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SPECTRA OF INFINITE PARAMETRIZED BANACH COMPLEXES

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ABSTRACT. The spectral mapping theorem $\sigma(r(a)) = r(\sigma(a))$ for Taylor and Słodkowski spectra is proved when the operator family a generates a nilpotent Lie algebra and the family of noncommuting rational functions (and its limits) r generates an infinite-dimensional projective and weak quasinilpotent Banach Lie algebra.

KEYWORDS: *Parametrized Banach complex, Banach Lie algebra, Słodkowski spectra.*

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1. INTRODUCTION

Taylor and Słodkowski spectra were defined in [14] and [13] for commutative operator families. When an operator family generates a finite dimensional solvable Lie algebra, some analogs of Taylor and Słodkowski spectra were considered in [1]–[3], [6], [11]. Spectra of such families are described by behavior of the finite parametrized (on the character space of the finite dimensional Lie algebra) Koszul complex. A spectral mapping theorem for Taylor spectrum σ of such families was obtained in [6] by A.S. Fainshtein. More precisely, let $a = (a_1, \dots, a_n)$ be a finite family in the algebra $B(X)$ of bounded linear operators on a complex Banach space X, and let a generates a finite dimensional nilpotent Lie algebra $E \subseteq B(X)$. Let us consider a finite family of polynomials $p = (p_1, \dots, p_m)$ of noncommuting variables $a_1, \dots, a_n$, i.e., a finite family in the enveloping algebra $U(E)$ of E. Let P be a Lie subalgebra of $U(E)$ generated by the family p, and let $p(a) = (p_1(a), \dots, p_n(a))$. If P is finite dimensional (and in this case P is a nilpotent Lie algebra), then $\sigma(p(a)) = p(\sigma(a))$ ([6]). But, often the Lie subalgebra P is infinite dimensional, which implies that spectra of such families p should be described by behavior of the infinite parametrized complexes.

In this paper, we consider Słodkowski and Taylor spectra for infinite parametrized Banach complexes. It is generally assumed that complexes depend on a parameter of some topological space Ω. Let $(\mathbf{X}, \mathbf{d})$ be a Banach complex parametrized on Ω. A family of so called Słodkowski spectra $\sigma_{\pi,k}$ of parametrized Banach complexes will be defined. If $(\mathbf{X}, \mathbf{d})$ is the finite Koszul complex associated with a commutative operator family or a family of operators generating a finite dimensional solvable Lie algebra, then these spectra are reduced to the known Słodkowski spectra of corresponding operator families ([13] and [2]). As a special case of parametrized Banach complexes, we consider a Banach complex $\mathbf{C}(\alpha)$ generated by a Banach module (X, α) over a Banach Lie algebra $\mathcal{F}$, where $\alpha : \mathcal{F} \to B(X)$ is the bounded representation of $\mathcal{F}$. This complex is parametrized on the space $\Delta(\mathcal{F})$ $(\subseteq \mathcal{F}^*)$ of bounded characters of $\mathcal{F}$. Let σ be one of Słodkowski spectra. The set $\sigma(\mathbf{C}(\alpha))$ is called *Słodkowski spectrum* of the representation α and is denoted by $\sigma(\alpha)$. If S is the set of topological Lie generators of $\mathcal{F}$, then Słodkowski spectrum $\sigma(\alpha(S))$ of the operator family $\alpha(S)$ is defined as the image of $\sigma(\alpha)$ by the canonical projection $\Delta(\mathcal{F}) \to \mathbb{C}^S, \lambda \mapsto (\lambda(s))_{s\in S}$.

Our main result is an infinite-dimensional version of Fainshtein's result. We prove the spectral mapping theorem

$$\sigma(r(a)) = r(\sigma(a))$$

for the family of limits of rational functions (in particular, polynomials, rational functions) $r = (r_i)_{i\in I}$ of noncommuting variables $a_1, \dots, a_n$ generated nilpotent Lie algebra E, when r generates a weak quasinilpotent Banach-Lie algebra $\mathcal{F}$ (all operators $\mathrm{ad}(r_i) \in B(\mathcal{F})$ are quasinilpotent) and $\mathcal{F}$ is a projective Banach space.

The structure of the paper is the following. In Section 2 we recall the ultrapower technique which is essentially used in the paper.

In Section 3, we prove that spectra $\sigma_{\pi,k}(\mathbf{X}, \mathbf{d})$ are stable by taking the functor $\hom(Y, \cdot)$, where Y is a projective Banach space, and by taking ultrapowers.

In Section 4, we define a cone $\mathrm{Con}_\beta(\mathbf{X}, \mathbf{d})$ of a bounded endomorphism β of a Banach complex $(\mathbf{X}, \mathbf{d})$ parametrized on Ω. The complex $\mathrm{Con}_\beta(\mathbf{X}, \mathbf{d})$ is parametrized on $\Omega \times \mathbb{C}$. If $\Pi : \Omega \times \mathbb{C} \to \Omega$ is the canonical projection, then we prove the projection theorem $\sigma_{\pi,k}(\mathbf{X}, \mathbf{d}) = \Pi(\sigma_{\pi,k}(\mathrm{Con}_\beta(\mathbf{X}, \mathbf{d})))$, which plays the central role in spectral mapping properties.

In Section 5, we consider Słodkowski spectra $\sigma_{\pi,k}(\alpha)$ for a bounded representation α of a Banach Lie algebra E. The sets $\sigma_{\pi,k}(\alpha)$ are weak precompact as subsets of the dual space E^*. It is known that for a finite dimensional commutative Lie algebra E, the spectrum $\sigma_{\pi,0}(\alpha)$ coincides with the approximate point spectrum $\sigma^{\mathrm{ap}}(\alpha)$ of α. We show that it is not true for infinite dimensional Lie algebra E. The difference is removed by taking ultrapowers of Banach spaces; more precisely, we introduce ultraspectra $\sigma^{\mathrm{u}}_{\pi,k}(\alpha)$ as a union of spectra $\sigma_{\pi,k}(\alpha_{\mathcal{U}})$ by all nontrivial ultrafilters $\mathcal{U}$, where $\alpha_{\mathcal{U}}$ is an ultrapower of the representation α. It is proved that $\sigma^{\mathrm{u}}_{\pi,0}(\alpha) = \sigma^{\mathrm{ap}}(\alpha)$, and, if $\dim(E) < \infty$, then $\sigma^{\mathrm{u}}_{\pi,k}(\alpha) = \sigma_{\pi,k}(\alpha)$, $0 \leqslant k \leqslant \infty$.

In Section 6, we state some projection properties for spectra $\sigma_{\pi,k}(\alpha)$ when E is a quasinilpotent Banach Lie algebra. It is proved that if F is a closed ideal of finite codimension in a quasinilpotent Banach Lie algebra E, and $\alpha[E, E]$ consists of quasinilpotent operators, then $\sigma_{\pi,k}(\alpha)|_F = \sigma_{\pi,k}(\alpha|_F)$, where $\alpha|_F$ is the restriction of α to F, and $\sigma_{\pi,k}(\alpha)|_F$ consists of all restrictions of functionals from spectrum $\sigma_{\pi,k}(\alpha)$ to F. For a finite-dimensional Lie subalgebra F of E, we also prove that if E is a projective Banach space, then $\sigma_{\pi,k}(\alpha)|_F \subseteq \sigma_{\pi,k}(\alpha|_F)$.

In Section 7, we introduce the Banach algebra A of "limits of rational functions" of variables generated nilpotent Lie algebra E, acting on an E-module. We also provide examples for projective and weak quasinilpotent Banach-Lie subalgebras in A.

Sections 8–10 are devoted to spectral mapping properties for subalgebras of the Banach algebra A.

2. PRELIMINARIES

As usual, $\mathbb{N}$ is the set of all positive integers, $\mathbb{C}$ is the field of complex numbers, and $\ell_1(S)$ is the (Banach) space of all absolutely summable complex functions on a set S. Let A, B, C, and D be arbitrary sets such that $C \subseteq B \subseteq A$, and let $f : B \to D$ be a function. Then, $f|_C$ denotes the restriction of f on C, and if f is extended up to a function $g : A \to D$, then we write $g = f|^A$. For any set of functions $\mathcal{L}$ defined on B with values in D we set $\mathcal{L}|_C = \{f|_C : f \in \mathcal{L}\}$. For complex normed spaces X and Y, the normed space of all bounded linear operators (with operator norm) from X into Y is denoted by $B(X, Y)$. Set $B(X) = B(X, X)$. The kernel and the image of an operator $T \in B(X, Y)$ are denoted by $\mathrm{N}(T)$ and $\mathrm{R}(T)$ respectively, and $T^* \in B(Y^*, X^*)$ denotes the dual operator. For a subset $M \subseteq A$ of an (associative) Banach algebra A with the identity element 1_A, we set $M^n = \{a_1 \cdots a_n : a_i \in M\}$. Then the union $\bigcup_n M^n$ is the multiplicative semigroup generated by M in A, denoted by $\mathrm{SG}(M)$. Let $\|M\| = \sup\{\|a\| : a \in M\}$ for a bounded set M and $\rho(M) = \lim_n \|M^n\|^{1/n}$. The number $\rho(M)$ is called the *(joint) spectral radius* of the set M ([12]).

The associative hull of M in any topological algebra A with identity is denoted by $\mathcal{P}(M)$, i.e. $\mathcal{P}(M)$ is the set of all polynomials $\{p(M)\}$ in A. We shall use more general "functions" (on M) in A than polynomials. We define the set of "rational functions" on M by follow Yu.V. Turovskii ([17]) as a collection of expressions $r^{(n)}(M)$, $n \in \mathbb{N}$, constructed as shown below. Let $\{r^{(0)}(M)\}$ be a collection of all polynomials. If the collection $\{r^{(n-1)}(M)\}$ has been defined, then we define an expression $r^{(n)}(M)$ as a polynomial of $r^{(n-1)}(M)$ and of $r^{(n-1)}(M)^{-1}$ if $r^{(n-1)}(M)$ is invertible. We say that a rational function of the form $r^{(n)}(M)$ has order n. A subalgebra $B \subseteq A$ is said to be a full subalgebra, if any invertible element $b \in B$ that is invertible in A is invertible in B. It is clear that full subalgebras are stable by taking arbitrary intersections, and the set of all rational functions $\mathcal{R}(M) = \bigcup_n \{r^{(n)}(M)\}$ is the full subalgebra in A generated by M. The closure of this subalgebra is called the *closed full hull* of M in A. For a homomorphism of unital topological algebras $h : A \to B$ one can easily observe that $h(r(M)) = r(h(M))$ for any rational function $r(M)$.

By a direct sum $X \oplus Y$ of Banach spaces X and Y we shall mean the ℓ_1-norm sum with $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in X \oplus Y$. The projective tensor product of Banach spaces X and Y is denoted by $X \widehat{\otimes} Y$. Let X be a Banach space and $n \in \mathbb{N}$. Assume that $X^{\widehat{\otimes} n}$ is $X \widehat{\otimes} \cdots \widehat{\otimes} X$ (n-times), S_n is the group of all permutations of the set $\{1,\ldots,n\}$, and $\varepsilon(\tau)$ is the sign of the permutation $\tau\in S_n$. Let $\delta_\tau\in B(X^{\widehat{\otimes}n})$, $\delta_\tau(x_1\otimes\cdots\otimes x_n)=x_{\tau(1)}\otimes\cdots\otimes x_{\tau(n)}$, $\tau\in S_n$. We define the exterior power Λ^nX of X as the image of the projection $A_n\in B(X^{\widehat{\otimes}n})$, $A_n=(n!)^{-1}\sum\limits_{\tau\in S_n}\varepsilon(\tau)\delta_\tau$. Assume that $x_1\wedge\cdots\wedge x_n=A_n(x_1\otimes\cdots\otimes x_n)$ and $X^{\widehat{\otimes}0}=\Lambda^0X=\mathbb{C}$. One can easily prove that, for a Banach space Y, there is an isometry between $B(\Lambda^nX,Y)$ and the space $C^n(X,Y)$ of all bounded skewsymmetric n-linear forms on X with values in Y. All Banach complexes considered are assumed to be cochain and nonnegative, i.e. a Banach complex $(\mathbf{X},\mathbf{d})$ is a sequence $0\longrightarrow X^0\xrightarrow{d^0}X^1\xrightarrow{d^1}\cdots\xrightarrow{d^{n-1}}X^n\xrightarrow{d^n}\cdots$, where X^n is a Banach space and $d^n\in B(X^n,X^{n+1})$, $d^{n+1}d^n=0$, $n\geqslant 0$. We omit the index n of d^n if it is not cause confusion. (Semi)normed spaces of cohomologies of the complex $(\mathbf{X},\mathbf{d})$ are denoted by $H^n(\mathbf{X},\mathbf{d})$, $n\geqslant 0$. Let Y be a Banach space. Then $B(Y,(\mathbf{X},\mathbf{d}))$:

$$0\longrightarrow B(Y,X^0)\xrightarrow{\beta^0}B(Y,X^1)\xrightarrow{\beta^1}\cdots\xrightarrow{\beta^{n-1}}B(Y,X^n)\xrightarrow{\beta^n}\cdots,$$

is a Banach complex, where $\beta^nT=d^n\cdot T$, $T\in B(Y,X^n)$. A Banach space Y is called *projective* if the complex $B(Y,(\mathbf{X},\mathbf{d}))$ is exact for any exact Banach complex $(\mathbf{X},\mathbf{d})$. The class of all projective Banach spaces is denoted by Proj. It is easy to prove that $Y\in\mathrm{Proj}$ if and only if for an epimorphism of Banach spaces $T:X\to Z$ and an operator $\varphi\in B(Y,Z)$ there exist $\psi\in B(Y,X)$ such that $T\cdot\psi=\varphi$. In particular, if $Y_1,Y_2\in\mathrm{Proj}$, then $Y_1\oplus Y_2\in\mathrm{Proj}$ and $Y_1\widehat{\otimes}Y_2\in\mathrm{Proj}$. By Proposition 4.3 from [15], $\ell_1(S)\in\mathrm{Proj}$ for arbitrary set S.

LEMMA 2.1. *Let $Y\in\mathrm{Proj}$ and $n\in\mathbb{N}$, then $\Lambda^nY\in\mathrm{Proj}$.*

Proof. It suffices to prove that for any epimorphism $T:X\to Z$ and an n-form $\omega\in C^n(Y,Z)$ there exists an n-form $\varsigma\in C^n(Y,X)$ such that $T\cdot\varsigma=\omega$. Let us proceed by induction on n. For any $y\in Y$ consider an $(n-1)$-form

$$\omega_y\in C^{n-1}(Y,Z),\quad \omega_y(y_1,\ldots,y_{n-1})=\omega(y,y_1,\ldots,y_{n-1})$$

and an operator $F\in B(Y,C^{n-1}(Y,Z))$, $F(y)=\omega_y$. By induction hypothesis, the operator $T_{n-1}:C^{n-1}(Y,X)\to C^{n-1}(Y,Z)$, $T_{n-1}h=T\cdot h$, is surjective. Hence there exists an operator $G\in B(Y,C^{n-1}(Y,X))$ such that $T_{n-1}\cdot G=F$. Let $\varsigma\in C^n(Y,X)$, $\varsigma(y_1,\ldots,y_n)=n^{-1}\sum\limits_{i=1}^n(-1)^{i-1}G(y_i)(y_1,\ldots,\widehat{y_i},\ldots,y_n)$, where $\widehat{y_i}$ means omission of the variable y_i. Then we have $(T\cdot\varsigma)(y_1,\ldots,y_n)=n^{-1}\sum\limits_{i=1}^n(-1)^{i-1}T_{n-1}(G(y_i))(y_1,\ldots,\widehat{y_i},\ldots,y_n)=\omega(y_1,\ldots,y_n)$. ∎

LEMMA 2.2. ([5], Lemma 1.2) Let X,Y,Z be Banach spaces and let $S\in B(X,Y)$, $T\in B(Y,Z)$, such that $TS=0$. Then, $\mathrm{R}(S)\neq\mathrm{N}(T)$ or $\mathrm{R}(T)$ is not closed, iff there exist bounded sequences $\{y_n\}\subset Y$ and $\{f_n\}\subset Y^*$ such that

$$\lim_n Ty_n=0,\quad \lim_n S^*f_n=0,\quad f_n(y_n)=1.$$

Let us remind some assertions about ultrapower technique. Let S be an infinite set and let $\mathcal{U}$ be a nontrivial (i.e. $\bigcap\limits_{M\in\mathcal{U}}M=\emptyset$) ultrafilter in S. The ultrafilter $\mathcal{U}$ is called to be $\aleph_0$-*incomplete* (see [10] and [4]) if there exists a countable partition $\{S_n : n \in \mathbb{N}\}$ of S such that $S_n \notin \mathcal{U}$ for each $n \in \mathbb{N}$. The filter of complements of finite subsets in $\mathbb{N}$ is called the *Fréchet filter*. Any nontrivial ultrafilter on $\mathbb{N}$ is $\aleph_0$-incomplete because it is majorized by the Fréchet filter. By [4], there exist $\aleph_0$-incomplete ultrafilters in any infinite set S. In the sequel, by an ultrafilter we will mean a nontrivial $\aleph_0$-incomplete ultrafilter, if not said otherwise. Let X be a Banach space and let $\ell_\infty(S, X)$ a Banach space of all bounded families $(x_s)_{s\in S}$ from X with sup-norm. For an ultrafilter $\mathcal{U}$ on I, let $N_{\mathcal{U}}(X)$ be a closed subspace in $\ell_\infty(S, X)$ which consists of all families $(x_s)_{s\in S}$ with $\lim\limits_{\mathcal{U}} x_s = 0$. The *ultrapower of* X *following* $\mathcal{U}$ is called the quotient space $X_{\mathcal{U}} = \ell_\infty(S, X)/N_{\mathcal{U}}(X)$. The coset of $(x_s)_{s\in S} \in \ell_\infty(S, X)$ in $X_{\mathcal{U}}$ is denoted by $[x_s]$. One can easily check that the norm $\|[x_s]\|$ is $\lim\limits_{\mathcal{U}} \|x_s\|$. The space X is contained in $X_{\mathcal{U}}$ as the subspace generated by constant families of $\ell_\infty(S, X)$, and $X_{\mathcal{U}} = X$ iff X is finite dimensional space (see Proposition 7 from [4]). For a subset $C \subseteq X$, the *ultrapower of* C *following* $\mathcal{U}$ is $C_{\mathcal{U}} = \{[c_s] \in X_{\mathcal{U}} : c_s \in C\}$. Let us consider ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on S and T respectively. Let $A_t = \{s \in S : (s, t) \in A\}$, $t \in T$, and $T_A = \{t \in T : A_t \in \mathcal{U}\}$ for $A \subseteq S \times T$. The production $\mathcal{U} \times \mathcal{V}$ is defined as the family of subsets $A \subseteq S \times T$ for which $T_A \in \mathcal{V}$. By [4], $\mathcal{U} \times \mathcal{V}$ is an ultrafilter.

Lemma 2.3. ([4]) *Let $\mathcal{U}$ and $\mathcal{V}$ be nontrivial ultrafilters on S and T, respectively. If one of them is $\aleph_0$-incomplete, then $\mathcal{U} \times \mathcal{V}$ is also $\aleph_0$-incomplete. Moreover, the canonical operator $X_{\mathcal{U}\times\mathcal{V}} \to (X_{\mathcal{U}})_{\mathcal{V}}$, $[x_{(s,t)}]_{(s,t)\in S\times T} \mapsto [[x_{(s,t)}]_{s\in S}]_{t\in T}$, is an isometric isomorphism.*

An arbitrary operator $T \in B(X, Y)$ between Banach spaces is extended up to $T_{\mathcal{U}} \in B(X_{\mathcal{U}}, Y_{\mathcal{U}})$, $T_{\mathcal{U}}[x_s] = [Tx_s]$ and $\|T_{\mathcal{U}}\| = \|T\|$. The following assertion was proved in Propositions 15, 16, 20, 22 from [4].

Lemma 2.4. *Let $T \in B(X, Y)$. Then $\mathrm{N}(T)_{\mathcal{U}} \subseteq \mathrm{N}(T_{\mathcal{U}})$, $\mathrm{R}(T_{\mathcal{U}}) \subseteq \mathrm{R}(T)_{\mathcal{U}}$ and $\overline{\mathrm{R}(T)} = Y \cap \overline{\mathrm{R}(T_{\mathcal{U}})}$. Moreover, the following statements are equivalent:*

(i) $\mathrm{R}(T)$ *is closed;*

(ii) $\mathrm{R}(T_{\mathcal{U}})$ *is closed;*

(iii) $\mathrm{N}(T)_{\mathcal{U}} = \mathrm{N}(T_{\mathcal{U}})$;

(iv) $\mathrm{R}(T_{\mathcal{U}}) = \mathrm{R}(T)_{\mathcal{U}}$.

3. SŁODKOWSKI SPECTRA

In this section we will introduce Słodkowski spectra for parametrized complexes and prove that these spectra are stable by taking the functor $\hom(Y,\cdot)$ (where Y is a projective Banach space) and by taking ultrapowers.

Let Ω be a topological space and $\mathbf{X} = \{X^n : n \geqslant 0\}$ be a collection of Banach spaces. Suppose that there exists a collection of continuous maps $\mathbf{d} = \{d^n : n \geqslant 0\}$, $d^n : \Omega \to B(X^n, X^{n+1})$, such that $(\mathbf{X}, \mathbf{d}(\lambda))$ is a Banach complex for each $\lambda \in \Omega$, where $\mathbf{d}(\lambda) = \{d^n(\lambda) : n \geqslant 0\}$. The collection of Banach complexes $(\mathbf{X}, \mathbf{d}(\lambda))$, $\lambda \in \Omega$, is called a *Banach complex parametrized on* Ω. Shortly, we say that an Ω-Banach complex $(\mathbf{X}, \mathbf{d})$ is given. A morphism $\mathbf{f} : (\mathbf{X}, \mathbf{d}) \to (\mathbf{Y}, \mathbf{d}')$ of Ω-Banach complexes is usually defined as a collection of continuous maps $\mathbf{f} = \{f^n : n \geqslant 0\}$, $f^n : \Omega \to B(X^n, Y^n)$, such that $\mathbf{f}(\lambda) : (\mathbf{X}, \mathbf{d}(\lambda)) \to (\mathbf{Y}, \mathbf{d}'(\lambda))$ is a morphism of Banach complexes. A short sequence of Ω-Banach complexes $0 \longrightarrow (\mathbf{X}, \mathbf{d}) \xrightarrow{\mathbf{f}} (\mathbf{Y}, \mathbf{d}') \xrightarrow{\mathbf{g}} (\mathbf{Z}, \mathbf{d}'') \longrightarrow 0$ is said to be exact if all sequences of Banach complexes

$$0 \longrightarrow (\mathbf{X}, \mathbf{d}(\lambda)) \xrightarrow{\mathbf{f}(\lambda)} (\mathbf{Y}, \mathbf{d}'(\lambda)) \xrightarrow{\mathbf{g}(\lambda)} (\mathbf{Z}, \mathbf{d}''(\lambda)) \longrightarrow 0, \quad \lambda \in \Omega,$$

are exact. Let $(\mathbf{X}, \mathbf{d})$ be a Ω-Banach complex. For any integer $p \geqslant 0$, we define the following set $\Sigma_p(\mathbf{X}, \mathbf{d}) = \{\lambda \in \Omega : H^p(\mathbf{X}, \mathbf{d}(\lambda)) \neq \{0\}\}$, where $H^p(\mathbf{X}, \mathbf{d}(\lambda))$ is the p-th cohomology space of $(\mathbf{X}, \mathbf{d}(\lambda))$. Let $\sigma_{\pi,k}(\mathbf{X}, \mathbf{d})$ be the set of all $\lambda \in \Omega$ for which $\lambda \in \bigcup_{p=0}^{k} \Sigma_p(\mathbf{X}, \mathbf{d})$ or $\mathrm{R}(d^k(\lambda))$ is not closed, where $0 \leqslant k \leqslant \infty$.

DEFINITION 3.1. The collection of all $\sigma_{\pi,k}(\mathbf{X}, \mathbf{d})$, $0 \leqslant k \leqslant \infty$, is called the *family of Słodkowski spectra* of the Ω-Banach complex $(\mathbf{X}, \mathbf{d})$. The set $\sigma_{\pi,\infty}(\mathbf{X}, \mathbf{d}) = \bigcup_{p=0}^{\infty} \Sigma_p(\mathbf{X}, \mathbf{d})$ is called *Taylor spectrum* of $(\mathbf{X}, \mathbf{d})$.

In the sequel, $\sigma(\mathbf{X}, \mathbf{d})$ will denote one of spectra $\sigma_{\pi,k}(\mathbf{X}, \mathbf{d}), 0 \leqslant k \leqslant \infty$, if not specified otherwise. If Y is a Banach space, then we have a new Ω-Banach complex $B(Y, (\mathbf{X}, \mathbf{d}(\lambda)))$ (with morphisms $\beta^p(\lambda)T = d^p(\lambda) \cdot T$), $\lambda \in \Omega$, denoted by $B(Y, (\mathbf{X}, \mathbf{d}))$.

THEOREM 3.2. *If* $(\mathbf{X}, \mathbf{d})$ *is an* Ω*-Banach complex and* Y *is a Banach space, then* $\sigma(\mathbf{X}, \mathbf{d}) \subseteq \sigma(B(Y, (\mathbf{X}, \mathbf{d})))$. *Moreover,*

$$\sigma(\mathbf{X}, \mathbf{d}) = \sigma(B(Y, (\mathbf{X}, \mathbf{d}))) \tag{3.1}$$

provided $Y \in \mathrm{Proj}$.

Proof. Let $\lambda \in \sigma(\mathbf{X}, \mathbf{d})$. By Lemma 2.2, there exist bounded sequences $\{x_n\} \subset X^p$ and $\{f_n\} \subset X^{p*}$ such that $\lim_n d^p(\lambda)x_n = 0$, $\lim_n d^{p-1}(\lambda)^* f_n = 0$, $f_n(x_n) = 1$, for some p. Let $f \in Y^*, \|f\| = 1$, and $y \in Y$, such that $f(y) = 1$. Then the sequences $\{f \otimes x_n\} \subset B(Y, X^p)$ and $\{F_n\} \subset B(Y, X^p)^*$ are bounded, where $F_n(u) = u^*(f_n)y$. Moreover, $\lim_n \beta^p(\lambda) f \otimes x_n = 0$ and $\lim_n \beta^{p-1}(\lambda)^* F_n = 0$. Whence, $\lambda \in \sigma(B(Y, (\mathbf{X}, \mathbf{d})))$ by virtue of Lemma 2.2.

Let $Y \in \mathrm{Proj}$ and $\lambda \notin \sigma(\mathbf{X}, \mathbf{d}) = \sigma_{\pi,k}(\mathbf{X}, \mathbf{d})$. Thus the complex $(\mathbf{X}, \mathbf{d}(\lambda))$ is exact in the first k terms and $\mathrm{R}(d^k(\lambda))$ is closed. Then $B(Y, (\mathbf{X}, \mathbf{d}(\lambda)))$ is also exact in the first k terms and $\mathrm{R}(\beta^k(\lambda)) \subseteq B(Y, \mathrm{R}(d^k(\lambda)))$. Moreover, if $T \in B(Y, \mathrm{R}(d^k(\lambda)))$, then there exists an operator $G \in B(Y, X^k)$ such that $d^k(\lambda) \cdot G = T$. So, $\beta^k(\lambda)G = T$, i.e. $\mathrm{R}(\beta^k(\lambda)) = B(Y, \mathrm{R}(d^k(\lambda)))$ and $\mathrm{R}(\beta^k(\lambda))$ is closed. Thus, $\lambda \notin \sigma_{\pi,k}(B(Y, (\mathbf{X}, \mathbf{d})))$ and we have proved (3.1). ∎

Let $(\mathbf{X}, \mathbf{d})$ be an Ω-Banach complex, $\mathcal{U}$ an ultrafilter on an infinite set S and let $d^n_{\mathcal{U}}(\lambda) = (d^n(\lambda))_{\mathcal{U}}$. Then we obtain a new Ω-Banach complex $(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$:

$$0 \longrightarrow X^0_{\mathcal{U}} \xrightarrow{d^0_{\mathcal{U}}(\lambda)} X^1_{\mathcal{U}} \xrightarrow{d^1_{\mathcal{U}}(\lambda)} \cdots \xrightarrow{d^{n-1}_{\mathcal{U}}(\lambda)} X^n_{\mathcal{U}} \xrightarrow{d^n_{\mathcal{U}}(\lambda)} \cdots, \quad \lambda \in \Omega,$$

which we call an *ultrapower of* $(\mathbf{X}, \mathbf{d})$.

LEMMA 3.3.

$$\sigma_{\pi,k}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}) = \bigcup_{p=0}^{k} \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}), \quad 0 \leqslant k \leqslant \infty.$$

Proof. By Definition 3.1, $\bigcup\limits_{p=0}^{k} \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}) \subseteq \sigma_{\pi,k}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$. For all p, $0 \leqslant p \leqslant k$, let $\lambda \notin \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$. Let us prove that $\mathrm{R}(d^k(\lambda)_{\mathcal{U}})$ is closed and consequently $\lambda \notin \sigma_{\pi,k}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$. Indeed, $\mathrm{N}(d^k(\lambda)_{\mathcal{U}}) = \mathrm{R}(d^{k-1}(\lambda)_{\mathcal{U}})$, in particular $\mathrm{R}(d^{k-1}(\lambda)_{\mathcal{U}})$ is closed. By Lemma 2.4, $\mathrm{R}(d^{k-1}(\lambda))$ is also closed and

$$\mathrm{R}(d^{k-1}(\lambda)) = X^k \cap \mathrm{R}(d^{k-1}(\lambda)_{\mathcal{U}}) = X^k \cap \mathrm{N}(d^k(\lambda)_{\mathcal{U}}) = \mathrm{N}(d^k(\lambda)).$$

Further, $\mathrm{N}(d^k(\lambda))_{\mathcal{U}} = \mathrm{R}(d^{k-1}(\lambda))_{\mathcal{U}} = \mathrm{R}(d^{k-1}(\lambda)_{\mathcal{U}}) = \mathrm{N}(d^k(\lambda)_{\mathcal{U}})$. By using Lemma 2.4 again, we obtain that $\mathrm{R}(d^k(\lambda)_{\mathcal{U}})$ is closed. ∎

THEOREM 3.4.

$$\sigma_{\pi,k}(\mathbf{X}, \mathbf{d}) = \sigma_{\pi,k}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}), \quad 0 \leqslant k \leqslant \infty.$$

Proof. Let $\lambda \in \Sigma_p(\mathbf{X}, \mathbf{d})$ for some p. If $\mathrm{R}(d^{p-1}(\lambda))$ is not closed, then by Lemma 2.4 $\mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}})$ is also not closed, i.e. $\lambda \in \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$. Thus, we can assume that $\mathrm{R}(d^{p-1}(\lambda))$ is closed. Let $x \in \mathrm{N}(d^p(\lambda)) \setminus \mathrm{R}(d^{p-1}(\lambda))$. By Lemma 2.4, $X^p \cap \mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}}) = \mathrm{R}(d^{p-1}(\lambda))$, thus $x \notin \mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}})$. But, $x \in \mathrm{N}(d^p(\lambda))_{\mathcal{U}}$ and $\mathrm{N}(d^p(\lambda))_{\mathcal{U}} \subseteq \mathrm{N}(d^p(\lambda)_{\mathcal{U}})$, i.e., $\lambda \in \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$.

Conversely, let $\lambda \in \sigma_{\pi,k}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$. By using Lemmas 2.4 and 3.3, we can assume that $\lambda \in \Sigma_p(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$ and images $\mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}})$, $\mathrm{R}(d^p(\lambda)_{\mathcal{U}})$ are closed for some p, $0 \leqslant p \leqslant k$. By Lemma 2.4, we have $\mathrm{N}(d^p(\lambda))_{\mathcal{U}} = \mathrm{N}(d^p(\lambda)_{\mathcal{U}})$, $\mathrm{R}(d^{p-1}(\lambda))_{\mathcal{U}} = \mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}})$. Let $[x_s] \in \mathrm{N}(d^p(\lambda)_{\mathcal{U}}) \setminus \mathrm{R}(d^{p-1}(\lambda)_{\mathcal{U}})$, where $x_s \in \mathrm{N}(d^p(\lambda))$. Then $x_{s_0} \notin \mathrm{R}(d^{p-1}(\lambda))$ for some $s_0 \in S$. Thus, $x_{s_0} \in \mathrm{N}(d^p(\lambda)) \setminus \mathrm{R}(d^{p-1}(\lambda))$, i.e., $\lambda \in \sigma_{\pi,k}(\mathbf{X}, \mathbf{d})$. ∎

COROLLARY 3.5. *If* $0 \longrightarrow (\mathbf{X}, \mathbf{d}) \xrightarrow{\mathbf{f}} (\mathbf{Y}, \mathbf{d}') \xrightarrow{\mathbf{g}} (\mathbf{Z}, \mathbf{d}'') \longrightarrow 0$ *is a short exact sequence of* Ω*-Banach complexes, then* $\sigma(\mathbf{Y}, \mathbf{d}') \subseteq \sigma(\mathbf{X}, \mathbf{d}) \cup \sigma(\mathbf{Z}, \mathbf{d}'')$.

Proof. Let $\lambda \in \sigma_{\pi,k}(\mathbf{Y}, \mathbf{d}')$. By using Theorem 3.4 and Lemma 3.3 we can assume that $\lambda \in \Sigma_i(\mathbf{Y}_{\mathcal{U}}, \mathbf{d}'_{\mathcal{U}})$ for some i, $0 \leqslant i \leqslant k$. Then $H^i(\mathbf{Y}_{\mathcal{U}}, \mathbf{d}'_{\mathcal{U}}(\lambda)) \neq 0$. It remains to use the long exact sequence of cohomologies

$$\cdots \longrightarrow H^i(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}(\lambda)) \longrightarrow H^i(\mathbf{Y}_{\mathcal{U}}, \mathbf{d}'_{\mathcal{U}}(\lambda)) \longrightarrow H^i(\mathbf{Z}_{\mathcal{U}}, \mathbf{d}''_{\mathcal{U}}(\lambda)) \longrightarrow \cdots$$

induced by the short exact sequence of complexes

$$0 \longrightarrow (\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}(\lambda)) \overset{\mathbf{f}(\lambda)_{\mathcal{U}}}{\longrightarrow} (\mathbf{Y}_{\mathcal{U}}, \mathbf{d}'_{\mathcal{U}}(\lambda)) \overset{\mathbf{g}(\lambda)_{\mathcal{U}}}{\longrightarrow} (\mathbf{Z}_{\mathcal{U}}, \mathbf{d}''_{\mathcal{U}}(\lambda)) \longrightarrow 0.$$

Then $H^i(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}}(\lambda)) \neq 0$ or $H^i(\mathbf{Z}_{\mathcal{U}}, \mathbf{d}''_{\mathcal{U}}(\lambda)) \neq 0$. ∎

4. THE PROJECTION PROPERTY

In this section, we prove a projection theorem for Słodkowski spectra of Ω-Banach complexes.

Let $(\mathbf{X}, \mathbf{d})$ be a Ω-Banach complex and $\beta = \{\beta_p \in B(X^p)\}$ a bounded endomorphism of $(\mathbf{X}, \mathbf{d})$ (i.e., $d^p(\lambda)\beta_p = \beta_{p+1}d^p(\lambda)$ for any $\lambda \in \Omega$ and p). Let $\beta - \mu = \{\beta_p - \mu \in B(X^p)\}$, $\mu \in \mathbb{C}$. The spectrum $\sigma(\beta)$ of β is defined as the union of ordinary spectra $\sigma(\beta_p)$. The Ω-Banach complex $\mathrm{Con}((\mathbf{X}, \mathbf{d}), \beta - \mu) = \{\mathrm{Con}((\mathbf{X}, \mathbf{d}(\lambda)), \beta - \mu), \lambda \in \Omega\}$ is called a *cone* of the endomorphism $\beta - \mu$, where $\mathrm{Con}((\mathbf{X}, \mathbf{d}(\lambda)), \beta - \mu)$ is the following Banach complex:

$$0 \longrightarrow X^0 \overset{\gamma^0(\lambda,\mu)}{\longrightarrow} X^1 \oplus X^0 \overset{\gamma^1(\lambda,\mu)}{\longrightarrow} \cdots \overset{\gamma^{p-1}(\lambda,\mu)}{\longrightarrow} X^p \oplus X^{p-1} \overset{\gamma^p(\lambda,\mu)}{\longrightarrow} \cdots,$$

where, $\gamma^p(\lambda, \mu)(x, y) = (d^p(\lambda)x, -d^{p-1}(\lambda)y + (\beta_p - \mu)x)$, $(x, y) \in X^p \oplus X^{p-1}$. If Ω reduces to one point, then this notion is reduced to the usual notion of cone of a complex. The collection of Banach complexes $\mathrm{Con}((\mathbf{X}, \mathbf{d}(\lambda)), \beta-\mu)$, $(\lambda, \mu) \in \Omega\times\mathbb{C}$ generates a $\Omega\times\mathbb{C}$-Banach complex (where $\Omega\times\mathbb{C}$ is equipped with the direct product topology) denoted by $\mathrm{Con}_\beta(\mathbf{X}, \mathbf{d})$.

LEMMA 4.1. *Let $\mathcal{U}$ be an ultrafilter. Then* $\mathrm{Con}_\beta(\mathbf{X}, \mathbf{d})_{\mathcal{U}} = \mathrm{Con}_{\beta_{\mathcal{U}}}(\mathbf{X}_{\mathcal{U}}, \mathbf{d}_{\mathcal{U}})$.

Proof. It is clear that the canonical linear operator

$$f_p : (X^p \oplus X^{p-1})_{\mathcal{U}} \to X^p_{\mathcal{U}} \oplus X^{p-1}_{\mathcal{U}}, \quad f_p[(x_s, y_s)] = ([x_s], [y_s])$$

is an isometric morphism. It remains to check that $f_{p+1}\gamma^p(\lambda, \mu)_{\mathcal{U}} = \gamma^p_{\mathcal{U}}(\lambda, \mu)f_p$, where $\gamma^p_{\mathcal{U}}(\lambda, \mu)$ is the differential of the complex $\mathrm{Con}((\mathbf{X}_{\mathcal{U}}, \mathbf{d}(\lambda)_{\mathcal{U}}), \beta_{\mathcal{U}} - \mu)$. ∎

LEMMA 4.2. *Assume that $\lambda \in \Sigma_p(\mathbf{X}, \mathbf{d})$ and $\mathrm{R}(d^{p-1}(\lambda))$ is closed. Then there exists $\mu \in \mathbb{C}$ such that $\lambda \in \Sigma_p(\mathrm{Con}((\mathbf{X}, \mathbf{d}), \beta - \mu))$ or $\mathrm{R}(\gamma^p(\lambda, \mu))$ is not closed.*

Proof. Let $T_p \in B(Z_p, X^{p+1})$, $T_p x^\sim = d^p(\lambda)x$, where $Z_p = X^p/\mathrm{R}(d^{p-1}(\lambda))$. Then $\mathrm{N}(T_p) \neq 0$ and $\beta_{p+1}T_p = T_p\beta^\sim_p$, where $\beta^\sim_p \in B(Z_p)$, $\beta^\sim_p x^\sim = (\beta_p x)^\sim$. Thus the kernel $\mathrm{N}(T_p)$ is invariant under $\beta^\sim_p$. By using that of the approximate point spectrum $\sigma^{\mathrm{ap}}(\beta^\sim_p)$ of $\beta^\sim_p$ is nonvoid, we see that there exist a number $\mu \in \mathbb{C}$ and a sequence of vectors $x^\sim_n \in \mathrm{N}(T_p)$, $\|x^\sim_n\| = 1$, such that $\lim_n(\beta^\sim_p - \mu)x^\sim_n = 0$. Then one may find a sequence of vectors $\{y_n\} \subset X^{p-1}$ such that $\lim_n(\beta_p - \mu)x_n - d^{p-1}(\lambda)y_n = 0$. A direct calculation shows that

$$\gamma^p(\lambda, \mu)(x_n, y_n) = (0, (\beta_p - \mu)x_n - d^{p-1}(\lambda)y_n) \to 0, \quad n \to \infty.$$

If $\lambda \notin \Sigma_p(\mathrm{Con}((\mathbf{X},\mathbf{d}),\beta-\mu))$, then $\mathrm{N}(\gamma^p(\lambda,\mu)) = \mathrm{R}(\gamma^{p-1}(\lambda,\mu))$ and for the norm $r_n = \|(x_n,y_n)^{\sim}\|$ of $(x_n,y_n)^{\sim} \in X^p \oplus X^{p-1}/\mathrm{R}(\gamma^{p-1}(\lambda,\mu))$ we have

$$\begin{aligned} r_n &= \inf_{(z,w)\in X^{p-1}\oplus X^{p-2}} \|(x_n,y_n) + (d^{p-1}(\lambda)z, -d^{p-2}(\lambda)w + (\beta_{p-1}-\mu)z)\| \\ &\geqslant \inf_{z\in X^{p-1}} \|x_n + d^{p-1}(\lambda)z\| = \|x_n^{\sim}\| = 1. \end{aligned}$$

Thus, $\inf\limits_{n\in\mathbb{N}} r_n \geqslant 1$ and $\lim\limits_n \gamma^p(\lambda,\mu)(x_n,y_n) = 0$, i.e., the image of the operator

$$X^p \oplus X^{p-1}/\mathrm{N}(\gamma^p(\lambda,\mu)) \to X^{p+1} \oplus X^p$$

induced by the operator $\gamma^p(\lambda,\mu)$ is not closed. Then the image of the operator $\gamma^p(\lambda,\mu)$ is also not closed. ∎

Theorem 4.3. *Let $(\mathbf{X},\mathbf{d})$ be a Ω-Banach complex and β be a bounded endomorphism of $(\mathbf{X},\mathbf{d})$. If $\Pi : \Omega\times\mathbb{C}\to\Omega$ is the canonical projection, then*

$$\sigma(\mathbf{X},\mathbf{d}) = \Pi(\sigma(\underset{\beta}{\mathrm{Con}}(\mathbf{X},\mathbf{d}))).$$

Proof. Let $\sigma = \sigma_{\pi,i}$, where $0 \leqslant i \leqslant \infty$, and let $\mathcal{U}$ an ultrafilter. By Theorem 3.4 and Lemma 4.1, we have $\sigma(\mathrm{Con}_\beta(\mathbf{X},\mathbf{d})) = \sigma(\mathrm{Con}_{\beta_\mathcal{U}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}))$, and also

$$\sigma(\underset{\beta_\mathcal{U}}{\mathrm{Con}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})) = \bigcup_{p=0}^{i} \Sigma_p(\underset{\beta_\mathcal{U}}{\mathrm{Con}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}))$$

by Lemma 3.3. Let $(\lambda,\mu) \in \Sigma_p(\mathrm{Con}_{\beta_\mathcal{U}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}))$ for some p, $0 \leqslant p \leqslant i$. Then $\lambda \in \Sigma_p(\mathrm{Con}((\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}),\beta_\mathcal{U}-\mu))$. By using the Słodkowski argument in the proof of Theorem 1.7 from [13], infer $\Sigma_p(\mathrm{Con}((\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}),\beta_\mathcal{U}-\mu)) \subseteq \Sigma_{p-1}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}) \cup \Sigma_p(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})$. Now, by using Theorem 3.4 again, we obtain $\lambda \in \sigma(\mathbf{X},\mathbf{d})$. Thus $\Pi(\sigma(\mathrm{Con}_\beta(\mathbf{X},\mathbf{d}))) \subseteq \sigma(\mathbf{X},\mathbf{d})$.

Let us prove the opposite inclusion. Let $\lambda \in \sigma(\mathbf{X},\mathbf{d})$. By Theorem 3.4 and Lemma 3.3, $\sigma(\mathbf{X},\mathbf{d}) = \sigma(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}) = \bigcup\limits_{k=0}^{i} \Sigma_k(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})$. Let p $(0 \leqslant p \leqslant i)$ be the lowest number such that $\lambda \in \Sigma_p(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})$. Then $\mathrm{R}(d^{p-1}(\lambda)_\mathcal{U})$ is closed. Indeed, if $p = 0$ then there is nothing to prove. Let $p > 0$ and $\mathrm{R}(d^{p-1}(\lambda)_\mathcal{U})$ is not closed. Then, by Definition 3.1, $\lambda \in \sigma_{\pi,p-1}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})$ and, by Lemma 3.3, $\lambda \in \bigcup\limits_{k=0}^{p-1} \Sigma_k(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})$, which contradicts to the choice of the number p. Now, using Lemma 4.2 there exists a $\mu \in \mathbb{C}$, such that $(\lambda,\mu) \in \sigma_{\pi,p}(\mathrm{Con}_{\beta_\mathcal{U}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})) \subseteq \sigma_{\pi,i}(\mathrm{Con}_{\beta_\mathcal{U}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U}))$. By Lemma 4.1 and Theorem 3.4, $\sigma_{\pi,i}(\mathrm{Con}_{\beta_\mathcal{U}}(\mathbf{X}_\mathcal{U},\mathbf{d}_\mathcal{U})) = \sigma_{\pi,i}(\mathrm{Con}_\beta(\mathbf{X},\mathbf{d}))$, i.e., $(\lambda,\mu) \in \sigma_{\pi,i}(\mathrm{Con}_\beta(\mathbf{X},\mathbf{d}))$. ∎

5. ULTRASPECTRA OF BANACH LIE ALGEBRA REPRESENTATIONS

In this section, we introduce the Banach complex generated by a Banach module over a Banach Lie algebra, parametrized on the character space of this Lie algebra. We obtain a condition of nonvoidness of spectra of this parametrized Banach complex.

A Banach Lie algebra (shortly, a B-L algebra) E is a Banach space and a Lie algebra with continuous Lie brackets $[\cdot,\cdot] : E \times E \to E, (a,b) \mapsto [a,b]$. A Banach module over a B-L algebra E (shortly, a Banach E-module) is a pair (X,α) consisting of a Banach space X and a continuous representation $\alpha : E \to B(X)$. A functional $\lambda \in E^*$ is called a *character* of E if $\lambda[E,E] = 0$. The space of all characters (with the weak topology) of a B-L algebra E is denoted by $\Delta(E)$ ($\subseteq E^*$). Let us consider the cochain Banach complex generated by the Banach E-module (X,α):

$$C^{\cdot}(\alpha) : 0 \to X \xrightarrow{d^0} B(E,X) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} B(\Lambda^n E, X) \xrightarrow{d^n} \cdots,$$

with the differential

$$\begin{aligned} d^n\omega(a_1 \wedge \cdots \wedge a_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \alpha(a_i)\omega(a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([a_i,a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{n+1}), \end{aligned}$$

where $\omega \in B(\Lambda^n E, X)$, $a_i \in E$. We denote the $\Delta(E)$-Banach complex $C^{\cdot}(\alpha - \lambda), \lambda \in \Delta(E)$, by $\mathbf{C}(\alpha)$. If (X,α) and (Y,β) are Banach E-modules and $\varphi : (X,\alpha) \to (Y,\beta)$ is a bounded E-homomorphism ($\varphi(\alpha(a)x) = \beta(a)\varphi(x)$), then for each $\lambda \in \Delta(E)$ it is induced a morphism of Banach complexes $\varphi^{\cdot}(\lambda) : C^{\cdot}(\alpha - \lambda) \to C^{\cdot}(\beta - \lambda)$, $\varphi^{\cdot}(\lambda)\omega = \varphi\omega$, i.e., there is a morphism of $\Delta(E)$-Banach complexes $\varphi^{\circ} : \mathbf{C}(\alpha) \to \mathbf{C}(\beta)$, where $\varphi^{\circ} = \{\varphi^{\cdot}(\lambda)\}$.

LEMMA 5.1. *Let $0 \to (X,\alpha) \xrightarrow{\varphi} (Y,\beta) \xrightarrow{\psi} (Z,\gamma) \to 0$ be an exact sequence of Banach E-modules and E-homomorphisms which is a $\mathbb{C}$-split sequence (i.e., $\mathrm{N}(\psi)$ is a complemented subspace). Then the sequence of $\Delta(E)$-Banach complexes*

$$0 \to \mathbf{C}(\alpha) \xrightarrow{\varphi^{\circ}} \mathbf{C}(\beta) \xrightarrow{\psi^{\circ}} \mathbf{C}(\gamma) \to 0$$

is exact.

Proof. By assumption, for each $\lambda \in \Delta(E)$ the sequence of Banach complexes $0 \to C^{\cdot}(\alpha - \lambda) \xrightarrow{\varphi^{\cdot}(\lambda)} C^{\cdot}(\beta - \lambda) \xrightarrow{\psi^{\cdot}(\lambda)} C^{\cdot}(\gamma - \lambda) \to 0$ is exact. But, this means that the required sequence of $\Delta(E)$-Banach complexes is exact. ∎

Definition 5.2. Let (X,α) be a Banach E-module and $\sigma(\mathbf{C}(\alpha))$ be one of spectra $\sigma_{\pi,k}(\mathbf{C}(\alpha))$, $0 \leqslant k \leqslant \infty$. We call this set a *Słodkowski spectrum* (the *Taylor spectrum*, if $k=\infty$) of the representation α and denote it by $\sigma(\alpha)$.

The point spectrum $\sigma^{\mathrm{p}}(\alpha)$ (approximate point spectrum $\sigma^{\mathrm{ap}}(\alpha)$) of a representation $\alpha : E \to B(X)$ is defined (see [11] and [17]) as the set of functions $\lambda : E \to \mathbb{C}$ for which there exists $x \in X$, such that, $\alpha(a)x = \lambda(a)x$ (there exists a net $(x_\gamma) \subseteq X, \|x_\gamma\| = 1$, such that, $(\alpha(a) - \lambda(a))x_\gamma \to 0$) for all $a \in E$, i.e., $\sigma^{\mathrm{p}}(\alpha) = \Sigma_0(\mathbf{C}(\alpha))$. If $\lambda \in \sigma^{\mathrm{ap}}(\alpha)$, then $\lambda \in E^*$ and $\lambda(a) \in \sigma(\alpha(a))$, $a \in E$. Moreover, $\alpha([a,b])x_\gamma \to 0$ for all $a,b \in E$, i.e. $\lambda \in \Delta(E)$. For any $S \subseteq B(X)$, we define $\sigma^{\mathrm{p}}(S)$ and $\sigma^{\mathrm{ap}}(S)$ as the corresponding spectra of the identical representation of the closed Lie subalgebra in $B(X)$ generated by S. It is clear that $\sigma^{\mathrm{p}}(\alpha(E)) \cdot \alpha = \sigma^{\mathrm{p}}(\alpha)$, $\sigma^{\mathrm{ap}}(\alpha(E)) \cdot \alpha = \sigma^{\mathrm{ap}}(\alpha)$ and $\sigma^{\mathrm{p}}(\alpha) \subseteq \sigma_{\pi,0}(\alpha) \subseteq \sigma^{\mathrm{ap}}(\alpha)$. If E is finite-dimensional, then $\sigma_{\pi,0}(\alpha) = \sigma^{\mathrm{ap}}(\alpha)$. In general, $\sigma_{\pi,0}(\alpha) \neq \sigma^{\mathrm{ap}}(\alpha)$. Indeed, let us consider the space $\ell_1 = \ell_1(\mathbb{N})$ with canonical base $\{f_n\}$ as a commutative B-L algebra and a bounded representation $\alpha : \ell_1 \to B(H), \alpha(f_n) = P_n$, in a separable Hilbert space H, where P_n is an orthogonal projection on the linear hull of the first n vectors of the canonical base $\{e_m\}$ of H. Then $\lim_m P_n e_m = 0, n \geqslant 1$, i.e., $0 \in \sigma^{\mathrm{ap}}(\alpha)$. But $0 \notin \sigma_{\pi,0}(\alpha)$. Indeed, take $x = \sum_{m=1}^{\infty} a_m e_m \in H$. Then

$$\|d^0 x\| = \sup_{n\in\mathbb{N}} \|(d^0 x) f_n\|_H = \sup_{n\in\mathbb{N}} \|P_n x\|_H = \sup_{n\in\mathbb{N}} \left(\sum_{m=1}^{n} |a_m|^2 \right)^{1/2} \geqslant \|x\|_H,$$

where $d^0 : H \to B(\ell_1, H)$, $(d^0 x) f_n = P_n x$, is the differential of the complex $C^{\cdot}(\alpha)$. Thus $\mathrm{R}(d^0)$ is closed, i.e., $0 \notin \sigma_{\pi,0}(\alpha)$.

Now, let $\mathcal{U}$ be an ultrafilter, and let $X_{\mathcal{U}}$ be the corresponding ultrapower of the Banach space X. Then the representation $\alpha : E \to B(X)$ induces the representation $\alpha_{\mathcal{U}} : E \to B(X_{\mathcal{U}}), \alpha_{\mathcal{U}}(a) = \alpha(a)_{\mathcal{U}}$, i.e., $(X_{\mathcal{U}}, \alpha_{\mathcal{U}})$ is also Banach E-module.

Definition 5.3. Let (X,α) be a Banach E-module and let σ be a Słodkowski spectrum. We define the *ultraspectrum* $\sigma^{\mathrm{u}}(\alpha)$ of the representation α as the union of spectra $\sigma(\alpha_{\mathcal{U}})$ by all ultrafilters $\mathcal{U}$, and we write $\sigma^{\mathrm{u}}(\alpha) = \sigma^{\mathrm{u}}_{\pi,k}(\alpha)$, if $\sigma = \sigma_{\pi,k}$. The union of all $\Sigma_0(\mathbf{C}(\alpha_{\mathcal{U}}))$ is called the *ultrapoint spectrum* $\sigma^{\mathrm{up}}(\alpha)$.

Lemma 5.4. *If* $\dim(E) < \infty$, *then* $C^{\cdot}(\alpha)_{\mathcal{U}} = C^{\cdot}(\alpha_{\mathcal{U}})$ *and* $\sigma^{\mathrm{u}}(\alpha) = \sigma(\alpha)$.

Proof. Let $\varphi_n : B(\Lambda^n E, X)_{\mathcal{U}} \to B(\Lambda^n E, X_{\mathcal{U}})$, $(\varphi_n[\omega_i])u = [\omega_i(u)], u \in \Lambda^n E$, be a linear operator. If $d^n_{\mathcal{U}}$ is the differential of the complex $C^{\cdot}(\alpha_{\mathcal{U}})$, then one can easily check that $\varphi_{n+1}(d^n)_{\mathcal{U}} = d^n_{\mathcal{U}}\varphi_n$. It remains to note that the condition $\dim(E) < \infty$ implies that φ_n is an isometriy for all n (see [10]). ∎

Theorem 5.5. *Let E be a* B-L *algebra and (X,α) a Banach E-module. Then*

$$\sigma^{\mathrm{u}}_{\pi,0}(\alpha) = \sigma^{\mathrm{up}}(\alpha) = \sigma^{\mathrm{ap}}(\alpha).$$

Proof. It is clear that $\sigma^{\mathrm{up}}(\alpha) \subseteq \sigma^{\mathrm{u}}_{\pi,0}(\alpha) \cap \sigma^{\mathrm{ap}}(\alpha)$. Let us prove that $\sigma^{\mathrm{ap}}(\alpha) \subseteq \sigma^{\mathrm{up}}(\alpha)$. Let $\lambda \in \sigma^{\mathrm{ap}}(\alpha(E))$. We have to prove that $\lambda\alpha \in \sigma^{\mathrm{up}}(\alpha)$. There exists a family of vectors $\{x_s\}_{s\in S} \subseteq X$, such that $\|x_s\| = 1$ and $\lim_{\mathcal{F}} (T - \lambda(T))x_s = 0$ for any $T \in \alpha(E)$, where $\mathcal{F}$ is some filter on the index set S. If $\mathcal{U}$ is an ultrafilter in S majorizing $\mathcal{F}$, then $\lim_{\mathcal{U}}(T - \lambda(T))x_s = 0, T \in \alpha(E)$. If $\mathcal{U}$ is a trivial filter, then there exists a joint eigenvector $x \in X$, $\|x\| = 1$, $(T - \lambda(T))x = 0$, $T \in \alpha(E)$. Then, for an $\aleph_0$-incomplete ultrafilter $\mathcal{V}$, $(\alpha(a)_{\mathcal{V}} - \lambda(\alpha(a)))[x] = 0$, $a \in E$, i.e., $d^0_{\mathcal{V}}(\lambda\alpha)[x] = 0$, where $d^0_{\mathcal{V}}(\lambda\alpha)$ is the differential of the complex $C^{\cdot}(\alpha_{\mathcal{V}} - \lambda\alpha)$, or $\lambda\alpha \in \Sigma_0(\mathbf{C}(\alpha_{\mathcal{V}}))$.

Now, suppose $\mathcal{U}$ is nontrivial (not necessarily $\aleph_0$-incomplete) ultrafilter. Then S is an infinite set. Let us replace $\mathcal{U}$ with an $\aleph_0$-incomplete ultrafilter. Let $\mathcal{V}$ be an $\aleph_0$-incomplete ultrafilter in $\mathbb{N}$. By Lemma 2.3, $\mathcal{U}\times\mathcal{V}$ is $\aleph_0$-incomplete. Now assume that $x_{(s,n)} = x_s$, $n \in \mathbb{N}$. Then, for any $T \in \alpha(E)$, we have $\lim_{\mathcal{U}\times\mathcal{V}}(T - \lambda(T))x_{(s,n)} = 0$, i.e., $(T_{\mathcal{U}\times\mathcal{V}} - \lambda(T))[x_{(s,n)}] = 0$ and $\lambda\alpha \in \sigma^{\mathrm{up}}(\alpha)$.

It remains to prove that $\sigma^{\mathrm{u}}_{\pi,0}(\alpha) \subseteq \sigma^{\mathrm{up}}(\alpha)$. Let $\lambda \in \sigma_{\pi,0}(\alpha_{\mathcal{U}}) \setminus \Sigma_0(\mathbf{C}(\alpha_{\mathcal{U}}))$, where $\mathcal{U}$ is an ultrafilter in some set S. Then there exists a sequence $\{[x^n_s]\} \subset X_{\mathcal{U}}, \|[x^n_s]\| = 1$, such that $\lim_n [(\alpha(a) - \lambda(a))x^n_s] = 0$. Let $\mathcal{V}$ be an ultrafilter in $\mathbb{N}$ majorizing the Fréchet filter. It is clear that $\lim_{\mathcal{V}}[(\alpha(a) - \lambda(a))x^n_s] = 0$. Let us consider the representation $(\alpha_{\mathcal{U}})_{\mathcal{V}} : E \to B((X_{\mathcal{U}})_{\mathcal{V}})$. By Lemma 2.3 $X_{\mathcal{U}\times\mathcal{V}} = (X_{\mathcal{U}})_{\mathcal{V}}$ and $\alpha_{\mathcal{U}\times\mathcal{V}} = (\alpha_{\mathcal{U}})_{\mathcal{V}}$. Let $[[x^n_s]] \in X_{\mathcal{U}\times\mathcal{V}}$. Then $\|[[x^n_s]]\| = \lim_{\mathcal{V}} \|[x^n_s]\| = 1$ and

$$\|[[(\alpha(a) - \lambda(a))x^n_s]]\| = \lim_{\mathcal{U}\times\mathcal{V}} \|(\alpha(a) - \lambda(a))x^n_s\| = \lim_{\mathcal{V}} \|[(\alpha(a) - \lambda(a))x^n_s]\| = 0.$$

Thus $d^0_{\mathcal{U}\times\mathcal{V}}(\lambda)[[x^n_s]] = 0$ or $\lambda \in \Sigma_0(\mathbf{C}(\alpha_{\mathcal{U}\times\mathcal{V}}))$. ∎

COROLLARY 5.6. *Let E be a solvable* B-L *algebra and (X, α) be a Banach E-module. Then the ultraspectrum $\sigma^{\mathrm{u}}(\alpha)$ is nonvoid.*

Proof. Indeed, by assumption, $\alpha(E)$ is a solvable Lie algebra of operators. By [7], $\sigma^{\mathrm{ap}}(\alpha(E)) \neq \emptyset$. Then, also, $\sigma^{\mathrm{ap}}(\alpha) \neq \emptyset$. According to Theorem 5.5, $\sigma^{\mathrm{u}}_{\pi,0}(\alpha) = \sigma^{\mathrm{ap}}(\alpha)$. It remains to note that $\sigma^{\mathrm{u}}_{\pi,0}(\alpha) \subseteq \sigma^{\mathrm{u}}(\alpha)$. ∎

THEOREM 5.7. *Let (X, α) be a Banach E-module. There exists an ultrafilter $\mathcal{U}$, such that*

$$\sigma^{\mathrm{p}}(\alpha_{\mathcal{U}}) = \sigma_{\pi,0}(\alpha_{\mathcal{U}}) = \sigma^{\mathrm{ap}}(\alpha).$$

In particular, $\sigma^{\mathrm{u}}_{\pi,0}(\alpha) = \sigma_{\pi,0}(\alpha_{\mathcal{U}})$.

Proof. Let S be the set of pairs $s = (N, n^{-1})$, where N is a finite subset in E and $n \in \mathbb{N}$. Assume $s_1 \leqslant s_2$, if $N_1 \subseteq N_2$ and $n_1 \leqslant n_2$, where $s_i = (N_i, n_i^{-1})$. Then $(S, \leqslant)$ is a partially ordered set and for any $s_1, s_2 \in S$, there exists $s_3 \in S$, such that $\sup\{s_1, s_2\} \leqslant s_3$. Thus the set of all sections $\Gamma(s)$, $s \in S$ ($\Gamma(s) = \{\gamma \in S : s \leqslant \gamma\}$), generates a filter base in S. Let $\mathcal{U}$ be an ultrafilter majorizing this filter base. Then $\mathcal{U}$ is $\aleph_0$-incomplete. Indeed, let $S_n = \{s \in S : s = (N, n^{-1})\}$, $n \in \mathbb{N}$. Undoubtedly, $S = \bigcup_n S_n$ and $S_n \cap \Gamma(s_n) = \emptyset$ for any $n \in \mathbb{N}$, where $s_n = (N, (n+1)^{-1})$, i.e., $S_n \notin \mathcal{U}$.

Let us prove that $\sigma^{\mathrm{ap}}(\alpha) \subseteq \sigma^{\mathrm{p}}(\alpha_{\mathcal{U}})$. Let $0 \in \sigma^{\mathrm{ap}}(\alpha)$. By definition, for every finite subset $N \subset E$ and $n \in \mathbb{N}$, there exists a vector $x \in X, \|x\| = 1$, such that $\|\alpha(N)x\| < n^{-1}$. Assume $x_s = x$, where $s = (N, n^{-1})$. Then for each $a \in E$, $\alpha(a)x_s \to 0$ following by the section filter in S. Then $\lim_{\mathcal{U}} \alpha(a)x_s = 0$ or

$\alpha(a)_{\mathcal{U}}[x_s] = 0$, $a \in E$, and $\|[x_s]\| = 1$, i.e. $0 \in \sigma^{\mathrm{p}}(\alpha_{\mathcal{U}})$. Thus $\sigma^{\mathrm{ap}}(\alpha) = \sigma^{\mathrm{p}}(\alpha_{\mathcal{U}})$. By Theorem 5.5, $\sigma_{\pi,0}(\alpha_{\mathcal{U}}) = \sigma^{\mathrm{u}}_{\pi,0}(\alpha)$. ∎

6. QUASINILPOTENT B-L ALGEBRAS

In this section, we establish projection properties of spectra $\sigma_{\pi,k}(\alpha)$, $0 \leqslant k \leqslant \infty$, of a representation α of a quasinilpotent B-L algebra E. A Banach-Lie algebra E with quasinilpotent operators $\mathrm{ad}(a) \in B(E), \mathrm{ad}(a)b = [a,b]$ $(a \in E)$ of the adjoint representation of E is called a *quasinilpotent* B-L algebra (see [18]). In the sequel, we shall use B-L algebras for which $\sigma(\mathrm{ad}(a)) = 0$ not for all $a \in E$ but only for elements $a \in S$ in some set S of Lie topological generators of E (i.e., the Lie subalgebra generated by S is dense in E). We call such algebras *weak quasinilpotent* B-L algebras. In practice, it is convenient to check weak quasinilpotentness than its quasinilpotentness, especially for finitely generated Banach Lie algebras (see Examples 7.7, 7.8).

For a B-L algebra E and for any $n \geqslant 0$, the space $\Lambda^n E$ is the Banach E-module via the representation $T_n : E \to B(\Lambda^n E)$, $(T_n a)(b_1 \wedge \cdots \wedge b_n) = \sum_{i=1}^{n} b_1 \wedge \cdots \wedge b_{i-1} \wedge \mathrm{ad}(a)b_i \wedge b_{i+1} \wedge \cdots \wedge b_n$. Moreover, if (X, α) is a Banach E-module, then the space $B(\Lambda^n E, X)$ is also a Banach E-module via the representation $\theta_n : E \to B(B(\Lambda^n E, X))$, $\theta_n(a) = L_{\alpha(a)} - R_{T_n(a)}$, where $L_{\alpha(a)}$ and $R_{T_n(a)}$ are the left and right multiplication operators.

LEMMA 6.1. *Let E be a* B-L *algebra. If $\sigma(\mathrm{ad}(a)) = \{0\}$ for some $a \in E$, then $\sigma(T_n(a)) = \{0\}$. Moreover, $\sigma(\theta_n(a)) = \sigma(\alpha(a))$ for a Banach E-module (X, α).*

Proof. Let $\mathrm{ad}_i(a) = 1 \otimes \cdots \otimes 1 \otimes \mathrm{ad}(a) \otimes 1 \otimes \cdots \otimes 1 \in B(E^{\widehat{\otimes} n}), 1 \leqslant i \leqslant n$, where $\mathrm{ad}(a)$ is in the i-th place, and let $S_n(a) = \sum_{i=1}^{n} \mathrm{ad}_i(a)$, a sum of mutually commuting operators. By assumption, $\sigma(\mathrm{ad}_i(a)) = \{0\}$. Then $\sigma(S_n(a)) = \{0\}$. One can easily check that $A_n S_n(a) = S_n(a) A_n$, where A_n is the projection on $\Lambda^n E$ defined in Section 2. Moreover, $T_n(a)$ is the restriction of the operator $S_n(a)$ to the invariant subspace $\Lambda^n E$. Consequently, $\sigma(T_n(a)) = \{0\}$.

Let (X, α) be a Banach E-module. Since $[L_{\alpha(a)}, R_{T_n(a)}] = 0$ and $R_{T_n(a)}$ is a quasinilpotent operator, it follows that $\sigma(\theta_n(a)) = \sigma(L_{\alpha(a)}) = \sigma(\alpha(a))$ by virtue of the spectral (for instance, Taylor spectrum of commuting families) mapping theorem. ∎

It is well known that (see 3.1 of [8]) the following formulas

$$d^n \theta_n(a) = \theta_{n+1}(a) d^n, \tag{6.1}$$

$$d^{n-1} i_n(a) + i_{n+1}(a) d^n = \theta_n(a), \tag{6.2}$$

$$\theta_{n-1}(a) i_n(b) - i_n(b) \theta_n(a) = i_n([a,b]) \tag{6.3}$$

are true, where d^n is the differential of the complex $C^{\cdot}(\alpha)$ and $i_n(a) : B(\Lambda^n E, X) \to B(\Lambda^{n-1} E, X)$, is defined by $(i_n(a)\omega)b = \omega(a \wedge b)$, which is a *homotopic operator.*

LEMMA 6.2. *Let E be a quasinilpotent* B-L *algebra. If $\lambda \in \sigma(\alpha)$, then $\lambda(a) \in \sigma(\alpha(a))$, $a \in E$. In particular, the spectrum $\sigma(\alpha)$ is precompact.*

Proof. By Lemma 6.1, $\sigma(\theta_n(a)) = \sigma(\alpha(a))$, $n \geqslant 0$. If $\lambda(a) \notin \sigma(\alpha(a))$, then $\lambda(a) \notin \bigcup_{n=0}^{\infty} \sigma(\theta_n(a))$. By (6.2), we have $d^{n-1}(\lambda)i_n(a)+i_{n+1}(a)d^n(\lambda) = \theta_n(a)-\lambda(a)$, where $d^n(\lambda)$ is the differential of the complex $C^{\cdot}(\alpha - \lambda)$. Since $\theta_n(a) - \lambda(a)$ is invertible, so $\lambda \notin \Sigma_n(\mathbf{C}(\alpha))$ by (6.1). This led to a contradiction, since $\lambda \in \sigma(\alpha)$. Thus $\lambda(a) \in \sigma(\alpha(a))$ for every $a \in E$. In particular, $\sigma(\alpha)$ is identified with the subset of $\prod_{a\in E} \sigma(\alpha(a))$, i.e., $\sigma(\alpha)$ is a weak precompact subset in E^*. ∎

LEMMA 6.3. *Let E be a* B-L *algebra and let (X, α) be a Banach E-module. Assume F is a closed ideal in E of codimension one and $e \in E \setminus F$. Then $C^n(E, X) = C^n(F, X) \oplus C^{n-1}(F, X)$ for $n \geqslant 0$. The subspace $C^n(F, X)$ is invariant under the operator $\theta_n(e)$ and*

$$C^{\cdot}(\alpha) = \mathrm{Con}(C^{\cdot}(\alpha|_F), \theta(e)),$$

where $\theta(e) = \{\theta_n(e)\}$.

Proof. By assumption, $E = \mathbb{C}\, e \oplus F$. We define a bounded linear operator

$$f_n : C^n(E, X) \to C^n(F, X) \oplus C^{n-1}(F, X), \quad f_n(\chi) = (\chi|_F, (i_n(e)\chi)|_F),$$

where $\chi|_F$ and $(i_n(e)\chi)|_F$ are restrictions of the corresponding forms to F. It is clear that $\mathrm{N}(f_n) = \{0\}$. Let $(\omega, \upsilon) \in C^n(F, X) \oplus C^{n-1}(F, X)$ and

$$\chi(c_1e + u_1, \ldots, c_ne + u_n) = \omega(u_1, \ldots, u_n) + \sum_{i=1}^{n}(-1)^{i+1}c_i\upsilon(u_1, \ldots \widehat{u}_i, \ldots, u_n),$$

where $u_i \in F, c_i \in \mathbb{C}$. One can easily check that $\chi \in C^n(E, X)$ and $\chi|_F = \omega$, $(i_n(e)\chi)|_F = \upsilon$. Thus $C^n(F, X)$ is identified with a complemented subspace in $C^n(E, X)$. Let us prove that $C^n(F, X)$ is invariant under the operator $\theta_n(e)$. Take $\omega \in C^n(F, X)$ and let

$$\xi(u_1, \ldots, u_n) = \alpha(e)\omega(u_1, \ldots, u_n) + \sum_{i=1}^{n}(-1)^{i+1}\omega([e, u_i], u_1, \ldots, u_n).$$

It is clear that $\xi \in C^n(F, X)$ and, if $\chi = f_n^{-1}(\omega, 0)$, then $(\theta_n(e)\chi)|_F = \xi$. Assume that $\theta_n(e)\omega = \xi$. Let d and d' be the differentials of complexes $C^{\cdot}(\alpha)$ and $C^{\cdot}(\alpha|_F)$, correspondingly. It is clear that $(d\chi)|_F = d'(\chi|_F)$, and by (6.1) $d'\theta_n(e)(\chi|_F) = \theta_n(e)d'(\chi|_F)$, $\chi \in C^n(E, X)$, i.e. $\theta(e) = \{\theta_n(e)\}$ is an endomorphism of the complex $C^{\cdot}(\alpha|_F)$. By (6.2),

$$(i_{n+1}(e)d\chi)|_F = (\theta_n(e)\chi)|_F - (di_n(e)\chi)|_F = \theta_n(e)(\chi|_F) - d'(i_n(e)\chi|_F).$$

Thus, $f_{n+1}d\chi = (d'(\chi|_F), -d'(i_n(e)\chi|_F) + \theta_n(e)(\chi|_F)) = \gamma f_n\chi$, where γ is the differential of the cone $\mathrm{Con}(C^{\cdot}(\alpha|_F), \theta(e))$, i.e., the family $\{f_n\}$ implements an isomorphisms of the required complexes. ∎

Remark 6.4. We can identify $C^n(E,X) = C^n(F,X) \oplus C^{n-1}(F,X)$ for each n. Then $C^n(F,X)$ is invariant under operators $\theta_n(a)$, $a \in E$, i.e. $C^n(F,X)$ is a closed E-submodule in $C^n(E,X)$. One may prove that $C^{n-1}(F,X)$ is also E-submodule. Indeed, if $\upsilon \in C^{n-1}(F,X)$, then there exists $\chi \in C^n(E,X)$ such that $\chi|_F = 0$ and $\upsilon = (i_n(e)\chi)|_F$. By using (6.3) we obtain that $\theta_{n-1}(a)\upsilon = (\theta_{n-1}(a)i_n(e)\chi)|_F = (i_n(e)\theta_n(a)\chi)|_F + (i_n([a,e])\chi)|_F = (i_n(e)\theta_n(a)\chi)|_F$ ($[a,e] \in F$), for each $a \in E$.

Theorem 6.5. *Let E be a quasinilpotent* B-L *algebra and F a closed ideal in E of finite codimension, and let (X,α) be a Banach E-module. Then $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$ and if $\alpha[E,E]$ consists of quasinilpotent operators, then $\sigma(\alpha)|_F = \sigma(\alpha|_F)$.*

Proof. It is clear that E/F is a quasinilpotent B-L algebra and, by Engel theorem (see 1.3.7 of [8]), it is a finite dimensional nilpotent Lie algebra. Thus we may suppose that the codimension of F is one. Let $\lambda \in \sigma(\alpha)$ and $e \in E \setminus F$. By Lemma 6.3, $C^{\cdot}(\alpha - \lambda) = \mathrm{Con}(C^{\cdot}((\alpha-\lambda)|_F), \theta(e) - \lambda(e))$. Then, $\lambda|_F \in \Delta(F)$ and $(\lambda|_F, \lambda(e)) \in \sigma(\mathrm{Con}_{\theta(e)}(\mathbf{C}(\alpha|_F)))$. By Theorem 4.3, $\lambda|_F \in \sigma(\mathbf{C}(\alpha|_F))$, i.e. $\lambda|_F \in \sigma(\alpha|_F)$.

Conversely, let $\mu \in \sigma(\alpha|_F)$ and $\alpha[E,E]$ consist of quasinilpotent operators. Since $[E,E] \subseteq F$ and F is a quasinilpotent B-L algebra, so by Lemma 6.2, $\mu(a) = 0$ for all $a \in [E,E]$. Then any linear extension of the functional μ up to E is a character of E. By Theorem 4.3, $(\mu, c) \in \sigma(\mathrm{Con}_{\theta(e)}\mathbf{C}(\alpha|_F))$ for some $c \in \mathbb{C}$. Let $\lambda(ze+u) = zc + \mu(u)$, $z \in \mathbb{C}$, $u \in F$. By Lemma 6.3, $\lambda \in \sigma(\alpha)$ and $\lambda|_F = \mu$. ∎

Now, we prove the inclusion $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$ for finite-dimensional Lie subalgebras F of E. We start with necessary lemmas.

Lemma 6.6. *Let E be a* B-L *algebra, (X,α) a Banach module and let $Y \in$* Proj. *Then, $\sigma(L_\alpha) = \sigma(\alpha)$, where $L_\alpha : E \to B(B(Y,X))$, $L_\alpha(a) = L_{\alpha(a)}$, is the left regular representation.*

Proof. By assumption, there exists a canonical isomorphism between Banach complexes $C^{\cdot}(L_\alpha)$ and $B(Y, C^{\cdot}(\alpha))$. It remains to use Theorem 3.2. ∎

Lemma 6.7. *Let E be a quasinilpotent* B-L *algebra, F a finite dimensional Lie subalgebra in E, and let (X,α) be a Banach E-module. If $E \in$* Proj, *then $\sigma(\theta_n|_F) = \sigma(\alpha|_F)$ for all n.*

Proof. By Engel theorem, F is a nilpotent Lie algebra. First, demonstrate that $\sigma(\theta_n|_F) = \sigma(L_\alpha|_F)$. Let $\delta_1 = L_\alpha|_F$ and $\delta_2 = -R_{T_n}|_F$ be the left and right regular representations of the Lie algebra F in the space $B(B(\Lambda^n E, X))$. Since $[\delta_1(a), \delta_2(b)] = 0$ for any $a, b \in F$, then the linear operator $\delta : F \times F \to B(B(\Lambda^n E, X))$, $\delta(a,b) = \delta_1(a) + \delta_2(b)$, is a representation of the Lie algebra $F \times F$. Let $M = \{(a,a) : a \in F\}$ be a Lie subalgebra in $F \times F$ and let $\iota : F \to M, \iota(a) = (a,a)$, be a canonical isomorphism of Lie algebras. We also have a Lie subalgebra $F \times \{0\} \subseteq F \times F$ and a canonical isomorphism $\varepsilon : F \to F \times \{0\}, \varepsilon(a) = (a,0)$. It is clear that $\theta_n|_F = \delta|_M \cdot \iota$ and $\delta_1 = \delta|_{F\times\{0\}} \cdot \varepsilon$.

Further, if $\lambda \in \sigma(\delta)$, then by [11] and by Lemma 6.1, $\lambda(0,a) \in \sigma(\delta(0,a)) = -\sigma(T_n(a)) = \{0\}$, $a \in F$. Then for each pair $(a,b) \in F \times F$, $\lambda(a,b) = \lambda(a,0)$. Thus $\lambda|_{F\times\{0\}} \cdot \varepsilon = \lambda|_M \cdot \iota$. By using Theorem 5 from [2] and Proposition 3.1 from[6], we have $\sigma(\theta_n|_F) = \sigma(\delta|_M) \cdot \iota = \sigma(\delta)|_M \cdot \iota = \{\lambda|_M \cdot \iota : \lambda \in \sigma(\delta)\} = \{\lambda|_{F\times\{0\}} \cdot \varepsilon :$

$\lambda \in \sigma(\delta)\} = \sigma(\delta|_{F\times\{0\}}) \cdot \varepsilon = \sigma(\delta_1)$, i.e. $\sigma(\theta_n|_F) = \sigma(L_\alpha|_F)$. But, by Lemma 2.1, $\Lambda^n E \in \mathrm{Proj}$, $n \geqslant 0$. Then by Lemma 6.6, $\sigma(L_\alpha|_F) = \sigma(\alpha|_F)$. ∎

THEOREM 6.8. *Let E be a quasinilpotent* B-L *algebra, and let F be a finite dimensional Lie subalgebra in E. If $E \in \mathrm{Proj}$, then $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$.*

Proof. It suffices to prove that if $0 \in \sigma(\alpha)$, then $0 \in \sigma(\alpha|_F)$. Let $\mathcal{U}$ be an ultrafilter on a set S. The following diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\beta}\uparrow & & \\
\cdots & \overset{\varepsilon_{\mathcal{U}}}{\to} & C^q(F, C^{n-1}(E,X)_{\mathcal{U}}) & \overset{\varepsilon_{\mathcal{U}}}{\to} & C^q(F, C^n(E,X)_{\mathcal{U}}) & \overset{\varepsilon_{\mathcal{U}}}{\to} & \cdots \\
 & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\beta}\uparrow & & \\
\cdots & \overset{\varepsilon_{\mathcal{U}}}{\to} & C^{q-1}(F, C^{n-1}(E,X)_{\mathcal{U}}) & \overset{\varepsilon_{\mathcal{U}}}{\to} & C^{q-1}(F, C^n(E,X)_{\mathcal{U}}) & \overset{\varepsilon_{\mathcal{U}}}{\to} & \cdots \\
 & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\beta}\uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

is commutative, where $\varepsilon_{\mathcal{U}}(\Phi) = d_{\mathcal{U}}\Phi, \Phi \in C^q(F, C^n(E,X)_{\mathcal{U}})$ ($d_{\mathcal{U}}$ is the differential of $C^{\cdot}(\alpha)_{\mathcal{U}}$) and β is the differential of the complex $C^{\cdot}((\theta_n|_F)_{\mathcal{U}})$. It is a bicomplex $\mathbf{B}$ with rows $B(\Lambda^q F, C^{\cdot}(\alpha)_{\mathcal{U}})$, $q \geqslant 0$, and columns $C^{\cdot}((\theta_n|_F)_{\mathcal{U}})$, $n \geqslant 0$. Since F is a finite dimensional Lie algebra,

$$H^n B(\Lambda^q F, C^{\cdot}(\alpha)_{\mathcal{U}}) = B(\Lambda^q F, H^n C^{\cdot}(\alpha)_{\mathcal{U}}).$$

Let $\beta : C^n(E,X)_{\mathcal{U}} \to B(F, C^n(E,X)_{\mathcal{U}}), (\beta[\omega_s])a = [\theta_n(a)\omega_s]$, be the differential of the n-th column of $\mathbf{B}$, and let $[\omega_s] \in C^n(E,X)_{\mathcal{U}}$ such that $d_{\mathcal{U}}[\omega_s] = 0$. If $I_{n-1} \in B(F, C^{n-1}(E,X)_{\mathcal{U}}), I_{n-1}(a) = i_n(a)_{\mathcal{U}}[\omega_s]$, then by (6.2), we deduce

$$\begin{aligned}
[\theta_n(a)\omega_s] &= [di_n(a)\omega_s] + [i_{n+1}(a)d\omega_s] = d_{\mathcal{U}} i_n(a)_{\mathcal{U}}[\omega_s] + i_{n+1}(a)_{\mathcal{U}} d_{\mathcal{U}}[\omega_s] \\
&= d_{\mathcal{U}} i_n(a)_{\mathcal{U}}[\omega_s] = \varepsilon_{\mathcal{U}}(I_{n-1})(a),
\end{aligned}$$

i.e., the induced operator of cohomologies

$$\beta^{\sim} : H^n(C^{\cdot}(\alpha)_{\mathcal{U}}) \to B(F, H^n(C^{\cdot}(\alpha)_{\mathcal{U}}))$$

is trivial for all n, $n \geqslant 0$. Let $\sigma = \sigma_{\pi,k}$ for some k, $0 \leqslant k \leqslant \infty$. Then $0 \in \bigcup_{p=0}^{k} \Sigma_p(\mathbf{C}(\alpha)_{\mathcal{U}})$ by Lemma 3.3 and Theorem 3.4. Let i, $0 \leqslant i \leqslant k$, be the lowest integer such that $0 \in \Sigma_i(\mathbf{C}(\alpha)_{\mathcal{U}})$. Now we use the method of "diagonal search" (see, for instance, the proof of Lemma 1.8 from [6]). Then, we have $0 \in \Sigma_j(\mathbf{C}((\theta_n|_F)_{\mathcal{U}}))$ for some n and j, $0 \leqslant j \leqslant i$. Thus $0 \in \sigma((\theta_n|_F)_{\mathcal{U}})$ for some n, and, by Lemma 5.4, $\sigma((\theta_n|_F)_{\mathcal{U}}) = \sigma(\theta_n|_F)$. Then, by Lemma 6.7, $\sigma(\theta_n|_F) = \sigma(\alpha|_F)$, i.e., $0 \in \sigma(\alpha|_F)$. ∎

Let E be a closed subspace in $B(X)$ generated by a family of mutually commuting operators $T' = \{T_\alpha : \alpha \in \Lambda\}$, i.e., E is a commutative B-L algebra. If T' is the bounded family, then there exists a bounded linear representation $\varepsilon : \ell_1(\Lambda) \to B(X)$, $\varepsilon(\sum a_\alpha e_\alpha) = \sum a_\alpha T_\alpha$, where $\{e_\alpha : \alpha \in \Lambda\}$ is the canonical base of the Banach space $\ell_1(\Lambda)$. Let us consider the injective maps $E^* \to \mathbb{C}^\Lambda$, $\lambda \mapsto (\lambda(T_\alpha))$, and $\ell_1(\Lambda)^* \to \mathbb{C}^\Lambda$, $\lambda \mapsto (\lambda(e_\alpha))$. We denote by $\sigma(T')$ and ℓ_1-$\sigma(T')$, correspondingly, the images of the spectra of the identical representation of E and the representation ε by means of these maps. By Lemma 6.2, the spectra $\sigma(T')$ and ℓ_1-$\sigma(T')$ are precompact in $\mathbb{C}^\Lambda$. Let $\sigma^{\mathrm{u}}(T')$ (ℓ_1-$\sigma^{\mathrm{u}}(T')$) be the union of $\sigma(T'_{\mathcal{U}})$ (ℓ_1-$\sigma(T'_{\mathcal{U}})$) by all ultrafilters $\mathcal{U}$, where $T'_{\mathcal{U}} = \{T_{\alpha\mathcal{U}} : \alpha \in \Lambda\}$. By Corollary 5.6, $\sigma^{\mathrm{u}}(T') \neq \emptyset$ and ℓ_1-$\sigma^{\mathrm{u}}(T'_{\mathcal{U}}) \neq \emptyset$.

Corollary 6.9. *Let $T' = \{T_\alpha : \alpha \in \Lambda\}$ and $T = \{T_\alpha : \alpha \in \Xi\}$, where $\Xi \subseteq \Lambda$ has finite complement $\Lambda \setminus \Xi = \{\alpha_1, \dots, \alpha_n\}$. Then*

$$\begin{aligned} \sigma(T) &= \sigma(T')|_\Xi, & \sigma^{\mathrm{u}}(T) &= \sigma^{\mathrm{u}}(T')|_\Xi, \\ \ell_1\text{-}\sigma(T) &= \ell_1\text{-}\sigma(T')|_\Xi, & \ell_1\text{-}\sigma^{\mathrm{u}}(T) &= \ell_1\text{-}\sigma^{\mathrm{u}}(T')|_\Xi. \end{aligned}$$

Moreover, ℓ_1-$\sigma^{\mathrm{u}}(T')|_{\Lambda\setminus\Xi} \subseteq \sigma(T_{\alpha_1}, \dots, T_{\alpha_n})$.

Proof. It suffices to note that a closed subspace in E generated by the family T has finite codimension, and to use Theorems 6.5 and 6.8. ∎

If T' is a finite family, then $\sigma(T') = \ell_1$-$\sigma(T') = \sigma^{\mathrm{u}}(T') = \ell_1$-$\sigma^{\mathrm{u}}(T')$ and the projection property for Taylor and Słodkowski spectra of a finite commutative operator family are known ([14] and [13]).

7. Dominated Banach algebras

Everywhere in the sequel, E is a finite-dimensional nilpotent Lie algebra, $U(E)$ is the universal enveloping algebra of E and (X, α) is a Banach E-module. In this section, we introduce some Banach algebras A of limits of rational functions on E acting on X, and we use Lie subalgebras in A which are B-L algebras with respect to some norm $\|\cdot\| \geqslant \|\cdot\|_A$. We call such algebras *B-L subalgebras* in A.

Definition 7.1. Let A be a Banach algebra with identity, containing E as Lie subalgebra. Assume that the full subalgebra in A generated by E is dense, i.e., $\overline{\mathcal{R}(E)} = A$. We say that A *dominates the module* (X, α) if there exists a bounded unital algebra homomorphism $\widetilde{\theta} : A \to B(B(\Lambda E, X))$ such that $\widetilde{\theta}|_E = \theta$.

We shall use the notation $A \succ (X, \alpha)$ if A dominates the E-module (X, α). It is clear that X makes into a Banach A-module if $A \succ (X, \alpha)$. We denote the corresponding bounded representation $A \to B(X)$ by $\widetilde{\alpha}$. Thus $\widetilde{\alpha}|_E = \alpha$. The elements of $\mathcal{R}(E)$ are called *rational functions* in A.

Lemma 7.2. *Let $A \succ (X, \alpha)$ and let $\mathcal{U}$ be an ultrafilter. Then $A \succ (X_{\mathcal{U}}, \alpha_{\mathcal{U}})$.*

Proof. By Definition 7.1, we have a continuous map $\widetilde{\theta}_{\mathcal{U}} : A \to B(B(\Lambda E, X)_{\mathcal{U}})$, $\widetilde{\theta}_{\mathcal{U}}(a) = \widetilde{\theta}(a)_{\mathcal{U}}$. But $B(\Lambda E, X)_{\mathcal{U}} = B(\Lambda E, X_{\mathcal{U}})$ and for each $u \in E$, $\widetilde{\theta}_{\mathcal{U}}(u) = \widetilde{\theta}(u)_{\mathcal{U}} = \theta(u)_{\mathcal{U}}$. It remains to note that $\theta(u)_{\mathcal{U}} = (L_{\alpha(u)} - R_{T(u)})_{\mathcal{U}} = L_{\alpha_{\mathcal{U}}(u)} - R_{T(u)} = \theta_{\mathcal{U}}(u)$ and $\widetilde{\theta}_{\mathcal{U}} = \widetilde{\theta_{\mathcal{U}}}$. ∎

Lemma 7.3. *If $A \succ (X, \alpha)$, then $C^\cdot(\alpha)$ is a complex of Banach A-modules.*

Proof. A member $C^p(E, X)$ of the complex $C^\cdot(\alpha)$ is a closed E-submodule in $B(\Lambda E, X)$ with respect to the representation θ. By (6.1), $d\widetilde{\theta}(a) = \widetilde{\theta}(a)d$ for each $a \in \mathcal{R}(E)$. Using Definition 7.1, infer that the latter is valid for each $a \in A$. It means that $C^\cdot(\alpha)$ is a complex of Banach A-modules. ∎

To explain our notion of dominated Banach algebra, we consider an algebra of convergent power series of a basis of E. Let $e = \{e_1, \dots, e_n\}$ be a basis of E such that the adjoint representation of E is reduced to the strictly triangular form. It is clear that for any tuple $r = (r_1, \dots, r_n)$ of positive real numbers, the basis $r^{-1}e = \{r_1^{-1}e_1, \dots, r_n^{-1}e_n\}$ has the same property. Let us consider different monomials of the form $v = e_{i_1} \cdots e_{i_s} \in U(E)$. Each element $u \in U(E)$ is represented as a linear combination of different monomials $u = a_1v_1 + \cdots + a_nv_m$. We introduce the degree of a monomial in the following way. Set $\deg(e_i) = \max\{k : e_i \in E^k\}$, where $E^1 = E, E^k = [E, E^{k-1}]$. The degree $\deg(v)$ of $v = e_{i_1} \cdots e_{i_s}$ is defined as the sum of degrees of multipliers. Let U_k be the set of elements of $U(E)$ presented as a sum of monomials of degree at least k. Using $[E^k, E^m] \subseteq E^{k+m}$ and nilpotentness of E, we obtain that $U^kU^m \subseteq U^{k+m}$ and $\dim(U^k/U^{k+1}) < \infty$. For each $u \in U(E)$, we set

$$\|u\| = \inf\{|a_1| + \cdots + |a_m| : u = a_1v_1 + \cdots + a_nv_m\}, \tag{7.1}$$

where the least lower bound is taken over all representations of u of indicated form. Moreover, in these representations of u, it suffices to consider only monomials with bounded (by a constant depending of u and n) degrees. Since the number of such different monomials is finite, the equality $\|u\| = 0$ is possible iff $u = 0$. Thus, the algebra $U(E)$ with $\|\cdot\|$ is a normed algebra. Let $A(e)$ be its completion. For a basis $r^{-1}e$, we get another Banach algebra $A(r^{-1}e)$.

LEMMA 7.4. *Let B be a Banach algebra and let $\alpha : E \to B$ be a Lie homomorphism. If the semigroup $\mathrm{SG}(r^{-1}\alpha(e))$ is bounded for some tuple r, then α is extended up to a bounded algebra homomorphism $\widetilde{\alpha} : A(r^{-1}e) \to B$.*

Proof. Let $C = \|\mathrm{SG}(r^{-1}\alpha(e))\|$, and let $\alpha|^{U(E)}$ be the canonical extension α. Then $\|\alpha|^{U(E)}(v)\| \leqslant C$ for a monomial $v = r_{i_1}^{-1}e_{i_1} \cdots r_{i_s}^{-1}e_{i_s} \in U(E)$. Whence, $\|\alpha|^{U(E)}(u)\| \leqslant C\|u\|$ for each $u \in U(E)$, by vitue of (7.1). Thus $\alpha|^{U(E)} : U(E) \to B$ is a bounded algebra homomorphism and it is extended up to a bounded homomorphism $\widetilde{\alpha} = \alpha|^{A(r^{-1}e)}$. ∎

In particular, the extension $\alpha|^{A(r^{-1}e)}$ is possible if $\rho(r^{-1}\alpha(e)) < 1$ or $\|r^{-1}\alpha(e)\| \leqslant 1$. Let $m = \dim(E/[E, E])$, $r_* = (r_1, \dots, r_m)$, $\mathbf{t} = (t_1, \dots, t_m)$, and let $N(r_*) = \Big\{u = \sum a_{\mathbf{t}}z^{\mathbf{t}} : \|u\|_{r_*} = \sum |a_{\mathbf{t}}|r_*^{\mathbf{t}} < \infty\Big\}$ be a Banach subalgebra of the algebra of holomorphic functions defined on the polydisk in $\mathbb{C}^m$ of multiradius r_* centered at zero. Since $\{e_{m+1}, \dots, e_n\}$ is a basis of $[E, E]$, there is a Lie homomorphism $\alpha : E \to N(r_*)$, $\alpha(e_i) = z_i$, $1 \leqslant i \leqslant m$, $\alpha(e_j) = 0$, $j > m$. It is clear that $\|r^{-1}\alpha(e)\|_{r_*} \leqslant 1$. By Lemma 7.4 we have a bounded homomorphism $A(r^{-1}e) \to N(r_*)$, $\sum a_{\mathbf{k}}e^{\mathbf{k}} \mapsto \sum a_{\mathbf{t}}z^{\mathbf{t}}$. But $N(r_*)$ is semisimple and $A(r^{-1}e)$ is commutative module the Jacobson radical $\mathrm{Rad}A(r^{-1}e)$ by [16], therefore the kernel of the latter homomorphism coincides with $\mathrm{Rad}A(r^{-1}e)$.

LEMMA 7.5. *Let (X, α) be a Banach E-module. If $\rho(r^{-1}\alpha(e)) < 1$ for some tuple r of positive numbers, then $A(r^{-1}e) \succ (X, \alpha)$.*

Proof. Let $e' = r^{-1}e$. By Lemma 7.4 and Definition 7.1, it suffices to prove that $\rho(\theta(e')) < 1$. As $\theta(u) = L_{\alpha(u)} - R_{T(u)}$ and $[L_{\alpha(u)}, R_{T(v)}] = 0$ for all $u, v \in E$, one can easily check that $\rho(\theta(e')) \leqslant \rho(L_{\alpha(e')}) + \rho(R_{T(e')}) \leqslant \rho(\alpha(e')) + \rho(T(e'))$. By Lemma 6.1, $T(E)$ is a nilpotent Lie algebra consisting of nilpotent operators which act on a finite-dimensional space ΛE. By Engel theorem, $T(E)$ generates a nilpotent algebra in $B(\Lambda E)$. Then $T(e')^k = 0$ for sufficiently large k. Thus, $\rho(T(e')) = 0$ and $\rho(\theta(e')) < 1$. ∎

Now, we give examples of B-L subalgebras in $A(r^{-1}e)$ which are projective and weak quasinilpotent. They will satisfy conditions of our main result on spectral mapping theorem to be proved in Section 10.

At first, we state the uniqueness of the expansion in a power series by e.

For brevity, we assume that E is Heisenberg algebra with basis $e = \{h, f, l\}$ such that $[h, f] = l$ and $[h, l] = [f, l] = 0$. By using exact finite-dimensional representations of E, we may assume that E is a Lie subalgebra of nilpotent operators in $B(V)$ for some finite-dimensional space V; moreover, the following conditions are fulfilled: $l^2 = hl = fl = 0$, and there exist linearly independent vectors $x, z \in V$ such that $x \in \mathrm{N}(f) \cap \mathrm{N}(l)$, $z \in \mathrm{N}(h) \cap \mathrm{N}(l)$, $h(x) = x'$, $f(z) = z'$, $x', z' \in \mathrm{N}(h) \cap \mathrm{N}(f)$ and x', z' are linearly independent. Let $V_m = V^{\widehat{\otimes} m}$, $m \geqslant 0$, $v^{\otimes m} = v \otimes \cdots \otimes v$, and let $\varsigma_m : U(E) \to B(V_m)$, $\varsigma_m(u) = \sum\limits_{i=1}^{m} 1 \otimes \cdots \otimes u \otimes \cdots \otimes 1$, $u \in E$ (u is situated in the i-th place), be a representation. We consider the following tensors $w_{tks} = x^{\otimes t} \otimes z^{\otimes k} \otimes y^{\otimes s}$, where $t, k, s \geqslant 0$, $y \in V$ such that $y' = l(y) \neq 0$. If we replace in w_{tks} one of the vectors x, z, y by x', z', y' respectively, then we shall denote obtained tensors by $w_{t'ks}, w_{tk's}, w_{tks'}$ respectively. If $m = t + k + s$, then one can easily check that $\varsigma_m(l^s) w_{tks} = s! w_{tks'}$ and $\varsigma_m(l^{s+1}) w_{tks} = 0$ as $l^2 = 0$. Similarly, we can prove that $\varsigma_m(h^p f^q l^s) w_{tks} = t!k!s! w_{t'k's'}$ if $p = t$, $q = k$, and $\varsigma_m(h^p f^q l^s) w_{tks} = 0$ if $p > t$ or $q > k$. Now let $p_\delta = \sum\limits_{t+k=\delta} a_{tk} h^t f^k$ be a nonzero homogeneous polynomial of degree δ. If $m = \delta + s$ and $\phi_{\delta s} = \sum\limits_{t+k=\delta} \overline{a_{tk}} w_{tks}$, then $\varsigma_m(p_\delta l^s)(\phi_{\delta s}) = s! \Big(\sum\limits_{t+k=\delta} t!k! |a_{tk}|^2 x'^{\otimes t} \otimes z'^{\otimes k} \Big) \otimes y'^{\otimes s}$. By assumption, x', z' are independent vectors. Then the tensor situated in brackets is nonzero. Thus, $\varsigma_m(p_\delta l^s)(\phi_{\delta s}) \neq 0$. In general case, we present any polynomial $p \in U(E)$ of variables h, f as a sum of homogeneous members $p = \sum\limits_{\gamma \geqslant \delta} p_\gamma$, where p_δ is the nonzero homogeneous polynomial of least degree δ presented in p. Then, $\varsigma_m(pl^s)(\phi_{\delta s}) = \varsigma_m(p_\delta l^s)(\phi_{\delta s}) \neq 0$.

Let $r = (r_1, r_2, r_3)$ be a tuple of positive numbers. Let $S_m = \varsigma_m(r^{-1}e)$ be the set of nilpotent operators. Then there exists a norm $\|\cdot\|_m$ in V_m such that the operator norm $\|S_m\|_m$ is less or equal to 1. Let $X = (\ell_1) \bigoplus\limits_{m \geqslant 0} V_m$ be the ℓ_1-norm sum of Banach spaces. If $\varsigma : U(E) \to B(X), \varsigma(u)(x_m) = (\varsigma_m(u) x_m)$, is a representation, then for $u = ar_1^{-1}h + br_2^{-1}f + cr_3^{-1}l \in E$, we have $\|\varsigma(u)(x_m)\| \leqslant \sum\limits_m (|a| + |b| + |c|) \|S_m\|_m \|x_m\|_m \leqslant (|a| + |b| + |c|) \|(x_m)\|$, i.e. $\|\varsigma(u)\| \leqslant |a| + |b| + |c|$. Then $\|r^{-1}\varsigma(e)\| \leqslant 1$. By Lemma 7.4, exists a bounded extension $\widetilde{\varsigma} : A(r^{-1}e) \to B(X)$.

LEMMA 7.6. *Let $u \in A(r^{-1}e)$. If u has an expansion $u = \sum a_{tks} h^t f^k l^s$ as an absolutely convergent power series then such expansion is unique.*

Proof. It suffices to show that if $u = \sum a_{tks} h^t f^k l^s = 0$, then all $a_{tks} = 0$. If l is absent in the expansion of u, then u defines usual holomorphic function $v(z) = \sum a_{tk} z_1^t z_2^k$ on the polydisk in $\mathbb{C}^2$ by means of the homomorphism $A(r^{-1}e) \to N(r_*)$, and $v(z) = 0$ on this polydisk. So, all $a_{lk} = 0$. Thus we can assume that $u \in \mathrm{Rad}A(r^{-1}e) = \overline{U(E)l}$, i.e. $u = \sum a_{tks} h^t f^k l^s$, $s \geqslant 1$. Assume that not all a_{tks} are equal to zero. Let s be the least degree of l in the expansion of u and let $u_n = \sum\limits_{t,k} a_{tkn} h^t f^k l^n$. Then $u = \sum\limits_{n \geqslant s} u_n$ and $u_s = \lim\limits_N \Big(\sum\limits_{i=\delta}^{N} p_i\Big) l^s$, where $p_i = \sum\limits_{t+k=i} a_{tk} h^t f^k$ is a homogeneous polynomial having degree i, and δ is the least degree of nonzero polynomials p_i. Let $x = (x_m) \in X$ such that $x_{\delta+s} = \phi_{\delta s}$, $x_m = 0$ if $m \neq \delta + s$. Then $y_{\delta+s} = \varsigma_{\delta+s}\Big(\Big(\sum\limits_{i=\delta}^{N} p_i\Big) l^s\Big) x_{\delta+s} = \varsigma_{\delta+s}(p_\delta l^s) x_{\delta+s} \neq 0$ and if $y = (y_m) \in X$ ($y_m = 0, m \neq \delta + s$), then $\widetilde{\varsigma}(u_s)x = \lim\limits_N \varsigma\Big(\sum\limits_{i=\delta}^{N} p_i l^s\Big) x = y$. But $\varsigma(l^{s+i})x = 0$, $i \geqslant 1$. Thus $\widetilde{\varsigma}(u_{s+i})x = 0$ and $\widetilde{\varsigma}(u)x = y \neq 0$, a contradiction. ∎

EXAMPLE 7.7. Let E be a Heisenberg algebra with a basis $e = \{h, f, l\}$ and consider the polynomials $p_1 = h^t$ ($t \geqslant 1$) and $p_2 = hf$. One can easily check that $(\mathrm{ad}(p_2))^n p_1 = (-t)^n p_1 l^n$ and the Lie subalgebra P in $U(E)$ generated by $\{p_1, p_2\}$ consists of all linear combinations of the form $\sum\limits_{n=-1}^{N} a_n p_1 l^n = a_{-1} p_2 + a_0 p_1 + \sum\limits_{n=1}^{N} a_n p_1 l^n$, $a_n \in \mathbb{C}, n \geqslant -1$. Thus P is an infinite dimensional Lie algebra. Let $\mathcal{F}$ be the subspace in $A(r^{-1}e)$ of all absolutely convergent series of the form $u = \sum\limits_{n=-1}^{\infty} a_n p_1 l^n$. Set $\|u\|_1 = \sum\limits_{n=-1}^{\infty} \|a_n p_1 l^n\| < \infty$. Then $\|\cdot\|_1$ is norm in $\mathcal{F}$ by virtue of Lemma 7.6. Undoubtedly, $\|\cdot\|_1 \geqslant \|\cdot\|$ and $\mathcal{F}$ furnished with the norm $\|\cdot\|_1$ is isomorphic to the space ℓ_1. Moreover, $\mathcal{F}$ is a B-L subalgebra in $A(r^{-1}e)$ with Lie generators p_1, p_2. Indeed, $[u, v] = \sum\limits_{n=0}^{\infty} t(a_n b_{-1} - a_{-1} b_n) p_1 l^{n+1}$ for $u = \sum\limits_{n=-1}^{\infty} a_n p_1 l^n$ and $v = \sum\limits_{n=-1}^{\infty} b_n p_1 l^n$ from $\mathcal{F}$, and since $[p_1 l^n, p_2] = t p_1 l^{n+1}$, so

$$\|[u,v]\|_1 \leqslant 2|b_{-1}|\,\|p_2\| \sum_{n=0}^{\infty} |a_n|\,\|p_1 l^n\| + 2|a_{-1}|\,\|p_2\| \sum_{n=0}^{\infty} |b_n|\,\|p_1 l^n\| \leqslant 2\|u\|_1\,\|v\|_1.$$

Further,

$$(\mathrm{ad}(p_1))^2 = 0 \quad \text{and} \quad (\mathrm{ad}(p_2))^m \Big(\sum_{n=-1}^{\infty} a_n p_1 l^n\Big) = (-t)^m \sum_{n=0}^{\infty} a_n p_1 l^{n+m}.$$

Then $\|(\mathrm{ad}(p_2))^m(u)\|_1 \leqslant \|(tl)^m\|\,\|u\|_1$, $u \in \mathcal{F}$. Since $l \in \mathrm{Rad}A(r^{-1}e)$, then

$$\|(\mathrm{ad}(p_2))^m\|_1^{1/m} \leqslant \|(tl)^m\|^{1/m} \to 0, \quad m \to \infty.$$

Thus $\mathcal{F}$ is a weak quasinilpotent B-L algebra isomorphic to the space ℓ_1.

EXAMPLE 7.8. Now let the polynomials $p_1 = h^2$, $p_2 = f^2$ of the base of Heisenberg algebra E and P be a Lie subalgebra in $U(E)$ generated by $\{p_1, p_2\}$. Note that P comprises all linear combinations $\sum_{n=0}^{N} S_n$, where $S_n = a_{2n}p_1 l^{2n} + b_{2n}p_2 l^{2n} + c_{2n+1}ql^{2n+1}$, $a_{2n}, b_{2n}, c_{2n+1} \in \mathbb{C}$, $q = 2hf - l$. Let $\mathcal{F}$ be a subspace in $A(r^{-1}e)$ of all series $u = \sum_{n=0}^{\infty} S_n$ such that $\|u\|_1 = \sum_{n=0}^{\infty} (\|a_{2n}p_1 l^{2n}\| + \|b_{2n}p_2 l^{2n}\| + \|c_{2n+1}ql^{2n+1}\|) < \infty$. As in Example 7.7 we can prove that $\mathcal{F}$ is a B-L subalgebra $A(r^{-1}e)$ isomorphic to the space ℓ_1. Moreover, $(\mathrm{ad}(p_1))^3 = (\mathrm{ad}(p_2))^3 = 0$. Thus, $\mathcal{F}$ is a weak quasinilpotent B-L algebra.

8. THE FORWARD SPECTRAL MAPPING PROPERTY

By follow R. Harte ([9]), we say that a joint spectrum σ has the forward spectral mapping property for generators a of a nilpotent Lie algebra and polynomials p if there is the inclusion $p(\sigma(a)) \subseteq \sigma(p(a))$. If the opposite inclusion is satisfied, then we say that σ has the backward spectral mapping property. In this section we prove the forward spectral mapping property for limits of rational functions generated a projective B-L algebra.

LEMMA 8.1. *Let $A \succ (X, \alpha)$. Then $\sigma(\widetilde{\theta}(a)) = \sigma(\widetilde{\alpha}(a))$ for all $a \in A$.*

Proof. Let M be a Lie algebra generated by the multiplication operators $L_{\alpha(u)}$, $R_{T(u)} \in B(B(\Lambda E, X))$, $u \in E$, and let $B = \overline{\mathcal{R}(M)}$. Since E is a nilpotent Lie algebra, so M is also nilpotent. By [16], the algebra B is commutative module the Jacobson radical Rad$B$. Then, $R_{T(u)} \in \mathrm{Rad}B$ for all $u \in E$. If $p = p(E)$ is a polynomial in A, then $\widetilde{\theta}(p) = L_{\tilde{\alpha}(p)} + Q$, where Q belongs to the ideal generated by $R_{T(u)}$, $u \in E$, i.e. $Q \in \mathrm{Rad}B$. If p is invertible in A, then $\widetilde{\theta}(p^{-1}) - L_{\tilde{\alpha}(p^{-1})} \in \mathrm{Rad}B$. Thus $\widetilde{\theta}(r^{(1)}) - L_{\tilde{\alpha}(r^{(1)})} \in \mathrm{Rad}B$ for any rational function $r^{(1)} = r^{(1)}(E)$ of first order in A. By induction on n one may easily prove that $\widetilde{\theta}(r^{(n)}) - L_{\tilde{\alpha}(r^{(n)})} \in \mathrm{Rad}B$ for each rational functions $r^{(n)}$ of n-th order in A. Take $a \in A$, then $a = \lim r_m$ is a limit of rational functions. Hence $\widetilde{\theta}(a) - L_{\tilde{\alpha}(a)} \in \mathrm{Rad}B$. It follows that the spectrum $\sigma_B(\widetilde{\theta}(a))$ with respect to the subalgebra B is equal to $\sigma_B(L_{\tilde{\alpha}(a)})$. Thus $\sigma(\widetilde{\theta}(a)) = \sigma_B(\widetilde{\theta}(a)) = \sigma(\widetilde{\alpha}(a))$. ∎

LEMMA 8.2. *Let $A \succ (X, \alpha)$, $\lambda \in \sigma_{\pi,\infty}(\alpha)$ and let $d(\lambda)$ be the differential of the complex $C^{\cdot}(\alpha - \lambda)$. If $\omega \in \mathrm{N}(d(\lambda)) \setminus \mathrm{R}(d(\lambda))$, then $(\widetilde{\theta}(r) - r(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$ for any rational function r in A.*

Proof. By using (6.1) and (6.2), we obtain that $(\widetilde{\theta}(p) - p(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$ for any polynomial p in A. By (6.1), $(\widetilde{\theta}(p^{-1}) - p(\lambda(E))^{-1})\omega \in \mathrm{R}(d(\lambda))$ if p is invertible in A. It follows that $(\widetilde{\theta}(r) - r(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$ for each rational function $r = r^{(1)}(E)$ of order 1.

Now, let $r = r^{(n)}(E)$ be a rational function in A of order n. We proceed by induction on n. Then $r = p(r_i, r_j^{-1})$ is a polynomial of rational functions r_i, r_j^{-1} $(i, j \in \Lambda)$ of $(n-1)$-th order and its inverses. By induction hypothesis $(\widetilde{\theta}(r_i) - r_i(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$, $i \in \Lambda$. If r_j is invertible, then using again the same argument, we obtain that $(\widetilde{\theta}(r_j^{-1}) - r_j(\lambda(E))^{-1})\omega \in \mathrm{R}(d(\lambda))$. Whence, $(\widetilde{\theta}(r) - r(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$ by virtue of (6.1) and (4.2). ∎

LEMMA 8.3. *Let $A \succ (X, \alpha)$ and $\lambda \in \sigma_{\pi,\infty}(\alpha)$. Then $r(\lambda(E)) \in \sigma(\widetilde{\alpha}(r))$ for any rational function r in A.*

Proof. By assumption, there exists $\omega \in C^k(E, X)$ such that $d(\lambda)\omega = 0$ and $\omega \notin \mathrm{R}(d(\lambda))$, where $d(\lambda)$ is the differential of the complex $C^\cdot(\alpha - \lambda)$. Let r be a rational function in A. By Lemma 8.2, $(\widetilde{\theta}(r) - r(\lambda(E)))\omega \in \mathrm{R}(d(\lambda))$. If $r(\lambda(E)) \notin \sigma(\widetilde{\theta}(r))$, then by (6.1) we would obtain that $\omega \in \mathrm{R}(d(\lambda))$. Thus $r(\lambda(E)) \in \sigma(\widetilde{\theta}(r))$. By Lemma 8.1, $r(\lambda(E)) \in \sigma(\widetilde{\alpha}(r))$. ∎

LEMMA 8.4. *Let $A \succ (X, \alpha)$ and $\lambda \in \sigma_{\pi,\infty}(\alpha)$. There exists a character $\lambda|^A$ of A such that $\lambda|^A|_E = \lambda$.*

Proof. Given a rational function $r = r(E)$ in A, we set $\lambda|^A(r) = r(\lambda(E))$. By Lemma 8.3, $\lambda|^A(r) \in \sigma(\widetilde{\alpha}(r))$. Thus $\lambda|^A : \mathcal{R}(E) \to \mathbb{C}$, $\lambda|^A(r) = r(\lambda(E))$, is a multiplicative linear functional. Moreover, $|\lambda|^A(r)| \leqslant \|\widetilde{\alpha}\| \, \|r\|$, $r \in \mathcal{R}(E)$, i.e., $\lambda|^A$ is a bounded map. ∎

For a Słodkowski spectrum σ, we denote $\sigma(\alpha)|^A = \{\lambda|^A : \lambda \in \sigma(\alpha)\}$.

Now, let $\mathcal{F}$ be a B-L subalgebra in A. By Definition 7.1, the space $B(\Lambda E, X)$ (in particular X) makes into a Banach $\mathcal{F}$-module. For any integer $p \geqslant 0$ and any ideal $I \subseteq E$, the space $C^p(I, X)$ is a closed $\mathcal{F}$-submodule in $B(\Lambda E, X)$(see Remark 6.4). Set $\widetilde{\theta}_{p,I}(u) = \widetilde{\theta}(u)|_{C^p(I,X)}$, $u \in A$, and $\widetilde{\theta}_{p,E} = \widetilde{\theta}_p$.

LEMMA 8.5. *Let $A \succ (X, \alpha)$ and $\mathcal{F}$ be a* B-L *subalgebra in A. Then*

$$\sigma(\widetilde{\theta}_{p,I}|_{\mathcal{F}}) \subseteq \sigma(\widetilde{\alpha}|_{\mathcal{F}}).$$

Proof. We proceed by induction on the pair $(p, \dim(I))$. Let J be an ideal in E such that $J \subseteq I$ and $\dim(I/J) = 1$. Let $u \in I \setminus J$. If $p = 0$, then there is nothing to prove. For $p \geqslant 1$, we use isomorphism

$$C^p(I, X) \to C^p(J, X) \oplus C^{p-1}(J, X), \quad \omega \mapsto (\omega|_J, (i_p(u)\omega)|_J),$$

defined in Lemma 6.3. If M is the kernel of the A-homomorphism $C^p(I, X) \to C^p(J, X)$, $\omega \mapsto \omega|_J$, then the map $\varphi : M \to C^{p-1}(J, X), \varphi(\omega) = (i_p(u)\omega)|_J$, is an A-isomorphism. Indeed, it suffices to prove that $\varphi(\theta(a)\omega) = \theta(a)\varphi(\omega)$ for all $\omega \in M$ and $a \in E$. By using (6.3), we obtain that

$$\varphi(\theta(a)\omega) = (\theta(a) i_p(u)\omega)|_J + (i_p([u, a])\omega)|_J = \theta(a)\varphi(\omega).$$

Thus we have an exact sequence of Banach A-modules (especially as $\mathcal{F}$-modules)

$$0 \to C^{p-1}(J, X) \xrightarrow{\varphi^{-1}} C^p(I, X) \longrightarrow C^p(J, X) \to 0,$$

which is $\mathbb{C}$-split. By Lemma 5.1, the sequence of $\Delta(\mathcal{F})$-Banach complexes

$$0 \to \mathbf{C}(\widetilde{\theta}_{p-1,J}|_{\mathcal{F}}) \longrightarrow \mathbf{C}(\widetilde{\theta}_{p,I}|_{\mathcal{F}}) \longrightarrow \mathbf{C}(\widetilde{\theta}_{p,J}|_{\mathcal{F}}) \to 0$$

is exact. Then, by Corollary 3.5, $\sigma(\widetilde{\theta}_{p,I}|_{\mathcal{F}}) \subseteq \sigma(\widetilde{\theta}_{p-1,J}|_{\mathcal{F}}) \cup \sigma(\widetilde{\theta}_{p,J}|_{\mathcal{F}})$ and by induction hypothesis $\sigma(\widetilde{\theta}_{p-1,J}|_{\mathcal{F}}) \cup \sigma(\widetilde{\theta}_{p,J}|_{\mathcal{F}}) \subseteq \sigma(\widetilde{\alpha}|_{\mathcal{F}})$, i.e. $\sigma(\widetilde{\theta}_{p,I}|_{\mathcal{F}}) \subseteq \sigma(\widetilde{\alpha}|_{\mathcal{F}})$. ∎

We introduce the bicomplex connected Banach complexes $C^{\cdot}(\widetilde{\alpha}|_{\mathcal{F}} - \mu)$ with $\mu \in \Delta(\mathcal{F})$, and $C^{\cdot}(\alpha|_I - \lambda)$ with $\lambda \in \Delta(I)$, by means of complexes $C^{\cdot}(\widetilde{\theta}_{p,I}|_{\mathcal{F}} - \mu)$, $p \geqslant 0$. The following diagram

$$\begin{array}{ccccc} & & \vdots & & \\ & & {\scriptstyle\beta_\mu}\uparrow & & \\ \cdots & \stackrel{\kappa_\lambda}{\to} & C^q(\mathcal{F}, C^p(I,X)) & \stackrel{\kappa_\lambda}{\to} & \cdots \\ & & {\scriptstyle\beta_\mu}\uparrow & & \\ & & \vdots & & \end{array}$$

is commutative, where $\kappa_\lambda(\Phi) = d^p(\lambda) \cdot \Phi$, $\Phi \in C^q(\mathcal{F}, C^p(I,X))$ ($d^p(\lambda)$ is the differential of the complex $C^{\cdot}(\alpha|_I - \lambda)$) and β_μ is the differential of the complex $C^{\cdot}(\widetilde{\theta}_{p,I}|_{\mathcal{F}} - \mu)$. Thus the latter diagram is a bicomplex with rows $C^q(\mathcal{F}, C^{\cdot}(\alpha|_I - \lambda))$, $q \geqslant 0$, and columns $C^{\cdot}(\widetilde{\theta}_{p,I}|_{\mathcal{F}} - \mu)$, $p \geqslant 0$, for which we use a notation $\mathrm{B}_{\lambda,\mu}(I, \mathcal{F}, X)$. The corresponding total complex is denoted by $\mathrm{Tot}_{\lambda,\mu}(I, \mathcal{F}, X)$. For any p, the bicomplex $\mathrm{B}_{\lambda,\mu}(I, \mathcal{F}, X)$ associates the cohomology complex

$$0 \to H^p(C^{\cdot}(\alpha|_I - \lambda)) \xrightarrow{\dot{\beta}_\mu} \cdots \xrightarrow{\dot{\beta}_\mu} H^p(C^q(\mathcal{F}, C^{\cdot}(\alpha|_I - \lambda))) \xrightarrow{\dot{\beta}_\mu} \cdots, \tag{8.1}$$

where $H^p(C^q(\mathcal{F}, C^{\cdot}(\alpha|_I - \lambda)))$, is the p-th cohomology space of the q-th row.

THEOREM 8.6. *Let $A \succ (X, \alpha)$, and let $\mathcal{F}$ be a* B-L *subalgebra in A. If $\mathcal{F} \in \mathrm{Proj}$, then*

$$\sigma(\alpha)|^A|_{\mathcal{F}} \subseteq \sigma^{\mathrm{u}}(\widetilde{\alpha}|_{\mathcal{F}}).$$

Proof. Let $\mathcal{U}$ be an ultrafilter on some set S, such that $\sigma^{\mathrm{p}}(\alpha_{\mathcal{U}}) = \sigma_{\pi,0}(\alpha_{\mathcal{U}}) = \sigma^{\mathrm{ap}}(\alpha)$ (Theorem 5.7), $\sigma = \sigma_{\pi,k}$, $0 \leqslant k \leqslant \infty$, $\lambda \in \sigma(\alpha)$, $\lambda|^A|_{\mathcal{F}} = \mu$ and let i be the lowest integer such that $\lambda \in \sigma_{\pi,i}(\alpha)$. If $i = 0$, then $\lambda \in \sigma^{\mathrm{ap}}(\alpha)$. Consequently, there exists a nonzero $[x_s] \in X_{\mathcal{U}}$ such that $(\alpha_{\mathcal{U}}(u) - \lambda(u))[x_s] = 0$, $u \in E$. By using the density of $\mathcal{R}(E)$ in A, and Lemmas 7.2 and 8.4, infer $(\widetilde{\alpha_{\mathcal{U}}}(f) - \lambda|^A(f))[x_s] = 0$, $f \in A$, i.e. $\mu \in \sigma_{\pi,0}(\widetilde{\alpha_{\mathcal{U}}}|_{\mathcal{F}})$. But $\widetilde{\alpha_{\mathcal{U}}} = \widetilde{\alpha}_{\mathcal{U}}$ by Lemma 7.2. Thus, $\mu \in \sigma^{\mathrm{u}}(\widetilde{\alpha}|_{\mathcal{F}})$.

Let $i > 0$. By Lemma 5.4, $\sigma_{\pi,i}(\alpha) = \sigma_{\pi,i}(\alpha_{\mathcal{U}})$, and, by Lemma 3.3, $\sigma_{\pi,i}(\alpha_{\mathcal{U}}) = \bigcup_{p=0}^{i} \Sigma_p(\mathbf{C}(\alpha_{\mathcal{U}}))$. Then $H^i(C^{\cdot}(\alpha_{\mathcal{U}} - \lambda))$ is a nontrivial Banach space (otherwise, by Definition 3.1, $\lambda \in \sigma_{\pi,i-1}(\alpha_{\mathcal{U}}) = \sigma_{\pi,i-1}(\alpha)$, which contradicts the choice of i). Let us consider the operator $\dot{\beta_{\mathcal{U}_\mu}} : H^i(C^{\cdot}(\alpha_{\mathcal{U}} - \lambda)) \to H^i(C^1(\mathcal{F}, C^{\cdot}(\alpha_{\mathcal{U}} - \lambda)))$ of the cohomology complex (8.1) for the bicomplex $\mathrm{B}_{\lambda,\mu}(E, \mathcal{F}, X_{\mathcal{U}})$. Let us prove that $\dot{\beta_{\mathcal{U}_\mu}} = 0$. Take $\omega \in \mathrm{N}(d^i_{\mathcal{U}}(\lambda)) \setminus \mathrm{R}(d^{i-1}_{\mathcal{U}}(\lambda))$, where $d^i_{\mathcal{U}}(\lambda)$ is the differential of the complex $C^{\cdot}(\alpha_{\mathcal{U}} - \lambda)$. For each $f \in \mathcal{F}$, $f = \lim_m r_n$ is a limit of rational functions in A. By Lemma 8.2,

$$\begin{aligned}\beta_{\mathcal{U}\mu}(\omega) f &= ((\widetilde{\theta_{\mathcal{U}}})_i(f) - \mu(f))\omega = \lim_m ((\widetilde{\theta_{\mathcal{U}}})_i(r_m) - \lambda|^A(r_m))\omega \\ &\in \overline{\mathrm{R}(d_{\mathcal{U}}^{i-1}(\lambda))} = \mathrm{R}(d_{\mathcal{U}}^{i-1}(\lambda)),\end{aligned}$$

where $\beta_{\mathcal{U}\mu}$ is the differential of the complex $C^{\cdot}((\widetilde{\theta_{\mathcal{U}}})_i|_{\mathcal{F}} - \mu)$. By assumption, $\mathcal{F} \in \mathrm{Proj}$. There exists an operator $\xi \in B(\mathcal{F}, C^{i-1}(E, X_{\mathcal{U}}))$ such that $d_{\mathcal{U}}^{i-1}(\lambda) \cdot \xi = \beta_{\mathcal{U}_\mu}(\omega)$, i.e., $\dot{\beta}_{\mathcal{U}_\mu}(\omega) = 0$.

From the condition $\lambda \notin \sigma_{\pi,i-1}(\alpha_{\mathcal{U}})$ it follows that $\lambda \notin \sigma_{\pi,i-1}(C^q(\mathcal{F}, \mathbf{C}(\alpha_{\mathcal{U}})))$ for all q, by Theorem 3.2. Now we can use the method of "diagonal search" (see Lemma 1.8 from [6]). Then $\mu \in \Sigma_j(\mathbf{C}((\widetilde{\theta_{\mathcal{U}}})_p|_{\mathcal{F}}))$ for some p and j, $j \leqslant i$, i.e., $\mu \in \sigma_{\pi,i}((\widetilde{\theta_{\mathcal{U}}})_p|_{\mathcal{F}})$. By Lemma 8.5, $\sigma_{\pi,i}((\widetilde{\theta_{\mathcal{U}}})_p|_{\mathcal{F}}) \subseteq \sigma_{\pi,i}(\widetilde{\alpha_{\mathcal{U}}}|_{\mathcal{F}}) = \sigma_{\pi,i}(\widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$. Thus $\mu \in \sigma^{\mathrm{u}}(\widetilde{\alpha}|_{\mathcal{F}})$. ∎

9. SPECTRA OF THE PAIR OF REPRESENTATIONS

In this section, the projection properties are established for spectra of the pair $(\alpha|_I, \widetilde{\alpha}|_{\mathcal{F}})$ of representations, which is essentially used in the spectral mapping theorem.

In Section 8 we have introduced $\Delta(I)\times\Delta(\mathcal{F})$-Banach complex $\mathrm{Tot}_{\lambda,\mu}(I, \mathcal{F}, X)$, $(\lambda, \mu) \in \Delta(I) \times \Delta(\mathcal{F})$, for which we use the denotation $\mathbf{Tot}(I, \mathcal{F}, X)$. One can identify topological spaces $\Delta(I \times \mathcal{F})$ and $\Delta(I) \times \Delta(\mathcal{F})$ by means of the bijective map $\tau \mapsto (\tau_I, \tau_{\mathcal{F}})$, where $\tau_I(u) = \tau(u, 0)$, $\tau_{\mathcal{F}}(f) = \tau(0, f)$. Thus, $\mathbf{Tot}(I, \mathcal{F}, X)$ is a $\Delta(I \times \mathcal{F})$-Banach complex.

DEFINITION 9.1. Let $\sigma(\mathbf{Tot}(I, \mathcal{F}, X))$ be a Słodkowski spectrum of $\mathbf{Tot}(I, \mathcal{F}, X)$. We call this set a *Słodkowski spectrum of the pair of representations* $(\alpha|_I, \widetilde{\alpha}|_{\mathcal{F}})$ denoted by $\sigma(\alpha|_I, \widetilde{\alpha}|_{\mathcal{F}})$.

Let us consider the cohomology complex (8.1) for $I = E$. By Lemma 7.3, $\mathrm{N}(d^p(\lambda))$ is a closed A (especially $\mathcal{F}$)-submodule in $C^p(E, X)$. The complex of Banach $\mathcal{F}$-modules generated by $\mathcal{F}$-module $(\mathrm{N}(d^p(\lambda)), \widetilde{\theta}|_{\mathcal{F}} - \mu)$ is a subcomplex of the p-th column of $\mathrm{B}_{\lambda,\mu}(E, \mathcal{F}, X)$. A $\mathcal{F}$-module structure of this complex is defined (see Section 6) by the representation

$$\Theta_\mu : \mathcal{F} \to B(B(\Lambda\mathcal{F}, \mathrm{N}(d^p(\lambda)))), \quad \Theta_\mu(f) = L_{(\tilde{\theta}-\mu)(f)} - R_{T(f)},$$

and let $I_p(f)$ be the homotopic operator of this complex. One can easily check that the image $\mathrm{R}(\kappa_\lambda)$ of the row differential κ_λ of $\mathrm{B}_{\lambda,\mu}(E, \mathcal{F}, X)$ is invariant under $\Theta_\mu(f)$ and $I_p(f)$. Thus the linear operators

$$\begin{aligned}\dot{\Theta}_\mu(f) &: H^p(C^q(\mathcal{F}, C^{\cdot}(\alpha - \lambda))) \to H^p(C^q(\mathcal{F}, C^{\cdot}(\alpha - \lambda))), \quad \dot{\Theta}_\mu(f)\Phi^{\sim} = (\Theta_\mu(f)\Phi)^{\sim}, \\ \dot{I}_p(f) &: H^p(C^q(\mathcal{F}, C^{\cdot}(\alpha - \lambda))) \to H^p(C^{q-1}(\mathcal{F}, C^{\cdot}(\alpha - \lambda))), \quad \dot{I}_p(f)\Phi^{\sim} = (I_p(f)\Phi)^{\sim}.\end{aligned}$$

are defined soundly. By using (6.1) and (6.2) for the complex generated by $\mathcal{F}$-module $(\mathrm{N}(d^p(\lambda)), \widetilde{\theta}|_{\mathcal{F}} - \mu)$ and passing to cohomologies, we obtain

$$\dot{\beta}_\mu \dot{\Theta}_\mu(f) = \dot{\Theta}_\mu(f)\dot{\beta}_\mu, \tag{9.1}$$

$$\dot{\beta}_\mu \dot{I}_p(f) + \dot{I}_{p+1}(f)\dot{\beta}_\mu = \dot{\Theta}_\mu(f). \tag{9.2}$$

Now let $\widetilde{\lambda}$ be a character of A such that $\widetilde{\lambda}|_E = \lambda$.

Lemma 9.2. *If $\mathcal{F} \in \mathrm{Proj}$ and $\mathrm{R}(d^{p-1}(\lambda))$ is closed, then*

$$\dot{\Theta}_\mu(f)\Phi^\sim = ((\widetilde{\lambda}(f) - \mu(f) - R_{T(f)})\Phi)^\sim.$$

Proof. By the definition of the operator $\dot{\Theta}_\mu(f)$, it suffices to prove that $(L_{\tilde{\theta}(f)}\Phi)^\sim = \widetilde{\lambda}(f)\Phi^\sim$, $\Phi \in C^q(\mathcal{F}, \mathrm{N}(d^p(\lambda)))$. By using the same argument as in the proof of Lemma 8.2, we infer that $(L_{(\tilde{\theta}-\tilde{\lambda})(f)}\Phi)(a) \in \overline{\mathrm{R}(d^{p-1}(\lambda))} = \mathrm{R}(d^{p-1}(\lambda))$ for each $a \in \Lambda^q\mathcal{F}$. By Lemma 2.1, there exists $\Psi \in C^q(\mathcal{F}, C^{p-1}(E,X))$ such that $\kappa_\lambda(\Psi) = d^{p-1}(\lambda) \cdot \Psi = L_{(\tilde{\theta}-\tilde{\lambda})(f)}\Phi$ or $(L_{(\tilde{\theta}-\tilde{\lambda})(f)}\Phi)^\sim = 0$. ∎

Theorem 9.3. *Let $A \succ (X,\alpha)$, $\mathcal{F}$ be a* B-L *subalgebra in A, $\lambda \in \sigma_{\pi,\infty}(\alpha)$, $\mu \in \Delta(\mathcal{F})$, and let $\mathcal{U}$ be an ultrafilter. Assume that $\mathcal{F} \in$*Proj *and $\mathcal{F}$ is a weak quasinilpotent* B-L *algebra. If $\lambda|^A|_{\mathcal{F}} \neq \mu$, then $(\lambda,\mu) \notin \sigma_{\pi,\infty}(\alpha_{\mathcal{U}}, \widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$.*

Proof. It suffices to prove that $\mathrm{Tot}_{\lambda,\mu}(E,\mathcal{F},X_{\mathcal{U}})$ is exact if $\lambda|^A|_{\mathcal{F}} \neq \mu$. It is well known (see, for instance, Lemma 1.7 of [6]) that the exactness of all cohomology complexes (8.1) of the bicomplex $\mathrm{B}_{\lambda,\mu}(E,\mathcal{F},X_{\mathcal{U}})$ imply the exactness of its total complex.

By Lemma 5.4, $\lambda \in \sigma_{\pi,\infty}(\alpha_{\mathcal{U}})$. Let k be the lowest integer such that $\lambda \in \Sigma_k(\mathbf{C}(\alpha_{\mathcal{U}}))$ and let $d_{\mathcal{U}}^{k-1}(\lambda)$ be the differential of the complex $C^\cdot(\alpha_{\mathcal{U}})$. Then $\mathrm{R}(d_{\mathcal{U}}^{k-1}(\lambda))$ is closed. Otherwise $\lambda \in \Sigma_{k-1}(\mathbf{C}(\alpha_{\mathcal{U}}))$ by Lemma 3.3 and this would contradict the choice of k. Assume that $\lambda|^A|_{\mathcal{F}} \neq \mu$ and let S be a set of Lie generators of $\mathcal{F}$ such that $\sigma(\mathrm{ad}(f)) = \{0\}$ for all $f \in S$. Then $\lambda|^A(f) \neq \mu(f)$ for some $f \in S$. Let $G(f) \in B(C^q(\mathcal{F}, C^p(E, X_{\mathcal{U}})))$, $G(f) = \lambda|^A(f) - \mu(f) - R_{T_q(f)}$. It is clear that $(\kappa_{\mathcal{U}})_\lambda G(f) = G(f)(\kappa_{\mathcal{U}})_\lambda$, where $(\kappa_{\mathcal{U}})_\lambda$ is the row differential of $\mathrm{B}_{\lambda,\mu}(E,\mathcal{F},X_{\mathcal{U}})$. It follows that if $G(f)$ is invertible, then $\dot{\Theta}_\mu(f)$ is also invertible by Lemma 9.2. Then, by (9.1) and (9.2), we would obtain that all cohomology complexes (8.1) are exact.

It remains to prove that $G(f)$ is an invertible operator. By assumption, $\mathrm{ad}(f) \in B(\mathcal{F})$ is quasinilpotent. By Lemma 6.1, $\sigma(R_{T_q(f)}) = \{0\}$ for all q. Thus $G(f)$ is invertible. ∎

Corollary 9.4. *Let $A \succ (X,\alpha)$, and let $\tau \in \sigma_{\pi,k}(\alpha_{\mathcal{U}}|_I, \widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$, where $\mathcal{F}$ is a* B-L *subalgebra in A. If $\mathcal{F} \in$*Proj*, then $\tau_I \in \sigma_{\pi,k+1}(\alpha|_I)$. But, if $\dim(\mathcal{F}) < \infty$, then $\tau_I \in \sigma_{\pi,k}(\alpha|_I)$. Moreover, if $\mathcal{F}$ is a weak quasinilpotent* B-L *algebra and $\tau \in \sigma_{\pi,k}(\alpha_{\mathcal{U}}, \widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$, then $\tau_E|^A|_{\mathcal{F}} = \tau_{\mathcal{F}}$.*

Proof. It suffices to prove that $\tau_I \in \sigma_{\pi,k+1}(\alpha_{\mathcal{U}}|_I)$ (respectively, if $\dim(\mathcal{F}) < \infty$, $\tau_I \in \sigma_{\pi,k}(\alpha_{\mathcal{U}}|_I)$). If $\tau_I \notin \sigma_{\pi,k+1}(\alpha_{\mathcal{U}}|_I)$ (respectively $\tau_I \notin \sigma_{\pi,k}(\alpha_{\mathcal{U}}|_I)$), then $C^\cdot(\alpha_{\mathcal{U}}|_I - \tau_I)$ is exact in first $k+1$ (respectively k) terms. By Lemma 2.1,

$\Lambda^q\mathcal{F} \in$Proj for all q. Then all rows of $\mathrm{B}_{\tau_I,\tau_\mathcal{F}}(I,\mathcal{F},X_\mathcal{U})$ are exact in first $k+1$ (respectively k) terms. By using the method of "diagonal search", we obtain that the same is true for the total complex $\mathrm{Tot}_{\tau_I,\tau_\mathcal{F}}(I,\mathcal{F},X_\mathcal{U})$. By Definition 3.1, $\tau \notin \sigma_{\pi,k}(\alpha_\mathcal{U}|_I, \widetilde{\alpha}_\mathcal{U}|_\mathcal{F})$. If $\dim(\mathcal{F}) < \infty$, then $\mathrm{Tot}_{\tau_I,\tau_\mathcal{F}}(I,\mathcal{F},X_\mathcal{U}) = \mathrm{Tot}_{\tau_I,\tau_\mathcal{F}}(I,\mathcal{F},X)_\mathcal{U}$ by Lemma 8 from [4]. By using Lemma 3.3 we obtain that $\tau \notin \sigma_{\pi,k}(\alpha_\mathcal{U}|_I, \widetilde{\alpha}_\mathcal{U}|_\mathcal{F})$.

Now, let $\mathcal{F}$ be a weak quasinilpotent B-L algebra and $\tau \in \sigma_{\pi,k}(\alpha_\mathcal{U}, \widetilde{\alpha}_\mathcal{U}|_\mathcal{F})$. Then $\tau_E \in \sigma_{\pi,k+1}(\alpha)$ and, by Lemma 8.4, there exists a character $\tau_E|^A$ of A such that $\tau_E|^A|_E = \tau_E$. By using Theorem 9.3, we obtain that $\tau_E|^A|_\mathcal{F} = \tau_\mathcal{F}$. ∎

The following lemma is a topological version of Lemma 5.2 in [6].

LEMMA 9.5. *Let $A \succ (X,\alpha)$, $\mathcal{F}$ be a* B-L *subalgebra in A, I and J ideals in E such that $J \subseteq I$, $\dim(I/J) = 1$, $\lambda \in \Delta(I)$ and $\mu \in \Delta(\mathcal{F})$, and let $u \in I\backslash J$. There exists a bounded endomorphism $\delta(u)$ of the Banach complex* $\mathrm{Tot}_{\lambda|_J,\mu}(J,\mathcal{F},X)$ *such that*

$$\mathrm{Tot}_{\lambda,\mu}(I,\mathcal{F},X) = \mathrm{Con}(\mathrm{Tot}_{\lambda|_J,\mu}(J,\mathcal{F},X), \delta(u) - \lambda(u)). \tag{9.3}$$

Moreover, $\sigma(\delta(u)) = \sigma(\alpha(u))$.

Proof. We denote the restriction of $\omega \in C^p(I,X)$ on $\Lambda^p J$ by $\omega|_J$. The following linear map

$$\kappa_{p,q} : C^q(\mathcal{F}, C^p(I,X)) \to C^q(\mathcal{F}, C^p(J,X)) \oplus C^q(\mathcal{F}, C^{p-1}(J,X)),$$
$$\kappa_{p,q}(\Phi) = (\Phi|_J, (i(u)\Phi)|_J),$$

implements a topological isomorphism by virtue of Lemma 6.3, where $\Phi|_J(h) = \Phi(h)|_J$, $(i(u)\Phi)|_J(h) = (i(u)\Phi(h))|_J$, $h \in \Lambda^p\mathcal{F}$. By definition, the differential $\gamma_{\lambda,\mu}$ of the complex $\mathrm{Tot}_{\lambda,\mu}(I,\mathcal{F},X)$ is given by the rule

$$\gamma_{\lambda,\mu}(\Phi) = \kappa_\lambda(\Phi) + (-1)^p\beta_\mu(\Phi), \quad \Phi \in C^q(\mathcal{F}, C^p(I,X)).$$

Let us find the components of $\kappa_{p+1,q}(\kappa_\lambda(\Phi))$ and $\kappa_{p,q+1}((-1)^p\beta_\mu(\Phi))$ in the corresponding decompositions. Let d_λ be the differential of $C^{\cdot}(\alpha|_I - \lambda)$. Then, $\kappa_\lambda(\Phi) = d_\lambda \cdot \Phi$ and

$$(d_\lambda \cdot \Phi)|_J(h) = d_\lambda(\Phi(h))|_J = d_{\lambda|_J}(\Phi(h)|_J) = (d_{\lambda|_J}(\Phi|_J))(h), \quad h \in \Lambda^p\mathcal{F},$$

i.e., $(d_\lambda \cdot \Phi)|_J = d_{\lambda|_J}(\Phi|_J)$. We use (6.2) to transform the second term in $\kappa_{p+1,q}(\kappa_\lambda(\Phi))$:

$$(i(u)(d_\lambda \cdot \Phi))|_J(h) = (i(u)d_\lambda\Phi(h))|_J = -d_{\lambda|_J}(i(u)\Phi(h))|_J + (\theta - \lambda)(u)(\Phi(h)|_J).$$

Thus,

$$\kappa_{p+1,q}(\kappa_\lambda(\Phi)) = (\kappa_{\lambda|_J}(\Phi|_J), -\kappa_{\lambda|_J}((i(u)\Phi)|_J) + (\theta - \lambda)(u)(\Phi|_J)).$$

Let us transform the components of $\kappa_{p,q+1}(\beta_\mu(\Phi)) = (\beta_\mu(\Phi)|_J, (i(u)\beta_\mu(\Phi))|_J)$:

$$\begin{aligned}\beta_\mu(\Phi)|_J(v_1,\dots,v_{q+1}) &= \sum_{i=1}^{q+1}(-1)^{i+1}(\widetilde{\theta} - \mu)(v_i)\Phi(v_1,\dots,\widehat{v}_i,\dots,v_{q+1})|_J \\ &\quad + \sum_{i<j}(-1)^{i+j}\Phi([v_i,v_j],v_1,\dots,\widehat{v}_i,\dots,\widehat{v}_j,\dots,v_{q+1})|_J \\ &= \beta_\mu(\Phi|_J)(v_1,\dots,v_{q+1}),\end{aligned}$$

i.e., $\beta_\mu(\Phi)|_J = \beta_\mu(\Phi|_J)$. To transform the second term, we consider the operator $\Gamma : A \to B(B(\Lambda I, X))$, $\Gamma(f) = [i(u), \widetilde{\theta}(f)]$. Demonstrate that, if $\omega \in C^p(I, X)$ such that $\omega|_J = 0$, then $(\Gamma(f)\omega)|_J = 0$ for all $f \in A$. Since $\mathcal{R}(E)$ is dense in A, it suffices to prove the latter for rational functions $r^{(n)}$, $n \geqslant 1$. We proceed by induction on n.

If $f \in E$, then $\Gamma(f) = i([u, f])$ by (6.3), and $[u, f] \in J$ by nilpotentness of E. Thus $(i([u, f])\omega)|_J = 0$. Let $f = p(e)$ be a polynomial in A of variables $e = \{e_1, \dots, e_n\} \subseteq E$. Assume firstly that $f = e_i^{s_i}$, $s_i > 1$, $1 \leqslant i \leqslant k$. Then $(\widetilde{\theta}(e_i^{s_i-1})\omega)|_J = 0$, and

$$(\Gamma(f)\omega)|_J = \theta(e_i)(\Gamma(e_i^{s_i-1})\omega)|_J + i([u, e_i])(\widetilde{\theta}(e_i^{s_i-1})\omega)|_J = \theta(e_i)(\Gamma(e_i^{s_i-1})\omega)|_J.$$

By using induction on s_i, we obtain that $(\Gamma(f)\omega)|_J = 0$. When f is arbitrary monomial of e, it suffices to use an induction on the length of f.

Let $f = r^{(n-1)}$ be a rational function of order $n - 1$ such that f is invertible and $(\Gamma(f)\omega)|_J = 0$. It is obvious that $(\Gamma(f^{-1})\omega)|_J = -(\widetilde{\theta}(f)^{-1}\Gamma(f)\widetilde{\theta}(f)^{-1}\omega)|_J$. By Remark 6.4, $C^p(I, X) = C^p(J, X) \oplus C^{p-1}(J, X)$ is the direct sum of A-invariant subspaces. Since $\omega|_J = 0$, so $\omega \in C^{p-1}(J, X)$ and $\widetilde{\theta}(f)^{-1}\omega \in C^{p-1}(J, X)$, i.e., $(\widetilde{\theta}(f)^{-1}\omega)|_J = 0$. We proceed by induction on s_i. By induction $(\Gamma(f)\widetilde{\theta}(f)^{-1}\omega)|_J = 0$. Then $\Gamma(f)\widetilde{\theta}(f)^{-1}\omega \in C^{p-1}(J, X)$. By using Remark 6.4, we have

$$\widetilde{\theta}(f)^{-1}(\Gamma(f)\widetilde{\theta}(f)^{-1}\omega) \in C^{p-1}(J, X),$$

i.e., $(\Gamma(f^{-1})\omega)|_J = 0$.

Let $f = p(a)$ be a polynomial in A of variables $a = \{a_1, \dots, a_k\}$, where $a_i = r^{(n-1)}$ or $a_i = (r^{(n-1)})^{-1}$ for some rational function $r^{(n-1)}$ of order $(n - 1)$. It suffices to assume that $f = a_i^{s_i}$, $s_i > 1$, $1 \leqslant i \leqslant k$. Then ω and $\widetilde{\theta}(a_i^{s_i-1})\omega \in C^{p-1}(J, X)$, i.e. $(\widetilde{\theta}(a_i^{s_i-1})\omega)|_J = 0$. We proceed by induction on s_i. By induction hypothesis $\Gamma(a_i^{s_i-1})(\widetilde{\theta}(a_i^{s_i-1})\omega)|_J = 0$. Then

$$(\Gamma(f)\omega)|_J = \widetilde{\theta}(a_i)(\Gamma(a_i^{s_i-1})\omega)|_J + \Gamma(a_i^{s_i-1})(\widetilde{\theta}(a_i^{s_i-1})\omega)|_J = \widetilde{\theta}(a_i)(\Gamma(a_i^{s_i-1})\omega)|_J = 0.$$

For $\Phi \in C^q(\mathcal{F}, C^p(I, X))$, we define the $(q + 1)$-form

$$(\Gamma \wedge \Phi)(f_1, \dots, f_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1}\Gamma(f_i)\Phi(f_1, \dots, \widehat{f_i}, \dots, f_{q+1}),$$

with $\Gamma \wedge \Phi \in C^{q+1}(\mathcal{F}, C^{p-1}(I, X))$. It follows that the correspondence

$$C^q(\mathcal{F}, C^p(J, X)) \to C^{q+1}(\mathcal{F}, C^{p-1}(J, X)), \Phi|_J \mapsto (-1)^p(\Gamma \wedge \Phi)|_J,$$

is a bounded linear operator denoted by Γ_p. Then

$$\begin{aligned}
&(i(u)\beta_\mu(\Phi))|_J(f_1, \dots, f_{q+1}) \\
&\quad = \sum_{i=1}^{q+1} (-1)^{i+1}(i(u)(\widetilde{\theta} - \mu)(f_i)\Phi(f_1, \dots, \widehat{f_i}, \dots, f_{q+1}))|_J \\
&\qquad + \sum_{i<j} (-1)^{i+j}(i(u)\Phi([f_i, f_j], f_1, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_{q+1}))|_J \\
&\quad = \beta_\mu((i(u)\Phi)|_J)(f_1, \dots, f_{q+1}) + (\Gamma \wedge \Phi)|_J(f_1, \dots, f_{q+1}).
\end{aligned}$$

Thus $\kappa_{p,q+1}(\beta_\mu(\Phi)) = (\beta_\mu(\Phi|_J), \beta_\mu((i(u)\Phi)|_J) + (-1)^p\Gamma_p(\Phi|_J))$ and for the differential $\gamma_{\lambda,\mu} = \kappa_\lambda + (-1)^p\beta_\mu$ of $\mathrm{Tot}_{\lambda,\mu}(I,\mathcal{F},X)$ the following equality (up to an isomorphism)

$$\gamma_{\lambda,\mu}(\Phi|_J,(i(u)\Phi)|_J) = (\gamma_{\lambda|_J,\mu}(\Phi|_J), -\gamma_{\lambda|_J,\mu}((i(u)\Phi)|_J) + (\delta(u)-\lambda(u))(\Phi|_J))$$

is valid, where $\delta(u) = L_{\theta(u)} + \sum_p \Gamma_p$. The operator $\delta(u)$ has a triangular operator matrix with diagonal elements which are equal to the same operator $L_{\theta(u)}$. Then $\sigma(\delta(u)) = \sigma(\theta(u)) = \sigma(\alpha(u))$ by Lemma 6.1. Moreover, it follows from the condition $\gamma^2_{\lambda,\mu} = 0$ that $\delta(u)$ is an endomorphism of the complex $\mathrm{Tot}_{\lambda|_J,\mu}(J,\mathcal{F},X)$. ∎

THEOREM 9.6. *Let $A \succ (X,\alpha)$, $\mathcal{F}$ be a* B-L *subalgebra in A, I and J ideals in E, such that $J \subseteq I$, $\dim(I/J) = 1$. If $\mathcal{F} \in$*Proj*, then $\sigma(\alpha|_I,\widetilde{\alpha}|_{\mathcal{F}})|_{J\times\mathcal{F}} = \sigma(\alpha|_J,\widetilde{\alpha}|_{\mathcal{F}})$. In particular, $\sigma(\alpha,\widetilde{\alpha}|_{\mathcal{F}})|_{\mathcal{F}} = \sigma(\widetilde{\alpha}|_{\mathcal{F}})$.*

Proof. By (9.3), $\sigma(\mathbf{Tot}(I,\mathcal{F},X)) \subseteq \sigma(\mathrm{Con}_{\delta(u)}(\mathbf{Tot}(J,\mathcal{F},X)))$, where $u \in I \setminus J$. Conversely, let $(\tau,c) \in \sigma(\mathrm{Con}_{\delta(u)}(\mathbf{Tot}(J,\mathcal{F},X)))$. By Theorem 4.3, $\tau \in \sigma(\mathbf{Tot}(J,\mathcal{F},X))$. By Definition 9.1, $\tau \in \sigma(\alpha|_J,\widetilde{\alpha}|_{\mathcal{F}})$. Then $\tau_J \in \sigma_{\pi,\infty}(\alpha|_J)$, otherwise we would obtain that all rows of $\mathrm{B}_{\tau_J,\tau_{\mathcal{F}}}(J,\mathcal{F},X)$ are exact ($\mathcal{F} \in$Proj), consequently $\mathrm{Tot}_{\tau_J,\tau_{\mathcal{F}}}(J,\mathcal{F},X)$ would be exact. Since $\alpha[I,I]$ consists of quasinilpotent operators and $[I,I] \subseteq J$, so $\tau_J[I,I] = 0$ by virtue of projection property (Corollary 5.5 of [6] and [11]). Thus any linear extension of τ_J up to a functional on I is a character. Let $\lambda \in I^*$, $\lambda|_J = \tau_J$, $\lambda(u) = c$. By Lemma 9.5, $\mathrm{Tot}_{\lambda,\tau_{\mathcal{F}}}(I,\mathcal{F},X) = \mathrm{Con}(\mathrm{Tot}_{\tau_J,\tau_{\mathcal{F}}}(J,\mathcal{F},X), \delta(u)-c)$, i.e., $(\tau,c) \in \sigma(\mathbf{Tot}(I,\mathcal{F},X))$.

By using Theorem 4.3 again, infer $\sigma(\alpha|_I,\widetilde{\alpha}|_{\mathcal{F}})|_{J\times\mathcal{F}} = \sigma(\alpha|_J,\widetilde{\alpha}|_{\mathcal{F}})$.

Let $E = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} = \{0\}$ be a chain of ideals in E such that $\dim(I_i/I_{i+1}) = 1$, $1 \leqslant i \leqslant n$. Then $\sigma(\alpha|_{I_i},\widetilde{\alpha}|_{\mathcal{F}})|_{I_{i+1}\times\mathcal{F}} = \sigma(\alpha|_{I_{i+1}},\widetilde{\alpha}|_{\mathcal{F}})$ and $\sigma(\alpha,\widetilde{\alpha}|_{\mathcal{F}})|_{\mathcal{F}} = \cdots = \sigma(\alpha|_{I_n},\widetilde{\alpha}|_{\mathcal{F}})|_{\mathcal{F}} = \sigma(\widetilde{\alpha}|_{\mathcal{F}})$. ∎

10. THE BACKWARD SPECTRAL MAPPING PROPERTY

In this section, we prove the backward spectral mapping property.

THEOREM 10.1. *Let $A \succ (X,\alpha)$ and let $\mathcal{F}$ be a weak quasinilpotent* B-L *algebra in A such that $\mathcal{F} \in$* Proj*. Then*

$$\sigma_{\pi,k}(\alpha)|^A|_{\mathcal{F}} \subseteq \sigma^{\mathrm{u}}_{\pi,k}(\widetilde{\alpha}|_{\mathcal{F}}) \subseteq \sigma_{\pi,k+1}(\alpha)|^A|_{\mathcal{F}}. \tag{10.1}$$

In particular, for the Taylor spectrum, we conclude that

$$\sigma_{\pi,\infty}(\alpha)|^A|_{\mathcal{F}} = \sigma^{\mathrm{u}}_{\pi,\infty}(\widetilde{\alpha}|_{\mathcal{F}}). \tag{10.2}$$

Moreover, if $\dim(\mathcal{F}) < \infty$, then $\mathcal{F}$ is a nilpotent Lie algebra and

$$\sigma_{\pi,k}(\alpha)|^A|_{\mathcal{F}} = \sigma_{\pi,k}(\widetilde{\alpha}|_{\mathcal{F}}). \tag{10.3}$$

Proof. The first inclusion in (10.1) follows from Theorem 8.6. Let $\mu \in \sigma_{\pi,k}(\widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$ for some ultrafilter $\mathcal{U}$. By Theorem 9.6, $\mu = \tau|_{\mathcal{F}}$ for some $\tau \in \sigma_{\pi,k}(\alpha_{\mathcal{U}},\widetilde{\alpha}_{\mathcal{U}}|_{\mathcal{F}})$. Then, $\tau_E \in \sigma_{\pi,k+1}(\alpha)$ and $\tau_E \in \sigma_{\pi,k}(\alpha)$ if $\dim(\mathcal{F}) < \infty$; moreover $\tau_E|^A|_{\mathcal{F}} = \mu$ by Corollary 9.4. Thus, $\mu \in \sigma_{\pi,k+1}(\alpha)|^A|_{\mathcal{F}}$, i.e., (10.1) has been proved and (10.2) follows from (10.1). If $\dim(\mathcal{F}) < \infty$, then $\mu \in \sigma_{\pi,k}(\alpha)|^A|_{\mathcal{F}}$ and $\sigma^{\mathrm{u}}_{\pi,k}(\widetilde{\alpha}|_{\mathcal{F}}) = \sigma_{\pi,k}(\widetilde{\alpha}|_{\mathcal{F}})$ by Lemma 5.4. By [16], the algebra A is commutative modulo $\mathrm{Rad}(A)$. Then $[\mathcal{F},\mathcal{F}] \subseteq \mathrm{Rad}A$ and consequently $\mathcal{F}$ is a solvable Lie algebra. Let S be a set of Lie generators such that $\sigma(\mathrm{ad}(s)) = \{0\}$ for all $s \in S$. Then $\mathrm{ad}(S)$ generates a Lie algebra consisting of nilpotent operators, i.e., $\mathcal{F}$ is a nilpotent Lie algebra. ∎

REMARK 10.2. Let $A \succ (X,\alpha)$ and let $\mathcal{F}$ be a finite-dimensional Lie subalgebra (not necessary weak quasinilpotent) in A. We have remarked in the proof of Theorem 10.1 that $\mathcal{F}$ should be solvable Lie algebra. By using Cartan subalgebras of $\mathcal{F}$ and (10.3), one can easily prove that an equality of the type (10.3) is also true for spectrum introduced in [1].

To obtain the classical version ($\sigma(r(a)) = r(\sigma(a))$) of our spectral mapping theorem (Theorem 10.1) we use the following. Let $\mathcal{F}$ be a B-L algebra, S a set of Lie generators, and let (X,α) be a Banach E-module. There is an injective continuous linear map $\widehat{S} : \Delta(\mathcal{F}) \to \mathbb{C}^S$, $\widehat{S}(\lambda) = (\lambda(s))_{s\in S}$. Set $\sigma(\alpha(S)) = \widehat{S}(\sigma^{\mathrm{u}}_{\pi,\infty}(\alpha))$. We call this set the *Taylor $\mathcal{F}$-spectrum* of the operator family $\alpha(S)$. Now let u be a set of Lie generators of E, $A \succ (X,\alpha)$, $\mathcal{F}$ a B-L subalgebra in A with a set of Lie generators r and let $a = \alpha(u)$, $r(a) = \widetilde{\alpha}(r)$. There is a bounded algebra homomorphism $A \to C(\sigma(a))$, $f \mapsto \varphi_f$, where $C(\sigma(a))$ is the algebra of all continuous functions on $\sigma(a)$ and $\varphi_f(\widehat{u}(\lambda)) = \lambda|^A(f)$, $\lambda \in \sigma^{\mathrm{u}}_{\pi,\infty}(\alpha)$. We identify φ_f with f. Under the assumptions of Theorem 10.1, we deduce

$$\begin{aligned}\sigma(r(a)) = \widehat{r}(\sigma^{\mathrm{u}}_{\pi,\infty}(\widetilde{\alpha}|_{\mathcal{F}})) = \widehat{r}(\sigma_{\pi,\infty}(\alpha)|^A|_{\mathcal{F}}) = \{(\lambda|^A(f))_{f\in r} : \lambda \in \sigma_{\pi,\infty}(\alpha)\}\\ = \{(f(\mu))_{f\in r} : \mu \in \sigma(a)\} = r(\sigma(a)).\end{aligned}$$

EXAMPLE 10.3. Let E be a Heisenberg algebra with a basis $e = \{h,f,l\}$ such that $[h,f] = l$ and $[h,l] = [f,l] = 0$, (X,α) a Banach E-module and let $a = \alpha(h), b = \alpha(f)$. If $r = (r_1,r_2,r_3)$ is a tuple of positive numbers such that $\rho(r^{-1}\alpha(e)) < 1$, then $A(r^{-1}e) \succ (X,\alpha)$ by Lemma 7.5. By using Examples 7.7, 7.8, and Theorem 10.1, we obtain the following equalities

$$\sigma(a^t, ab) = \{(\lambda^t, \lambda\mu) : (\lambda,\mu) \in \sigma(a,b)\}, \quad \sigma(a^2, b^2) = \{(\lambda^2,\mu^2) : (\lambda,\mu) \in \sigma(a,b)\}.$$

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