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STRUCTURE AND ISOMORPHISM CLASSIFICATION OF COMPACT QUANTUM GROUPS $A_u(Q)$ AND $B_u(Q)$

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ABSTRACT. We classify the compact quantum groups $A_{\rm u}(Q)$ (respectively, $B_{\rm u}(Q)$) up to isomorphism when $Q > 0$ (respectively, when $Q\overline{Q} \in \mathbb{R}I_n$). We show that the general $A_{\rm u}(Q)$'s and $B_{\rm u}(Q)$'s for arbitrary Q have explicit decompositions into free products of these special $A_{\rm u}(Q)$'s and $B_{\rm u}(Q)$'s.

KEYWORDS: *Compact quantum groups, universal quantum groups, Woronowicz C^*-algebras, Hopf algebras, free products.*

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1. INTRODUCTION

Recall ([13] and [16]) that a *compact matrix quantum group* is a pair $G = (A, u)$ of a unital C^*-algebra A and of a matrix system u of generators u_{ij}, $i, j = 1, \dots, n$, that satisfies the following two axioms:

(1) there is a unital C^*-homomorphism $\Phi : A \to A \otimes A$ such that $\Phi(u_{ij}) = \sum\limits_{k=1}^{n} u_{ik} \otimes u_{kj}$ for each i, j;

(2) the matrices $u = (u_{ij})$ and $u^{\rm t}$ are invertible in $M_n(\mathbb{C}) \otimes A$.

In [3], [7] and [6], we constructed for each $Q \in GL(n, \mathbb{C})$ two families $A_{\rm u}(Q)$ and $B_{\rm u}(Q)$ of compact matrix quantum groups in the sense of Woronowicz ([14]). The compact quantum groups $A_{\rm u}(Q)$ and $B_{\rm u}(Q)$ are defined in terms of generators u_{ij}, $i, j = 1, \dots n$, by the relations

$$A_{\rm u}(Q) : u^* u = I_n = u u^*, \quad u^{\rm t} Q \overline{u} Q^{-1} = I_n = Q \overline{u} Q^{-1} u^{\rm t};$$
$$B_{\rm u}(Q) : u^* u = I_n = u u^*, \quad u^{\rm t} Q u Q^{-1} = I_n = Q u Q^{-1} u^{\rm t},$$

where $u = (u_{ij})$. The $A_u(Q)$'s are universal in the sense that every compact matrix quantum group is a quantum subgroup of $A_u(Q)$ for some $Q > 0$. Similarly, the $B_u(Q)$'s are universal in the sense that every compact matrix quantum group with self conjugate fundamental representation is a quantum subgroup of $B_u(Q)$ for some Q. The subscript u denotes "universality". For $Q > 0$ (respectively, Q with $Q\overline{Q} \in \mathbb{R}I_n$), Bănică determined in [2] (respectively [1]) the fusion rings of the irreducible representations of the quantum group $A_u(Q)$ (respectively, $B_u(Q)$). Note that he used $A_u(F)$ (respectively, $A_o(F)$) to denote $A_u(Q)$ (respectively $B_u(Q)$), where $Q = F^*F$ (respectively $Q = F^*$). Although these quantum groups are of a different nature from the well known quantum groups obtained from ordinary Lie groups by deformation quantization ([4], [5], [4a] and [15]); they share many properties of the ordinary Lie groups. Not only they are fundamental objects in the framework of compact matrix quantum groups of Woronowicz ([13]), they are also useful objects in the study of intrinsic quantum group symmetries, such as ergodic quantum group symmetries on operator algebras and quantum automorphism groups of noncommutative spaces (cf. [12], [10]). Although much is known about $A_u(Q)$ and $B_u(Q)$, some basic questions are still left unanswered ([9]). For instance, for different Q's, how do the $A_u(Q)$'s (respectively, $B_u(Q)$'s) differ from each other? Do some of these quantum groups constitute building blocks for the two families of quantum groups in an appropriate sense?

The purpose of this paper is to answer these questions. Recall that the fundamental representation u of $A_u(Q)$ (respectively, $B_u(Q)$) is irreducible if and only if Q is positive (respectively, $Q\overline{Q} \in \mathbb{R}I_n$); see [12] (respectively, [1]). When these conditions are satisfied, we classify $A_u(Q)$ and $B_u(Q)$ up to isomorphism, and show that they are not free products or tensor products or crossed products (see [6] and [8] for these constructions). We show that the general $A_u(Q)$'s and $B_u(Q)$'s for arbitrary Q are free products of these special $A_u(Q)$'s and $B_u(Q)$'s, and we give their explicit decompositions in terms of free products.

In the following, the word "morphisms" means morphisms between compact quantum groups (cf. [6]).

1. THE QUANTUM GROUPS $A_u(Q)$ FOR POSITIVE Q

Let $Q \in GL(n, \mathbb{C})$. Then $A_u(Q) = A_u(cQ)$ for any nonzero number c. For a positive matrix Q, we can normalize it so that $\mathrm{Tr}(Q) = \mathrm{Tr}(Q^{-1})$.

THEOREM 1.1. *Let $Q \in GL(n, \mathbb{C})$ and $Q' \in GL(n', \mathbb{C})$ be positive matrices normalized as above with eigenvalues $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$ and $q'_1 \geqslant q'_2 \geqslant \cdots \geqslant q'_{n'}$ respectively. Then:*

(1) *$A_u(Q)$ is isomorphic to $A_u(Q')$ if and only if*

(i) *$n = n'$, and*

(ii) *$(q_1, q_2, \ldots, q_n) = (q'_1, q'_2, \ldots, q'_n)$ or $(q_n^{-1}, q_{n-1}^{-1}, \ldots, q_1^{-1}) = (q'_1, q'_2, \ldots, q'_n)$.*

(2) *$A_u(Q)$ is not a free product. That is, if $A_u(Q) = A * B$ is a free product of Woronowicz C^*-algebras A and B, then either $A = A_u(Q)$ or $B = A_u(Q)$.*

Proof. Clearly, we may assume $n, n' \geqslant 2$.

(1) Since $A_{\rm u}(Q)$ is similar to $A_{\rm u}(VQV^{-1})$ for $V \in U(n)$ (cf. [3]), we may assume up to isomorphism that Q and Q' are in diagonal form, say,

$$Q = \mathrm{diag}(q_1, q_2, \dots, q_n), \quad Q' = \mathrm{diag}(q'_1, q'_2, \dots, q'_{n'}).$$

We claim that every non-trivial irreducible representation of the quantum group $A_{\rm u}(Q)$ other than u and $\overline{u}$ has a dimension greater than n.

In [2], the irreducible representations of the quantum group $A_{\rm u}(Q)$ are parameterized by the free monoid $\mathbb{N} * \mathbb{N}$ with generators α and β and anti-multiplicative involution $\overline{\alpha} = \beta$ (the neutral element is e with $\overline{e} = e$). The classes of u and $\overline{u}$ are α and β, respectively. Let d_x be the dimension of irreducible representations in the class $x \in \mathbb{N} * \mathbb{N}$. By Theorem 1 of [2], we have the following dimension formula:

$$d_x d_y = \sum_{\substack{x=ag \\ \overline{g}b=y}} d_{ab}.$$

Hence $d_{x\alpha^{k+l}y} = d_{x\alpha^k} d_{\alpha^l y}$ for $k, l \geqslant 1$. This identity prevails when we change α to β. From these we see that apart from the trivial class e, the classes with smaller dimensions are those words x in which the powers of α and β are equal to 1. Moreover, we infer from [2] that for any word x, d_x does not change when we exchange α and β in x. Hence the minimal dimension is among $d_\alpha, d_{\alpha\beta}, d_{\alpha\beta\alpha}, \dots,$ so we now concentrate on these numbers. Let $f(1), f(2), f(3), \dots$ be this sequence and let $f(0) = 1 = d_e$. Then applying the above dimension formula to $d_{\alpha\beta\cdots\alpha\beta} d_\alpha$ and $d_{\alpha\beta\cdots\alpha\beta\alpha} d_\beta$, we get $f(k+1) = nf(k) - f(k-1)$, $k \geqslant 1$, noting that $f(1) = d_\alpha = d_\beta = n$. Since $n \geqslant 2$, we have

$$f(k+1) - f(k) = (n-1)f(k) - f(k-1) \geqslant f(k) - f(k-1) \geqslant \cdots \geqslant f(1) - f(0) > 0.$$

Hence $d_\alpha = d_\beta = n < d_x$ for $x \neq \alpha, \beta, e$. This proves our claim.

We introduced in [11] F-matrices for classes of irreducible representations of a compact quantum group, based on Woronowicz ([13]). If v is an irreducible representation with F-matrix F_v in the sense of [13], then the F-matrix $F_{[v]}$ for the class $[v]$ of v is the diagonal matrix with eigenvalues of F_v arranged in decreasing order on the diagonal. This is an invariant of the class $[v]$. We have $F_{[\overline{v}]} = \mathrm{diag}(\lambda_m^{-1}, \lambda_{m-1}^{-1}, \dots, \lambda_1^{-1})$ if $F_{[v]} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$. (Warning: $F_{\overline{v}} = (F_v^{\rm t})^{-1}$.)

Now for the quantum group $A_{\rm u}(Q)$, we have $F_{[u]} = F_{\rm u} = Q^{\rm t} = Q$ (cf. Remark 1.5.(3) of [3]). Therefore, if the quantum groups $A_{\rm u}(Q)$ and $A_{\rm u}(Q')$ are isomorphic to each other, the F-matrices for the *classes* of the irreducible representations u and $\overline{u}$ (of minimal dimension n) of $A_{\rm u}(Q)$ correspond to those of $A_{\rm u}(Q')$ (cf. Lemma 4.2 of [11]). Whence we have conditions (i) and (ii) in the theorem. Conversely, assume conditions (i) and (ii) are satisfied. If $Q = Q'$, there is nothing to prove. So we assume that $(q_n^{-1}, q_{n-1}^{-1}, \dots, q_1^{-1}) = (q'_1, q'_2, \dots, q'_n)$. As $F_{[\overline{u}]} = \mathrm{diag}(q_n^{-1}, q_{n-1}^{-1}, \dots, q_1^{-1})$, we can choose a unitary representation v in the class $[\overline{u}]$ of $\overline{u}$ (non-unitary in general) so that $F_v = F_{[\overline{u}]}$. Then the entries of v generate the same algebra $A_{\rm u}(Q)$ as those of u and they satisfy the relations for $A_{\rm u}(Q')$. It is now clear that $A_{\rm u}(Q)$ and $A_{\rm u}(Q')$ are isomorphic to each other,

(2) Suppose $A_{\rm u}(Q) = A * B$. By the classification of irreducible representations of the quantum groups $A * B$ in [6], the representation u is a tensor product of non-trivial irreducible representations of the quantum groups A and B. Also by [6], each representation in this tensor product is also an irreducible representation

of $A_{\rm u}(Q)$. From the claim in the proof of (1) above, we deduce that $A_{\rm u}(Q)$ has no irreducible representation of dimension 1 other than the trivial one. Therefore there is only one term in the tensor product. That is, u is a representation of the quantum group A or B. Whence $A_{\rm u}(Q) = A$ or $A_{\rm u}(Q) = B$. ∎

REMARKS 1.2. (i) As a corollary of the proof above, we have the following rigidity result for $A_{\rm u}(Q)$ (a similar result holds for $B_{\rm u}(Q)$ in the next section). Let $Q, Q' \in GL(n, \mathbb{C})$ be positive, normalized as above. If $A_{\rm u}(Q')$ is a quantum subgroup of $A_{\rm u}(Q)$ given by a surjection $\pi : A_{\rm u}(Q) \to A_{\rm u}(Q')$, then $Q' = VQV^{-1}$ or $Q' = V(Q^{\rm t})^{-1}V^{-1}$ for some $V \in U(n)$ and hence π is an isomorphism. To see this, first $Su'S^{-1} = \pi(u)$ or $SQ'^{1/2}\overline{u'}Q'^{-1/2}S^{-1} = \pi(u)$ for some $V \in U(n)$, which satisfy the relations for $A_{\rm u}(Q)$. The assertion follows from the irreducibility of the representation u'.

(ii) By the same method, one can also prove that $A_{\rm u}(Q)$ is neither a tensor product, nor a crossed product (cf. [8]). This remark also applies to $B_{\rm u}(Q)$ blow.

(iii) Although $A_{\rm u}(Q)$ is a universal analog of $U(n)$ for compact quantum groups ([7], [6] and [3]), the proof in the above shows that $A_{\rm u}(Q)$ has no nontrivial irreducible representations of dimension 1 when $n \geqslant 2$, a property in sharp contrast to $U(n)$. Further study of the irreducible representations of the quantum group $A_{\rm u}(Q)$ gives evidences that $A_{\rm u}(Q)$ may be a *simple* compact quantum group in an appropriate sense (work in progress).

2. THE QUANTUM GROUPS $B_u(Q)$ WITH $Q\overline{Q} \in \mathbb{R}I_n$

The quantum group $B_{\rm u}(Q)$ has only one irreducible representation of minimal dimension among the non-trivial ones (cf. [1]). If $B_{\rm u}(Q)$ is isomorphic to $B_{\rm u}(Q')$, then the fundamental representation of $B_{\rm u}(Q)$ corresponds to that of $B_{\rm u}(Q')$ under the isomorphism. Using the irreducibility of the fundamental representations along with the defining relations for $B_{\rm u}(Q)$ and $B_{\rm u}(Q')$ we immediately obtain part (1) of the following theorem (we need not consider the F-matrices for this). The proof of part (2) of the theorem is similar to the proof of part (2) of Theorem 1.1 above and will be omitted.

THEOREM 2.1. *Let $Q \in GL(n, \mathbb{C})$ and $Q' \in GL(n', \mathbb{C})$ be matrices such that $Q\overline{Q} \in \mathbb{R}I_n$ and $Q'\overline{Q'} \in \mathbb{R}I_{n'}$. Then:*

(1) *$B_{\rm u}(Q)$ is isomorphic to $B_{\rm u}(Q')$ if and only if*

(i) *$n = n'$, and*

(ii) *there exist $S \in U(n)$ and $c \in \mathbb{C}^*$ such that $Q = zS^{\rm t}Q'S$.*

(2) *$B_{\rm u}(Q)$ is not a free product. That is, if $B_{\rm u}(Q) = A * B$ is a free product of Woronowicz C^*-algebras A and B, then either $A = B_{\rm u}(Q)$ or $B = B_{\rm u}(Q)$.*

EXPLICIT PARAMETRIZATION OF THE ISOMORPHISM CLASSES OF $B_u(Q)$. Contrary to Theorem 1.1, Theorem 2.1 above does not give an explicit parametrization of the isomorphism classes of the $B_{\rm u}(Q)$'s. Assume the normalization $\mathrm{Tr}(QQ^*) = \mathrm{Tr}((QQ^*)^{-1})$ (or, equivalently, $\mathrm{Tr}(\overline{Q}Q^{\rm t}) = \mathrm{Tr}((\overline{Q}Q^{\rm t})^{-1})$). Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant$

λ_n be the eigenvalues of QQ^* (therefore of $\overline{Q}Q^t$). Using the defining relations of $B_u(Q)$, we have that the antipode κ satisfies

$$\kappa^2(u) = \kappa(Q^{-1}u^t Q) = Q^{-1}Q^t u(Q^{-1})^t Q,$$

where u is the fundamental representation of $B_u(Q)$. Hence from the assumption that $Q\overline{Q} = cI_n$ we get $F_u = \overline{Q}Q^t$ (also cf. Remark 1.5.(3) of [3]). Since $u = Q^*\overline{u}Q^{*-1}$, one has $F_{[u]} = F_{[\overline{u}]}$ and hence

$$F_{[u]} = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_n^{-1}, \lambda_{n-1}^{-1}, \ldots, \lambda_1^{-1}) = F_{[\overline{u}]}.$$

Let $Q^* = U|Q^*| = U\sqrt{QQ^*}$ be the polar decomposition. We have that

$$cI_n = Q\overline{Q} = |Q^*|U^*|Q^*|^t U^t.$$

Taking determinants of both sides of

$$cI_n|Q^*|^{-1} = U^*|Q^*|^t U^t,$$

and using the relationship between the eigen values of $F_{[u]}$ and those of $|Q^*|$, we get $c^n = 1$. Note that c is real ($cI_n = Q\overline{Q} = \overline{Q}Q = \overline{Q\overline{Q}}$), we see that $c = \pm 1$ for n even and $c = 1$ for n odd.

Conversely, we claim that $Q\overline{Q} = cI_n$ with $c = \pm 1$ for n even and $c = 1$ for n odd implies that $\mathrm{Tr}(QQ^*) = \mathrm{Tr}((QQ^*)^{-1})$. To see this, let $r > 0$ be such that $\mathrm{Tr}(Q_1Q_1^*) = \mathrm{Tr}((Q_1Q_1^*)^{-1})$, where $Q_1 = rQ$. Then the above analysis applied to $B_u(Q_1)$ yields $Q_1\overline{Q_1} = cI_n$, i.e. $r^2Q\overline{Q} = cI_n$. Hence $r = 1$.

Let us summarize this discussion in the following:

PROPOSITION 2.2. *Let $Q \in GL(n, \mathbb{C})$ be a matrix such that $Q\overline{Q} \in \mathbb{R}I_n$. Then the condition* $\mathrm{Tr}(QQ^*) = \mathrm{Tr}((QQ^*)^{-1})$ *is equivalent to $Q\overline{Q} = \pm I_n$ for n even and $Q\overline{Q} = I_n$ for n odd.*

It would be interesting to find an elementary proof of the above fact (i.e. a proof without using quantum group theory).

Since $B_u(rQ) = B_u(Q)$ for any non-zero r, we will assume the normalization $Q\overline{Q} = \pm I_n$ below to find a parametrization of the equivalent classes of $B_u(Q)$. We note that if both Q and Q' are so normalized and that $Q' = zS^tQS$ for some non-zero complex number z, then a straightforward computation shows that z has modulus 1. So we can restrict the z in Theorem 2.1 (1) to complex numbers of modulus 1.

We will need the following easy lemma.

LEMMA 2.3. *Let $Q' = zS^tQS$, where z is a number of modulus 1, S is a unitary scalar matrix of the same size as Q. Let $Q = U|Q|$ and $Q' = U'|Q'|$ be the polar decompositions of Q and Q'. Then $|Q'| = S^{-1}|Q|S$ and $U' = zS^tUS$.*

Proof. The identity $|Q'| = S^{-1}|Q|S$ follows from $Q'^*Q' = S^{-1}Q^*QS$.

Now, on the one hand we have

$$Q' = U'|Q'| = U'S^{-1}|Q|S,$$

on the other hand we have

$$Q' = zS^tQS = zS^tU|Q|S.$$

Hence $U'S^{-1}|Q|S = zS^tU|Q|S$, and $U' = zS^tUS$. ∎

From the analysis preceding Proposition 2.2, we can assume that the eigenvalues of $|Q|$ are ordered either

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant \mu_k^{-1} \geqslant \cdots \geqslant \mu_2^{-1} \geqslant \mu_1^{-1}$$

or

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant 1 \geqslant \mu_k^{-1} \geqslant \cdots \geqslant \mu_2^{-1} \geqslant \mu_1^{-1}$$

according to $n = 2k$ or $n = 2k+1$. In virtue of Lemma 2.3 and Theorem 2.1 (1), we can assume that either

$$|Q| = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_k, \mu_k^{-1}, \ldots, \mu_2^{-1}, \mu_1^{-1})$$

or

$$|Q| = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_k, 1, \mu_k^{-1}, \ldots, \mu_2^{-1}, \mu_1^{-1})$$

is in diagonal form. So $Q = U \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_2^{-1}, \mu_1^{-1})$. Then from the normalization $Q\overline{Q} = \pm I_n$ and Lemma 2.3, we immediately obtain the following more explicit form of Theorem 2.1 (1):

THEOREM 2.4. *The isomorphism classes of $B_{\mathrm{u}}(Q)$ are given by*

$$(U, (\mu_1, \mu_2, \ldots, \mu_k)), \quad \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant 1,$$

where $U \in U(n)$ is a solution of the equation

$$\begin{aligned} &U \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_k, \mu_k^{-1}, \ldots, \mu_2^{-1}, \mu_1^{-1}) \\ &\qquad = c \cdot \operatorname{diag}(\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_k^{-1}, \mu_k, \ldots, \mu_2, \mu_1) U^{\mathrm{t}}, \end{aligned}$$

or the equation

$$\begin{aligned} &U \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_k, 1, \mu_k^{-1}, \ldots, \mu_2^{-1}, \mu_1^{-1}) \\ &\qquad = \operatorname{diag}(\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_k^{-1}, 1, \mu_k, \ldots, \mu_2, \mu_1) U^{\mathrm{t}} \end{aligned}$$

according to $n = 2k$, where $c = \pm 1$, or $n = 2k+1$. The pairs $(U, (\mu_1, \mu_2, \ldots, \mu_k))$ and $(U', (\mu'_1, \mu'_2, \ldots, \mu'_k))$ represent the same class if and only if $(\mu_1, \mu_2, \ldots, \mu_k) = (\mu'_1, \mu'_2, \ldots, \mu'_k)$ and $U' = zS^{\mathrm{t}}US$ for a $z \in \mathbb{T}$ and a stabilizing unitary matrix S:

$$S \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_2^{-1}, \mu_1^{-1}) S^{-1} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_2^{-1}, \mu_1^{-1}).$$

We can now easily recover the classification in [12] for $SU_q(2)$ from the above. First we note that $C(SU_q(2)) = B_{\mathrm{u}}(Q)$ with normalized $Q = \begin{bmatrix} 0 & -s\sqrt{|q|}^{-1} \\ \sqrt{|q|} & 0 \end{bmatrix}$, where $s = q^{-1}|q|$ (cf. Section 5 of [1] or [6]). In this case $Q = U|Q|$ with

$$|Q| = \begin{bmatrix} \sqrt{|q|}^{-1} & 0 \\ 0 & \sqrt{|q|} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -s \\ 1 & 0 \end{bmatrix}.$$

So the parametrization for $C(SU_q(2))$ in terms of Theorem 2.4 is $\left(U, \sqrt{|q|}^{-1}\right)$, which is equivalent to saying that they are non-isomorphic to each other for $q \in [-1, 1] \backslash \{0\}$ (cf. Theorem 3.1 of [11]).

Our work in progress shows that the quantum groups $B_{\mathrm{u}}(Q)$ are *simple* when $Q\overline{Q} \in \mathbb{R} I_n$.

3. THE QUANTUM GROUPS $A_u(Q)$ AND $B_u(Q)$ FOR ARBITRARY Q

In the following, h will denote the Haar measure of the ambient quantum group. Note that $A_u(Q) = C(\mathbb{T})$ and $B_u(Q) = C^*(\mathbb{Z}/2\mathbb{Z})$ for $Q \in GL(1, \mathbb{C})$.

THEOREM 3.1. *Let $Q \in GL(n, \mathbb{C})$. Let*

$$u = S \operatorname{diag}(m_1 w_1, m_2 w_2, \dots, m_k w_k) \cdot S^{-1}$$

be a decomposition of u of $A_u(Q)$ into unitary isotypical components $m_j w_j$, $j = 1, \dots, k$, for some $S \in U(n)$. Let Q_j be the positive matrix $h(w_j^t \overline{w}_j)$. Then

$$A_u(Q) \cong A_u(Q_1) * A_u(Q_2) * \cdots * A_u(Q_k).$$

Proof. Let $E = S^t Q \overline{S}$. Then the second set of relations for $A_u(Q)$ becomes

$$w^t E \overline{w} E^{-1} = I_n = E \overline{w} E^{-1} w^t, \quad \text{i.e. } w^t E \overline{w} = E,$$

where $w = \operatorname{diag}(v_1, v_2, \dots, v_k)$, $v_j = m_j w_j$, $j = 1, \dots, k$. Block decompose E according to w, say, $E = (E_{ij})_{i,j=1}^k$. Then the above set of relations becomes $v_i^t E_{ij} \overline{v}_j = E_{ij}$ where $i, j = 1, \dots, k$. By Lemma 1.2 of [3], $(v_i^t)^{-1} = \widetilde{Q}_i \overline{v}_i \widetilde{Q}_i^{-1}$, where $\widetilde{Q}_i = \operatorname{diag}(Q_i, \dots, Q_i)$ (m_i copies). Hence $\widetilde{Q}_i^{-1} E_{ij} \overline{v}_j = \overline{v}_i \widetilde{Q}_i^{-1} E_{ij}$. Since the w_i's are mutually inequivalent irreducible representations, we deduce that $E_{ij} = 0$ for $i \neq j$, and that E_{jj} is a matrix of the form $(c_{rs}^j Q_j)_{r,s=1}^{m_j}$ for some complex scalars c_{rs}^j. From these, a computation shows that the entries of the matrix $\widetilde{u} = S \operatorname{diag}(m_1 u_1, m_2 u_2, \dots, m_k u_k) S^{-1}$ satisfy the defining relations for $A_u(Q)$, where u_j is the fundamental representation of $A_u(Q_j)$, $j = 1, \dots, k$. Hence there is a surjection π from $A_u(Q)$ to $A_u(Q_1) * A_u(Q_2) * \cdots * A_u(Q_k)$ such that $\pi(u) = \widetilde{u}$. That is $\pi(w_j) = u_j$, $j = 1, \dots, k$.

Again by Lemma 1.2 of [3] and the properties of free product Woronowicz C^*-algebras ([6]), there is a surjection ρ from $A_u(Q_1) * A_u(Q_2) * \cdots * A_u(Q_k)$ to $A_u(Q)$ such that $\rho(u_j) = w_j$, $j = 1, \dots, k$. This is the inverse of π. ∎

COROLLARY 3.2. (i) *Let $Q = \operatorname{diag}(e^{i\theta_1} P_1, e^{i\theta_2} P_2, \dots, e^{i\theta_k} P_k)$, with positive matrices P_j and distinct angles $0 \leqslant \theta_j < 2\pi$, ($j = 1, \dots, k$, $k \geqslant 1$). (Note that every normal matrix is unitarily equivalent to one such, unique up to permutation of the indices j.) Then*

$$A_u(Q) \cong A_u(P_1) * A_u(P_2) * \cdots * A_u(P_k).$$

(ii) *If $Q \in GL(2, \mathbb{C})$ is a non-normal matrix, then $A_u(Q) = C(\mathbb{T})$.*

(iii) *For $Q \in GL(2, \mathbb{C})$, $A_u(Q)$ is either isomorphic to $C(\mathbb{T})$, or $C(\mathbb{T}) * C(\mathbb{T})$, or $A_u(\operatorname{diag}(1, q))$ with $0 < q \leqslant 1$.*

Proof. (i) Let S and E be as in the proof of Theorem 3.1. Since we do not have an explicit formula for the Haar measure of $A_u(Q)$, we must determine the matrices Q_j in Theorem 3.1 by other means.

We have an evident surjection

$$\pi : A_u(Q) \to A_u(P_1) * A_u(P_2) * \cdots * A_u(P_k)$$

such that $(\pi(u_{ij})) = \operatorname{diag}(u_1, \dots, u_k)$, where $u = (u_{ij})$ is the fundamental representation of $A_u(Q)$ and u_j is the fundamental representation of $A_u(P_j)$, $j = 1, \dots, k$.

That is, the free product quantum group of the $A_{\mathrm{u}}(P_j)$'s is a quantum subgroup of $A_{\mathrm{u}}(Q)$ (cf. [6] for the terminology). Since the u_j's are mutually inequivalent representations (cf. Theorem 3.10 of [6]), the multiplicities of the irreducible constituents of u are all equal to one and the matrix E in the proof of Theorem 3.1 is of the form

$$E = \mathrm{diag}(c_1 Q_1, \ldots, c_l Q_l)$$

for some $l \leqslant k$, $c_1, \ldots, c_l \in \mathbb{C}^*$, $Q_1, \ldots, Q_l > 0$. Since the angles θ_j are distinct and E is unitarily equivalent to Q, we must have $l = k$, $|c_j| Q_j = P_j$ and $c_j |c_j|^{-1} = \mathrm{e}^{\mathrm{i}\theta_j}$ after a possible permutation of the indices j. (Note that permutation of the indices j does not change the quantum group $B_{\mathrm{u}}(T_1) * \cdots * B_{\mathrm{u}}(T_k)$.) We conclude the proof by noting that $A_{\mathrm{u}}(Q_j) = A_{\mathrm{u}}(|c_j| Q_j) = A_{\mathrm{u}}(P_j)$.

(ii) Since Q is not positive, the fundamental representation u of $A_{\mathrm{u}}(Q)$ is reducible (cf. 3.1 of [1]). Since Q is not normal, we deduce from (i) and the proof of Theorem 3.1 that u is equivalent to a representation of the form $2w_1$ (i.e. $m_1 = 2$), where w_1 is an irreducible representation of dimension 1.

(iii) This follows from (i) and (ii) (cf. also Theorem 1.1). ∎

THEOREM 3.3. *Let $Q \in GL(n, \mathbb{C})$. Then the fundamental representation u of $B_{\mathrm{u}}(Q)$ has a unitary isotypical decomposition of the form*

$$u = S\, \mathrm{diag}(m_1 w_1, m_1 \widetilde{w}_1, m_2 w_2, m_2 \widetilde{w}_2, \ldots \\ \ldots, m_k w_k, m_k \widetilde{w}_k, m_1' w_1', m_2' w_2', \ldots, m_l' w_l') S^{-1}$$

for some $S \in U(n)$, where the w_i's are not self-conjugate, $\widetilde{w}_i = P_i^{1/2} \overline{w}_i P_i^{-1/2}$, $P_i = h(w_i^{\mathrm{t}} \overline{w}_i)$, the w_j''s are self-conjugate, $i = 1, \ldots, k$, $j = 1, \ldots, l$, $k, l \geqslant 0$. Let Q_j be such that $w_j' = Q_j^* \overline{w'}_j Q_j^{*-1}$. Then the $Q_j \overline{Q}_j$'s are nonzero scalar matrices and

$$B_{\mathrm{u}}(Q) \cong A_{\mathrm{u}}(P_1) * A_{\mathrm{u}}(P_2) * \cdots * A_{\mathrm{u}}(P_k) * B_{\mathrm{u}}(Q_1) * B_{\mathrm{u}}(Q_2) * \cdots * B_{\mathrm{u}}(Q_l).$$

Proof. Note that the fundamental representation u of $B_{\mathrm{u}}(Q)$ is self-conjugate: $u = Q^* \overline{u} Q^{*-1}$. Hence, in the isotypical decomposition of u, if an irreducible component is not self-conjugate, then its conjugate representation also appears (with the same multiplicity as the former). By Lemma 1.2 of [3], we deduce that each $P_i^{1/2} \overline{w}_i P_i^{-1/2}$ is a unitary representation. Hence u has a decomposition as stated in the theorem. Since the w_j''s are irreducible, $Q_j \overline{Q}_j$ are scalar matrices as in [1].

Let $E = S^{\mathrm{t}} Q S$. Then the second set of relations for $B_{\mathrm{u}}(Q)$ becomes

$$w^{\mathrm{t}} E w E^{-1} = I_n = E w E^{-1} w^{\mathrm{t}}, \quad \text{i.e. } w^{\mathrm{t}} E w = E,$$

where

$$w = \mathrm{diag}(v_1, \widetilde{v}_1, v_2, \widetilde{v}_2, \ldots, v_k, \widetilde{v}_k, v_1', v_2', \ldots, v_l'),$$

$v_i = m_i w_i$, $\widetilde{v}_i = m_i \widetilde{w}_i$, $v_j' = m_j' w_j'$, $i = 1, \ldots, k, j = 1, \ldots, l$. Block decompose E according to w, we find that E is of the form

$$E = \mathrm{diag}\left(\begin{bmatrix} 0 & X_1 \\ Y_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & X_2 \\ Y_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & X_k \\ Y_k & 0 \end{bmatrix}, Z_1, Z_2, \ldots, Z_l \right),$$

and the above set of relations takes the form

$$v_i^{\mathrm{t}} X_i \widetilde{v}_i = X_i, \quad \widetilde{v}_i^{\mathrm{t}} Y_i v_i = Y_i, \quad v_j'^{\mathrm{t}} Z_j v_j = Z_j.$$

By the assumptions in Theorem 3.3, we have that (cf. proof of Theorem 3.1)

$$(w_i^{\mathrm{t}})^{-1} = P_i \overline{w}_i P_i^{-1}, \quad (\widetilde{w}_i^{\mathrm{t}})^{-1} = \overline{P}_i^{-1/2} w_i \overline{P}_i^{1/2}, \quad (w_j'^{\mathrm{t}})^{-1} = Q_j w_j' Q_j^{-1}.$$

Hence we have

$$X_i = (x_{rs}^i P_i^{1/2})_{r,s=1}^{m_i}, \quad Y_i = (y_{rs}^i \overline{P}_i^{-1/2})_{r,s=1}^{m_i}, \quad Z_j = (z_{rs}^j Q_j)_{r,s=1}^{m_j'}$$

for $x_{rs}^i, y_{rs}^i, z_{rs}^j \in \mathbb{C}$. From these, a computation then shows that the entries of the matrix

$$\widetilde{u} = S\,\mathrm{diag}(m_1 u_1, m_1 \widetilde{u}_1, m_2 u_2, m_2 \widetilde{u}_2, \ldots, m_k u_k, m_k \widetilde{u}_k, m_1' u_1', m_2' u_2', \ldots, m_l' u_l') S^{-1}$$

satisfy the defining relations for $B_{\mathrm{u}}(Q)$, where u_i (respectively u_j') is the fundamental representation of $A_{\mathrm{u}}(P_i)$ (respectively $B_{\mathrm{u}}(Q_j)$) and $\widetilde{u}_i = P_i^{1/2} \overline{u}_i P_i^{-1/2}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. Hence there is a surjection π from $B_{\mathrm{u}}(Q)$ onto $A_{\mathrm{u}}(P_1) * A_{\mathrm{u}}(P_2) * \cdots * A_{\mathrm{u}}(P_k) * B_{\mathrm{u}}(Q_1) * B_{\mathrm{u}}(Q_2) * \cdots * B_{\mathrm{u}}(Q_l)$, such that $\pi(u) = \widetilde{u}$. That is, $\pi(w_i) = u_i$ and $\pi(w_j') = u_j'$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. As in the proof of Theorem 3.1, π is an isomorphism. ∎

COROLLARY 3.4. (i) *Let* $Q = \mathrm{diag}(T_1, T_2, \ldots, T_k)$ *be a matrix such that* $T_j \overline{T}_j = \lambda_j I_{n_j}$, *where the* λ_j*'s are distinct non-zero real numbers (the sizes* n_j *need not be different),* $j = 1, \ldots, k$, $k \geqslant 1$. *Then*

$$B_{\mathrm{u}}(Q) \cong B_{\mathrm{u}}(T_1) * B_{\mathrm{u}}(T_2) * \cdots * B_{\mathrm{u}}(T_k).$$

(ii) *Let* $Q = \begin{bmatrix} 0 & T \\ q\overline{T}^{-1} & 0 \end{bmatrix}$, *where* $T \in GL(n, \mathbb{C})$ *and* q *is a complex but non-real number. Then* $B_{\mathrm{u}}(Q)$ *is isomorphic to* $A_{\mathrm{u}}(|T|^2)$ *under the map* π *which sends the entries of the fundamental representation* u *of* $B_{\mathrm{u}}(Q)$ *to the entries of the matrix* $\mathrm{diag}(u_1, u_2)$, *where* u_1 *is the fundamental representation of* $A_{\mathrm{u}}(|T|^2)$ *and* $u_2 = |T| \overline{u}_1 |T|^{-1}$, *the unitary representation equivalent of* $\overline{u}_1$.

Proof. (i) Let S and E be as in the proof of Theorem 3.3. We have an evident surjection from $B_{\mathrm{u}}(Q)$ onto the free product $B_{\mathrm{u}}(T_1) * B_{\mathrm{u}}(T_2) * \cdots * B_{\mathrm{u}}(T_k)$ sending the matrix entries of the fundamental representation u of $B_{\mathrm{u}}(Q)$ to entries of $\mathrm{diag}(u_1, \ldots, u_k)$, where u_j is the fundamental representation of $B_{\mathrm{u}}(T_j)$, $j = 1, \ldots, k$. Since the u_j's are mutually inequivalent self-conjugate representations and none of them is conjugate to *another* (cf. Theorem 3.10 of [6]), Theorem 3.3 implies that the pieces $A_{\mathrm{u}}(P_i)$ do not appear in the decomposition of $B_{\mathrm{u}}(Q)$ and that the multiplicities $m_j' = 1$. Therefore, the matrix E in the proof of Theorem 3.3 has the form

$$E = \mathrm{diag}(Z_1, Z_2, \ldots, Z_l)$$

for some $l \leqslant k$, where $Z_j = z_j Q_j$, $z_j \in \mathbb{C}^*$, and the $Q_j \overline{Q}_j$'s are scalar matrices with $j = 1, \ldots, l$. Hence,

$$E\overline{E} = \mathrm{diag}(c_1 I_{n_1'}, \ldots, c_l I_{n_l'})$$

for some $c_j \in \mathbb{C}^*$, where n'_j is the size of the matrix Q_j, $j = 1, \dots, l$. From the unitary equivalence

$$E\overline{E} = S^{\mathrm{t}} Q\overline{Q}\,\overline{S} = S^{\mathrm{t}}\,\mathrm{diag}(\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k})\overline{S}$$

and the fact that the λ_j's are distinct, we must have $l = k$, $c_j = \lambda_j$ and $n'_j = n_j$, up to a possible permutation of the indices j. Now $E\overline{E} = S^{\mathrm{t}} Q\overline{Q}\,\overline{S}$ takes the form

$$\mathrm{diag}(c_1 I_{n_1}, \dots, c_l I_{n_k}) = S^{\mathrm{t}}\,\mathrm{diag}(c_1 I_{n_1}, \dots, c_l I_{n_k})\overline{S}.$$

Then from the assumption that the c_j's are distinct we deduce again that S is a block diagonal matrix $S = \mathrm{diag}(S_1, \dots, S_k)$ with $S_j \in U(n_j)$, $j = 1, \dots, k$. Hence

$$E = S^{\mathrm{t}} Q S = \mathrm{diag}(S_1^{\mathrm{t}} T_1 S_1, \dots, S_k^{\mathrm{t}} T_k S_k),$$

and therefore $Z_j = z_j Q_j = S_j^{\mathrm{t}} T_j S_j$, $j = 1, \dots, k$. Hence by Theorem 2.1, $B_{\mathrm{u}}(Q_j)$ is isomorphic to $B_{\mathrm{u}}(T_j)$. The proof is finished by appealing to Theorem 3.3.

(ii) Let $T = U|T|$ be the polar decomposition of T. Then

$$Q = \begin{bmatrix} 0 & T \\ q\overline{T}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & |T| \\ q\overline{|T|}^{-1} & 0 \end{bmatrix} \begin{bmatrix} U^{\mathrm{t}} & 0 \\ 0 & 1 \end{bmatrix}.$$

By Theorem 2.1, we can assume $T > 0$ from now on. Then

$$(Q^*)^{-1} = \begin{bmatrix} 0 & T^{-1} \\ \overline{q}^{-1}\overline{T} & 0 \end{bmatrix}.$$

Let $u = \mathrm{diag}(u_1, u_2)$, where u_1 and u_2 are the unitary representations as given in the statement of Corollary 3.4. Then, a quick computation shows that

$$Q^*\overline{u}(Q^*)^{-1} = u,$$

that is, $u^{-1} = u^* = Q^{-1} u^{\mathrm{t}} Q$. Hence we have a surjection π from $B_{\mathrm{u}}(Q)$ onto $A_{\mathrm{u}}(T^2)$ as in the statement of Corollary 3.4 (ii). Then Theorem 3.10 of [6] and Theorem 3.3 above implies that $B_{\mathrm{u}}(Q)$ is isomorphic to $A_{\mathrm{u}}(P_1)$ for some $P_1 > 0$. Now the rigidity of $A_{\mathrm{u}}(P_1)$ (see Remark (1.2) in Section 1) implies that $A_{\mathrm{u}}(P_1)$, and therefore $B_{\mathrm{u}}(Q)$, is isomorphic to $A_{\mathrm{u}}(T^2)$. ∎

CONCLUDING REMARKS. 3.5. (i) Using Theorem 1.1 (respectively Theorem 2.1 along with [6], one can show that the decomposition in Theorem 3.1 (respectively Theorem 3.3) is unique in the evident sense.

(ii) Theorems 1.2, 2.1, 3.1 and 3.3 solve Problem 1.1 of [9].

(iii) Using the same method as in Theorems 3.1 and 3.3, we see that intersections of quantum groups of the form $A_{\mathrm{u}}(Q)$ (respectively $B_{\mathrm{u}}(Q)$) in the sense of [9] does not give rise to non-trivial finite quantum groups. This solves Problem 2.4 of [9].

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