J. OPERATOR THEORY
48(2002), 615–619

STRONGLY REDUCTIVE ALGEBRAS ARE SELFADJOINT

BEBE PRUNARU

Communicated by Norberto Salinas

ABSTRACT. Let $B(H)$ denote the algebra of all bounded linear operators on some complex Hilbert space H. A unital subalgebra $\mathcal{A} \subset B(H)$ is said to be strongly reductive if, whenever $\{P_\lambda\}$ is a net of orthogonal projections in $B(H)$ such that $\|(1 - P_\lambda)TP_\lambda\| \to 0$ for all $T \in \mathcal{A}$, then the same holds true for all T in the C^*-algebra generated by $\mathcal{A}$ in $B(H)$. In this paper we prove that the norm-closure of every strongly reductive algebra is selfadjoint.

KEYWORDS: *Strongly reductive algebras, invariant subspaces.*

MSC (2000): Primary 47L10, 46L05; Secondary 47A15, 47A65.

1. INTRODUCTION

Let H be a complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H. A unital *subalgebra* $\mathcal{A} \subset B(H)$ is called *strongly reductive* if for each net $\{P_\lambda\}$ of orthogonal projections in $B(H)$ such that $\|(1 - P_\lambda)TP_\lambda\| \to 0$ for all $T \in A$, the same holds true for all T in the C^*-algebra generated by $\mathcal{A}$ in $B(H)$. In other words, any net of almost invariant projections for $\mathcal{A}$ is almost reducing. In the case when $\mathcal{A}$ is norm separable, the above definition is easily seen to be equivalent with that obtained by replacing nets with sequences. An *operator* $T \in B(H)$ is said to be *strongly reductive* if the norm-closed algebra generated by T in $B(H)$ is strongly reductive.

Strongly reductive operators have been introduced by K.J. Harrison in [8], who proved that if T is strongly reductive, then its spectrum $\sigma(T)$ has empty interior and does not separate the plane. In [3], the authors obtain an invariant subspace result for algebras generated by strongly reductive operators, under some extra assumptions. In [1], C. Apostol, C. Foiaş and D. Voiculescu proved that every strongly reductive operator is normal. This, together with the above mentioned result of Harrison shows that the norm-closed algebra generated by a strongly reductive operator is selfadjoint. In [2] the authors proved the more general result that the norm-closure of every commutative norm-separable strongly reductive algebra is selfadjoint. The separability hypothesis was subsequently removed by D. Hadwin ([6]). Further study of strongly reductive separable algebras has been done in [5].

In this paper we prove the following general result:

THEOREM 1.1. *The norm-closure of every strongly reductive algebra is selfadjoint.*

In the proof of this theorem we shall make use of several known results about strongly reductive algebras obtained in the above mentioned papers. Another basic tool in the proof will be the Brown-Lomonosov lemma, which enables us to show that $\mathcal{A}$ contains sufficiently many finite rank operators. This will make possible to employ a certain result from [10] about reductive algebras with many finite rank operators, to deduce that the weak closure of some restriction of $\mathcal{A}$ is selfadjoint. A standard approximation procedure will then show that $\mathcal{A}$ contains all compact operators from the C^*-algebra it generates, which will imply that $\mathcal{A}$ is indeed self-adjoint.

2. SOME PRELIMINARY RESULTS

We will now introduce some notations and recall several results that will be used in the proof of Theorem 1.1. Let $K(H)$ denote the ideal of all compact operators on H and let $\pi : B(H) \to B(H)/K(H)$ denote the Calkin projection. For any operator $T \in B(H)$, its essential (Calkin) norm will be denoted by $\|T\|_{\mathrm{e}}$. For any pair of vectors $x, y \in H$, the rank-one operator $x \otimes y \in K(H)$ is defined by $(x \otimes y)z = (z, y)x$, for any $z \in H$. If $1 \leqslant n \leqslant \infty$, then $H^{(n)}$ is the orthogonal sum of n copies of H, and if $T \in B(H)$ then $T^{(n)} = T \oplus T \oplus \cdots$ on $H^{(n)}$. If $\mathcal{S} \subset B(H)$ then $\mathcal{S}^{(n)} = \{T^{(n)} : T \in \mathcal{S}\}$, and $C^*(\mathcal{S})$ will denote the C^*-algebra generated by $\mathcal{S}$ in $B(H)$.

Two $*$-representations $\Phi_j : \mathcal{A} \to B(H_j)$, $j = 1, 2$, of a C^*-algebra $\mathcal{A}$ are said to be *approximately equivalent* if there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators from H_1 onto H_2 such that the operators $U_n\Phi_1(a)U_n^* - \Phi_2(a)$ are compact and converge in norm to 0, for every $a \in \mathcal{A}$. Proofs of the following two results can be found in [5].

LEMMA 2.1. (see [5], Corollary 4.6) *If $\mathcal{A} \subset B(H)$ is a norm-separable strongly reductive algebra then for any $*$-representation $\Phi : C^*(\mathcal{A}) \to B(\widetilde{H})$ which is approximately equivalent with the identity representation, the algebra $\Phi(\mathcal{A})$ is strongly reductive.*

LEMMA 2.2. (see [5], Lemma 4.2) *Let $\mathcal{A} \subset B(H)$ be a norm-separable strongly reductive algebra acting on a separable Hilbert space. Then the norm-closure of its image $\pi(\mathcal{A})$ in the Calkin algebra is selfadjoint.*

The following result about strongly reductive algebras will be also used in the proof of Theorem 1.1.

Lemma 2.3. *Suppose $\mathcal{A} \subset B(H)$ is a norm-separable, strongly reductive algebra acting on a separable Hilbert space, and let $u, v \in H$ such that $|(Tu, v)| \leqslant \|T\|_{\mathrm{e}}$ for any $T \in \mathcal{A}$. Then the same holds true for all $T \in C^*(\mathcal{A})$.*

Proof. Let φ be a continuous functional on $\pi(C^*(\mathcal{A}))$ with $\|\varphi\| \leqslant 1$ such that $\varphi(\pi(T)) = (Tu, v)$ for every $T \in \mathcal{A}$. It then follows from Wittstock's Theorem ([12]) that there exist a separable Hilbert space $\widetilde{H}$, vectors $\xi, \eta \in \widetilde{H}$ with $\|\xi\|$, $\|\eta\| \leqslant 1$, and a $*$-representation $\Psi : \pi(C^*(\mathcal{A})) \to B(\widetilde{H})$ such that $\varphi(\pi(T)) = (\Psi(\pi(T))\xi, \eta)$ for every $T \in C^*(\mathcal{A})$. In particular, we have $(Tu, v) = (\Psi(\pi(T))\xi, \eta)$ for every $T \in \mathcal{A}$. Let $\Phi : C^*(\mathcal{A}) \to B(H \oplus \widetilde{H})$ be the representation defined by $\Phi(T) = T \oplus \Psi(\pi(T))$, for every $T \in C^*(\mathcal{A})$.

If $M_0 = \{Tu \oplus \Psi(\pi(T))\xi : T \in \mathcal{A}\}$ then its closure M is an invariant subspace for $\Phi(\mathcal{A})$ and moreover, the vector $v \oplus (-\eta)$ is orthogonal on M. By Voiculescu's Theorem ([11]), Φ and the identity representation of $C^*(\mathcal{A})$ are approximately equivalent. By Lemma 2.1, $\Phi(\mathcal{A})$ is therefore strongly reductive. In particular, the space M is reducing for $\Phi(\mathcal{A})$, therefore $(Tu, v) = (\Psi(\pi(T))\xi, \eta)$ for every $T \in C^*(\mathcal{A})$. From this, we immediately infer that $|(Tu, v)| \leqslant \|T\|_{\mathrm{e}}$ for every $T \in C^*(\mathcal{A})$. The proof of this lemma is finished. ∎

The next result that will be used in the proof of Theorem 1.1 is a slightly modified version of the Brown-Lomonosov Lemma ([4], [9]). Its proof is quite similar to that given in the above mentioned papers.

Theorem 2.4. *Let $\mathcal{A} \subset B(H)$ be a norm-closed unital subalgebra, and let $H_0 \subset H$ be a closed invariant subspace for $\mathcal{A}$. Let $\Omega \subset H_0$ be a closed ball such that $0 \notin \Omega$. Then either there exist nonzero vectors $u, v \in H_0$ such that $|(Tu, v)| \leqslant \|T\|_{\mathrm{e}}$ for every $T \in \mathcal{A}$, or there exist a vector $x \in \Omega$ and a finite rank idempotent $F \in \mathcal{A}$ such that $Fx = x$.*

Recall that a weakly closed unital subalgebra $\mathcal{A}$ of $B(H)$ is called reductive if every invariant subspace for $\mathcal{A}$ is also reducing. Obviously, the weak closure of any strongly reductive algebra is reductive. Another result that we shall use in the proof of Theorem 1.1 is the following theorem, due to E. Nordgren and P. Rosenthal:

Theorem 2.5. (cf. [10]) *Suppose $\mathcal{A} \subset B(H)$ is a reductive algebra such that the linear span of the ranges of all finite rank operators in $\mathcal{A}$ is dense in H. Then $\mathcal{A}$ is selfadjoint.*

3. PROOF OF THE MAIN THEOREM

We are now ready to prove our main result.

Proof of Theorem 1.1. Let us fix a strongly reductive algebra $\mathcal{A} \subset B(H)$. First of all, we may assume that both $\mathcal{A}$ and H are separable in norm. The general (nonseparable) case follows from this particular one exactly as in the proof of [6, Theorem 8.4]. Moreover, we may and shall assume that $\mathcal{A}$ is norm-closed. By Lemma 2.2, the norm-closure of $\pi(\mathcal{A})$ is selfadjoint, therefore in order to prove the theorem it suffices to show that $\mathcal{A}$ contains all compact operators from $C^*(\mathcal{A})$. The proof of this assertion will be accomplished through a sequence of steps.

Step 1. Let us consider the C^*-algebra $\mathcal{B} = C^*(\mathcal{A}) \cap K(H)$. Then there exists an orthogonal decomposition of H as $H = H_0 \oplus \sum\limits_{j\geqslant 1}^{\oplus} H_j^{(m(j))}$ with $1 \leqslant m(j) < \infty$, such that the C^*-algebra $\mathcal{B}$ can be written (up to a unitary equivalence) as the direct sum $\mathcal{B} = \{0\} \oplus \sum\limits_{j\geqslant 1}^{\oplus} (K(H_j))^{(m(j))}$. Moreover, any operator $T \in C^*(\mathcal{A})$ can be written as $T = T_0 \oplus \sum\limits_{j\geqslant 1}^{\oplus} T_j^{(m(j))}$ with $T_j \in B(H_j)$ for $j \geqslant 0$. Therefore, the proof of the theorem will be finished once we show that $\mathcal{A}$ contains all operators $T \in B(H)$ of the form $T = 0 \oplus (\cdots \oplus 0 \oplus T_j^{(m(j))} \oplus 0 \oplus \cdots)$ with $T_j \in K(H_j)$ and $j \geqslant 1$.

Step 2. Let $j \geqslant 1$, let $1 \leqslant k \leqslant m(j)$, and let $H_{j,k} = \{x = x_1 \oplus \cdots \oplus x_{m(j)} \in H_j^{(m(j))} : x_i = 0 \text{ for } i \neq k\}$. We will show that for any vector $x \in H_{j,k}$ and any $\varepsilon > 0$, there exist $y \in H_{j,k}$ with $\|x - y\| < \varepsilon$ and a finite rank idempotent $F \in \mathcal{A}$ such that $Fy = y$. Assume that the above assertion is false. It then follows from Theorem 2.4, applied to the invariant subspace $H_{j,k} \subset H$, that there exist nonzero vectors $u, v \in H_{j,k}$ such that $|(Tu, v)| \leqslant \|T\|_{\mathrm{e}}$ for every $T \in \mathcal{A}$. According to Lemma 2.3, the same holds true for every $T \in C^*(\mathcal{A})$. This implies that $(Tu, v) = 0$ for every $T \in K(H_j)$, hence either $u = 0$ or $v = 0$. This finishes the proof of Step 2.

Step 3. The previous step shows that the linear span of the ranges of all finite-rank operators in $\mathcal{A}$ is dense in $H \ominus H_0$. Applying Theorem 2.5, we infer that the weak closure of the restriction of $\mathcal{A}$ to $H \ominus H_0$ is selfadjoint. Let us now fix $j \geqslant 1$. According to Step 2, there exist a non-zero vector $y \in H_j$ and a finite rank idempotent $F \in \mathcal{A}$ such that $F\widetilde{y} = \widetilde{y}$, where $\widetilde{y} = y \oplus 0 \oplus \cdots \in H_j^{(m(j))}$. It then follows immediately that $F(y \otimes y)^{(m(j))} = (y \otimes y)^{(m(j))}$. Since $(y \otimes y)^{(m(j))} \in C^*(\mathcal{A})$, there exists a net $\{T_\lambda\}_\lambda$ in $\mathcal{A}$ such that $T_\lambda^* \to (y \otimes y)^{(m(j))}$ strongly on $H \ominus H_0$. Since F has finite rank, the net $\{FT_\lambda\}_\lambda$ converges to $(y \otimes y)^{(m(j))}$ in norm on the whole space H, therefore $(y \otimes y)^{(m(j))} \in \mathcal{A}$. Since the restriction of $\mathcal{A}$ to $H_{j,1}$ has no nontrivial invariant subspaces, a standard argument shows that for any $j \geqslant 1$, $\mathcal{A}$ contains all compact operators $T \in B(H)$ of the form $T = 0 \oplus (\cdots \oplus 0 \oplus T_j^{(m(j))} \oplus 0 \oplus \cdots)$ with $T_j \in K(H_j)$. The proof of the theorem is complete. ∎

Theorem 1.1 together with Theorem 9 and Corollary 10 from [7] imply the following characterization of selfadjoint operator algebras:

COROLLARY 3.1. *Let H be a complex Hilbert space and let $\mathcal{A} \subset B(H)$ be a unital subalgebra. Suppose that if $\{x_\lambda\}$ and $\{y_\lambda\}$ is any pair of bounded nets of vectors from H such that $(Tx_\lambda, y_\lambda) \to 0$ for any $T \in \mathcal{A}$, then $(T^*x_\lambda, y_\lambda) \to 0$ for any $T \in \mathcal{A}$. Then the norm-closure of $\mathcal{A}$ is selfadjoint.*

Acknowledgements. In a preliminary version of this paper, Theorem 1.1 was proved only for separable algebras. The author would like to thank Professor Don Hadwin for his valuable suggestion regarding the nonseparable case.

REFERENCES

1. C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Strongly reductive operators are normal, *Acta Sci. Math. (Szeged)* **38**(1976),261–263.
2. C. APOSTOL, C. FOIAŞ, D. VOICULESCU, On strongly reductive algebras, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 633–641.
3. C. APOSTOL, C.K. FONG, Invariant subspaces for algebras generated by strongly reductive operators, *Duke Math. J.* **42**(1975), 495–498.
4. S. BROWN, Lomonosov's theorem and essentially normal operators, *New Zealand J. Math.* **23**(1994), 11–18.
5. C. FOIAŞ, C. PASNICU, D. VOICULESCU, Weak limits of almost invariant projections, *J. Operator Theory* **2**(1979), 79–93.
6. D. HADWIN, An asymptotic double commutant theorem for C^*-algebras, *Trans. Amer. Math. Soc.* **244**(1978), 273–297.
7. D. HADWIN, Approximately hyperreflexive algebras, *J. Operator Theory* **28**(1992), 51–64.
8. K.J. HARRISON, Strongly reductive operators, *Acta Sci. Math. (Szeged)* **37**(1975), 205–212.
9. V. LOMONOSOV, An extension of Burnside's theorem to infinite dimensional spaces, *Israel J. Math.* **75**(1991), 329–339.
10. E. NORDGREN, P. ROSENTHAL, Algebras containing unilateral shifts or finite rank operators, *Duke Math. J.* **40**(1973), 419–424.
11. D. VOICULESCU, A non-commutative Weyl-von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.
12. G. WITTSTOCK, Ein operatorwertiger Hahn-Banach Satz, *J. Funct. Anal.* **40**(1981), 127–150.

BEBE PRUNARU
Institute of Mathematics
of the Romanian Academy
PO Box 1–764
RO–70700 Bucharest
ROMANIA

E-mail: bprunaru@imar.ro

Received July 6, 2000.