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# COMMUTATORS OF OPERATORS ON BANACH SPACES

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ABSTRACT. We study the commutators of operators on a Banach space $\mathfrak{X}$ to gain insight into the non-commutative structure of the Banach algebra $\mathcal{B}(\mathfrak{X})$ of all (bounded, linear) operators on $\mathfrak{X}$. First we obtain a purely algebraic generalization of Halmos's theorem that each operator on an infinite-dimensional Hilbert space is the sum of two commutators. Our result applies in particular to the algebra $\mathcal{B}(\mathfrak{X})$ for $\mathfrak{X} = c_0$, $\mathfrak{X} = C([0,1])$, $\mathfrak{X} = \ell_p$, and $\mathfrak{X} = L_p([0,1])$, where $1 \leqslant p \leqslant \infty$. Then we show that each weakly compact operator on the $p^{\text{th}}$ James space $\mathfrak{J}_p$, where $1 < p < \infty$, is the sum of three commutators; a key step in the proof of this result is a characterization of the weakly compact operators on $\mathfrak{J}_p$ as the set of operators which factor through a certain reflexive, complemented subspace of $\mathfrak{J}_p$.

KEYWORDS: *Commutators, operators on Banach spaces, Banach algebras, traces.*

MSC (2000): Primary 47B47; Secondary 47L10.

## 1. INTRODUCTION

Throughout this note all vector spaces and algebras are assumed to be over the field $\mathbb{C}$ of complex numbers.

The *commutator* of a pair of elements $A$ and $B$ in an algebra A is given by

$$[A, B] := AB - BA \in \mathcal{A}.$$

We use the term *operator* to denote a bounded, linear map between Banach spaces. For Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, we denote by $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ the collection of all operators from $\mathfrak{X}$ to $\mathfrak{Y}$. We write $\mathcal{B}(\mathfrak{X})$ instead of $\mathcal{B}(\mathfrak{X}, \mathfrak{X})$; this is a unital Banach algebra with identity $I_{\mathfrak{X}}$ (the identity operator on $\mathfrak{X}$). Our aim is to gain insight into the non-commutative structure of $\mathcal{B}(\mathfrak{X})$ by studying the commutators of operators on $\mathfrak{X}$. However, some of our methods are purely algebraic, and so we shall work in that generality whenever possible.

The first important contribution to the study of commutators is due to A. Wintner ([19]) who in 1947 proved the following theorem.

THEOREM 1.1. (Wintner) *The identity element* $\mathbb{1}_{\mathcal{A}}$ *in a unital, normed algebra* $\mathcal{A}$ *is not a commutator, that is, there are no elements* $A$ *and* $B$ *in* $\mathcal{A}$ *such that* $\mathbb{1}_{\mathcal{A}} = [A, B]$. ∎

Like much good mathematics, Wintner's theorem has its roots in physics. Indeed, it was prompted by the fact that the linear maps $P$ and $Q$ representing the quantum-mechanical momentum and position, respectively, satisfy the commutation relation

$$[P, Q] = -\frac{\mathrm{i}h}{2\pi} I,$$

where $h$ is Planck's constant and $I$ is the identity operator on the underlying Hilbert space. Wintner's theorem implies that the linear maps $P$ and $Q$ cannot both be bounded (and this is in fact Wintner's original statement; however, as observed by P.R. Halmos, Wintner's proof immediately generalizes to yield Theorem 1.1, above).

As is also the case for much of the good mathematics that has grown out of physics, the study of commutators of operators on Hilbert space has subsequently developed a life of its own, completely independent of its quantum-mechanical origin. Halmos has shown that each operator on an infinite-dimensional Hilbert space is the sum of two commutators ([8]), and A. Brown and C. Pearcy have characterized the set of operators on an infinite-dimensional Hilbert space which are commutators ([1]). A nice exposition of these and some related results is given in Chapter 24 of Halmos's book ([9]).

The present note can be seen as a further step away from the quantum-mechanical origin of the study of commutators of operators. Indeed, we replace the Hilbert space by a Banach space $\mathfrak{X}$ and study the commutators of operators on $\mathfrak{X}$.

The note is organized as follows. In Section 2 we introduce traces and explain their relation to commutators. In Section 3 the concept of a Mityagin decomposition of a unital algebra is defined, and we show that each element in a unital algebra with a Mityagin decomposition is the sum of two commutators. Examples of algebras with a Mityagin decomposition include $\mathcal{B}(\mathfrak{X})$ for $\mathfrak{X} = c_0$, $\mathfrak{X} = C([0, 1])$, $\mathfrak{X} = \ell_p$, and $\mathfrak{X} = L_p([0, 1])$, where $1 \leqslant p \leqslant \infty$. In Section 4 we consider the commutators of operators on the $p^{\text{th}}$ James space $\mathfrak{J}_p$ for $1 < p < \infty$. It is known that the ideal $\mathcal{W}(\mathfrak{J}_p)$ of weakly compact operators on $\mathfrak{J}_p$ has codimension 1 in $\mathcal{B}(\mathfrak{J}_p)$. This implies that the identity operator on $\mathfrak{J}_p$ has distance at least 1 to any sum of commutators. Modifying techniques developed by G.A. Willis, we show that an operator on $\mathfrak{J}_p$ is weakly compact if and only if it factors through a certain reflexive, complemented subspace of $\mathfrak{J}_p$. This characterization enables us to deduce that each weakly compact operator on $\mathfrak{J}_p$ is the sum of three commutators. In particular it follows that each trace on $\mathcal{B}(\mathfrak{J}_p)$ is a scalar multiple of the character on $\mathcal{B}(\mathfrak{J}_p)$ induced by the quotient homomorphism of $\mathcal{B}(\mathfrak{J}_p)$ onto $\mathcal{B}(\mathfrak{J}_p)/\mathcal{W}(\mathfrak{J}_p)$.

## 2. TRACES AND COMMUTATORS

DEFINITION 2.1. Let $\mathcal{A}$ be an algebra. A linear map $\tau : \mathcal{A} \to \mathbb{C}$ satisfying
$$\tau(AB) = \tau(BA), \quad A, B \in \mathcal{A}$$
is called a *trace on* $\mathcal{A}$.

We write $\operatorname{com}\mathcal{A}$ for the linear subspace of $\mathcal{A}$ spanned by its commutators, that is,
$$\operatorname{com}\mathcal{A} := \left\{ \sum_{k=1}^{n} [A_k, B_k] : A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{A},\ n \in \mathbb{N} \right\}.$$
If $\operatorname{com}\mathcal{A} = \mathcal{A}$, then we say that $\mathcal{A}$ is *spanned by its commutators.*

Traces are closely related to commutators and the subspace they span as the following easy observations show.

PROPOSITION 2.2. *Let $\mathcal{A}$ be an algebra. A linear map $\tau : \mathcal{A} \to \mathbb{C}$ is a trace if and only if* $\operatorname{com}\mathcal{A} \subseteq \ker \tau$.

COROLLARY 2.3. *Let $\mathcal{A}$ be an algebra. Then*
$$\operatorname{com}\mathcal{A} = \bigcap \big\{ \ker \tau : \tau : \mathcal{A} \to \mathbb{C} \textit{ is a trace} \big\}.$$
*In particular $\mathcal{A}$ has no non-zero traces if and only if $\mathcal{A}$ is spanned by its commutators.*

Like commutators, traces are extensively studied in the Hilbert-space case, that is, for $C^*$-algebras and von Neumann algebras, but for other non-commutative (Banach) algebras, little is known about traces. We deduce results about traces in Corollary 3.3, Proposition 3.7, and Corollary 4.8, below.

## 3. COMMUTATORS IN ALGEBRAS WITH MITYAGIN DECOMPOSITIONS

DEFINITION 3.1. A *Mityagin decomposition* of a unital algebra $\mathcal{A}$ consists of a (necessarily infinite) set $\mathbb{I}$ with a selected point $k_0 \in \mathbb{I}$, a bijection $\rho : \mathbb{I} \to \mathbb{I} \setminus \{k_0\}$, and elements $L, R, A_k$, and $B_k$, $k \in \mathbb{I}$, of $\mathcal{A}$ satisfying:

(i) $B_k A_m = \delta_{k,m} \mathbb{1}_{\mathcal{A}}$ for all $k, m \in \mathbb{I}$, where $\delta_{k,m}$ is Kronecker's delta symbol;

(ii) $C = 0$ is the only element in $\mathcal{A}$ such that $B_k C = 0$ for each $k \in \mathbb{I}$;

(iii) for each $k \in \mathbb{I}$, $B_k L = B_{\rho(k)}$ and
$$B_k R = \begin{cases} 0 & \text{for } k = k_0, \\ B_{\rho^{-1}(k)} & \text{for } k \in \mathbb{I} \setminus \{k_0\}; \end{cases}$$

(iv) for each element $C$ in A, there is an element $\widetilde{C}$ in A such that $B_k \widetilde{C} = C B_k$ for each $k \in \mathbb{I}$.

The definition of a Mityagin decomposition is constructed to extract the key properties of the algebra $\mathcal{B}(\mathfrak{X})$ for a Banach space $\mathfrak{X}$ which is weakly infinitely divisible in the sense of B.S. Mityagin (see [14], Section 7.1, p. 94).

Mityagin decompositions are relevant for our purposes because they enable us to generalize Halmos's proof ([8]) that each operator on an infinite-dimensional Hilbert space is the sum of two commutators to a purely algebraic situation; Example 3.8 (iii), below, shows that our result does indeed imply Halmos's theorem.

Theorem 3.2. *Each element in a unital algebra with a Mityagin decomposition is the sum of two commutators.*

*Proof.* Let $\mathcal{A}$ be a unital algebra with a Mityagin decomposition, and take $\mathbb{I}, k_0, \rho, L, R, A_k$, and $B_k$, $k \in \mathbb{I}$, as in Definition 3.1. A straightforward calculation using properties (iii) and (i) yields that

$$B_k(LR - RL) = \left\{ \begin{matrix} B_{k_0} & \text{for } k = k_0, \\ 0 & \text{for } k \in \mathbb{I} \setminus \{k_0\} \end{matrix} \right\} = B_k(A_{k_0}B_{k_0}), \quad k \in \mathbb{I}.$$

This implies that $LR - RL = A_{k_0}B_{k_0}$ by (ii). Let $C \in \mathcal{A}$ be given, and take $\widetilde{C} \in \mathcal{A}$ such that $B_k\widetilde{C} = CB_k$ for each $k \in \mathbb{I}$ as in (iv). Then it follows that

$$B_k\widetilde{C}L = CB_kL = CB_{\rho(k)} = B_{\rho(k)}\widetilde{C} = B_kL\widetilde{C}, \quad k \in \mathbb{I},$$

and hence $\widetilde{C}$ commutes with $L$ by (ii). We conclude that

$$\begin{aligned}[L, R\widetilde{C}] + [B_{k_0}, A_{k_0}C] &= LR\widetilde{C} - R\widetilde{C}L + B_{k_0}A_{k_0}C - A_{k_0}CB_{k_0} \\ &= (LR - RL)\widetilde{C} + C - A_{k_0}B_{k_0}\widetilde{C} = C,\end{aligned}$$

as desired. ∎

Corollary 3.3. *There are no non-zero traces on a unital algebra with a Mityagin decomposition.*

Remark 3.4. For normed algebras, Theorem 1.1 implies that Theorem 3.2 is optimal in terms of the number of commutators required in the sum.

Our next result (Proposition 3.7) serves two purposes:

(i) it provides us with a large stock of examples of Banach spaces $\mathfrak{X}$ such that $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition (e.g., see Examples 3.8 and 3.9);

(ii) its proof highlights the intuition behind the definition of a Mityagin decomposition of $\mathcal{B}(\mathfrak{X})$ for a Banach space $\mathfrak{X}$: $(A_kB_k)_{k\in\mathbb{I}}$ is a family of pairwise orthogonal projections which decomposes $\mathfrak{X}$ into a "direct sum" of subspaces isomorphic to $\mathfrak{X}$, and with respect to this decomposition, the operators $L$ and $R$ act as a "left shift" and a "right shift", respectively, and $\widetilde{C}$ is the "diagonal operator" induced by the operator $C$.

First we require a definition.

Definition 3.5. Let $\mathfrak{E}$ be a Banach space with a normalized, unconditional basis $(e_n)_{n=1}^\infty$ of unconditional constant 1 (see [12], p. 18, for the definition).

We say that the basis $(e_n)_{n=1}^\infty$ is *shiftable* if it is equivalent to the basic sequence $(e_n)_{n=2}^\infty$ in the sense of Definition 1.a.7 from [12].

For each sequence $(\mathfrak{X}_n)_{n=1}^\infty$ of Banach spaces, we set

$$\Big(\bigoplus_{n=1}^\infty \mathfrak{X}_n\Big)_{\mathfrak{E}} := \Big\{(x_n)_{n=1}^\infty : x_n \in \mathfrak{X}_n \text{ for each } n \in \mathbb{N} \text{ and } \sum_{n=1}^\infty \|x_n\|e_n \text{ is norm-convergent in } \mathfrak{E}\Big\}.$$

This is a Banach space with respect to the coordinatewise-defined vector space operations and the norm

$$\left\|(x_n)_{n=1}^{\infty}\right\| := \left\|\sum_{n=1}^{\infty} \|x_n\| e_n\right\|, \quad (x_n)_{n=1}^{\infty} \in \Big(\bigoplus_{n=1}^{\infty} \mathfrak{X}_n\Big)_{\mathfrak{E}}.$$

In the case where $\mathfrak{X}_1 = \mathfrak{X}_2 = \cdots \stackrel{\mathrm{def}}{=} \mathfrak{X}$, we write $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ instead of $\big(\bigoplus_{n=1}^{\infty} \mathfrak{X}_n\big)_{\mathfrak{E}}$.

Symmetric and subsymmetric bases are easy examples of shiftable bases. On the other hand, W.T. Gowers has constructed a Banach space which has an unconditional basis, but is not isomorphic to its hyperplanes (see [7]), so in particular its basis cannot be shiftable.

Shiftable bases are useful because they guarantee the existence of left and right shift operators on $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ as the following elementary lemma shows.

LEMMA 3.6. *Let $\mathfrak{E}$ be a Banach space with a normalized, unconditional basis $(e_n)_{n=1}^{\infty}$ of unconditional constant* 1. *The basis $(e_n)_{n=1}^{\infty}$ is shiftable if and only if, for each Banach space $\mathfrak{X}$, there are operators $L'$ and $R'$ on $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ satisfying*

$$L'(x_n)_{n=1}^{\infty} = (x_2, x_3, \ldots) \quad \textit{and} \quad R'(x_n)_{n=1}^{\infty} = (0, x_1, x_2, \ldots), \tag{3.1}$$

*for each $(x_n)_{n=1}^{\infty} \in \mathfrak{E}(\mathbb{N}, \mathfrak{X})$.*

PROPOSITION 3.7. *Let $\mathfrak{X}$ be a Banach space such that $\mathfrak{X}$ is isomorphic to $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ for some Banach space $\mathfrak{E}$ with a shiftable, normalized, unconditional basis of unconditional constant* 1. *Then the algebra $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition.*

*It follows that each operator on $\mathfrak{X}$ is the sum of two commutators and that there are no non-zero traces on $\mathcal{B}(\mathfrak{X})$.*

*Proof.* (cf. Lemma 8 from [14]) For each $k \in \mathbb{N}$, let $J_k : \mathfrak{X} \to \mathfrak{E}(\mathbb{N}, \mathfrak{X})$ and $P_k : \mathfrak{E}(\mathbb{N}, \mathfrak{X}) \to \mathfrak{X}$ denote the canonical $k^{\text{th}}$ coordinate embedding and projection, respectively. Take operators $L'$ and $R'$ on $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ satisfying (3.1), and take an isomorphism $U : \mathfrak{X} \to \mathfrak{E}(\mathbb{N}, \mathfrak{X})$. Define $\mathbb{I} := \mathbb{N}$, $k_0 := 1$, $\rho : k \mapsto k+1, \mathbb{N} \to \mathbb{N} \setminus \{1\}$, $L := U^{-1}L'U$, $R := U^{-1}R'U$, $A_k := U^{-1}J_k$, and $B_k := P_k U$, $k \in \mathbb{N}$. Then it is straightforward to check that properties (i)–(iii) in Definition 3.1 hold. Moreover, for $C \in \mathcal{B}(\mathfrak{X})$, let

$$\operatorname{diag}(C) : (x_n)_{n=1}^{\infty} \longmapsto (Cx_n)_{n=1}^{\infty}, \quad \mathfrak{E}(\mathbb{N}, \mathfrak{X}) \longrightarrow \mathfrak{E}(\mathbb{N}, \mathfrak{X}),$$

be the diagonal operator on $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ induced by $C$, and set $\widetilde{C} := U^{-1} \operatorname{diag}(C) U$. Then $B_k \widetilde{C} = C B_k$ for each $k \in \mathbb{N}$, verifying property (iv). ∎

EXAMPLE 3.8. Suppose that one of the following three conditions holds:
(i) $\mathfrak{E} = c_0$ and either $\mathfrak{X} = c_0$ or $\mathfrak{X} = C([0,1])$;
(ii) $\mathfrak{E} = \ell_p$ and either $\mathfrak{X} = \ell_p$ or $\mathfrak{X} = L_p([0,1])$, where $1 \leqslant p < \infty$;
(iii) $\mathfrak{E} = \ell_2$ and $\mathfrak{X}$ is an infinite-dimensional Hilbert space.
Then the standard basis of $\mathfrak{E}$ satisfies the conditions in Proposition 3.7, and $\mathfrak{X}$ is isomorphic to $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$. It follows that $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition and that each operator on $\mathfrak{X}$ is the sum of two commutators.

The following example will be important for us in Section 4.

EXAMPLE 3.9. Let $\mathfrak{E}$ be a Banach space with a normalized, symmetric basis of symmetric constant 1 (see [12], p. 113 for the definition), and let $(\mathfrak{X}_n)_{n=1}^{\infty}$ be a sequence of finite-dimensional Banach spaces such that

$$\sup\big\{d_{\mathrm{BM}}(\mathfrak{X}_m \oplus \mathfrak{X}_n, \mathfrak{X}_{m+n}) : m, n \in \mathbb{N}\big\} < \infty,$$

where $d_{\mathrm{BM}}$ is the Banach-Mazur distance. (In particular $\dim \mathfrak{X}_m + \dim \mathfrak{X}_n = \dim \mathfrak{X}_{m+n}$ for each pair $(m, n)$ of natural numbers.) Set $\mathfrak{X} := \big(\bigoplus_{n=1}^{\infty} \mathfrak{X}_n\big)_{\mathfrak{E}}$. Then $\mathfrak{X}$ is isomorphic to $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ by Corollary 7 (i) from [3], so that $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition, and each operator on $\mathfrak{X}$ is the sum of two commutators.

EXAMPLE 3.10. Proposition 3.7 remains true if we replace the $\mathfrak{E}$-direct sum $\mathfrak{E}(\mathbb{N}, \mathfrak{X})$ by the corresponding $\ell_\infty$-direct sum. Indeed, suppose that $\mathfrak{X}$ is a Banach space isomorphic to

$$\ell_\infty(\mathbb{N}, \mathfrak{X}) := \big\{(x_n)_{n=1}^{\infty} : x_n \in \mathfrak{X} \text{ for each } n \in \mathbb{N} \text{ and } \sup\big\{\|x_n\| : n \in \mathbb{N}\big\} < \infty\big\}.$$

Then, exactly as in the proof of Proposition 3.7, we can show that $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition. In particular it follows that $\mathcal{B}(\ell_\infty)$ has a Mityagin decomposition and that each operator on $\ell_\infty$ is the sum of two commutators. The same holds for $L_\infty([0, 1])$ because $L_\infty([0, 1])$ is isomorphic to $\ell_\infty$ (see [15]).

EXAMPLE 3.11. Let $\mathfrak{X} := \ell_p \oplus c_0$ or $\mathfrak{X} := \ell_p \oplus \ell_q$, where $p, q \in [1, \infty[$ are distinct. Then $\mathcal{B}(\mathfrak{X})$ does not have a Mityagin decomposition. This is shown by Mityagin in [14], Section 7.2A, p. 95, for $\mathfrak{X} = \ell_2 \oplus c_0$, but his proof applies verbatim to all the spaces $\mathfrak{X}$ listed above.

Examples 3.8 and 3.11 show that the class of Banach spaces $\mathfrak{X}$ such that $\mathcal{B}(\mathfrak{X})$ has a Mityagin decomposition is not closed under finite direct sums. However, the class of Banach spaces $\mathfrak{X}$ such that the commutators span $\mathcal{B}(\mathfrak{X})$ is indeed closed under finite direct sums.

PROPOSITION 3.12. *Let $n \geqslant 2$ be an integer, and let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be Banach spaces such that $\mathcal{B}(\mathfrak{X}_1), \dots, \mathcal{B}(\mathfrak{X}_n)$ are spanned by their commutators. Then the algebra $\mathcal{B}(\mathfrak{X}_1 \oplus \cdots \oplus \mathfrak{X}_n)$ is spanned by its commutators. Moreover, if there is a natural number $N$ such that each operator on $\mathfrak{X}_k$ is the sum of $N$ commutators for each $k \in \{1, \dots, n\}$, then each operator on $\mathfrak{X}_1 \oplus \cdots \oplus \mathfrak{X}_n$ is the sum of $N + 1$ commutators.*

*Proof.* Set $\mathfrak{X} := \mathfrak{X}_1 \oplus \cdots \oplus \mathfrak{X}_n$. There is a standard algebra isomorphism between $\mathcal{B}(\mathfrak{X})$ and the set of $(n \times n)$-matrices $(T_{k,m})_{k,m=1}^{n}$ with $T_{k,m} \in \mathcal{B}(\mathfrak{X}_m, \mathfrak{X}_k)$. We shall identify operators on $\mathfrak{X}$ with $(n \times n)$-matrices via this isomorphism.

Let $T = (T_{k,m})_{k,m=1}^{n} \in \mathcal{B}(\mathfrak{X})$ be given. Take a natural number $N$ such that, for each $k \in \{1, \dots, n\}$, there are operators $A_1^{(k)}, \dots, A_N^{(k)}, B_1^{(k)}, \dots, B_N^{(k)}$ on $\mathfrak{X}_k$ satisfying

$$T_{k,k} = \sum_{m=1}^{N} [A_m^{(k)}, B_m^{(k)}].$$

Then it follows that

$$\sum_{m=1}^{N}[A_m, B_m] = \begin{pmatrix} T_{1,1} & 0 & \cdots & 0 \\ 0 & T_{2,2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & T_{n,n} \end{pmatrix},$$

where we have introduced

$$A_m := \begin{pmatrix} A_m^{(1)} & 0 & \cdots & 0 \\ 0 & A_m^{(2)} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_m^{(n)} \end{pmatrix} \quad \text{and} \quad B_m := \begin{pmatrix} B_m^{(1)} & 0 & \cdots & 0 \\ 0 & B_m^{(2)} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & B_m^{(n)} \end{pmatrix}$$

for each $m \in \{1, \dots, N\}$. Set

$$C_{k,m} := \begin{cases} kI_{\mathfrak{X}_k} & \text{for } k = m, \\ 0 & \text{for } k \neq m, \end{cases} \quad \text{and} \quad D_{k,m} := \begin{cases} 0 & \text{for } k = m, \\ T_{k,m}/(k-m) & \text{for } k \neq m. \end{cases}$$

Then the matrices $C := (C_{k,m})_{k,m=1}^n$ and $D := (D_{k,m})_{k,m=1}^n$ satisfy:

$$[C, D] = \begin{pmatrix} 0 & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & 0 & \cdots & T_{2,n} \\ \vdots & & \ddots & \vdots \\ T_{n,1} & T_{n,2} & \cdots & 0 \end{pmatrix},$$

and hence we have:

$$T = (T_{k,m})_{k,m=1}^n = \sum_{m=1}^{N}[A_m, B_m] + [C, D],$$

as desired. ∎

Prompted by Proposition 3.7, A. Villena has raised the following question.

QUESTION 3.13. Let $\mathfrak{X}$ be a Banach space such that $\mathfrak{X}$ is isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$. Are there any non-zero traces on $\mathcal{B}(\mathfrak{X})$?

DEFINITION 3.14. A unital algebra $\mathcal{A}$ is *properly infinite* if there are elements $A_1, A_2, B_1$, and $B_2$ in $\mathcal{A}$ satisfying $B_k A_m = \delta_{k,m} \mathbb{1}_{\mathcal{A}}$ for $k, m \in \{1, 2\}$.

Definition 3.1 (i) shows that a unital algebra with a Mityagin decomposition is properly infinite.

For a Banach space $\mathfrak{X}$, we have: $\mathcal{B}(\mathfrak{X})$ is properly infinite if and only if $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$. This suggests the following algebraic generalization of Villena's question.

QUESTION 3.15. Is it true that there are no non-zero traces on each unital, properly infinite algebra?

Let $\mathcal{A}$ be a unital, properly infinite $C^*$-algebra. T. Fack has shown that each self-adjoint element in $\mathcal{A}$ can be written as a sum of five commutators ([6], Theorem 2.1). It follows that $\mathcal{A}$ is spanned by its commutators, and thus the answer to Question 3.15 is positive at least for $C^*$-algebras. Recently, Fack's result has been improved and simplified by C. Pop who has given a very short and elegant proof that each element in a unital, properly infinite $C^*$-algebra is the sum of two commutators (see [16]).

4. COMMUTATORS OF OPERATORS ON JAMES SPACES

Throughout this section, we fix a real number $p$ in the open interval $]1,\infty[$.

Definition 4.1. For each sequence $x = (\alpha_k)_{k=1}^\infty$ of complex numbers, set

$$\|x\|_{\mathfrak{J}_p} := \sup\Bigg\{\bigg(\sum_{m=1}^{n-1} |\alpha_{k_m} - \alpha_{k_{m+1}}|^p\bigg)^{\frac{1}{p}} : n, k_1, \ldots, k_n \in \mathbb{N},\ n \geqslant 2,\ k_1 < k_2 < \cdots < k_n\Bigg\}.$$

The $p^{\text{th}}$ *James space* is

$$\mathfrak{J}_p := \big\{(\alpha_k)_{k=1}^\infty : \alpha_k \in \mathbb{C} \text{ for each } k \in \mathbb{N},\ \big\|(\alpha_k)_{k=1}^\infty\big\|_{\mathfrak{J}_p} < \infty, \text{ and } \alpha_k \to 0 \text{ as } k \to \infty\big\}.$$

Then $\big(\mathfrak{J}_p, \|\cdot\|_{\mathfrak{J}_p}\big)$ is a Banach space with respect to the coordinatewise-defined vector space operations. It has a monotone basis $(e_k)_{k=1}^\infty$ given by $e_k = (\delta_{k,m})_{m=1}^\infty$ for each $k \in \mathbb{N}$, and this basis is shrinking, so that the biorthogonal functionals $(f_k)_{k=1}^\infty$ associated with $(e_k)_{k=1}^\infty$ form a basis of the dual space $\mathfrak{J}_p^*$.

A fundamental property of $\mathfrak{J}_p$ is that it is *quasi-reflexive*, that is, the canonical image of $\mathfrak{J}_p$ in its bidual space $\mathfrak{J}_p^{**}$ has codimension 1; for $p = 2$, this is shown in [10]. An immediate consequence of the quasi-reflexivity of $\mathfrak{J}_p$ is that the ideal $\mathcal{W}(\mathfrak{J}_p)$ of weakly compact operators has codimension 1 in $\mathcal{B}(\mathfrak{J}_p)$, and hence the quotient homomorphism from $\mathcal{B}(\mathfrak{J}_p)$ onto $\mathcal{B}(\mathfrak{J}_p)/\mathcal{W}(\mathfrak{J}_p)$ gives rise to a *character* on $\mathcal{B}(\mathfrak{J}_p)$, that is, a surjective algebra homomorphism $\varphi : \mathcal{B}(\mathfrak{J}_p) \to \mathbb{C}$. Specifically, $\varphi$ is given by

$$\varphi(\zeta I_{\mathfrak{J}_p} + T) = \zeta, \quad \zeta \in \mathbb{C},\ T \in \mathcal{W}(\mathfrak{J}_p). \tag{4.1}$$

It is shown in [11] that $\mathcal{W}(\mathfrak{J}_p)$ is the only maximal ideal in $\mathcal{B}(\mathfrak{J}_p)$; in particular $\varphi$ is the only character on $\mathcal{B}(\mathfrak{J}_p)$. The aim of this section is to show that the only traces on $\mathcal{B}(\mathfrak{J}_p)$ are the scalar multiples of $\varphi$.

It follows from Corollary 2.3 that $\mathcal{B}(\mathfrak{J}_p)$ is not spanned by its commutators. More precisely, we have the following result.

Proposition 4.2. *The identity operator on $\mathfrak{J}_p$ has distance 1 to the linear subspace spanned by the commutators of $\mathcal{B}(\mathfrak{J}_p)$ :*

$$\inf\big\{\|I_{\mathfrak{J}_p} - T\| : T \in \operatorname{com}\mathcal{B}(\mathfrak{J}_p)\big\} = 1.$$

*Proof.* It is clear that the distance is at most 1 because 0 is a commutator. Conversely, suppose that $T \in \operatorname{com}\mathcal{B}(\mathfrak{J}_p)$. Then $\varphi(T) = 0$, and hence

$$\|I_{\mathfrak{J}_p} - T\| \geqslant \big|\varphi(I_{\mathfrak{J}_p} - T)\big| = 1. \qquad ∎$$

W.J. Davis, T. Figiel, W.B. Johnson, and A. Pełczyński have shown that an operator is weakly compact if and only if it factors through a reflexive Banach space ([5]). Our first aim is to show that all weakly compact operators on $\mathfrak{J}_p$ factor through the *same* reflexive Banach space and that this reflexive Banach space can be chosen to be a complemented subspace of $\mathfrak{J}_p$ (see Theorem 4.3, below).

For each natural number $n$, let $\mathfrak{J}_p^{(n)}$ denote the $n$-dimensional linear subspace of $\mathfrak{J}_p$ spanned by the first $n$ basis vectors $e_1, \ldots, e_n$, and set

$$\mathfrak{J}_p^{(\infty)} := \left(\bigoplus_{n=1}^{\infty} \mathfrak{J}_p^{(n)}\right)_{\ell_p}.$$

Then $\mathfrak{J}_p^{(\infty)}$ is reflexive, and $\mathfrak{J}_p$ contains a complemented subspace isomorphic to $\mathfrak{J}_p^{(\infty)}$. (For $p = 2$, this is shown by P.G. Casazza, Bor-Luh Lin, and R.H. Lohman in Section 1 from [4]; the general case is treated in Proposition 4.4 (iv) from [11].) Moreover, since the Banach-Mazur distance between $\mathfrak{J}_p^{(m)} \oplus \mathfrak{J}_p^{(n)}$ and $\mathfrak{J}_p^{(m+n)}$ is uniformly bounded in $(m, n) \in \mathbb{N}^2$, Example 3.9 shows that

$$\mathfrak{J}_p^{(\infty)} \cong \ell_p(\mathbb{N}, \mathfrak{J}_p^{(\infty)}). \tag{4.2}$$

For Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, let $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$ denote the set of operators on $\mathfrak{X}$ which factor through $\mathfrak{Y}$, that is,

$$\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X}) := \big\{ST : T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}),\ S \in \mathcal{B}(\mathfrak{Y}, \mathfrak{X})\big\}.$$

It follows from (4.2) and Corollary 3.7 in [11] that $\mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p)$ is an ideal in $\mathcal{B}(\mathfrak{J}_p)$. By Propositions 4.4 (iv) and 4.18 in [11], $\mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p)$ is a dense subset of $\mathcal{W}(\mathfrak{J}_p)$. Modifying techniques developed by G.A. Willis in [18], we shall show that these two sets are actually equal.

THEOREM 4.3. *An operator on $\mathfrak{J}_p$ is weakly compact if and only if it factors through $\mathfrak{J}_p^{(\infty)}$:*

$$\mathcal{W}(\mathfrak{J}_p) = \mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p).$$

*Proof.* Only the inclusion $\mathcal{W}(\mathfrak{J}_p) \subseteq \mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p)$ requires a proof. We proceed as in the proof of Proposition 6 from [18]. Each weakly compact operator $T$ on $\mathfrak{J}_p$ can be written as $T = K + R$, where $K$ is a compact operator on $\mathfrak{J}_p$ and $R$ is an operator on $\mathfrak{J}_p$ whose "matrix" $(R_{k,m})_{k,m=1}^{\infty} := \big(f_k(Re_m)\big)_{k,m=1}^{\infty}$ satisfies:

$$\sum_{m=1}^{\infty} R_{k,m} = 0, \quad k \in \mathbb{N}$$

and there are only finitely many non-zero entries in each row and column of $(R_{k,m})_{k,m=1}^{\infty}$. This follows from Lemma 2.1 in [13] for $p = 2$ and from equation (4.4) in [11] in the general case.

For each strictly increasing sequence $(j_n)_{n=1}^{\infty}$ of natural numbers, Willis considers in [18] an idempotent operator $S[j_n]$. This operator is related to the idempotent operator $P_{\mathbf{j}}$ defined in Section 4 from [11] by $S[j_n] = I_{\mathfrak{J}_p} - P_{\mathbf{j}}$, where $j_0 := 0$ and $\mathbf{j} := (j_n)_{n=0}^{\infty}$. In particular it follows from Proposition 4.4 in [11] that

$$I_{\mathfrak{J}_p} - S[j_n] \in \mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p). \tag{4.3}$$

Continuing along the lines of Willis's proof of Proposition 6 in [18], we can construct operators $L$ and $V$ on $\mathfrak{J}_p$ and strictly increasing sequences $(i_n)_{n=1}^\infty$ and $(i'_n)_{n=1}^\infty$ of natural numbers such that

$$\begin{aligned} &\big(I_{\mathfrak{J}_p} - S[i_n]\big)(K-L) = K-L, \qquad &&\big(I_{\mathfrak{J}_p} - S[i_n+1]\big)L = L,\\ &\big(I_{\mathfrak{J}_p} - S[i'_n]\big)(R-V) = R-V, \qquad &&\big(I_{\mathfrak{J}_p} - S[i'_n+1]\big)V = V. \end{aligned}$$

(The construction of the operator $V$ requires the observations that there is a formal inclusion operator from $\ell_p$ into $\mathfrak{J}_p$ and that Willis's lemma ([18], p. 255) is valid for any $p \in \,]1,\infty[$ provided that the number 2 is replaced by $p$ throughout.) By (4.3), this implies that $T \in \mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p)$ because $\mathrm{fac}_{\mathfrak{J}_p^{(\infty)}}(\mathfrak{J}_p)$ is an ideal in $\mathcal{B}(\mathfrak{J}_p)$. ∎

We note in passing the following consequence of Theorem 4.3. For $p = 2$, it is due to P.G. Casazza ([2], Corollary VII) who used it as a key step in his proof that $\mathfrak{J}_2$ is primary.

COROLLARY 4.4. *A complemented subspace of $\mathfrak{J}_p$ is reflexive if and only if it is isomorphic to a complemented subspace of $\mathfrak{J}_p^{(\infty)}$.*

*Proof.* The reflexivity of $\mathfrak{J}_p^{(\infty)}$ implies that each subspace of $\mathfrak{J}_p^{(\infty)}$ is reflexive. Consequently, each subspace of $\mathfrak{J}_p$ isomorphic to a subspace of $\mathfrak{J}_p^{(\infty)}$ is reflexive.

Conversely, let $P : \mathfrak{J}_p \to \mathfrak{J}_p$ be an idempotent operator with reflexive image. Then $P$ is weakly compact, and hence we can take operators $T : \mathfrak{J}_p \to \mathfrak{J}_p^{(\infty)}$ and $S : \mathfrak{J}_p^{(\infty)} \to \mathfrak{J}_p$ such that $P = ST$. It follows from Lemma 3.6 (ii) in [11] that the operator $TSTS : \mathfrak{J}_p^{(\infty)} \to \mathfrak{J}_p^{(\infty)}$ is idempotent and that the images of $P$ and $TSTS$ are isomorphic. ∎

LEMMA 4.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces such that $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{Y}$ and $\mathcal{B}(\mathfrak{Y})$ is spanned by its commutators. Then each operator in $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$ is a sum of commutators of operators in $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$. Moreover, if there is a natural number $N$ such that each operator in $\mathcal{B}(\mathfrak{Y})$ is the sum of $N$ commutators, then each operator in $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$ is the sum of $N+1$ commutators of operators in $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$.*

*Proof.* Let $R \in \mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$ be given, and take $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$ and $S \in \mathcal{B}(\mathfrak{Y},\mathfrak{X})$ such that $R = ST$. Since $TS$ is an operator on $\mathfrak{Y}$, we can find a natural number $N$ and operators $A_1,\dots,A_N, B_1,\dots,B_N$ on $\mathfrak{Y}$ such that $TS = \sum_{k=1}^N [A_k,B_k]$. The fact that $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{Y}$ implies that there are operators $U : \mathfrak{X} \to \mathfrak{Y}$ and $V : \mathfrak{Y} \to \mathfrak{X}$ with $UV = I_{\mathfrak{Y}}$. For each $k \in \{1,\dots,N\}$, set $C_k := VA_kU$ and $D_k := VB_kU$, and set $C_{N+1} := SU$ and $D_{N+1} := VT$. Then $C_1,\dots,C_{N+1}, D_1,\dots,D_{N+1}$ belong to $\mathrm{fac}_{\mathfrak{Y}}(\mathfrak{X})$, and we have that

$$\sum_{k=1}^{N+1}[C_k,D_k] = \sum_{k=1}^{N}(VA_kB_kU - VB_kA_kU) + ST - VTSU = ST = R,$$

as desired. ∎

Theorem 4.6. *Each weakly compact operator on $\mathfrak{J}_p$ is the sum of three commutators of weakly compact operators on $\mathfrak{J}_p$.*

*Proof.* Each operator on $\mathfrak{J}_p^{(\infty)}$ is the sum of two commutators by (4.2) and Proposition 3.7. Now the result follows from Theorem 4.3 and Lemma 4.5 because $\mathfrak{J}_p^{(\infty)}$ is isomorphic to a complemented subspace of $\mathfrak{J}_p$. ∎

Corollary 4.7. $\mathcal{W}(\mathfrak{J}_p) = \operatorname{com}\mathcal{W}(\mathfrak{J}_p) = \operatorname{com}\mathcal{B}(\mathfrak{J}_p)$.

Corollary 4.8. *A map $\tau : \mathcal{B}(\mathfrak{J}_p) \to \mathbb{C}$ is a trace if and only if it is a scalar multiple of the character $\varphi$ given by* (4.1).

*Proof.* It is clear that a scalar multiple of a character is a trace. Conversely, suppose that $\tau$ is a trace. Then $\mathcal{W}(\mathfrak{J}_p) \subseteq \ker\tau$ by Proposition 2.2 and Corollary 4.7. This implies that

$$\tau(\zeta I_{\mathfrak{J}_p} + T) = \zeta\,\tau(I_{\mathfrak{J}_p}) = \tau(I_{\mathfrak{J}_p})\cdot\varphi(\zeta I_{\mathfrak{J}_p} + T), \quad \zeta \in \mathbb{C},\ T \in \mathcal{W}(\mathfrak{J}_p),$$

and therefore $\tau = \tau(I_{\mathfrak{J}_p})\cdot\varphi$. ∎

Question 4.9. Is there a weakly compact operator on $\mathfrak{J}_p$ which is *not* the sum of any two commutators? In other words, is the upper bound 3 on the number of commutators in the sum obtained in Theorem 4.6 optimal?

The results presented so far might convey the impression that the commutators of operators on a Banach space $\mathfrak{X}$ always span a "large" subspace of $\mathcal{B}(\mathfrak{X})$ and thus that there are only "few" traces on $\mathcal{B}(\mathfrak{X})$. We conclude with an example to show that this is not true in general.

Example 4.10. In [17], C.J. Read constructs a Banach space $\mathfrak{R}$ with the following properties:

(i) there is an ideal $\mathcal{I}$ in $\mathcal{B}(\mathfrak{R})$ of codimension 1;

(ii) the ideal $\mathcal{W}(\mathfrak{R})$ of weakly compact operators on $\mathfrak{R}$ has infinite codimension in $\mathcal{B}(\mathfrak{R})$;

(iii) the "square" $\mathcal{I}^2 := \operatorname{span}\{ST : S, T \in \mathcal{I}\}$ of $\mathcal{I}$ is contained in $\mathcal{W}(\mathfrak{R})$.

It follows that

$$\operatorname{com}\mathcal{B}(\mathfrak{R}) = \operatorname{com}\mathcal{I} \subseteq \mathcal{I}^2 \subseteq \mathcal{W}(\mathfrak{R}),$$

and hence $\operatorname{com}\mathcal{B}(\mathfrak{R})$ has infinite codimension in $\mathcal{B}(\mathfrak{R})$. In particular there are infinitely many linearly independent traces on $\mathcal{B}(\mathfrak{R})$.

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