_{i=l(n)+1}^{l(n+1)}e_{ii}\Big)$, where $l(n+1)=l(n)+1+m(n)(j(n)-l(n)-1)$. This gives (iv). We also have

$$x_{n+1}^*d(a)x_{n+1}=\sum a_i^*(n)d(a)a_i(n)=\sum_i a_i^*(n)d_n(a)a_i(n).$$

Thus (i) follows. Clearly $x_{n+1}^* d(a) x_m = 0$ for $m \leqslant n$ and $a \in A$, and $\|x_n^* e_m\| \leqslant 2^{-n}$ for $m = 1, 2, \ldots, n$. Thus (ii) and (iii) follows.

Now we have a sequence satisfying (i)-(iv). Note that (i) implies that

$$\sum \|x_n^* x_n - \psi_n(1)\| < \varepsilon.$$

Let $x = \sum x_n$. The sum converges strictly (by (iii) and (iv)) and $\|x\| \leqslant 1$. Then

$$\begin{aligned}\varphi(a) - x^* d(a) x &= (\varphi(a) - \psi(a)) + (\psi(a) - x^* d(a) x)\\ &= (\varphi(a) - \psi(a)) + \sum (\psi_n(a) - x_n^* d(a) x_n) + \sum_{n \neq m} x_n^* d(a) x_m.\end{aligned}$$

The first term is in $\mathrm{S}B \otimes \mathcal{K}$ and has norm less than ε for every $a \in S$. The second term has norm less than ε and is in $\mathrm{S}B \otimes \mathcal{K}$, since each $\psi_n(a) - x_n^* d(a) x_n \in \mathrm{S}B \otimes \mathcal{K}$ and $\|\psi_n(a) - x_n^* d(a) x_n\| < 2^{-n}\varepsilon$, for every $a \in S$. The last term is zero by (ii). So $\varphi(a) - x^* d(a) x \in \mathrm{S}B \otimes \mathcal{K}$ and $\|\varphi(a) - x^* d(a) x\| < 2\varepsilon$ for all $a \in S$. Note that the linear span of S is dense in A. Therefore $\varphi(a) - x^* d(a) x \in \mathrm{S}B \otimes \mathcal{K}$ for all $a \in A$. ∎

1.6. Lemma. *Let A be a nuclear unital C^*-subalgebra of a unital separable simple C^*-algebra B ($1_A = 1_B$) and let $d : A \to M(B \otimes \mathcal{K})$ be defined by $d(a) = \operatorname{diag}(a, a, \ldots, a, \ldots)$. Then for every unital contractive completely positive linear map $\varphi : A \to M(B \otimes \mathcal{K})$ there exists a sequence $\{v_n\}$ of isometries in $M(B \otimes \mathcal{K})$ such that*

$$\varphi(a) - v_n^* d(a) v_n \in B \otimes \mathcal{K} \quad \textit{and} \quad \|\varphi(a) - v_n^* d(a) v_n\| \to 0$$

for all $a \in A$.

Proof. The proof is exactly the same as that of 1.5 but we use 1.2 instead of 1.3. ∎

1.7. Remark. In the conditions of Theorem 1.5 put $p_n = v_n v_n^*$. It is easy to verify that if φ is a homomorphism, then $p_n d(a) - d(a) p_n \in \mathrm{S}B \otimes \mathcal{K}$ and $\lim_{n \to \infty} \|p_n d(a) - d(a) p_n\| = 0$. For example, given any self-adjoint $a \in A_{\mathrm{sa}}$, if φ is a homomorphism, we have $p_n d(a) p_n d(a) p - p_n d(a)^2 p_n \in \mathrm{S}B \otimes \mathcal{K}$ for all $a \in A_{\mathrm{sa}}$. This implies that $p_n d(a)(1 - p_n) d(a) p_n \in \mathrm{S}B \otimes \mathcal{K}$. Therefore $p_n d(a)(1 - p_n) \in \mathrm{S}B \otimes \mathcal{K}$. So $p_n d(a) - d(a) p_n \in \mathrm{S}B \otimes \mathcal{K}$ for all $a \in A_{\mathrm{sa}}$.

1.8. Definition. Let E_1 and E_2 be Hilbert B-modules. Let $L(E_1), L(E_2)$, $L(E_1, E_2)$ be the set of all bounded module maps with adjoints from E_1 to E_1, from E_2 to E_2 and from E_1 to E_2, respectively. Let $\varphi_1 : A \to L(E_1)$ and $\varphi_2 : A \to L(E_2)$ be two linear maps. Let $\varepsilon \geqslant 0$ and X be a subset of A. We write $\varphi_1 \doteq_\varepsilon \varphi_2$ on X, if there is a unitary $U \in L(E_1, E_2)$ such that

$$U^* \varphi_1(a) U - \varphi_2(a) \in \mathcal{K}(E_2), \quad \text{and} \quad \|U^* \varphi_1(a) U - \varphi_2(a)\| \leqslant \varepsilon$$

for all $a \in X$. Note that, by [36], if $E = H_B$ and B is σ-unital, then $L(E) = M(B \otimes \mathcal{K})$, $\mathcal{K}(E) = B \otimes \mathcal{K}$. Furthermore, if $E_1 = E_2$, we write $\varphi_1 \equiv_\varepsilon \varphi_2$ on X, if

$$\varphi_1(a) - \varphi_2(a) \in \mathcal{K}(E_1) \quad \text{and} \quad \|\varphi_1(a) - \varphi_2(a)\| \leqslant \varepsilon$$

for all $a \in X$.

1.9. Let A and B be two C^*-algebras (with A unital). An extension of A by B is *absorbing* if τ is unitarily equivalent to $\tau\oplus\sigma$ for any trivial unital extension σ (of A by B).

1.10. THEOREM. *Let A, B and d be as in Lemma 1.5 (or in Lemma 1.6). Then the unital trivial essential extension of A by $\mathrm{S}B\otimes\mathcal{K}$ (respectively, by $B\otimes\mathcal{K}$) given by d is absorbing.*

Proof. We will show the case that A, B and d are in 1.5 by applying 1.5. The case that A, B and d are in 1.6 is proved the exactly same way but by applying 1.6.

We will use the same argument as in the proof of Theorem 6 in [36]. Let $\varphi : A \to M(\mathrm{S}B\otimes\mathcal{K})$ be a monomorphism. One can define a unital monomorphism $\varphi_\infty = \bigoplus_1^\infty \varphi : A \to M(\mathrm{S}B\otimes\mathcal{K})$, by dividing the identity of $M(\mathrm{S}B\otimes\mathcal{K})$ into (countably) infinitely many copies of projections in $M(\mathrm{S}B\otimes\mathcal{K})$ which are equivalent to the identity (of $M(\mathrm{S}B\otimes\mathcal{K})$). Fix an $\varepsilon > 0$ and a compact subset X. Applying 1.5, we obtain an isometry $s \in M(\mathrm{S}B\otimes\mathcal{K})$ such that $\varphi_\infty \equiv_\varepsilon s^*ds$ on X. As in Remark 1.7, $ss^*d(a) - d(a)ss^* \in \mathrm{S}B\otimes\mathcal{K}$ and $\|ss^*d(a) - d(a)ss^*\| < \varepsilon$ for all $a \in X$. Let $p = ss^*$. Note that $M(\mathrm{S}B\otimes\mathcal{K}) \cong pM(\mathrm{S}B\otimes\mathcal{K})p$. We have $s\varphi_\infty s^* \equiv_\varepsilon pdp$ on X. Therefore $\varphi_\infty \doteq_\varepsilon pdp$ on X. On the other hand, $d \equiv_{2\varepsilon} pdp + (1-p)d(1-p)$ on X because $pd(a) - d(a)p \in \mathrm{S}B\otimes\mathcal{K}$ and $\|pd(a) - d(a)p\| < \varepsilon$ for all $a \in X$. Denoting $(1-p)d(1-p) : A \to pM(\mathrm{S}B\otimes\mathcal{K})p$ by d_0, we conclude that $d \doteq_{3\varepsilon} \varphi_\infty \oplus d_0$ on X and $\varphi\oplus d \doteq_{3\varepsilon} \varphi\oplus\varphi_\infty\oplus d_0$ on X. Since $\varphi_\infty\oplus d_0$ is unitarily equivalent to $\varphi\oplus\varphi_\infty\oplus d_0$, it follows that $d \doteq_{6\varepsilon} \varphi\oplus d$ on X. ∎

1.11. LEMMA. *Let A be a unital separable C^*-algebra, B and C be (non-unital) σ-unital C^*-algebras. Suppose that $d : A \to M(B\otimes\mathcal{K})$ is a monomorphism which gives a unital trivial absorbing extension of A by $B\otimes\mathcal{K}$ and suppose that $\varphi : B \to C$ is a homomorphism such that φ maps a (countable) approximate identity to an approximate identity of C. Let $\varphi' : M(B\otimes\mathcal{K}) \to M(C\otimes\mathcal{K})$ be the extension of $\varphi\otimes\mathrm{id}_{\mathcal{K}}$. Then $\varphi'\circ d$ gives a unital trivial absorbing extension of A by $C\otimes\mathcal{K}$.*

Proof. Let $h : A \to B(\ell^2)$ be a unital faithful representation such that $\pi\circ h : A \to B(\ell^2) \to B(\ell^2)/\ell^2$ is injective. Note that such a unital faithful representation always exists. If $h(A)\cap\mathcal{K} \neq 0$, replace h by $j_1\circ\bigoplus_1^\infty h$, where $j_1 : \bigoplus_1^\infty B(\ell^2) \to B(\ell^2)$ is a monomorphism. Let $j_2 : B(\ell^2) \to M(B\otimes\mathcal{K})$ and $j_3 : B(\ell^2) \to M(C\otimes\mathcal{K})$ be the monomorphisms by identifying $B(\ell^2)$ with scalar operators in $M(B\otimes\mathcal{K})$ ($= L(H_B)$) and with scalar operators in $M(C\otimes\mathcal{K})$ ($= L(H_C)$), respectively. It follows from Theorem 6 in [36] that $d_1 = j_2\circ h$ gives an essential unital trivial absorbing extension of A by B. So there is a unitary $U \in M(A\otimes\mathcal{K})$ such that

$$U^*d_1(a)U - d(a) \in B\otimes\mathcal{K}$$

for all $a \in A$. Let $W = \varphi'(U)$. Then

$$W^*\varphi'\circ d_1(a)W - \varphi'\circ d(a) \in C\otimes\mathcal{K}$$

for all $a \in A$. Since $\varphi'\circ d_1 = j_3\circ h$, by [36], $\varphi'\circ d_1$ is absorbing. So is $\varphi'\circ d$. ∎

1.12. Theorem. *Let A be a unital separable nuclear C^*-algebra which is a unital C^*-subalgebra of a unital C^*-algebra B. Let $d(a) = \mathrm{diag}(a, a, \dots, a)$ be a diagonal map from A to $M(B \otimes \mathcal{K})$.*

(i) *Let τ_1 be the unital essential trivial extension of A by $B \otimes \mathcal{K}$ given by d. If either A is simple or B is simple, then τ_1 is absorbing.*

(ii) *Regard d as diagonal map from A to $M(C([0,1], B) \otimes \mathcal{K}$ (see 1.4) and let τ_2 be the unital essential trivial extension of A by $C([0,1], B) \otimes \mathcal{K}$ given by d. If either A is simple or B is simple, then τ_2 is absorbing.*

(iii) *Regard d as the diagonal map from A to $M(\mathrm{S}B \otimes \mathcal{K})$ (see 1.4) and let τ_3 be the unital essential trivial extension of A by $\mathrm{S}B \otimes \mathcal{K}$ given by d. If either A or B is simple, then τ_3 is absorbing.*

Proof. For (iii), if A is simple, we view d as the diagonal map from A to $M(\mathrm{S}A \otimes \mathcal{K})$. Then, by 1.10, d gives an absorbing extension of A by $\mathrm{S}A \otimes \mathcal{K}$. It then follows from 1.11 that τ_1 is absorbing.

If B is simple, it follows from 1.10 that τ_3 is absorbing.

The same argument shows that (i) holds.

For (ii), consider the embedding from B to $C([0,1], B)$ which maps B to the constant functions. Then (ii) follows from (i) and 1.11. ∎

The following shows that, for general C^*-algebra, d does not give an absorbing extension.

1.13. Theorem. *Let A be a unital separable nuclear C^*-algebra. Suppose that $d : a \to \mathrm{diag}(a, a, \dots, a, \dots)$ is the diagonal map from A to $M(A \otimes \mathcal{K})$. Then the unital essential extension of A by $A \otimes \mathcal{K}$ defined by d is an absorbing extension if and only if A is simple.*

Proof. The "if" part follows from (i) of 1.12.

To prove the "only if" part, we assume that A is not simple. Let $\gamma : A \to D$ be a unital surjective homomorphism which is not injective and $\gamma_1 = \gamma \otimes 1 : A \otimes \mathcal{K} \to D \otimes \mathcal{K}$. Denote by $\gamma_1 : M(A \otimes \mathcal{K}) \to M(D \otimes \mathcal{K})$ the surjective extension (see [60]). Consider a faithful representation $\pi_0 : A \to B(\ell^2)$. Let $h : B(\ell^2) \to M(A \otimes \mathcal{K})$ be the map identifying $B(\ell^2)$ with the subalgebra of scalar (infinite) matrices in $M(A \otimes \mathcal{K})$. Set $\pi_1 = h \circ \pi_0$. Then π_1 gives a strong unital essential trivial extension of A by $A \otimes \mathcal{K}$. Note that $\gamma_1 \circ \pi_1 : A \to M(D \otimes \mathcal{K})$ is a monomorphism but $\gamma_1 \circ d : A \to M(D \otimes \mathcal{K})$ is not injective. So d can not be absorbing. ∎

2. SOME K-THEORETICAL COMPUTATION

2.1. DEFINITION. Let A be a nuclear C^*-algebra. Let $\mathrm{PK}(A,B)$ be (the equivalence classes of) those extensions in $\mathrm{Ext}(A,B)$ such that the six-term exact sequences associated with these extensions breaks into two *pure* group extensions:

$$0 \to \mathrm{K}_0(B) \to \mathrm{K}_0(E) \to \mathrm{K}_0(A) \to 0$$

and

$$0 \to \mathrm{K}_1(B) \to \mathrm{K}_1(E) \to \mathrm{K}_1(A) \to 0.$$

Note that $\mathrm{PK}(A,B)$ is a subgroup of $\mathrm{Ext}(A,B)$.

Recall that by a pure extension of abelian groups

$$0 \to G_0 \to G \to G_1 \to 0$$

we mean that every finitely generated subgroup of G_1 lifts, or equivalently any torsion element of G_1 lifts to a torsion element of G of the same order (see [7]). We identify $\mathrm{Ext}(A,B)$ with $\mathrm{KK}^1(A,B)$ and set $\mathrm{KL}^1(A,B) = \mathrm{KK}^1(A,B)/\mathrm{PK}(A,B)$. We will also identify $\mathrm{KK}(A,B)$ with $\mathrm{KK}^1(A,\mathrm{S}B)$ and define

$$\mathrm{KL}(A,B) = \mathrm{KL}^1(A,\mathrm{S}B).$$

A C^*-algebra A is said to have The Universal Coefficient Theorem (UCT), if for any σ-unital C^*-algebra B, one has

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta} \mathrm{KK}^*(A,B) \xrightarrow{\gamma} \mathrm{Hom}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \longrightarrow 0$$

where δ has degree 1 and γ has degree 0. All C^*-algebras in the so called "bootstrap" class $\mathcal{N}$ of Rosenberg and Schochet satisfy UCT (see [67]). Let C_n be a commutative C^*-algebra with $\mathrm{K}_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{K}_1(C_n) = \{0\}$. Following Dădârlat and Loring ([15]), we let

$$\underline{K}(A) = \mathrm{K}_0(A) \oplus \mathrm{K}_0(A \otimes C(\mathbb{S}^1)) \bigoplus_{n \geqslant 2} \mathrm{K}_0(A \otimes C_n \otimes C(\mathbb{S}^1)).$$

If A satisfies the UCT, then A is KK-equivalent to a commutative C^*-algebra. So by [15] if A satisfies the UCT, then A satisfies the Universal Multi-Coefficient Theorem (UMCT), i.e., for any σ-unital C^*-algebra B, one has

$$0 \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta} \mathrm{KK}(A,B) \xrightarrow{\Gamma} \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B)) \longrightarrow 0$$

(see [15]), where $\mathrm{Pext}^1_{\mathbb{Z}}(\mathrm{K}_i(A),\mathrm{K}_i(B))$ is the subgroup of $\mathrm{ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ of all pure extensions of $\mathrm{K}_i(A)$ by $\mathrm{K}_i(B)$, $i = 0,1$. So, if A satisfies the UCT, Γ gives an isomorphism from $\mathrm{KL}^1(A,B)$ to $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B))$. In particular, if $\varphi,\psi : A \to B$ are two homomorphisms such that $\varphi \otimes \mathrm{id}_{F_n}$ and $\psi \otimes \mathrm{id}_{F_n}$ induce the same map from $\mathrm{K}_0(A \otimes F_n)$ to $\mathrm{K}_0(B \otimes F_n)$ for each n, where $F_1 = C(\mathbb{S}^1)$ and $F_n = C(\mathbb{S}^1 \times C_n)\tilde{}$ with $n = 1,2,\ldots,$, then $[\varphi] = [\psi]$ in $\mathrm{KL}(A,B)$. Conversely, if $[\varphi] = [\psi]$ in $\mathrm{KL}(A,B)$, then $\varphi \otimes \mathrm{id}_{F_n}$ and $\psi \otimes \mathrm{id}_{F_n}$ induce the same map on $\mathrm{K}_0(A \otimes F_n)$ for all n. This fact will be used in the proof of 2.6.

2.2. DEFINITION. Let C be a C^*-algebra. We denote by $\mathrm{Aut}(C)$ the set of automorphisms on C and by $\mathrm{Aut}_0(C)$ the path connect component of $\mathrm{Aut}(C)$ containing the id_C.

Consider an essential extension:

$$0 \to B \to E \to A \to 0.$$

Let $U \in M(B)$ be a unitary such that $U^*xU \in E$ for all $x \in E$. Then $\alpha(x) = \mathrm{ad}(U)(x) = U^*xU$, for $x \in E$, defines an automorphism on E. Moreover, the map $\overline{\alpha}$, defined by $\overline{\alpha}(\pi(x)) = \mathrm{ad}(\pi(U))(\pi(x))$ for $x \in E$, is an automorphism on A. Denote by E_α and A_α the C^*-subalgebras generated by U and E, and by $\pi(U)$ and A, respectively. This also gives an extension $\sigma_\alpha : A_\alpha \to M(B)/B$. We will use these notation in the rest of this paper. It is also clear that $B \times_\alpha \mathbb{Z}$ is a (closed) ideal of $E \times_\alpha \mathbb{Z}$. Furthermore, the quotient is isomorphic to $A \times_{\overline{\alpha}} \mathbb{Z}$. Thus we have

$$0 \to B \times_\alpha \mathbb{Z} \to E \times_\alpha \mathbb{Z} \to A \times_{\overline{\alpha}} \mathbb{Z} \to 0.$$

2.3. DEFINITION. Fix a C^*-algebra B and a nuclear C^*-subalgebra A of B. Let $\alpha : A \to B$ be a monomorphism. Define

$$M_\alpha(A, B) = \{f \in C([0,1], B) : f(0) \in A \text{ and } f(1) = \alpha(f(0))\}.$$

Then we have the following short exact sequence:

$$0 \to \mathrm{S}B \to M_\alpha(A, B) \to A \to 0.$$

If $A = B$, we will denote $M_\alpha(A, A)$ by $M_\alpha(A)$.

Let $\tau_\alpha : A \to M(\mathrm{S}B)/\mathrm{S}B$ be the essential extension determined by the above short exact sequence. Let $[\tau_\alpha]$ be the element in $\mathrm{Ext}(A, \mathrm{S}B) = \mathrm{KK}^1(A, \mathrm{S}B)$ represented by τ_α. Identifying $\mathrm{KK}^1(A, \mathrm{S}B)$ with $\mathrm{KK}(A, B)$, we define $[\alpha] \in \mathrm{KK}(A, B)$ by the image of $[\tau_\alpha]$. Let also $[\tau_\alpha]$ be the element in $\mathrm{KL}^1(A, \mathrm{S}B)$ which is the image of τ_α. Identifying $\mathrm{KL}^1(A, \mathrm{S}B)$ with $\mathrm{KL}(A, B)$, we defined $[\alpha] \in \mathrm{KL}(A, B)$ by the image of $[\tau_\alpha]$.

2.4. REMARK. Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$, and $B = A \times_\alpha \mathbb{Z}$. There is an action $\beta = \hat{\alpha}$ of $\mathbb{S}^1$ on B; we may regard β as an action of $\mathbb{R}$ on B with $\mathbb{Z}$ acting trivially. By Takai duality we have $B \times_\beta \mathbb{S}^1 \cong A \otimes \mathcal{K}$. By 10.3.2 of [4], $B \times_\beta \mathbb{R} \cong M_{\hat{\beta}}(B \times_\beta \mathbb{S}^1)$. Now the Connes' Thom isomorphism gives $\mathrm{K}_i(B \times_\beta \mathbb{Z}) \cong \mathrm{K}_{1-i}(B) = \mathrm{K}_{1-i}(A \times_\alpha \mathbb{Z})$. Thus there is a natural ismorphism from $\mathrm{K}_i(M_\alpha(A))$ to $\mathrm{K}_{1-i}(A \times_\alpha \mathbb{Z})$. This isomorphism can be used to prove the Pimsner-Voiculescu Exact Sequence for cross products (see 10.4 in [4]). We will use this isomorphism to prove the following easy fact.

2.5. LEMMA. *Let E be an essential extension*

$$0 \to B \to E \to A \to 0$$

of A by B, where A and B are σ-unital C^*-algebras. Let $U \in M(B)$ be a unitary with $U^*cU \in E$, for all $c \in E$, and $\alpha(c) = U^*cU$ for all $c \in E$. Then the six-term exact sequence in K*-theory given by*

$$0 \to B \times_\alpha \mathbb{Z} \to E \times_\alpha \mathbb{Z} \to A \times_{\overline{\alpha}} \mathbb{Z} \to 0$$

gives the following commutative diagram

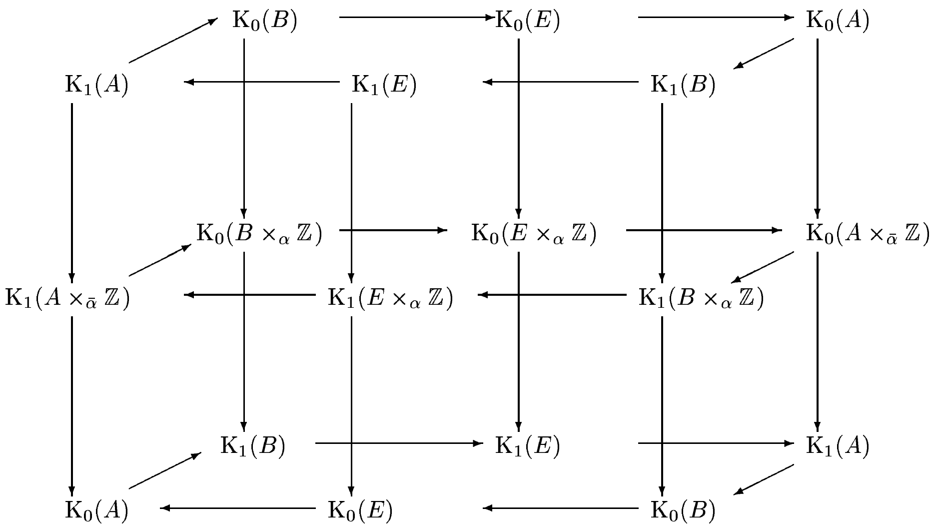

Figure 1.

where the top and bottom planes are the six-term exact sequence given by

$$0 \to B \to E \to A \to 0$$

and the middle plane is the six-term exact sequence given by

$$0 \to B \times_\alpha \mathbb{Z} \to E \times_\alpha \mathbb{Z} \to A \times_{\overline{\alpha}} \mathbb{Z} \to 0$$

and all vertical columns are exact in the middle.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & SB & \to & M_\alpha(B) & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & SE & \to & M_\alpha(E) & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & SA & \to & M_{\overline{\alpha}}(A) & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This commutative diagram gives the commutative diagram in Figure 2.

For any C^*-algebra C, let $\beta : C \to C$ be an automorphism. There is an isomorphism φ from $K_i(M_\beta(C))$ onto $K_{1-i}(C \times_\beta \mathbb{Z})$ (see the previous remark and

10.4 in [4]) such that the following diagram commutes

$$
\begin{array}{ccccccccc}
\mathrm{K}_0(C) & & \longrightarrow & & \mathrm{K}_1(M_\beta(C)) & & \longrightarrow & & \mathrm{K}_1(C) \\
 & \searrow & & & \downarrow & & & \swarrow & \\
 & & \mathrm{K}_0(C) & \to & \mathrm{K}_0(C\times_\beta \mathbb{Z}) & \to & \mathrm{K}_1(C) & & \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
 & & \mathrm{K}_0(C) & \leftarrow & \mathrm{K}_1(C\times_\beta \mathbb{Z}) & \leftarrow & \mathrm{K}_1(C) & & \\
 & \nearrow & & & \uparrow & & & \nwarrow & \\
\mathrm{K}_0(C) & & \longleftarrow & & \mathrm{K}_0(M_\beta(C)) & & \longleftarrow & & \mathrm{K}_1(C)
\end{array}
$$

for any C^*-algebra C. Then the diagram in the lemma follows (see also the proof of Lemma 2.1 in [50] for some more detail). ∎

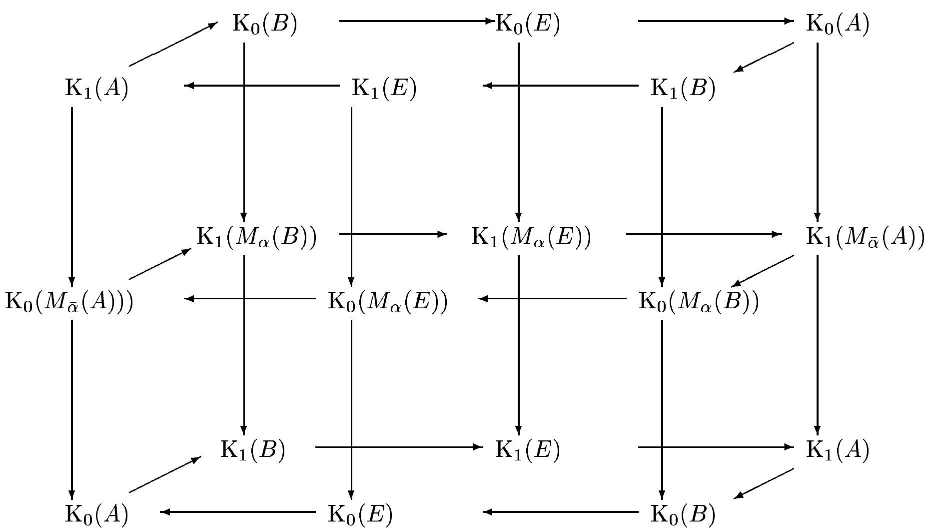

Figure 2.

2.6. Lemma. *Let B be a unital C^*-algebra and A be a unital nuclear separable C^*-algebra with the* UCT *which is a unital C^*-subalgebra of B and let $\alpha : A \to B$ and $\beta : A \to B$ be two monomorphisms such that $[\alpha] = [\beta]$ in* $\mathrm{KL}(A, B)$. *Suppose that there is a unitary $U \in C^{\mathrm{b}}(C(0,1), M(B\otimes\mathcal{K})_\sigma)$ (see* 1.4*) such that*

$$U^*\mathrm{diag}(f, a, a, \ldots)U - \mathrm{diag}(g, a, a, \ldots) \in \mathrm{S}B\otimes\mathcal{K}$$

for all $a \in A$ and for all $f \in M_\alpha(A, B)$ and $g \in M_\beta(A, B)$ with $f(0) = a$, $f(1) = \alpha(a)$, $g(0) = a$ and $g(1) = \beta(a)$. Then, for each $t \in (0,1)$, $[\Gamma_t] = [\mathrm{id}_{\mathrm{E}}]$ in $\mathrm{KL}(E, E)$, *where $E = D+B\otimes\mathcal{K}$, $D = \{\mathrm{diag}(a, a, \ldots, a, \ldots) : a \in A\}$ and $\Gamma_t(x) = U(t)^*xU(t)$ for $x \in E$.*

Proof. It is easy to see that Γ_t is an automorphism on E for each $t \in (0,1)$. We first show that $(\Gamma_t)_* = \mathrm{id}_{\mathrm{K}_0(E)}$. We want to show that, for any projection

$p \in M_m(E)$, $[\Gamma_t(p)] = [p]$ in $\mathrm{K}_0(E)$. To save notation, by replacing $M_m(A)$ by A and $M_m(B)$ by B, we may assume that $p \in E$.

Fix $t_0 \in (0,1)$. Since $U(t_0) \in M(B \otimes \mathcal{K})$, $\Gamma_{t_0}(p)$ is equivalent to p in E, if $p \in B \otimes \mathcal{K}$. So we may assume that $p \notin B \otimes \mathcal{K}$.

We claim that there is a projection $q = \mathrm{diag}(p_1, p_1, \ldots,) \in D$, where $p_1 \in A$ is a projection, such that $p - q \in B \otimes \mathcal{K}$. To see this, note that there is a positive element $a \in A$ such that

$$\mathrm{diag}(a, a, \ldots) - p \in B \otimes \mathcal{K}.$$

Set $c = \mathrm{diag}(a, a, \ldots)$. Then $c^2 - c \in B \otimes \mathcal{K}$. This implies that $\|a^2 - a\| \to 0$. This happens only when $a^2 = a$. The claim follows.

Thus there is $N > 1$ such that

$$\|(1 - e_N)(p - q)\| < \frac{1}{8}$$

where $e_n = \sum_{j=1}^{n} 1_B \otimes e_{ii}$ and $\{e_{ij}\}$ is the matrix unit for $\mathcal{K}$. Then, from a standard computation, there is a projection $p' \in (1 - e_N)E(1 - e_N)$ and a projection $p'' \in e_N E e_N$ such that

$$\|(1 - e_N)q - p'\| < \frac{1}{2} \quad \text{and} \quad \|p' + p'' - p\| < \frac{1}{2}.$$

So p is unitarily equivalent to $p' + p''$ and p' is unitarily equivalent to $(1 - e_N)q$ in E. Thus we may assume that $p = (1 - e_N)q + p''$. Since $[U^*(t_0)p''U(t_0)] = [p'']$, it suffices to show that, in the *group* $\mathrm{K}_0(E)$, $[U^*(t_0)(1 - e_N)qU(t_0)] = [(1 - e_N)q]$. Therefore, to save notation, we may further assume that $p = (1 - e_N)q$. Define $f(t) \in M_\alpha(A, B) \subset C([0,1], B)$ and $g(t) \in M_\beta(A, B) \subset C([0,1], B)$ such that

$$f(0) = p_1, \quad f(t) = \alpha(p_1) \quad \text{for } t \in [t_0, 1],$$

and

$$g(0) = p_1, \quad g(t) = \beta(p_1), \quad \text{for } t \in [t_0, 1]$$

(f and g are not necessary projections). Therefore, using the fact that $[\alpha]|\mathrm{K}_0(A) = [\beta]|\mathrm{K}_0(A)$, we have, in $\mathrm{K}_0(E)$,

$$[U^*(t_0)\alpha(p_1)U(t_0)] = [\alpha_1(p_1)], \quad [\alpha(p_1)] = [\beta(p_1)]$$

and

$$[U^*(t_0)(e_N - e_1)qU(t_0)] = [(e_N - e_1)q].$$

From these identities, in the *group* $\mathrm{K}_0(E)$, to show $[U^*(t_0)(1 - e_N)qU(t_0)] = [(1 - e_N)q]$, it suffices to show that

$$\begin{aligned}
&[U^*(t_0)(1 - e_N)qU(t_0)] + [U^*(t_0)(e_N - e_1)qU(t_0)] + [U^*(t_0)\alpha(p_1)U(t_0)] \\
&\qquad = [(1 - e_N)q] + [(e_N - e_1)q(e_N - e_1)] + [\beta(p_1)].
\end{aligned}$$

Set

$$c(p_1, f, g)(t) = U^*(t)\mathrm{diag}(f(t), p_1, \ldots)U(t) - \mathrm{diag}(g(t), p_1, \ldots).$$

Then $c(p_1, f, g) \in \mathrm{S}B \otimes \mathcal{K}$. Since $\mathrm{diag}(g(t), p_1, p_1, \ldots, p_1, \ldots) \in C([0,1], E)$,

$$U^*(t)\mathrm{diag}(f(t), p_1, \ldots,)U(t) \in C([0,1], E).$$

Therefore, for all $t \in [t_0, 1)$, $U^*(t)\mathrm{diag}(f(t), p_1, \ldots,)U(t)$ are projections in E and they are all equivalent in E. Since $\|c(p_1, f, g)(t)\| \to 0$ as $t \to 1$,

$$\|U^*(t)\mathrm{diag}(\alpha(p_1), p_1, \ldots)U(t) - \mathrm{diag}(\beta(p_1), p_1, \ldots)\| < \frac{1}{2}$$

for all $t \geqslant t_1$, where $1 > t_1 \geqslant t_0$. This implies that, when $1 > t \geqslant t_1$,

$$[U^*(t)\mathrm{diag}(\alpha(p_1), p_1, \ldots)U(t)] = [\mathrm{diag}(\beta(p_1), p_1, \ldots)]$$

in $\mathrm{K}_0(E)$. Since, from the above, for $t \in [t_0, 1)$,

$$[U(t)^*\mathrm{diag}(\alpha(p_1), p_1, \ldots)U(t)] = [U(t_0)^*\mathrm{diag}(\alpha(p_1), p_1, \ldots)U(t_0)],$$

we have

$$[U(t_0)^*\mathrm{diag}(\alpha(p_1), p_1, \ldots)U(t_0)] = [\mathrm{diag}(\beta(p_1), p_1, \ldots)]$$

in $\mathrm{K}_0(E)$, or equivalently

$$[U(t_0)^*(1 - e_1)qU(t_0)] + [U(t_0)^*\alpha(p_1)U(t_0)] = [(1 - e_1)q] + [\beta(p_1)]$$

in $\mathrm{K}_0(E)$. By the above, this implies that

$$[U^*(t_0)pU(t_0)] = [p]$$

for any $t_0 \in (0, 1)$. This proves that $(\Gamma_t)_* = \mathrm{id}|\mathrm{K}_0(E)$ for all $t \in (0, 1)$.

Now let F_n be as in 2.1. Put $A_n = A \otimes F_n$, $B_n = B \otimes F_n$, $E_n = E \otimes F_n$, $\alpha_n = \alpha \otimes \mathrm{id}_{F_n}$, $\beta_n = \beta \otimes \mathrm{id}_{F_n}$, $W_n(t) = U(t) \otimes 1_{F_n}$ and $\Gamma_{t,n} = \Gamma_t \otimes \mathrm{id}_{F_n}$. It is clear that $M_{\alpha_n}(A_n, B_n) = M_{\alpha_n}(A, B) \otimes F_n$ and $M_{\beta_n}(A_n, B_n) = M_{\beta_n}(A, B) \otimes F_n$. Note that, since $[\alpha] = [\beta]$ in $\mathrm{KL}(A, B)$, $[\alpha_n]|\mathrm{K}_0(A_n) = [\beta_n]|\mathrm{K}_0(A_n)$. We also have $\Gamma_{t,n}(c) = W_n(t)^*cW_n(t)$ for $c \in E_n$ and

$$W_n(t)^*\mathrm{diag}(f_n, c, \ldots)W_n(t) - \mathrm{diag}(g_n, c, \ldots) \in \mathrm{S}(B_n) \otimes \mathcal{K}$$

for all $f_n \in M_{\alpha_n}(A_n, B_n)$, $g_n \in M_{\beta_n}(A_n, B_n)$ with $f_n(0) = g_n(0) = c$ and for all $c \in E_n$. Thus, by applying what we have proved, we conclude that $(\Gamma_{t,n})_* = \mathrm{id}_{\mathrm{K}_0(E_n)}$. Therefore, by the last part of 2.1, $[\Gamma_t] = [\mathrm{id}_E]$ in $\mathrm{KL}(E, E)$ for every $t \in (0, 1)$. ∎

2.7. Proposition. *Let A be a unital separable nuclear C^*-algebra with the* UCT *and let E be a unital essential trivial extension of A by $B \otimes \mathcal{K}$, where B is σ-unital, let $U \in M(B \otimes \mathcal{K})$ be a unitary with $Ux - xU \in B \otimes \mathcal{K}$ for all $x \in E$ and let $\alpha = \mathrm{ad}(U) \in \mathrm{Aut}(E)$. Suppose that $[\alpha] = [\mathrm{id}_E]$ in $\mathrm{KL}(E, E)$, $A_\alpha = A \otimes C(\mathbb{S}^1)$ and σ_α is the extension of $A \otimes C(\mathbb{S}^1)$ (given by α as in* 2.2*) by $B \otimes \mathcal{K}$. Then $[\sigma_\alpha] = 0$ in $\mathrm{KL}^1(C(\mathbb{S}^1) \otimes A, B)$.*

Proof. To save notation, we assume that $B = B \otimes \mathcal{K}$. We note that there is a continuous path of unitaries $U_t \in M(B)$ (by [56])) such that $U_0 = 1_{M(B)}$ and $U_1 = U$. Define $\alpha_t(b) = U_t^*bU_t$ for $b \in B$. So $\alpha|B \in \mathrm{Aut}_0(B)$. In particular, $\alpha|B$ is homotopy to id_B. So, by 10.5.2 in [4],

$$\mathrm{K}_i(B \times_\alpha \mathbb{Z}) = \mathrm{K}_0(B) \oplus \mathrm{K}_1(B).$$

Since $Ux - xU \in B$, $\overline{\alpha} = \mathrm{id}_A$. Therefore $A \times_{\overline{\alpha}} \mathbb{Z} = A \otimes C(\mathbb{S}^1)$. Note that the six-term exact sequence in K-theory given by the short exact sequence

$$0 \to B \to E \to A \to 0$$

yields two splitting short exact sequences

$$0 \to \mathrm{K}_0(B) \to \mathrm{K}_0(E) \to \mathrm{K}_0(A) \to 0 \quad \text{and} \quad 0 \to \mathrm{K}_1(B) \to \mathrm{K}_1(E) \to \mathrm{K}_1(A) \to 0.$$

Since $\alpha_{*i} = \mathrm{id}_{\mathrm{K}_i(E)}$, $i = 0, 1$, by the Pimsner-Voiculescu exact sequence, we have the following short exact sequence:

$$0 \to \mathrm{K}_i(E) \to \mathrm{K}_i(E \times_\alpha \mathbb{Z}) \to \mathrm{K}_{i+1}(E) \to 0, \quad i = 0, 1.$$

Since $[\alpha] = [\mathrm{id}_E]$ in $\mathrm{KL}(E, E)$, the above extensions are pure. This is because

$$0 \to \mathrm{K}_i(E) \to \mathrm{K}_{i+1}(M_\alpha(E)) \to \mathrm{K}_{i+1}(E) \to 0, \quad i = 0, 1$$

is pure. Therefore, by performing a diagram chase in the big diagram of 2.5, we see that the map from $\mathrm{K}_0(E \times_\alpha \mathbb{Z})$ to $\mathrm{K}_0(A \times_{\overline{\alpha}} \mathbb{Z})$ is surjective and the map from $\mathrm{K}_1(E \times_\alpha \mathbb{Z})$ to $\mathrm{K}_1(A \times_{\overline{\alpha}} \mathbb{Z})$ is also surjective. Hence the six-term exact sequence in K-theory determined by the short exact sequence

$$0 \to B \times_\alpha \mathbb{Z} \to E \times_\alpha \mathbb{Z} \to A \times_{\overline{\alpha}} \mathbb{Z} \to 0$$

yields two short exact sequences

$$0 \to \mathrm{K}_0(B) \oplus \mathrm{K}_1(B) \to \mathrm{K}_i(E \times_\alpha \mathbb{Z}) \to \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \to 0, \quad i = 0, 1.$$

Furthermore, using the fact that

$$0 \to \mathrm{K}_i(E) \to \mathrm{K}_i(E \times_\alpha \mathbb{Z}) \to \mathrm{K}_{i+1}(E) \to 0, \quad i = 0, 1,$$

is pure, from the big diagram of 2.5, we conclude that the extension

$$0 \to \mathrm{K}_0(B) \oplus \mathrm{K}_1(B) \to \mathrm{K}_i(E \times_\alpha \mathbb{Z}) \to \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \to 0, \quad i = 0, 1$$

is pure.

Let $h : E \times_\alpha \mathbb{Z} \to E_\alpha$ (see 2.2) be the surjective map. It is clear that h gives the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & B \times_\alpha \mathbb{Z} & \to & E \times_\alpha \mathbb{Z} & \to & A \times_{\overline{\alpha}} \mathbb{Z} & \to & 0 \\
 & & \downarrow h_J & & \downarrow h & & \downarrow \overline{h} & & \\
0 & \to & B & \to & E_\alpha & \to & A_\alpha & \to & 0.
\end{array}$$

Therefore, from the above, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{K}_0(B \times_\alpha \mathbb{Z}) & & & \longrightarrow & \mathrm{K}_0(E \times_\alpha \mathbb{Z}) & \longrightarrow & & & \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \\
 & \searrow & & & \downarrow & & & \swarrow & \\
 & & \mathrm{K}_0(B) & \to & \mathrm{K}_0(E_\alpha) & \to & \mathrm{K}_0(A_\alpha) & & \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
 & & \mathrm{K}_1(A_\alpha) & \leftarrow & \mathrm{K}_1(E_\alpha) & \leftarrow & \mathrm{K}_1(B) & & \\
 & \nearrow & & & \uparrow & & & \nwarrow & \\
\mathrm{K}_1(A) \oplus \mathrm{K}_0(A) & & & \longleftarrow & \mathrm{K}_1(E \times_\alpha \mathbb{Z}) & \longleftarrow & & & \mathrm{K}_1(B \times_\alpha \mathbb{Z}).
\end{array}$$

Note, since $A \times_{\overline{\alpha}} \mathbb{Z} = A \otimes C(\mathbb{S}^1)$ and $A_\alpha = A \otimes C(\mathbb{S}^1)$, that $\overline{h}$ is an isomorphism. Therefore the map from $\mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ to $\mathrm{K}_0(A_\alpha)$ in the above diagram is an isomorphism. Since, in the above diagram, the map from $\mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ to $\mathrm{K}_1(B \times_\alpha \mathbb{Z})$ is zero, the map from $\mathrm{K}_0(A_\alpha)$ to $\mathrm{K}_1(B)$ is also zero. Similarly, the map from $\mathrm{K}_1(A_\alpha)$ to $\mathrm{K}_0(B)$ is also zero. Thus, the six-term exact sequence given by the extension

$$0 \to B \to E_\alpha \to A_\alpha \to 0$$

breaks into

$$0 \to \mathrm{K}_i(B) \to \mathrm{K}_i(E_\alpha) \to \mathrm{K}_i(A_\alpha) \to 0, \quad i = 0, 1.$$

Therefore, from the UCT, we know that $[\tau_\alpha] \in \mathrm{ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A_\alpha), \mathrm{K}_*(B))$. Since

$$0 \to \mathrm{K}_i(B \times_\alpha \mathbb{Z}) \to \mathrm{K}_i(E \times_\alpha \mathbb{Z}) \to \mathrm{K}_1(A) \oplus \mathrm{K}_0(A) \to 0$$

is pure, from the above 12-term commutative diagram, one checks easily that

$$0 \to \mathrm{K}_i(B) \to \mathrm{K}_i(E_\alpha) \to \mathrm{K}_i(A_\alpha) \to 0, \quad i = 0, 1$$

is also pure. We conclude that $[\tau_\alpha] = 0$ in $\mathrm{KL}^1(A \otimes C(\mathbb{S}^1), B)$. ∎

3. LIFTING AUTOMORPHISMS

3.1. DEFINITION. Consider again the situation in 2.2. A is a unital C^*-subalgebra of A_α. Let $\tau_0 : A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ be a unital trivial extension of A by $B \otimes \mathcal{K}$. Define $\tau_1(a) = \mathrm{diag}(\sigma_\alpha(a), \tau_0(a))$ for all $a \in A$ and $\tau_1(\pi(U)) = \mathrm{diag}(\pi(U), 1)$. This gives a unital essential extension of A_α by $B \otimes \mathcal{K}$. We say that σ_α *absorbs any essential trivial extension* of A (by $B \otimes \mathcal{K}$) if τ_1 is unitarily equivalent to σ_α for every such τ_0. Note that, in the above, if τ_0 is absorbing, then τ_1 always absorbs any trivial essential extensions of A. This fact will be used in the proof of 3.4 and 4.3.

3.2. THEOREM. *Let A be a C^*-algebra. Then every automorphism $\alpha \in \mathrm{Aut}_0(A)$ is approximately inner.*

Proof. The case that A is separable is proved in [44] which follows from a result of Pedersen and a result of Kadison and Ringrose. Now we reduce the general case to the separable case. Fix a finite subset $\mathcal{F} \subset A$. Let α_t $(t \in [0, 1]) \in \mathrm{Aut}(A)$ be a continuous path of automorphisms such that $\alpha_0 = \alpha$ and $\alpha_1 = \mathrm{id}_A$. Let $\{t_n\}$ be a dense subset of $(0, 1)$. Let $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{F}_2 = \alpha(\mathcal{F}_1)$, $\mathcal{F}_3 = \alpha(\mathcal{F}_2) \cup \alpha_{t_1}(\mathcal{F}_2)$, $\mathcal{F}_4 = \alpha(\mathcal{F}_3) \cup \alpha_{t_1}(\mathcal{F}_3) \cup \alpha_{t_2}(\mathcal{F}_3)$, $\mathcal{F}_5 = \alpha(\mathcal{F}_4) \cup \alpha_{t_1}(\mathcal{F}_4) \cup \alpha_{t_2}(\mathcal{F}_4) \cup \alpha_{t_3}(\mathcal{F}_4), \ldots$ Let B be the C^*-subalgebra of A generated by $\{\mathcal{F}_n\}$. Then B is a separable C^*-algebra and α, α_{t_n} are invariant on B. This implies that $\alpha|B \in \mathrm{Aut}_0(B)$. It follows from the separable case that, for any $\varepsilon > 0$, there is a unitary $U \in B \subset A$ such that

$$\|\alpha(x) - U^* x U\| < \varepsilon$$

for all $x \in \mathcal{F}$. ∎

3.3. THEOREM. (Theorem 1.4 in [69]) *Let A be a separable nuclear C^*-algebra which satisfies the Universal Coefficient Theorem and B be a σ-unital C^*-algebra. Suppose that $x \in \mathrm{Ext}(A, B)$. Then x is in the closure of zero if and only if $x \in \mathrm{PK}(A, B)$.*

The statement in [69] is about (stably) quasidiagonal extensions. However, the above statement is actually proved in 1.4 in [69]. By the definition, an essential extension $\tau : A \to M(B)/B$, $[\tau]$ *is in the closure of zero* if and only if the following holds: For any essential trivial absorbing extension $\tau_0 : A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$, there are trivial extensions $\tau_n : A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ such that

$$\lim_{n \to \infty} \tau_n(a) = \tau(a) \oplus \tau_0(a)$$

for all $a \in A$.

3.4. THEOREM *Let A be a unital nuclear separable C^*-algebra which satisfies the* UCT *and B be a σ-unital C^*-algebra. Let $\tau : A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ be a unital essential trivial absorbing extension of A by $B \otimes \mathcal{K}$, let E be the C^*-algebra determined by this unital essential trivial extension, let $U \in M(B \otimes \mathcal{K})$ with $U^*x - xU \in B \otimes \mathcal{K}$ for $x \in E$ and let $\alpha = \mathrm{ad}(U) \in \mathrm{Aut}(E)$. Suppose that $\sigma_\alpha : A_\alpha \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ induced by α (see* 2.2*) absorbs any trivial extensions of A (see* 3.1*). Then α is approximately inner if $[\alpha] = [\mathrm{id}_E]$ in* $\mathrm{KL}(E, E)$.

Proof. (1) First, we show that there exists a continuous path of unitaries $V_t \in M(B \otimes \mathcal{K})$ ($t \in [0, 1]$) such that $V_0 = 1$, $V_t x - xV_t \in B \otimes \mathcal{K}$ for all $x \in E$, and if $\beta = \mathrm{ad}(V_1)$ then $A_\beta = C(\mathbb{S}^1) \otimes A$, and $\sigma_\beta : C(\mathbb{S}^1) \otimes A :\to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ is a unital essential trivial absorbing extension.

There is an absorbing unital trivial essential extension $\tau_0' : C([0, 1]) \otimes A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$. Let $C = \{f \in C([0, 1]) \otimes A : f(0) = f(1)\}$. Then $C \cong C(\mathbb{S}^1) \otimes A$. Thus there is an absorbing unital trivial essential extension $\tau_0 : C(\mathbb{S}^1) \otimes A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ which satisfies the following: There is a continuous path of unitaries $V_t \in M(B \otimes \mathcal{K})$ ($t \in [0, 1]$) such that $[\pi(V_t), \tau_0(1 \otimes a)] = 0$ for all $a \in A$, $\pi(V_1) = \tau_0(z \otimes 1_A)$ and $V_0 = 1$, where z is the canonical unitary generator of $C(\mathbb{S}^1)$. To see the existence of τ_0', we first obtain a unital monomorphism $h_1 : C([0, 1]) \otimes A \to B(\ell^2)$ such that $h_1(C([0, 1]) \otimes A) \cap \mathcal{K} = \emptyset$ (see the proof of 1.11) and then let $\varphi : B(\ell^2) \to M(B \otimes \mathcal{K})$ be the map by identifying $B(\ell^2)$ with the subalgebra of scalar matrices in $M(B \otimes \mathcal{K})$. By Theorem 6 in [36], we know that τ_0' given by $\varphi \circ h_1$ is an absorbing unital trivial essential extension. Let E' be the C^*-algebra given by the (splitting) short exact sequence determined by $\tau_0|A$

$$0 \to B \otimes \mathcal{K} \to E' \to A \to 0.$$

Since $\tau_0|A$ and τ are absorbing, they are unitarily equivalent. So, without loss of generality, we may assume that $E = E'$. This proves (1).

(2) There is an automorphism $\beta : E \to E$ such that β and α are in the same path component in $\mathrm{Aut}(E)$, $A_\beta = C(\mathbb{S}^1) \otimes A$ and $\sigma_\beta : C(\mathbb{S}^1) \otimes A \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ is trivial and absorbing.

Let V_1 be as in (1) and let $W = \mathrm{diag}(U, V_1)$. Note that A_α is a quotient of $C(\mathbb{S}^1) \otimes A$. There is a surjective map $h : A_\alpha \to A$ such that $h|A = \mathrm{id}_A$ (we regard A as a unital C^*-subalgebra of A_α). Consider $\sigma : A_\alpha \to M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ defined by $\sigma(a) = \mathrm{diag}(\sigma_\alpha(a), h(a))$ for $a \in A_\alpha$ (see 2.2 for σ_α). Since σ_α absorbs any trivial extensions by A, there is a unitary $Z \in M(B \otimes \mathcal{K})$ such that $\sigma_\alpha = \mathrm{ad}(Z) \circ \sigma$. So $\pi(U) = \mathrm{ad}(Z)(\mathrm{diag}(\pi(U), 1))$. Now set $W_t = \mathrm{ad}(Z)(\mathrm{diag}(U, V_t))$, where V_t is as in (1). Let $\beta_t = \mathrm{ad}(W_t)$ and $\beta = \mathrm{ad}(W_1)$. By (1), we have $A_\beta = C(\mathbb{S}^1) \otimes A$ and σ_β is trivial absorbing extension. Since $\beta_0 = \alpha$, we see (2) holds.

(3) Now we prove the theorem. By (2) and 3.2, we may further assume that σ_α is a unital essential trivial absorbing extension of $C(\mathbb{S}^1) \otimes A$ by $B \otimes \mathcal{K}$. Now let β be as in (1). Note that $\beta = \mathrm{ad}(V_1)$ and $\mathrm{ad}(V_1) \in \mathrm{Aut}_0(E)$. It follows from 2.7 that σ_β and σ_α represent the same element in $\mathrm{KL}^1(A \otimes C(\mathbb{S}^1), B)$. Thus, by 3.3, since both σ_β and σ_α are absorbing, there are unitaries $Z_n \in M(B \otimes \mathcal{K})$ such that

$$\|Z_n^* V_1 Z_n - U + a_n\| \to 0$$

as $n \to \infty$, where $a_n \in B \otimes \mathcal{K}$. Set $W_n = Z_n^* V_1 Z_n$. For any $0 < \delta < 1/2$, suppose that

$$\|W_n - U + a_n\| < \delta.$$

Set $W_n' = (W_n + a_n)[(W_n + a_n)^*(W_n + a_n)]^{-1/2}$. Note $(W_n + a_n)^*(W_n + a_n)$ is invertible. Note also that

$$(W_n)^*W_n' - 1_E \in B \otimes \mathcal{K}.$$

For any $\varepsilon > 0$, with a sufficiently small δ,

$$\|W_n' - U\| < \frac{\varepsilon}{2}.$$

Thus

$$\|\mathrm{ad}(W_n') - \alpha\| < \varepsilon.$$

Therefore, to show that α is approximately inner, it suffices to show that $\mathrm{ad}(W_n')$ is approximate inner (as an automorphism on E) for every such W_n' with sufficiently large n. Let $Y = (W_n)^*W_n'$. Then Y is a unitary in $1 + B \otimes \mathcal{K} \subset E$. Note that $\mathrm{ad}(W_n') = \mathrm{ad}(W_n) \circ \mathrm{ad}(Y)$ and $\mathrm{ad}(W_n) \in \mathrm{Aut}_0(E)$. By 3.2, $\mathrm{ad}(W_n)$ is approximately inner. Therefore $\mathrm{ad}(W_n')$ is approximately inner. ∎

4. STABLE APPROXIMATE UNITARY EQUIVALENCE

For the rest of the paper, we will denote by $\mathcal{B}$ the class of all separable, nuclear, C^*-algebras which satisfy the UCT.

4.1. DEFINITION. Let A be a C^*-algebra and B be a unital C^*-algebra. Two homomorphisms $\varphi, \psi : A \to B$ are said to be *stably approximately unitarily equivalent* if, for any monomorphism $h : A \to B$, $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there exists an integer $n > 0$ and a unitary $U \in M_{n+1}(B)$ such that

$$\|U^*\mathrm{diag}(\varphi(a), h(a), h(a), \ldots, h(a))U - \mathrm{diag}(\psi(a), h(a), h(a), \ldots, h(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $h(a)$ repeats n times on the both diagonals.

4.2. Let A be a separable unital nuclear C^*-subalgebra of a unital C^*-algebra B (we assume that $1_A = 1_B$). We will denote by τ_∞ the unital trivial extension of A by $\mathrm{S}B \otimes \mathcal{K}$ defined by the unital map $d(a) = \mathrm{diag}(a, a, \ldots, a, \ldots)$ from A to $M(\mathrm{S}B \otimes \mathcal{K})$. Here we identify B with constant functions in $C([0,1], B)$. If either A is simple or B is simple, then by 1.10, d gives an absorbing trivial extension of A by $\mathrm{S}B \otimes \mathcal{K}$.

4.3. THEOREM *Let B be a unital C^*-algebra and let A be a unital C^*-algebra in $\mathcal{B}$ which is a unital C^*-subalgebra of B. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then α and β are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $\mathrm{KK}(A, B)$ and if either A is simple or B is simple.*

Proof. For any unital monomorphism $h : A \to B$, to simplify notation, without loss of generality, we may assume that the monomorphism is the embedding. By replacing α and β by $\alpha \oplus h$ and $\beta \oplus h$, without loss of generality, we may assume that α and β are injective. Let $\tau_1 = \tau_\alpha$ and $\tau_2 = \tau_\beta$ be the extensions of A by $\mathrm{S}B$ as in 2.3 and let τ_∞ be as in 4.2. Then, if either A is simple or B is simple, by the Universal Coefficient Theorem ([67]) and 1.10 there is a unitary $U \in M(\mathrm{S}B \otimes \mathcal{K}) = C^{\mathrm{b}}((0,1), M(B \otimes \mathcal{K})_\sigma)$ (see 1.4) such that

$$\pi(U)^*\mathrm{diag}(\tau_1, \tau_\infty)\pi(U) = \mathrm{diag}(\tau_2, \tau_\infty),$$

where $\pi : M(\mathrm{S}B)\otimes\mathcal{K}) \to M(\mathrm{S}B\otimes\mathcal{K})/\mathrm{S}B\otimes\mathcal{K}$ is the quotient map. For any $a \in A$, let $f \in M_\alpha(A,B)$ and $g \in M_\beta(A,B)$ with $f(0) = a$, $f(1) = \alpha(a)$, $g(0) = a$ and $g(1) = \beta(a)$. Set

$$c(a,f,g)(t) = U^*(t)\mathrm{diag}(f(t),a,a,\ldots)U(t) - \mathrm{diag}(g(t),a,a,\ldots), \quad t \in (0,1).$$

Then we have $c(a,f,g) \in \mathrm{S}B\otimes\mathcal{K}$. In particular, $\|c(a,f,g)(t)\| \to 0$ as $t \to 1$ or $t \to 0$.

Let

$$D = \{\mathrm{diag}(a,a,\ldots,a,\ldots) : a \in A\},$$

where a repeats infinitely many times. Let $E = D + B\otimes\mathcal{K}$. For each $t \in (0,1)$ define $\Gamma_t(x) = U^*(t)xU(t)$ for $x \in E$. Then Γ_t is an automorphism on D. It follows from Lemma 2.6 that $[\Gamma_t] = [\mathrm{id}_E]$ in $\mathrm{KL}(E,E))$ for each $t \in (0,1)$. Let $\pi_0 : E \to E/B\otimes\mathcal{K}$. Then $\pi_0\circ\Gamma_t = \pi_0$. By 1.11, $\pi_0\circ d$ gives an absorbing trivial extension of A by $B\otimes\mathcal{K}$. We now apply Theorem 3.4. Note $U(t) \in M(B\otimes\mathcal{K})$ for each $t \in (0,1)$ and also note that E is a C^*-subalgebra of $M(B\otimes\mathcal{K})$ given by the diagonal map d from A to $M(B\otimes\mathcal{K})$. Furthermore, by replacing $U(t)$ by $\mathrm{diag}(U(t),1)$ and τ_∞ by $\mathrm{diag}(\tau_\infty,\tau_\infty)$, we may assume that σ_{Γ_t} absorbs any trivial essential extensions of A (by $B\otimes\mathcal{K}$); see 3.1. So, by 3.4, Γ_t is approximately inner. Thus, for any $\varepsilon > 0$, any finite subset $\mathcal{G} \subset E$ and for each $t \in (0,1)$, there is a unitary $W_t \in E$ such that

$$\|W_t^* g W_t - U^*(t)gU(t)\| < \frac{\varepsilon}{32}$$

for all $g \in \mathcal{G}$. Since $W_t \in E$, there is $W_t' \in E$ such that

$$\|W_t - W_t'\| < \frac{\varepsilon}{32}$$

and

$$\Big[1-\Big(1_B\otimes\sum_{i=1}^k e_{ii}\Big)\Big]W_t' = W_t'\Big[1-\Big(1_B\otimes\sum_{i=1}^k e_{ii}\Big)\Big] = \mathrm{diag}(0,\ldots,0,w,w,\ldots,w,\ldots),$$

where the diagonal element has k zero for some integer $k > 0$ and $w \in A$ is a unitary (and $\{e_{ij}\}$ is a matrix unit for $\mathcal{K}$).

Now fix a finite subset $\mathcal{F} = \{a_1,a_2,\ldots,a_l\}$ in the unit ball of A. Let $f_i \in C([0,1],B)$ such that $f_i(0) = a_i$ and $f_i(1) = \alpha(a_i)$, and $g_i \in C([0,1],B)$ such that $g_i(0) = a_i$ and $g_i(1) = \beta(a_i)$, $i = 1,2,\ldots,l$. We also use a_i for the constant functions in $C([0,1],B)$. Choose t close to 1 so that

$$\|f_i(t) - f_i(1)\| < \frac{\varepsilon}{4}, \quad \|g_i(t) - g_i(1)\| < \frac{\varepsilon}{4}$$

and

$$\|c(a_i,f_i,g_i)(t)\| < \frac{\varepsilon}{16}, \quad i = 1,2,\ldots,l.$$

With sufficiently large $\mathcal{G}$, we have

$$\|(W_t')^*\mathrm{diag}(f_i(t),a_i,a_i,\ldots,a_i,\ldots)W_t' - U(t)^*\mathrm{diag}(f_i(t),a_i,a_i,\ldots,a_i,\ldots)U(t)\| < \frac{\varepsilon}{8}$$

for $i = 1, 2, \ldots, l$. Then

$$\begin{aligned}\|(W_t')^*\mathrm{diag}(f_i(t), a_i, a_i, \ldots)W_t' - \mathrm{diag}(g_i(t), a_i, a_i, \ldots,)\| &\leqslant \frac{\varepsilon}{8} + \|c(a_i, f_i, g_i)(t)\| \\ &< 3\frac{\varepsilon}{16}\end{aligned}$$

for $i = 1, 2, \ldots, l$. Let $w_1 = \Big(1_B \otimes \sum_{i=1}^{k} e_{ii}\Big)W_t'\Big(1_B \otimes \sum_{i=1}^{k} e_{ii}\Big)$. Note that w_1 is a unitary in $M_k(B)$. We have

$$\|w_1^*\mathrm{diag}(f_i(t), a_i, a_i, \ldots, a_i)w_1 - \mathrm{diag}(g_i(t), a_i, a_i, \ldots, a_i))\| < 3\frac{\varepsilon}{16}$$

for $i = 1, 2, \ldots, l$, where a_i repeats $k-1$ times on both diagonals.

Finally,

$$\|w_1^*\mathrm{diag}(\alpha(a_i), a_i, a_i, \ldots, a_i)w_1 - \mathrm{diag}(\beta(a_i), a_i, a_i, \ldots, a_i)\| < \frac{\varepsilon}{4} + 3\frac{\varepsilon}{16} + \frac{\varepsilon}{4} < \varepsilon$$

for $i = 1, 2, \ldots, l$.

From the first sentence of this proof, we conclude that α and β are stably approximately unitarily equivalent. ∎

4.4. PROPOSITION. (cf 5.4 in [66]) *Let B be a unital C^*-algebra and let A be a unital C^*-algebra in $\mathcal{B}$ which is a unital C^*-subalgebra of B. Let $\alpha : A \to B$ and $\beta : A \to B$ be two unital homomorphisms. Suppose that α and β are stably approximately unitarily equivalent. Then $[\alpha] = [\beta]$ in* $\mathrm{KL}(A, B)$.

Proof. We assume that

$$\|u_n^*\mathrm{diag}(\alpha(a), a, \ldots, a)u_n - \mathrm{diag}(\beta(a), a, \cdots, a)\| \to 0$$

for all $a \in A$ as $n \to \infty$, where a repeats K_n times and $u_n \in M_{\mathrm{K}_n+1}(B)$ are unitaries. Set $\alpha_n : A \to M_{\mathrm{K}_n+1}(B)$ by $\alpha_n(a) = \mathrm{ad}(u_n) \circ \mathrm{diag}(\alpha(a), a, \ldots, a)$ and $\beta_n : A \to M_{\mathrm{K}_n+1}(B)$ by $\beta_n(a) = \mathrm{diag}(\beta(a), a, \ldots, a)$. Let τ_{α_n} and τ_{β_n} be the extension of A by $\mathrm{S}B \otimes \mathcal{K}$ determined by $M_{\alpha_n}(A, M_{\mathrm{K}_n+1}(B))$ and $M_{\beta_n}(A, M_{\mathrm{K}_n+1}(B))$, respectively. Let $f_n(t) \in M_{\alpha_n}(A, M_{\mathrm{K}_n+1}(B))$ and $g_n(t) \in M_{\beta_n}(A, M_{\mathrm{K}_n+1}(B))$ with $f_n(0) = g_n(0)$; then

$$\|f_n(1) - g_n(1)\| = \|\alpha_n(f(0)) - \beta_n(f_n(0))\| \to 0$$

(as $n \to \infty$). This implies that (view τ_{α_n} and τ_{β_n} as maps from A to $M(\mathrm{S}B \otimes \mathcal{K})/\mathrm{S}B \otimes \mathcal{K})$)

$$\|\tau_{\alpha_n}(a) - \tau_{\beta_n}(a)\| \to 0$$

for all $a \in A$. Set $\gamma : A \to M(B \otimes \mathcal{K})$ by $\gamma(a) = \mathrm{diag}(a, a, \ldots)$ for $a \in A$ and $\tau_\infty = \pi \circ \gamma$, where $\pi : M(\mathrm{S}B \otimes \mathcal{K}) \to M(\mathrm{S}B \otimes \mathcal{K})/\mathrm{S}B \otimes \mathcal{K}$ is the quotient map.

From the above, we obtain unitaries $v_n \in M(\mathrm{S}B \otimes \mathcal{K})$ such that

$$\lim_{n\to\infty} \mathrm{ad}(v_n) \circ \mathrm{diag}(\tau_\alpha(a), \tau_\infty(a)) = \mathrm{diag}(\tau_\beta(a), \tau_\infty(a))$$

for all $a \in A$. Let τ_∞^{-1} and τ_β^{-1} be extension such that $[\tau_\infty^{-1}] = -[\tau_\infty]$ and $[\tau_\beta^{-1}] = -[\tau_\beta]$ in $\mathrm{Ext}(A, \mathrm{S}B \otimes \mathcal{K})$. Then

$$\begin{aligned}&\lim_{n\to\infty} \mathrm{ad}(v_n) \circ \mathrm{diag}(\tau_\alpha(a), \tau_\infty(a)) \oplus \tau_\infty^{-1}(a) \oplus \tau_\beta^{-1}(a) \\ &\qquad = \mathrm{diag}(\tau_\beta(a), \tau_\infty(a)) \oplus \tau_\infty^{-1}(a) \oplus \tau_\beta^{-1}(a)\end{aligned}$$

for all $a \in A$. Since $[\tau_\infty \oplus \tau_\infty^{-1}] = 0$ and $[\tau_\beta \oplus \tau_\infty] + [\tau_\infty^{-1} \oplus \tau_\beta^{-1}] = 0$ in $\mathrm{Ext}(A, \mathrm{S}B \otimes \mathcal{K})$, by [69] (see 3.3), $[\tau_\alpha + \tau_\beta^{-1}] = 0$ in $\mathrm{KL}^1(A, \mathrm{S}B \otimes \mathcal{K})$, or $[\tau_\alpha] = [\tau_\beta]$ in $\mathrm{KL}^1(A, \mathrm{S}B \otimes \mathcal{K})$. So $[\alpha] = [\beta]$ in $\mathrm{KL}(A, B)$. ∎

4.5. THEOREM. *Let A be a unital C^*-algebra in $\mathcal{B}$ and let B be a unital simple C^*-algebra which contains A as a unital C^*-subalgebra. Suppose that there are C^*-subalgebras $A_n \subset A$ with finitely generated $\mathrm{K}_i(A_n)$ with $i = 0, 1$ such that each A_n satisfies the* UCT *and $\bigcup_{n=1}^{\infty} A_n$ is dense in A. Let $\alpha, \beta : A \to B$ be homomorphisms. Then α and β are stably approximately unitarily equivalent if and only if $[\alpha] = [\beta]$ in* $\mathrm{KL}(A, B)$.

Proof. The "only if" part follows from 4.4. Let $\alpha_n = \alpha|A_n$ and $\beta_n = \beta|A_n$. Since A is unital, $1_A \in A_n$ for all large n. So without loss of generality we may assume that $1_A \in A_n$ for all n. From the surjective map

$$\mathrm{KL}(A, B) \to \varprojlim \mathrm{KL}(A_n, B)$$

(see [15] for example), since $[\alpha] = [\beta]$ in $\mathrm{KL}(A, B)$, we know, for all sufficiently large n, $[\alpha_n] = [\beta_n]$ in $\mathrm{KL}(A_n, B)$. Since $\mathrm{K}_i(A_n)$ is finitely generated and A_n has UCT, we conclude that $[\alpha_n] = [\beta_n]$ in $\mathrm{KK}(A_n, B)$ for all large n. By 4.3, α_n and β_n are stably approximately unitarily equivalent for all large n. It follows that α and β are stably approximately unitarily equivalent. ∎

The above actually holds if we assume that A is simple without assuming that B is simple. This is because the map from A_n to B factors through A so the corresponding diagonal map is in fact absorbing.

4.6. DEFINITION. Now we introduce a class of C^*-algebras which has certain "bounded K-theoretical stable rank". Fix $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$. We say a unital C^*-algebra $A \in \mathcal{C}_{(l,b,M)}$, if

(a) for any projections $p, q \in M_K(A)$ with $[p] = [q]$ in $\mathrm{K}_0(A)$, $p \oplus 1_{M_{\mathrm{KL}(A)}}$ is Murry-von Neumann equivalent to $q \oplus 1_{M_{\mathrm{KL}(A)}}$ for all K;

(b) the canonical map $U(M_l(A))/U_0(M_l(A)) \to \mathrm{K}_1(A)$ is surjective;

(c) the exponential length of $M_m(A)$, $\mathrm{cel}(M_m(A)) \leqslant b$ for all m;

(d) if $k > 0$ and $-l[1_A] \leqslant kx \leqslant l[1_A]$, then $-lMk[1_A] \leqslant x \leqslant lMk[1_A]$ for all $x \in \mathrm{K}_0(A)$.

Every purely infinite simple C^*-algebra is in $\mathcal{C}_{(l,b,M)}$. By some results of Rieffel ([64]), all C^*-algebras of stable rank one satisfy (a) and (b). Every C^*-algebra A of real rank zero has exponential length bounded by π, i.e., $\mathrm{cel}(A) \leqslant \pi$ (see [47], also 6.7 in [61]). It is easy to see that if $\mathrm{K}_0(A)$ is weakly unperforated, then A satisfies (d). Therefore every unital C^*-algebra with real rank zero, stable rank one and weakly unperforated K_0 is in $\mathcal{C}_{(l,b,M)}$, for any $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$. Fix $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$. Let $B_n \in \mathcal{C}_{(l,b,M)}$ and $C = \prod_{n=1}^{\infty} (B_n)$. Then $\mathrm{K}_0(C) = \prod_b \mathrm{K}_0(B_n)$ and $\mathrm{K}_1(C) = \prod \mathrm{K}_1(B_n)$, where

$$\prod_b \mathrm{K}_0(B_n) = \{\{x_n\} : x_n = [p_n] - [q_n],\ p_n, q_n \in M_k(B_n) \text{ projections with } k \text{ independent of } n\}.$$

(see [31] for example and [32]). Therefore, it is easy to see that C satisfies (a), (b) and (d) with the same l and M. It is also easy to see that C satisfies (c) with the same b (see [31]). Therefore $C \in \mathcal{C}_{(l,b,M)}$. Let $C_0 = \bigoplus_n B_n$ and let $\pi : C \to C/C_0$ be the quotient map. Since every projection, partial isometry and unitary in C/C_0 can be lifted to a projection, partial isometry and unitary, it is rather easy to check that that $C/C_0 \in \mathcal{C}_{(l,b,M)}$. For example, if $p, q \in M_K(C/C_0)$ and $[p] = [q]$ in $\mathrm{K}_0(C/C_0)$. Then, there are projections $\{p_n\}, \{q_n\} \in M_K(C)$ such that $\pi(\{p_n\}) = p$, $\pi(\{q_n\}) = q$ and $[p_n] = [q_n]$ in $\mathrm{K}_0(B_n)$ for $n \geqslant N$ and for some integer $N > 0$. So we may assume that $p_n = q_n$ if $n < N$. Then $p_n \oplus 1_{M_{\mathrm{KL}}(B_n)}$ is equivalent to $q_n \oplus 1_{M_{\mathrm{KL}}(B_n)}$. Therefore C/C_0 satisfies (a) for l. In the special case that each B_n has real rank zero and stable rank one, it is known that both $\prod B_n$ and C/C_0 have stable rank one and real rank zero. It is also easy to see that if each $\mathrm{K}_0(B_n)$ is weakly unperforated so are $\mathrm{K}_0(C)$ and $\mathrm{K}_0(C/C_0)$.

Though for most applications the C^*-algebra A has the "K-theoretically locally finite" condition in 4.5, we would like to include the following theorem which does not require A to be "K-theoretically locally finite". We believe however that 4.7 holds without these additional conditions (condition (d) is not used in the proof of 4.7 but used in 4.8 and 5.3).

4.7. Theorem. *Let $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$. Let B be a unital C^*-algebra in $\mathcal{C}_{(l,b,M)}$ and let A be a unital simple C^*-algebra in $\mathcal{B}$ which is a unital C^*-subalgebra of B. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then α and β are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $\mathrm{KL}(A, B)$.*

Proof. Let $\sigma : A \to M(\mathrm{S}B \otimes \mathcal{K})/\mathrm{S}B \otimes \mathcal{K})$ such that $[\sigma] = -[\beta]$ in $\mathrm{KK}(A, B)$ ($= \mathrm{Ext}(A, \mathrm{S}B)$) and $\gamma = \alpha \oplus \sigma$. In $\mathrm{KK}(A, B)$, $[\gamma]$ gives an element in $\bigoplus_{i=0,1} \mathrm{Pext}(\mathrm{K}_i(A), \mathrm{K}_{i+1}(B))$ which is represented by the following pure extensions:

$$0 \to \mathrm{K}_{i+1}(B) \to H_i \xrightarrow{p} \mathrm{K}_i(A) \to 0, \quad i \in \mathbb{Z}/2\mathbb{Z}.$$

Let $\mathrm{K}_1(A)$ be generated by $g_1, g_2, \ldots, g_n, \ldots$ and let $G_n^{(1)} = (g_1, g_2, \ldots, g_n)$ the subgroup generated by $g_1, \ldots, g_n$. There is an injective homomorphism $j_n : G_n^{(1)} \to H_1$ such that $p \circ j_n = \mathrm{id}_{G_n^{(1)}}$. Note that $j_n(g_k) - j_i(g_k) \in \mathrm{K}_0(B)$ for any $n, i \geqslant k$. Suppose that $j_n(g_k) - j_k(g_k)$ is represented by a difference of two projections in $M_{l(k,n)}(B)$. Denote by $L(n) = \max\{l(k,n) : k \leqslant n\}$. Let $B^L = \prod M_{L(n)}(B)$. Since $B \in \mathcal{C}_{(l,b,M)}$, it is easy to check (using (a)) that

$$\mathrm{K}_0(B^L) = \{\{x_n\} : x_n = [p_n] - [q_n], \text{ where } p_n, q_n \in M_{\mathrm{KL}(n)}(B) \text{ for some integer } K > 0\}.$$

From the definition of $L(n)$, we conclude that, for any $g \in G_k^{(1)}$,

$$(0, \ldots, 0, j_{k+1}(g) - j_k(g), j_{k+2}(g) - j_k(g), \ldots, j_n(g) - j_k(g), \ldots) \in \mathrm{K}_0(B^L).$$

Let $\alpha_\infty = \{\alpha\}, \beta_\infty = \{\beta\} : A \to \prod_n M_{L(n)}(B) = B^L$ with $\{\alpha\} = (\alpha, \alpha, \ldots)$. Note α_∞ and β_∞ give essential extensions of A by $\mathrm{S}(B^L)$ by the "mapping tori" as before. Let $\tilde{\sigma} : A \to M(\mathrm{S}B \otimes \mathcal{K})$ be a completely positive map such that $\pi \circ \tilde{\sigma} = \sigma$, where $\pi : M(\mathrm{S}B \otimes \mathcal{K}) \to M(\mathrm{S}B \otimes \mathcal{K})/\mathrm{S}B \otimes \mathcal{K}$. It is easy to check that $\{\tilde{\sigma}\}$ maps A

to $M(\mathrm{S}(B^L)\otimes\mathcal{K})$ (note that $\prod M((\mathrm{S}M_{L(n)}(B))\otimes\mathcal{K})\neq M(\mathrm{S}(B^L)\otimes\mathcal{K})$ in general, but the image of $\{\tilde\sigma\}$ is in $M(\mathrm{S}(B^L)\otimes\mathcal{K})$). This shows that $\sigma_\infty : A\to M(\mathrm{S}B^L\otimes\mathcal{K})/\mathrm{S}B^L\otimes\mathcal{K}$. It is then clear that $[\sigma_\infty]=-[\beta_\infty]$ in $\mathrm{KK}(A,B^L)$. Furthermore, we have $\gamma_\infty=\{\gamma\} : A\to M(\mathrm{S}B^L\otimes\mathcal{K})/\mathrm{S}B^L\otimes\mathcal{K}$ and $[\gamma_\infty]=[\alpha_\infty]-[\beta_\infty]$ in $\mathrm{KK}(A,B^L)$. In fact, $[\gamma_\infty]$ give the following pure extension

$$0\longrightarrow \mathrm{K}_{i+1}(B^L)\longrightarrow F_i\xrightarrow{p}\mathrm{K}_i(A)\longrightarrow 0,$$

where $F_i=\{\{g\}+\{g_n\} : g\in H_i, \{g_n\}\in \mathrm{K}_{i+1}(B^L)\}$. Let $\kappa_n : \mathrm{K}_1(A)\to H_1$ be a map (not necessary additive) such that $(\kappa_n)|G_n^{(1)}=j_n$. By the construction above, $\{\kappa_n\}$ maps $\mathrm{K}_1(A)$ into F_1. Let $\Pi : B^L\to B^L/\bigoplus_n M_{L(n)}(B)$ be the quotient map. Then $[\Pi\circ\alpha_\infty]-[\Pi\circ\beta_\infty]=[\Pi\circ\gamma_\infty]\in\bigoplus_{i=0,1}\mathrm{Pext}(\mathrm{K}_i(A),\mathrm{K}_{i+1}(B^L/\bigoplus_n M_{L(n)}(B))$. It is easy to check that

$$\mathrm{K}_i\Big(B^L/\bigoplus_n M_{L(n)}(B)\Big)=\mathrm{K}_i(B^L)/\bigoplus\mathrm{K}_i(B),\quad i=0,1.$$

Note that $(\Pi)_*\circ(\{\kappa_n\})$ is an injective homomorphism from $\mathrm{K}_1(A)$ into $F_1/\oplus\mathrm{K}_0(B)$. This implies that $[\Pi\circ\gamma_\infty]$ gives a trivial element in $\mathrm{Pext}\big(\mathrm{K}_1(A), \mathrm{K}_0\big(B^L/\bigoplus_n M_{L(n)}(B)\big)\big)$. On the other hand, since $B\in\mathcal{C}_{(l,b,M)}$, every unitary in $U_0(M_n(B))$ is connected to the identity by a continuous path of unitaries with length no more than b. From this, one checks easily (also using (b) in 4.6) that

$$\mathrm{K}_1(B^L)=\prod\mathrm{K}_1(B).$$

Thus, a similar but easier argument shows that $[\Pi\circ\gamma_\infty]$ gives also a trivial element in $\mathrm{Pext}\big(\mathrm{K}_0(A),\mathrm{K}_1\big(B^L/\bigoplus_n M_{L(n)}(B)\big)\big)$ (we do not need to worry about the size of $j_n(g)-g_k(g)$ as in the K_0 case, since $\mathrm{K}_1(B^L)=\prod\mathrm{K}_1(B)$). This implies that

$$[\Pi\circ\alpha_\infty]=[\Pi\circ\beta_\infty]$$

in $\mathrm{KK}\big(A,B^L/\bigoplus_n M_{L(n)}(B)\big)$.

Note also that if $E : A\to B$ is a unital embedding, then $E_n : A\to M_{L(n)}(B)$ defined by sending a to $\mathrm{diag}(E(a),\dots,E(a))$ gives a unital embedding $\tilde E : A\to B^L$ and a unital embedding $\overline{E}=\Pi\circ\tilde E : A\to B^L/\bigoplus M_{L(n)}(B)$. From 4.3, for any $\varepsilon>0$ and any finite subset $\mathcal{F}\subset A$, we obtain a unitary $U\in B^L/\bigoplus M_{L(n)}(B)$ and an integer $k>0$ such that

$$\|U^*\mathrm{diag}(\Pi\circ\alpha_\infty(a),\overline{E}(a),\dots,\overline{E}(a))U-\mathrm{diag}(\Pi\circ\beta_\infty(a),\overline{E}(a),\dots,\overline{E}(a))\|<\frac{\varepsilon}{2}$$

for all $a\in\mathcal{F}$ (where $\overline{E}(a)$ repeats k times). It is easy to see (see 1.3 in [46]) that there are unitaries $U_n\in M_K(M_{L(n)}(B))$ such that $\Pi(\{U_n\})=U$. We have, for sufficiently large n,

$$\|U_n^*\mathrm{diag}(\alpha(a),E(a),\dots,E(a))U_n-\mathrm{diag}(\beta(a),E(a),\dots,E(a))\|<\varepsilon$$

for all $a\in\mathcal{F}$ (where $E(a)$ repeats $K(L(n))$ times). This implies that α is stably approximately unitarily equivalent to β. ∎

In the proof of 4.7, in fact $\mathrm{Pext}\big(\,\cdot\,,\prod_n \mathrm{K}_1(B)/\oplus \mathrm{K}_1(B)\big)=0$ (see [34]).

The main point of the following theorem is that the integer n does not depend on φ, ψ or B.

4.8. Theorem. *Let A be a unital simple C^*-algebra in $\mathcal{B}$. Let $l\geqslant 1$, $b\geqslant \pi$ and $M\geqslant 1$. For any $\varepsilon>0$ and any finite subset $\mathcal{F}\subset A$ there exists an integer $n>0$ satisfying the following: for any unital C^*-algebra $B\in\mathcal{C}_{(l,b,M)}$ if φ, ψ, $\sigma : A\to B$ are homomorphisms with $[\varphi]=[\psi]$ in $\mathrm{KL}(A,B)$, $i=0,1$, where σ is unital, then there is a unitary $u\in M_{n+1}(B)$ such that*

$$\|u^*\mathrm{diag}(\varphi(a),\sigma(a),\dots,\sigma(a))u-\mathrm{diag}(\psi(a),\sigma(a),\dots,\sigma(a))\|<\varepsilon$$

for all $a\in\mathcal{F}$, where $\sigma(a)$ repeats n times.

Proof. Suppose that the theorem is false. Then there are $\varepsilon_0>0$ and a finite subset $\mathcal{F}\subset A$ satisfying: for any n, there are unital C^*-algebras B_n and monomorphisms φ_n, ψ_n, $\sigma_n : A\to B_n$ with $[\varphi_n]=[\psi_n]$ in $\mathrm{KL}(A,B_n)$ such that

$$\inf\{\sup_{\mathcal{F}}\{\|u^*\mathrm{diag}(\varphi(a),\sigma_n(a),\dots,\sigma_n(a))u-\mathrm{diag}(\psi_n(a),\sigma_n(a),\dots,\sigma_n(a))\|\}\geqslant\varepsilon_0$$

where the infimum is taken over all unitaries $u\in M_n(B_n)$, and where $\sigma_n(a)$ repeats n times.

Set $C_0=\bigoplus_{n=1}^{\infty}B_n$ and $C=\prod_{n=1}^{\infty}B_n$. Define Φ, Ψ, $\Sigma : A\to C$ by $\Phi(a)=\{\varphi_n(a)\}$, $\Psi(a)=\{\psi_n(a)\}$ and $\Sigma(a)=\{\sigma_n(a)\}$ for $a\in A$. Since $B_n\in\mathcal{C}_{(l,b,M)}$, one computes (see 2.7 in [32] and also 2.9 in [31]),

$$\mathrm{K}_0\Big(\prod B_n\Big)=\prod_b \mathrm{K}_0(B_n),\quad \text{(see also 4.6)};\qquad \mathrm{K}_1\Big(\prod B_n\Big)=\prod \mathrm{K}_1(B_n),$$

$$\mathrm{K}_i\Big(\prod B_n,\mathbb{Z}/k\mathbb{Z}\Big)\subset\prod \mathrm{K}_i(B_n,\mathbb{Z}/k\mathbb{Z}),\quad k>0,\ i=0,1.$$

From [15], two homomorphisms $h_1,h_2 : A\to B$ induce the same element in $\mathrm{KL}(A,B)$ if and only if they induce the same homomorphisms from $\mathrm{K}_i(A,\mathbb{Z}/k\mathbb{Z})$ into $\mathrm{K}_i(B,\mathbb{Z}/k\mathbb{Z})$, $i=0,1$, $k=0,1,\dots$.

From this we conculde, since $[\varphi_n]=[\psi_n]$ in $\mathrm{KL}(A,B_n)$, that $[\Phi]=[\Psi]$ in $\mathrm{KL}(A,\prod B_n)$. It follows that $[\Phi]=[\Psi]$ in $\mathrm{KL}(A,C)$. Let $\pi : C\to C/C_0$ be the quotient map and set $\overline{\Phi}=\pi\circ\Phi$, $\overline{\Psi}=\pi\circ\Psi$ and $\overline{\Sigma}=\pi\circ\Sigma$. So $[\overline{\Phi}]=[\overline{\Psi}]$ in $\mathrm{KL}(A,C/C_0)$.

From 4.6, $C/C_0\in\mathcal{B}_{(l,b,M)}$. Thus, by 4.7, there exists an integer $N>0$ and a unitary $u\in M_{N+1}(C/C_0)$ such that

$$\|u^*\mathrm{diag}(\overline{\Phi}(a),\overline{\Sigma}(a),\dots,\overline{\Sigma}(a))u-\mathrm{diag}(\overline{\Psi}(a),\overline{\Sigma}(a),\dots,\overline{\Sigma}(a))\|<\frac{\varepsilon_0}{3}$$

for all $a\in\mathcal{F}$, where $\overline{\Sigma}(a)$ repeats N times. It is easy to see (see 1.3 in [46]) that there is a unitary $U\in C$ such that $\pi(U)=u$. Therefore, for each $a\in\mathcal{F}$, there exists $c_a\in M_{N+1}(C_0)$ such that

$$\|U^*\mathrm{diag}(\Phi(a),\Sigma(a),\dots,\Sigma(a))U-\mathrm{diag}(\Psi(a),\Sigma(a),\dots,\Sigma(a))+c_a\|<\frac{\varepsilon_0}{3}$$

where $\Sigma(a)$ repeats N times. Write $U = \{u_n\}$, where $u_n \in M_{N+1}(B_n)$ are unitaries. Since $c_a \in M_{N+1}(C_0)$ and $\mathcal{F}$ is finite, there is $N_0 > 0$ such that

$$\|u_n^* \mathrm{diag}(\varphi_n(a), \sigma_n(a), \ldots, \sigma_n(a))u_n - \mathrm{diag}(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a))\| < \frac{\varepsilon_0}{2}$$

for all $a \in \mathcal{F}$, where σ_n repeats N times, provided that $n \geqslant N_0$. So, if $n \geqslant \max(N, N_0)$, we have

$$\|v_n^* \mathrm{diag}(\varphi_n(a), \sigma_n(a), \ldots, \sigma_n(a))v_n - \mathrm{diag}(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a))\| < \frac{\varepsilon_0}{2} < \varepsilon_0$$

for all $a \in \mathcal{F}$, where $\sigma_n(a)$ repeats n times and $v = u_n \oplus 1_{M_{n-N_0}}$ is a unitary in $M_n(B_n)$, a contradiction. ∎

4.9. COROLLARY. *Let A be a unital simple C^*-algebra in $\mathcal{B}$ which satisfies the "*K*-theoretically locally finite" condition in* 4.5, *or $A \in \mathcal{C}_{(l,b,M)}$ for some $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$, and let $\alpha : A \to A$ be an endomorphism. Then α is stably approximately unitarily equivalent to id_A if and only if $[\alpha] = [\mathrm{id}]$ in* $\mathrm{KL}(A, A)$.

In the special case where C^*-algebras are purely infinite simple, we have the following.

4.10. THEOREM. *Let A and B be unital separable nuclear purely infinite simple C^*-algebras with the* UCT *and $\varphi, \psi : A \to B$ be two homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $[\varphi] = [\psi]$ in* $\mathrm{KL}(A, B)$.

Proof. By repeated application of 2.4 in [62], we may write, for any integer $n > 0$,

$$\varphi = \mathrm{diag}(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{n+1}) \quad \text{and} \quad \psi = \mathrm{diag}(\psi_1, \psi_2, \psi_3, \ldots, \psi_{n+1}),$$

where φ_i and ψ_i are homomorphisms from A to B, and φ_i and ψ_i have "trivializing factorization" (see 2.1 in [62]) for $i = 2, 3, \ldots, n+1$. It follows from 2.2 in [62] that all φ_i and ψ_i with $i \geqslant 2$ are approximately unitarily equivalent. So without loss of generality we may write

$$\varphi = \mathrm{diag}(\varphi_1, \varphi_2, \varphi_2, \ldots, \varphi_2) \quad \text{and} \quad \psi = \mathrm{diag}(\psi_1, \varphi_2, \varphi_2, \ldots, \varphi_2),$$

where φ_2 repeats n times. Note also, since φ_2 has trivializing factorization, $[\varphi_2] = 0$ in $\mathrm{KL}(A, B)$. This implies that $[\varphi_1] = [\psi_1]$ in $\mathrm{KL}(A, B)$. Then we see that 4.8 applies. ∎

4.11. REMARK. A similar version of 4.10 is contained in Section 4 of [63]. It is the key "uniqueness theorem" for the classification of nuclear purely infinite simple C^*-algebras. For further discussion, see 5.5 and 5.6.

5. APPROXIMATELY MULTIPLICATIVE MORPHISMS

Recent developments in C^*-algebra theory require the study of maps that are only approximately multiplicative. In this section, we show that a version of 4.7 holds for maps which are approximately multiplicative.

5.1. Definition. Let A and B be C^*-algebras, let $L : A \to B$ be a contractive completely positive linear map, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. L is said to be *$\mathcal{F}$-ε-multiplicative*, if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$.

5.2. Definition. Let A be a unital C^*-algebra and C_n be as in 2.1. One can choose C_n to be the mapping cone of a degree n map of $C_0(\mathbb{R})$. Let $\mathcal{P}(A)$ be the set of all projections in $\bigcup\limits_{m,n=1} M_m(A \otimes \widetilde{C}_n \otimes C(\mathbb{S}^1))$. For a finite subset $\mathcal{P} \subset \mathcal{P}$, as in 1.4 in [46], there are a finite subset $\mathcal{G}(\mathcal{P}) \subset A$ and $\delta(\mathcal{P}) > 0$ such that if B is any unital C^*-algebra and $L : A \to B$ is a contractive completely positive linear map which is $\mathcal{G}(\mathcal{P})$-$\delta(\mathcal{P})$-multiplicative, then L defines a map $[L]$ from $\mathcal{P}$ into $\underline{\mathrm{K}}(B)$. Note if $p, q \in \mathcal{P}$, $p, q \in M_m(A \otimes \widetilde{C}_n \otimes C(\mathbb{S}^1))$ and $[p] = [q]$, there is a partial isometry $v \in M_{m+k}(A \otimes \widetilde{C}_n \otimes C(\mathbb{S}^1))$ such that

$$v^*v = \mathrm{diag}(p, 1_A \otimes 1_k) \quad \text{and} \quad vv^* = \mathrm{diag}(q, 1_A \otimes 1_k).$$

We require that $\mathcal{G}(\mathcal{P})$ is so large that $v \in M_{m+k}(\mathcal{G}(\mathcal{P}))$ and

$$\|L\otimes\mathrm{id}(v^*v) - (L\otimes\mathrm{id}(v)^*)(L\otimes\mathrm{id}(v))\| \quad \text{and} \quad \|L\otimes\mathrm{id}(vv^*) - (L\otimes\mathrm{id}(v))(L\otimes\mathrm{id}(v)^*)\|$$

are sufficiently small so that $[L]$ is well defined on $\overline{\mathcal{P}}$, where $\overline{\mathcal{P}}$ is the image of $\mathcal{P}$ in $\underline{\mathrm{K}}(A)$. It is worth noting that, if $\mathrm{K}_i(A)$ is torsion free ($i = 0, 1$), then one only needs to consider the projections in $\bigcup\limits_{m=1} M_m(A \otimes C(\mathbb{S}^1))$.

5.3. Theorem. *Let A be a unital simple C^*-algebra in $\mathcal{B}$, $l \geqslant 1$, $b \geqslant \pi$ and $M \geqslant 1$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a positive number $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \mathcal{P}(A)$ and an integer $n > 0$ satisfying the following: for any unital C^*-algebra $B \in \mathcal{C}_{(l,b,M)}$ if φ, ψ, $\sigma : A \to B$ are three $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with*

$$[\varphi]|_{\overline{\mathcal{P}}} = [\psi]|_{\overline{\mathcal{P}}}$$

and σ is unital, then there is a unitary $u \in M_{n+1}(B)$ such that

$$\|u^*\mathrm{diag}(\varphi(a), \sigma(a), \ldots, \sigma(a))u - \mathrm{diag}(\psi(a), \sigma(a), \ldots, \sigma(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $\sigma(a)$ repeats n times.

Proof. Suppose that the theorem is false. Then there are $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subset A$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{\mathcal{G}_n\}$ which is dense in the unit ball of A, a sequence of finite subsets $\{\mathcal{P}_n\}$ of $\mathcal{P}(A)$ with $\bigcup\limits_{n=1}^{\infty} \mathcal{P}_n = \mathcal{P}(A)$, a sequence of $\{k(n)\}$

of integers and sequences $\{\varphi_n\}$, $\{\psi_n\}$ and $\{\sigma_n\}$ of $\mathcal{G}_n$-δ_n-multiplicative positive linear maps from A to B_n with $[\varphi_n]|\overline{\mathcal{P}}_n = [\psi]|\overline{\mathcal{P}}_n$ satisfying the following:

$$\inf\{\sup\{\|u^*\mathrm{diag}(\varphi_n(a), \sigma_n(a), \ldots, \sigma_n(a))u - \mathrm{diag}(\psi_n(a),\sigma_n(a), \ldots, \sigma_n(a))\| \\ : a \in \mathcal{F}\} \geqslant \varepsilon_0$$

where $\sigma_n(a)$ repeats $k(n)$ times and the infimum is taken over all unitaries in $M_{k(n)+1}(B_n)$.

Set $C_0 = \bigoplus_{n=1}^{\infty} B_n$ and $C = \prod_{n=1}^{\infty} B_n$. Define Φ, Ψ, $\Sigma : A \to C$ by $\Phi(a) = \{\varphi_n(a)\}$, $\Psi(a) = \{\psi_n(a)\}$ and $\Sigma(a) = \{\sigma_n(a)\}$ for $a \in A$. Let $\pi : C \to C/C_0$ be the quotient map and set $\overline{\Phi} = \pi \circ \Phi$, $\overline{\Psi} = \pi \circ \Psi$ and $\overline{\Sigma} = \pi \circ \Sigma$. Note that $\overline{\Phi}, \overline{\Psi}$ and $\overline{\Sigma}$ are monomorphisms. Given an element $p \in \mathcal{P}_k$, for some k, we claim that

$$[\overline{\Phi}(p)] = [\overline{\Psi}(p)].$$

As in 4.8 (see 2.7 in [32] and also 2.9 [31]), we have

$$\mathrm{K}_0\Big(\prod B_n\Big) = \prod_b \mathrm{K}_0(B_n), \quad \mathrm{K}_1\Big(\prod B_n\Big) = \prod \mathrm{K}_1(B_n), \tag{5.1}$$

$$\mathrm{K}_i\Big(\prod B_n, \mathbb{Z}/k\mathbb{Z}\Big) \subset \prod \mathrm{K}_i(B_n, \mathbb{Z}/k\mathbb{Z}), \quad k > 0,\ i = 0, 1. \tag{5.2}$$

We also have

$$\mathrm{K}_0(C/C_0) = \Big(\prod_b \mathrm{K}_0(B_n)\Big)/\bigoplus \mathrm{K}_0(B_n),$$
$$\mathrm{K}_1(C/C_0) = \Big(\prod \mathrm{K}_1(B_n)\Big)/\bigoplus \mathrm{K}_1(B_n)$$

and

$$\mathrm{K}_i(C/C_0, \mathbb{Z}/k\mathbb{Z}) \subset \Big(\prod \mathrm{K}_i(B_n, \mathbb{Z}/k\mathbb{Z})\Big)/\bigoplus \mathrm{K}_i(B_n, \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1.$$

It is rather easy to see that

$$[\overline{\Phi}(p)] = [\overline{\Psi}(p)]$$

if p is a projection in $M_m(A)$ (see below). If p is represented by an element in $\mathrm{K}_1(A)$ or in $\bigoplus \mathrm{K}_i(A, \mathbb{Z}/k\mathbb{Z})$ with $k > 0$, let $\overline{z} = [\overline{\Phi}(p)]$ and $\overline{y} = [\overline{\Psi}(p)]$. Then, there are $\{z_n\}$ and $\{y_n\}$ in $\prod \mathrm{K}_1(B_n)$ or in $\prod \mathrm{K}_i(B_n, \mathbb{Z}/k\mathbb{Z})$ such that $\pi_{*i}(\{z_n\}) = \overline{z}$ and $\pi_{*i}(\{y_n\}) = \overline{y}$, where π_{*i} is the corresponding quotient map. Since

$$[\varphi_n]|\overline{\mathcal{P}}_n = [\psi_n]|\overline{\mathcal{P}}_n,$$

we have

$$[\varphi_n]([p]) = [\psi_n]([p]) = x_n$$

for all sufficiently large n.

From (5.1) and (5.2) above, we have

$$\pi_{*i}(\{y_n\}) = \pi_{*i}(\{x_n\}) = \pi_{*i}(\{z_n\}).$$

Thus, $[\overline{\Phi}(p)] = [\overline{\Psi}(p)]$ in $\mathrm{K}_1(C/C_0)$ or in $\mathrm{K}_i(C/C_0, \mathbb{Z}/k\mathbb{Z})$. It follows that

$$[\overline{\Phi}] = [\overline{\Psi}] \ \text{ in } \ \mathrm{KL}(A, C/C_0)\,(= \mathrm{Hom}_\Lambda(\mathrm{K}_*(A), \mathrm{K}_*(C/C_0))).$$

From 4.6, $C/C_0 \in \mathcal{C}_{(l,b,M)}$. Applying 4.8, we obtain an integer N and a unitary $u \in M_{N+1}(C/C_0)$ such that

$$\|u^*\mathrm{diag}(\overline{\Phi}(a), \overline{\Sigma}(a), \ldots, \overline{\Sigma}(a))u - \mathrm{diag}(\overline{\Psi}(a), \overline{\Sigma}(a), \ldots, \overline{\Sigma}(a))\| < \frac{\varepsilon_0}{3}$$

for all $a \in \mathcal{F}$, where $\overline{\Sigma}(a)$ repeats N times. As in the proof of 4.8, there is a unitary $U \in M_{N+1}(C)$ such that $\pi(U) = u$ and for each $a \in \mathcal{F}$ there exists $c_a \in M_{N+1}(C_0)$ such that

$$\|U^*\mathrm{diag}(\Phi(a), \Sigma(a), \ldots, \Sigma(a))U - \mathrm{diag}(\Psi(a), \Sigma(a), \ldots, \Sigma(a)) + c_a\| < \frac{\varepsilon_0}{3}$$

where $\Sigma(a)$ repeats N times. Write $U = \{u_n\}$, where $u_n \in M_{N+1}(B_n)$ are unitaries. Since $c_a \in M_{N+1}(C_0)$ and $\mathcal{F}$ is finite, there is $N_0 > 0$ such that

$$\|u_n^*\mathrm{diag}(\varphi_n(a), \sigma_n(a), \ldots, \sigma_n(a))u_n - \mathrm{diag}(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a))\| < \frac{\varepsilon_0}{2}$$

for all $a \in \mathcal{F}$, where σ_n repeats N times. The proof concludes as in the proof of 4.8. ∎

5.4. Remark. If in 5.3 $\mathrm{K}_i(A)$ is torsion free, $i = 0, 1$, then, clearly from the proof, we can replace $\mathcal{P}(A)$ by the set of projections in $\bigcup\limits_{m=1} M_m(A \otimes C(\mathbb{S}^1))$, and if further $\mathrm{K}_1(A) = 0$, we can replace $\mathcal{P}(A)$ by the set of projections in $\bigcup\limits_{m=1} M_m(A)$ and the condition that $B \in \mathcal{C}(l, b, M)$ can be replaced by (a) in 4.6 only.

5.5. Theorem. *Let A be a unital separable nuclear purely infinite simple C^*-algebra with the* UCT. *For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a positive number $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset \mathcal{P}(A)$ satisfying the following: for any unital purely infinite simple C^*-algebra B if $\varphi, \psi : A \to B$ are $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with*

$$[\varphi]|_{\overline{\mathcal{P}}} = [\psi]|_{\overline{\mathcal{P}}}$$

then there is a unitary $u \in B$ such that

$$\|u^*\varphi(a)u - \psi(a)\| < \varepsilon.$$

Proof. The proof is similar to that of 4.10. We sketch it as follows. For any $\eta > 0$, any finite subset $\mathcal{G}_1 \subset A$ and any integer $n > 0$, by 2.4 in [62], id_A and $\mathrm{diag}(h_0, h_1, h_1, \ldots, h_1)$ are approximately the same within η on $\mathcal{G}_1$, where $h_0 : A \to pAp$ is a monomorphism, p is a projection in A, $h_1 : A \to C$ is also a monomorphism, C is a C^*-subalgebra of A which is isomorphic to O_2 and h_1 repeats n times. One should note that a $\mathcal{G}_1$-η-multiplicative contractive completely positive linear map from O_2 to another unital C^*-algebra is close to a monomorphism, if $\mathcal{G}_1$ is sufficiently large and η is sufficiently small. From [65], we also know that two homomorphisms from O_2 are approximately unitarily equivalent. So, as in 4.10, without loss of generality, we may assume that

$$\varphi = \mathrm{diag}(\varphi_1, h_2, h_2, \ldots, h_2) \quad \text{and} \quad \psi = \mathrm{diag}(\psi_1, h_2, h_2, \ldots, h_2),$$

where φ_1, $\psi_1 : A \to qBq$ are two $\mathcal{G}$-δ-multiplicative completely positive linear maps and $h_2 : eBe$ are unital homomorphisms with $[h_2] = 0$ in $\mathrm{KL}(A, B)$. Therefore 5.3 applies. ∎

5.6. REMARK. The condition that B is purely infinite simple can be eased a bit. It suffices to assume that $B \in \mathcal{C}_{(l,b,M)}$ with the additional condition that $U(B)/U_0(B) = \mathrm{K}_1(B)$. The last condition is used in [65] to show that two homomorphisms from O_2 are approximately unitarily equivalent. Theorem 5.5 could serve as the key "uniqueness theorem" for classifying nuclear purely infinite simple C^*-algebras. In [63] (see also [38]), a different uniqueness theorem is used. Theorem 5.5 together with an "existence theorem" (i.e. given an element $\alpha \in \mathrm{KL}(A, B)$ and a finite subset $\mathcal{P} \subset \mathcal{P}(A)$, there is an almost multiplicative morphism $L : A \to B$ such that $[L]|\overline{\mathcal{P}} = \alpha|\overline{\mathcal{P}}$) will give the classification theorem of Kirchberg and Phillips: two (unital) separable nuclear purely infinite simple C^*-algebras with the UCT and with the same K-theory are isomorphic. A stronger version of this "existence theorem" is included in Section 3 of [63].

Evidently, our main results also apply to the case in which the simple C^*-algebras are stable rank one. Applications of the results in Sections 4 and 5 to simple C^*-algebras of stable rank one will appear in subsequent papers. In the first subsequent paper we give a characterization of rational simple AF-algebras (see [53]).

First note added in proof. George A. Elliott pointed to the author that in fact the assumption that B is simple in 1.5 can be replaced, for example, by the assumption that the embedding from A to B is full. It was also brought to the author's attention by Marius Dădârlat and Soren Eilers that the above fact can be used to replace the assumption that A or B is simple in Section 4 by the assumption that the embedding (from A to B) is full.

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REFERENCES

1. C. AKEMANN, G.K. PEDERSEN, Complications of semicontinuity in C^*-algebra theory, *Duke Math. J.* **40**(1973), 785–795.
2. C. AKEMANN, G.K. PEDERSEN, J. TOMIYAMA, Multipliers of C^*-algebras, *J. Funct. Anal.* **13**(1973), 277–301.
3. W. ARVESON, Notes on extensions of C^*-algebras, *Duke Math. J.* **44**(1977), 329–355.
4. B. BLACKADAR, K-*Theory for Operator Algebras*, Springer-Verlag, New York 1986.
5. B. BLACKADAR, E. KIRCHBERG, Generalized inductive limits of finite-dimensional C^*-algebras, *Math. Ann.* **307**(1997), 343–380.
6. L.G. BROWN, Stable isomorphisms of hereditary subalgebras of C^*-algebras, *Pacific J. Math.* **71**(1997), 335–348.
7. L.G. BROWN, M. DĂDÂRLAT, Extensions of C^*-algebras and quasidiagonality, *J. London Math. Soc.* **53**(1996), 582–600.
8. L.G. BROWN, R. DOUGLAS, P. FILLMORE, Unitary equivalence modulo the compact operators and extensions of C^*-algebras, in *Proceedings of the Conference on*

Operator Theory, Lecture Notes in Math., vol. 345, Springer, New-York 1973, pp. 56–128.
9. L.G. Brown, R. Douglas, P. Fillmore, Extensions of C^*-algebras and K-homology, *Ann. of Math.* **105**(1977), 265–324.
10. M.-D. Choi, E. Effros, The completely positive lifting problem for C^*-algebras, *Ann. of Math.* **104**(1976), 585–609.
11. A. Connes, An analogue of the Thom isomorphism for cross products of a C^*-algebra by an action of $\mathbb{R}$, *Adv. Math.* **39**(1981), 31–55.
12. M. Dădârlat, Approximately unitarily equivalent morphisms and inductive limit C^*-algebras, K-*theory* **9**(1995), 117–137.
13. M. Dădârlat, Reduction to dimension three of local spectra of real rank zero C^*-algebras, *J. Reine Angew. Math.* **460**(1995), 189–212.
14. M. Dădârlat, G. Gong, A classification result for approximately homogeneous C^*-algebras of real rank zero, *Geom. Funct. Anal.* **7**(1997), 646–711
15. M. Dădârlat, T. Loring, A universal multi-coefficient theorem for the Kasparov groups, *Duke Math. J.*, **84**(1996), 355–377.
16. G.A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra* **38**(1976), 29–44.
17. G.A. Elliott, A classification of certain simple C^*-algebras, in *Quantum and Non-Commutative Analysis*, Kluwer Acad. Press, Dordrecht 1993, pp. 373–385
18. G.A. Elliott, On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443**(1993), 179–219.
19. G.A. Elliott, The classification problem for amenable C^*-algebras, in *Proc. ICM* 1994, Zurich, Birkhauser verlag, Basel, pp. 922–932.
20. G.A. Elliott, D.E. Evans, The structure of irrational rotation C^*-algebras, *Ann. of Math.* **138**(1993), 477–501.
21. G.A. Elliott, G. Gong, On inductive limits of matrix algebras over two-tori, *Amer. J. Math.* **118**(1996), 263–290.
22. G.A. Elliott, G. Gong, On the classification of C^*-algebras of real rank zero. II, *Ann. of Math.* **144**(1996), 497–610.
23. G.A. Elliott, G. Gong, H. Lin, C. Pasnicu, Abelian C^*-subalgebras of C^*-algebras of real rank zero and inductive limit C^*-algebras, *Duke Math. J.* **85**(1996), 511–554.
24. G.A. Elliott, M. Rørdam, Classification of certain infinite simple C^*-algebras. II, *Comment. Math. Helvetici* **70**(1995), 615–638.
25. P. Friis, M. Rørdam, Almost commuting self-adjoint matrices—a short proof of Huaxin Lin's theorem, *J. Reine Angew. Math.* **47**(1996), 121–131.
26. G. Gong, Approximation by dimension drop algebras and classification, *C.R. Math. Rep. Acad. Sci. Canada* **16**(1994), 40–44.
27. G. Gong, On inductive limits of matrix algebras over higher dimensional spaces. I, *Math. Scand.* **80**(1997), 41–55.
28. G. Gong, On inductive limits of matrix algebras over higher dimensional spaces. II, *Math. Scand.* **80**(1997), 56-100.
29. G. Gong, Classification of C^*-algebras of real rank zero and unsuspended E-equivalence types, *J. Funct. Anal.* **152**(1998), 281–329.
30. G. Gong, On the classification of simple inductive limit C^*-algebras. I: The Reduction Theorem, preprint.
31. G. Gong, H. Lin, Almost multiplicative morphisms and almost commuting matrices, *J. Operator Theory* **40**(1998), 217–275.
32. G. Gong, H. Lin, Asymptotic morphisms and K-theory, *Internat. J. Math.* **11**(2000), 983–1000.
33. G. Cong, H. Lin, Classification of homomorphisms from $C(x)$ to simple C^*-algebras of real rank zero, *Acta Math. Sinica* (Engl. Ser.) **16**(2000), 181–206.

34. A. HULANICKI, The structure of the factor group of the unrestricted sum by the restrict sum of Abelian groups, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **10**(1962), 77–80.
35. R. KADISON, J. RINGROSE, Derivations and automorphism of operator algebras, *Comm. Math. Phys.* **4**(1967), 32–63.
36. G.G. KASPAROV, Hilbert C^*-modules: Theorem of Stinespring and Voiculescu, *J. Operator Theory* **4**(1980), 133–150.
37. E. KIRCHBERG, Lectures on the classification of simple purely infinite C^*-algebras, preprint.
38. E. KIRCHBERG, The classification of purely infinite C^*-algebras using Kasparov's theory, third draft.
39. E. KIRCHBERG, N.C. PHILLIPS, Embedding of exact C^*-algebras and continuous fields in the Cuntz algebra O_2, *J. Reine Angew. Math.* 525 (2000), 55–94.
40. H. LIN, Skeleton C^*-subalgebras, *Canad. J. Math.* **44**(1992), 324–341.
41. H. LIN, Approximation by normal elements with finite spectra in simple AF-algebras, *J. Operator Theory* **31**(1994), 83–98.
42. H. LIN, C^*-algebra Extensions of $C(X)$, *Mem. Amer. Math. Soc.* **115** no. 550 (1995).
43. H. LIN, Approximation by normal elements with finite spectra in C^*-algebras of real rank zero, *Pacific J. Math.* **173**(1996), 443–489.
44. H. LIN, Classification of C^*-algebras of real rank zero with zero K_1, *J. Operator Theory* **35**(1996), 147–178.
45. H. LIN, Almost commuting selfadjoint matrices and applications, *Fields Inst. Commun.*, vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 193–234.
46. H. LIN, Almost multiplicative morphisms and some applications, *J. Operator Theory* **37**(1997), 121–154.
47. H. LIN, Extensions of $C(X)$ by simple C^*-algebras of real rank zero, *Amer. J. Math.* **119**(1997), 1263–1289.
48. H. LIN, Homomorphisms from $C(X)$ into C^*-algebras, *Canad. J. Math.* **49**(1997), 963–1009.
49. H. LIN, Almost commuting unitaries and classification of purely infinite simple C^*-algebras, *J. Funct. Anal.* **155**(1998), 1–24.
50. H. LIN, Classification of simple C^*-algebras with a unique trace, *Amer. J. Math.* **120**(1998), 1289–1315.
51. H. LIN, Lifting automorphisms, *K-theory* **16**(1999), 105–127.
52. H. Lin, Homomorphisms from C^*-algebras of continuous traces, *Math. Scand.* **86** (2000), 249–272.
53. H. LIN, Tracially AF C^*-algebras, *Trans. amer. Math. Soc.* **353**(2001), 693–722
54. H. LIN, N.C. PHILLIPS, Almost multiplicative morphisms and Cuntz-algebra O_2, *Internat. J. Math.* **6**(1995), 625–643.
55. H. LIN, H. SU, Classification of direct limits of generalized Toeplitz algebras, *Pacific J. Math.* **181**(1997), 89–140.
56. J. MINGO, K-theory and multipliers of stable C^*-algebras, *Trans. Amer. Math. Soc.* **299**(1987), 225–230.
57. V.I. PAULSEN, *Completely Bounded Maps and Dilations*, Pitnam Res. Notes Math., vol. 146, Sci. Tech. Harlow, 1986.
58. G.K. PEDERSEN, Lifting groups of automorphisms, *Sympos. Math.* **20**(1976), 161–167.
59. G.K. PEDERSEN, *C^*-algebras and their Automorphism Groups*, Academic Press, London–New York–San Francisco, 1979.
60. G.K. PEDERSEN, SAW^*-algebras and corona C^*-algebras, contributions to non-commutative topology, *J. Operator Theory* **15**(1986), 15–32.
61. N.C. PHILLIPS, A survey of exponential rank, C^*-algebras: 1943–1993 — A fifty year celebration, *Contemp. Math.* **167**(1994), 353–399.

62. N.C. Phillips, Approximate unitarily equivalence of homomorphisms from odd Cuntz algebras, *Fields Inst. Commun.*, vol. 13, Amer. Math. Soc, Providence, RI, 1997, pp. , 243–255.
63. N.C. Phillips, A classification theorem for nuclear purely infinite simple C^*-algebras, *Doc. Math.* **5**(2000), 49–114.
64. M. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc.* **46**(1983), 301–333.
65. M. Rørdam, Classification of inductive limits of Cuntz algebras, *J. Reine Angew. Math.* **440**(1993), 175–200.
66. M. Rørdam, Classification of certain infinite simple C^*-algebras, *J. Funct. Anal.* **131**(1995), 415–458.
67. J. Rosenberg, C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, *Duke Math. J.* **55**(1987), 431–474.
68. N. Salinas, Relative quasidiagonality and KK-theory, *Houston J. Math.* **18**(1992), 97–116.
69. C.L. Schochet, The fine structure of the Kasparov groups. II: relative quasidiagonality, preprint, 1996.
70. K. Thomsen, On isomorphisms of inductive limits of C^*-algebras, *Proc. Amer. Math. Soc.* **113**(1991), 947–953.
71. D. Voiculescu, A non-commutative Weyl-von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.

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Second note added in proof. This article was first written in 1997 and reported in December 1997 at Victoria Canadian Winter meeting. Since then there have been several developments related to this paper. The main theorem had been applied to the classification of simple nuclear C^*-algebras which have tracial rank zero (or, equivalently, tracially AF C^*-algebras) (see, for example, in the appended list of references, [2a], [1a], [3a] and [4a]) It was observed (by M. Dădârlat and S. Eilers) that simplicity conditions in Theorem 4.3 can be replaced by the condition that A is embedded "fully" in B. They also show that A does not need to require to satisfying the UCT. A rather elementary proof of Theorem 4.3 can be found in [5a]. It is recently showed by the author that $[\alpha] = [\beta]$ in $\mathrm{KK}(A, B)$ can be replaced by $[\alpha] = [\beta]$ in $\mathrm{KL}(A, B)$ which is also a necessary condition.

APPENDED REFERENCES

1a. M. DĂDÂRLAT, S. EILERS, *On the classification of nuclear C^*-algebras*, preprint, 1998.

2a. H. LIN, Classification of simple tracially AF C^*-algebras, *Canad. J. Math.* **53**(2001), 161–194.

3a. H. LIN, Simple C^*-algebras with locally bounded irreducible representations, *J. Funct. Anal.* **187**(2001), 42-69.

4a. H. LIN, Classification of simple C^*-algebras with tracial topological rank zero, MSRI preprint, 2000.

5a. H. LIN, *An Introduction to the Classification of Amenable C^*-Algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.