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ANALOGUES OF COMPOSITION OPERATORS ON NON-COMMUTATIVE H^p-SPACES

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Abstract. We briefly review the theory of non-commutative H^p-spaces and suggest a possible non-commutative analogue of the disc algebra. We then pass to the theory of composition operators and proceed to identify some of the basic algebraic principles underlying the theory of these operators on classical Hardy spaces. This framework is then generalised to the non-commutative setting. Here we succeed in describing what may be regarded as non-commutative analogues of composition operators. Building on these ideas it is shown that even in this very general context one yet finds what effectively constitutes operator theoretic remnants of the Littlewood Subordination Principle (see Proposition 2.4 and Theorem 4.12). In conclusion we investigate the connection between linear isometries on non-commutative H^p spaces and analogues of composition operators.

Keywords: *Subdiagonal algebra, H^p-spaces, non-commutative composition operator, irreducible representation, Jordan morphism, linear isometry.*

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1. INTRODUCTION

The classical Hardy spaces $H^p(\mathbb{D})$, $1 \leqslant p < \infty$, where $\mathbb{D}$ is the open unit disc are Banach spaces of analytic functions on the unit disc which satisfy the condition

$$\sup_{0<r<1} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t < \infty.$$

The Banach algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on the unit disc appears as a dense subspace of each of these spaces. By taking radial limits each of these spaces may be identified with the subspaces $H^p(\mathbb{T})$ of $L^p(\mathbb{T})$ consisting of

p-integrable functions whose negative Fourier coefficients vanish. (Here $\mathbb{T} = \partial\mathbb{D}$ denotes the circle group.) In fact, if indeed $1 < p < \infty$, $H^p(\mathbb{T})$ appears as a topologically complemented subspace of $L^p(\mathbb{T})$. For details see the book of Hoffman ([15]). Analogues of these spaces may variously be defined on $\mathbb{D}^n$ (n Cartesian copies of the open disc), $\mathbb{D}_n$ (the open unit ball of $\mathbb{C}^n$), and $\mathbb{P}^+$ (the upper half-plane). For a concise description of these spaces see ([35], pp 8–9).

The theory of composition operators is a dynamic and rapidly growing field which may be studied in the many different contexts. In the present context of the spaces $H^p(E)$, a bounded linear operator C is called a *composition operator* if there exists a transformation $T : E \to E$ such that $C(f) = f \circ T$ for each $f \in H^p(E)$.

We may then write $C = C_T$ with the transformation T being thought of as the *symbol* of the composition operator. In the setting of the spaces $H^p(\mathbb{D})$ the theory of composition operators relates very closely to the theory of analytic self-maps on the unit disc. The fact that any analytic self-map on the open disc induces a composition operator on $H^p(\mathbb{D})$, $1 \leqslant p < \infty$, is a consequence of the classical Littlewood Subordination Principle which basically tells us that if $T(0) = 0$, then

$$\int_0^{2\pi} |f(T(r\mathrm{e}^{\mathrm{i}t}))|^p \mathrm{d}t \leqslant \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t$$

for each $0 < p < \infty$, each $f \in H^p(\mathbb{D})$ and each $0 \leqslant r < \infty$. (The restriction $T(0) = 0$ may be circumvented by bringing the Möbius transformation

$$S : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{T(0) - z}{1 - \overline{T(0)}z}$$

into play.) The interested reader is referred to [17] for further details. In the multivariable context of $E = \mathbb{D}^n$ and $E = \mathbb{D}_n$ the theory of composition operators exhibits a connection to the theory of Carleson measures rather than general analytic self-maps on E. Further details may be found in for example [35]. (Since it is not the purpose of this paper to present either a detailed introduction to the theory, nor an exhaustive bibliography, we content ourselves with presenting the references [17] and [35] as representative samples of what may be found in the literature. Of course many other excellent references like [34] and [8] abound.)

To put the present theory in context we present a brief discussion of the evolution of the "non-commutative" theory. (In doing so the fair apportioning of credit will not always be easy. Sincere apologies are offered to any who feel slighted in the process.) A fuller discussion of the evolution may be found in [24]. In the fifties the theory of Hardy spaces developed in two directions. On the one hand there is the work of Masani and Wiener on matrix valued functions ([26], [27]) and on the other that of Helson and Lowdenslager in the setting of compact groups with ordered duals ([13]). This eventually led to the weak* Dirichlet algebras of functions of Srinivasan and Wang ([36]). The first step towards the approach of non-selfadjoint operator algebras seems to have been the work of Kadison and Singer on triangular operator algebras ([19]). In a watershed paper, Arveson in 1967 ([1]) introduced the notion of a subdiagonal algebra in order to unify the perspectives of [13], [19], and [26]. Basically, a subdiagonal algebra is a subalgebra of a von Neumann algebra which plays the role of a non-commutative analogue of a weak* Dirichlet algebra and which has many of the structural properties of the space $H^\infty(\mathbb{D})$. With the concept of a non-commutative H^∞ in place, one may then appeal to the theory of non-commutative L^p spaces associated with some von Neumann algebra, and by analogy with the commutative setting, define the H^p's to be the closure of H^∞ in the relevant $L^p(\mathcal{M})$-spaces.

Building on the work of Arveson, the first explicit systematic study of non-commutative analogues of Hardy spaces (within the framework under present consideration) seems to have been the work of Zsidó ([39], [40], [41]). However his work was in the specific context of spectral subspaces relative to the action of some group of $*$-automorphisms. The refinement of a more general formalism for such objects appears to have been a subsequent partly independent development. In the nature of such situations, aspects of the more recent theory seem to be more general rediscoveries of earlier work by Zsidó. Compare for example Theorem 5.2 from [24] (an important step in the non-commutative Riesz-Bochner theory) and Lemma 4.1 from [41]. The theory of non-commutative H^p-spaces has in the interim matured to the point where a description and analysis of analogues of composition operators appears to be plausible. In for example [24] one finds a Riesz factorisation theorem, a Riesz-Bochner theorem on the existence and boundedness of harmonic conjugates, direct sum decompositions and duality. In a follow-up paper, a non-commutative concept of a BMO was introduced ([25]). On a related note the development of the concept of a Shilov boundary of an arbitrary operator space ([14], [5]) opens up many new opportunities for further enriching the theory along classical lines (see for example the discussion of the Shilov boundary of $H^\infty(\mathbb{D})$ in [15]). (Building on ideas of Arveson ([2], [3]) it seems to have been Hamana ([14]) who first introduced the concept of a Shilov boundary of an arbitrary operator space. For a full discussion of the evolution of the idea, see the paper of Blecher ([5]).)

The objective of this present paper is then to identify and study a plausible non-commutative concept of a composition operator within the structure of non-commutative H^p-spaces in a way that canonically extends the classical theory. As such it is the fourth in a series of papers aimed at initiating a theory of composition operators in a wide range of non-commutative contexts. See also [20], [22] and [21]. Besides taking a step towards the eventual characterisation of linear isometries on non-commutative H^p-spaces, the results in this paper establish a very general systematic framework (a kind of template) within which one may study a wide range of more specialised settings. In fact, despite the generality of these results, many of the standard results regarding composition operators on $H^p(\mathbb{D})$-spaces may be canonically extracted from this present cycle of results. Future possibilities include the study of this problem in the category of operator spaces. Given the additional structure in this category like the notion of a (non-commutative) Shilov boundary ([14], [5]), it is possible that some of the assymetries of the present effort may be eliminated and that a rich and varied theory awaits the brave soul who dares to venture in these uncharted waters.

Given a unital Banach algebra $\mathcal{A}$ with identity $\mathbb{I}$, under the term *irreducible representation* of $\mathcal{A}$ we understand a continuous homomorphism $\pi : \mathcal{A} \to B(X)$, where $B(X)$ is the set of all bounded linear operators on some Banach space X, such that $\pi(\mathcal{A})$ is an irreducible subalgebra of $B(X)$ in the sense of admitting only trivial invariant subspaces. The *(Jacobson) radical*, $\mathrm{rad}(\mathcal{A})$, of $\mathcal{A}$ is the intersection of the kernels of all such representations of $\mathcal{A}$. (See also 4.2.16 and 4.3.1 in [29].)

A Jordan morphism between two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear mapping $\varphi : \mathcal{A} \to \mathcal{B}$ which preserves the Jordan product, specifically $\varphi(ab+ba) = \varphi(a)\varphi(b)+\varphi(b)\varphi(b)$ for all $a, b \in \mathcal{A}$. What seems to be a meta-theorem in the theory of composition operators is that in a wide variety of contexts a bounded operator on a function space is a composition operator precisely when it is *multiplicative* in the sense of behaving like a homomorphism whenever this makes sense. In passing to the non-commutative context this principle of multiplicativity seems to translate to precisely the preservation of the Jordan product.

Unless otherwise specified, we will generally assume $\mathcal{M} \subset B(\mathfrak{h})$ ($\mathfrak{h}$ a Hilbert space) to be a von Neumann algebra equipped with a finite normalised faithful normal trace τ. At an intuitive level this construct of course fulfills the role of a non-commutative $L^\infty(\mathbb{T})$. If now we denote the set of orthogonal projections in $\mathcal{M}$ by $\mathcal{P}_{\mathcal{M}}$ and the operators affiliated to $\mathcal{M}$ by $\overline{\mathcal{M}}$, the set of τ-measurable operators affiliated to $\mathcal{M}$ is defined to be

$$\widetilde{\mathcal{M}} = \{a \in \overline{\mathcal{M}} : \forall\, \varepsilon > 0\ \exists\, e \in \mathcal{P}_{\mathcal{M}} \text{ such that } e(\mathfrak{h}) \subset D(a) \text{ and } \tau(\mathbb{I} - e) < \varepsilon)\}.$$

We may then write $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ and $L^p(\mathcal{M}, \tau) = \{a \in \widetilde{\mathcal{M}} : \tau(|a|^p) < \infty\}$, $1 \leqslant p < \infty$ ([28]). It is well known that this definition coincides with other traditional formulations of non-commutative L^p-spaces (see for example [10], [38]). Since in the present context there is therefore no danger of confusion we will simply write $L^p(\mathcal{M}) = L^p(\mathcal{M}, \tau)$, $1 \leqslant p \leqslant \infty$. As in the commutative setting an application of the non-commutative Hölder inequality ([10], Theorem 4.2) to the fact that $\tau(\mathbb{I}) = 1$ yields the observation that L^p injects continuously into L^q whenever $1 \leqslant q \leqslant p \leqslant \infty$. For any subset $\mathcal{S}$ of $L^p(\mathcal{M})$ we will write $\mathcal{S}^*$ for the set $\{a \in L^p(\mathcal{M}) : a^* \in \mathcal{S}\}$. Now, given a von Neumann algebra $\mathcal{M}$ and a von Neumann subalgebra $\mathcal{N}$, an *expectation* from $\mathcal{M}$ onto $\mathcal{N}$ is defined to be a positive identity preserving linear operator $\Phi : \mathcal{M} \to \mathcal{N}$ which satisfies the property $\Phi(xy) = x\Phi(y)$ for all $x \in \mathcal{N}$ and $y \in \mathcal{M}$. A comprehensive survey of the fundamentals of the theory of expectations may be found in Section 6 of [1]. We are now finally ready to introduce the non-commutative analogue of $H^\infty(\mathbb{T})$.

Definition 1.1. Let $\mathcal{A}$ be a *weak**-closed unital subalgebra of $\mathcal{M}$, and Φ a faithful normal expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. We deem $\mathcal{A}$ to be a *finite maximal subdiagonal subalgebra* of $\mathcal{M}$ with respect to Φ if:

(i) $\mathcal{A} + \mathcal{A}^*$ is *weak** dense in $\mathcal{M}$;
(ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{A}$;
(iii) $\tau \circ \Phi = \tau$.

To see that the work of Zsidó on spectral subspaces (referred to earlier) falls within the ambit of this formalism, we refer the interested reader to the discussion on p. 115 of [40] and to for example Theorem 5.1 of [41]. In this regard the context of say [40] may be identified as a special case of the present context by setting $B^\infty = \Phi$ and $X^B((0, 1]) = \mathcal{A}$. (Here we have used the notation of [40].)

Observe that in the present context a subalgebra $\mathcal{A}$ of the type defined above is automatically maximal among those subalgebras satisfying (i) and (ii) ([9]). Given a finite maximal subdiagonal subalgebra of $\mathcal{M}$ we will by analogy with the classical setting write $H^\infty(\mathcal{M}) = \mathcal{A}$. If now $\mathcal{M}$ appears as the double commutant of some a priori given concrete unital C^*-algebra $\mathcal{C}$, then such a $\mathcal{C}$ presents a reasonable non-commutative analogue of $C(\mathbb{T})$, and hence by analogy with the classical setting $\widetilde{\mathcal{A}} = H^\infty(\mathcal{M}) \cap \mathcal{C}$ may then be regarded as a *non-commutative disc algebra.* By the non-commutative Hölder inequality ([10], Theorem 4.2) $L^\infty(\mathcal{M})$, and hence $H^\infty(\mathcal{M})$, appears as a subalgebra of each $L^p(\mathcal{M})$, $1 \leqslant p < \infty$, since $\tau(\mathbb{I}) = 1$. We may therefore define $H^p(\mathcal{M})$, $1 \leqslant p < \infty$, to be the $\|\cdot\|_p$-closure of $H^\infty(\mathcal{M})$ in $L^p(\mathcal{M})$. Again by Theorem 4.2 of [10] it is then easy to see that more generally H^p injects continuously into H^q whenever $1 \leqslant q \leqslant p \leqslant \infty$. This framework proves to be general enough to cover a wide range of perspectives, yet retains sufficient structure to admit of a relatively detailed theory. The interested reader is referred to [24] for further concrete examples.

Now let $H^\infty_0(\mathcal{M}) = \{x \in H^\infty : \Phi(x) = 0\}$. It is an easy consequence of the multiplicative condition on Φ that H^∞_0 is in fact an ideal in H^∞. For $1 \leqslant p < \infty$ we may now define $H^p_0(\mathcal{M})$ to be the $\|\cdot\|_p$-closure of H^∞_0 in $L^p(\mathcal{M})$. In direct analogy with the classical case, $L^p(\mathcal{M})$, $1 < p < \infty$, may be decomposed into a topological direct sum $L^p(\mathcal{M}) = H^p \oplus (H^p_0)^* = H^p_0 \oplus L^p(\mathcal{D}) \oplus (H^p_0)^*$. If $p = 2$ the decomposition is even orthogonal. (A proof for the case $p = 2$ may be found in [23] and for the case $p \neq 2$ in [24].) For any $1 \leqslant p \leqslant \infty$ the spaces H^p and H^p_0 may be described by the following useful formulae ([32], Section 3):

$$H^p = H^1 \cap L^p(\mathcal{M}) = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in H^\infty_0\} \tag{1.1}$$

$$H^p_0 = H^1_0 \cap L^p(\mathcal{M}) = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in H^\infty\}. \tag{1.2}$$

Now, suppose that $\mathcal{M}$ indeed appears as the double commutant of some a priori given unital C^*-algebra $\mathcal{C}$. Given $H^\infty(\mathcal{M})$, we denote $H^\infty \cap \mathcal{C}$, the non-commutative analogue of the disc algebra, by $\widetilde{\mathcal{A}}$. In the classical setting where indeed $\mathcal{C} = C(\mathbb{T})$ and where $\widetilde{\mathcal{A}} = A(\mathbb{D})$ is the disc algebra, one form of the classical theorem of F. and M. Riesz asserts that if $A(\mathbb{D})$ is regarded as a subspace of $C(\mathbb{T})$, then on canonically embedding $L^1(\mathbb{T})$ (and hence $H^1_0 \subset L^1$) in the continuous dual of $C(\mathbb{T})$, H^1_0 appears as precisely the annihilator of $A(\mathbb{D})$ (see for example Theorem 1.1, [30]). Amongst other facts this of course implies that in this setting H^1_0 appears as a weak*-closed subspace of the dual of $C(\mathbb{T})$. However, even in a commutative context, important aspects of the above result fail in more general settings, and hence a full non-commutative analogue of this result is possibly too much to hope for. (See for example the discussion on p. 56 and Exercise 11 (d), p. 60, in the book of Hoffman ([5]).) Nevertheless, in spite of these difficulties, a weaker version of this identification of the annihilator of $\widetilde{\mathcal{A}}$ does even in the non-commutative context seem to be contained in the identities listed above. (To avoid confusion we will denote the commutant of $\mathcal{C}$ by $\mathcal{C}'$ and the continuous dual by $\mathcal{C}^\sharp$.)

THEOREM 1.2. *Let $\mathcal{M}$, $\mathcal{C}$, $H^\infty(\mathcal{M})$ and $\widetilde{\mathcal{A}} = H^\infty(\mathcal{M}) \cap \mathcal{C}$ be as before. Let $L^1(\mathcal{M})$ (and hence $H^1_0(\mathcal{M}) \subset L^1(\mathcal{M})$) be canonically embedded in $\mathcal{C}^\sharp$ by identifying each $a \in L^1(\mathcal{M})$ with the functional $x \to \tau(ax)$. If, as a subspace of $L^\infty(\mathcal{M}) = \mathcal{M}$, the space $\widetilde{\mathcal{A}}$ is $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$-dense in $H^\infty(\mathcal{M})$, then, with respect to this embedding, H^1_0 is $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-dense in*

$$\widetilde{\mathcal{A}}^\perp = \{\omega \in \mathcal{C}^\sharp : \omega(a) = 0 \textit{ for every } a \in \widetilde{\mathcal{A}}\}$$

(the polar of $\widetilde{\mathcal{A}}$) and is a relatively $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-closed subspace of $L^1(\mathcal{M})$ (i.e. $H_0^1 = \widetilde{\mathcal{A}}^\perp \cap L^1(\mathcal{M})$).

(Recall that, by standard results, the process of identifying each $a \in L^1(\mathcal{M})$ with the functional $x \to \tau(ax)$ identifies $L^1(\mathcal{M})$ with the predual $\mathcal{M}_\sharp$ of $\mathcal{M}$ ([38], Proposition II.15). Each of these functionals in turn restrict without change of norm to an ultra-weakly continuous functional on $\mathcal{C}$ ([18], 7.4.2 and 10.1.11).)

Proof. We first show that H_0^1 is a relatively $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-closed subspace of $L^1(\mathcal{M})$ under the given hypothesis. To this end, suppose that we are given $a \in L^1(\mathcal{M})$ and $\{a_\lambda\} \subset H_0^1$ such that

$$\tau(a_\lambda x) \to \tau(ax) \quad \text{for all } x \in \mathcal{C}. \tag{1.3}$$

By (1.2) above, we then have that

$$0 = \tau(a_\lambda x) \to \tau(ax) = 0 \quad \text{for each } x \in H^\infty \cap \mathcal{C}.$$

Now, since $a \in L^1(\mathcal{M})$, it follows from the discussion preceding the proof that the functional $x \to \tau(ax)$ is $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$-continuous on all of $\mathcal{M}$. Thus, by the hypothesis, the canonical extension of this functional to $\mathcal{M}$ annihilates not just $\widetilde{\mathcal{A}} = H^\infty(\mathcal{M}) \cap \mathcal{C}$, but $H^\infty(\mathcal{M})$, the $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-closure of $\widetilde{\mathcal{A}}$. Hence, by (1.2) above, we must have that $a \in H_0^1$.

To show that H_0^1 is $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-dense in $\widetilde{\mathcal{A}}^\perp$, the bipolar theorem assures us that we need only show that the polar of H_0^1 in $\mathcal{C}$ is precisely $\widetilde{\mathcal{A}}$. Now, if $a \in \widetilde{\mathcal{A}}$, then by (1.1) above $\tau(xa) = 0$ for every $x \in H_0^\infty$. However, since $a \in \widetilde{\mathcal{A}} \subset \mathcal{M}$, the functional $x \to \tau(xa)$ is continuous on $L^1(\mathcal{M})$ (see for example Theorem 4.2 in [10]). Thus, since H_0^∞ is $\|\cdot\|_1$-dense in H_0^1, it easily follows that $\tau(xa) = 0$ for every $x \in H_0^1$.

Conversely, suppose we are given $a \in \mathcal{C}$ such that $\tau(xa) = 0$ for all $x \in H_0^1$. Since $H_0^\infty \subset H_0^1$, it then trivially follows from (1.1) that $a \in H^\infty$, i.e. that $a \in \widetilde{\mathcal{A}} = H^\infty(\mathcal{M}) \cap \mathcal{C}$. ∎

For the sake of completeness we note the existence of what appears to be a rather different non-commutative version of the F. and M. Riesz theorem in the context of spectral subspaces of one-parameter groups of $*$-automorphisms. (See Theorem 5.3, [4] and Theorem 4.1, [40].)

2. SURVEY OF CLASSICAL FRAMEWORK

Our main objective in this section is not just to survey the general theory of composition operators on classical Hardy spaces, but to do so with a view to identifying basic algebraic structure underlying and characterising the theory in the various classical settings. Success in obtaining such an abstract algebraic description of the basic theory in these contexts could then justifiably serve as the foundation for extending the theory in a natural way to significantly more general contexts. The encapsulation of the essentials of the theory in a context encompassing many as yet uninvestigated specialised settings could in turn serve as a useful tool for the development of the theory in such settings.

PROPOSITION 2.1. *The point evaluations δ_z, $z \in \mathbb{D}$, of $H^\infty(\mathbb{D})$ correspond to precisely those irreducible representations (multiplicative functionals) of $H^\infty(\mathbb{D})$ which extend continuously to some (to all) $H^p(\mathbb{D})$, $1 \leqslant p < \infty$. (See for example Proposition 2 in* [17]*.)*

From pp. 8–9 of [35] it is clear that at least the one direction of the above equivalence holds for the cases $H^p(E)$ where $E = \mathbb{D}_n, \mathbb{D}^n$ or $\mathbb{P}^+$ (the upper half-plane) as well.

PROPOSITION 2.2. ([35], 3.1.1, 3.2.1) *For each of the cases $E = \mathbb{D}_n, \mathbb{D}^n$ or $\mathbb{P}^+$, a bounded linear map $C : H^p(E) \to H^p(E)$, $1 \leqslant p < \infty$, is a composition operator if and only if given any $z \in E$ we may find $w \in E$ so that $\delta_z \circ C = \delta_w$. (In the case $E = \mathbb{D}$ the above two conditions are equivalent to C acting multiplicatively on H^p.)*

In, at least, the case of $H^\infty(\mathbb{D})$ the argument of Proposition 3 from [17] may easily be adapted to show that:

PROPOSITION 2.3. *For any bounded linear operator $C : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ the following are equivalent:*

(i) *C is a composition operator on H^∞;*

(ii) *C is a multiplicative map (hence a homomorphism since $H^\infty(\mathbb{D})$ is a Banach algebra) whose adjoint maps the set $\{\delta_z : z \in \mathbb{D}\}$ back into itself;*

(iii) *C is an identity preserving homomorphism which on composition with an irreducible representation of $H^\infty(\mathbb{D})$ which extends continuously to some (alternatively, to all) $H^p(\mathbb{D})$, $1 \leqslant p < \infty$, yields another irreducible representation of the same type.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious whereas the equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 2.1. To see that (i) follows from (ii) observe that if (ii) holds we may define a map $T : \mathbb{D} \to \mathbb{D}$ by $T(z) = w_z$ where for any given $z \in \mathbb{D}$, $w_z \in \mathbb{D}$ is selected so that $\delta_z \circ C = \delta_{w_z}$. It is then an exercise to show that $(Cf)(z) = (f \circ T)(z)$ for every $z \in \mathbb{D}$ and every $f \in H^\infty(\mathbb{D})$. ∎

The above proposition affords a mechanism of identifying those operators which appear as composition operators. Turning our attention to identifying the transformations which induce composition operators we obtain results of the following type:

PROPOSITION 2.4. *Let $T : \mathbb{D} \to \mathbb{C}$ be given. Then the following are equivalent:*

(i) *for some $1 \leqslant p < \infty$ T induces a composition operator $C_T : H^p \to H^p$;*

(ii) *T induces a composition operator $C_T : H^\infty \to H^\infty$ which extends continuously to $H^p \to H^p$ for some $1 \leqslant p < \infty$;*

(iii) *for each $1 \leqslant q < \infty$ T induces a composition operator $C_T : H^q \to H^q$.*

Proof. The proof follows from for example Proposition 3 in [17] and the remark following it. Observe that if (i) holds then T must map $\mathbb{D}$ back into itself (by Proposition 2.1) and hence C_T will map H^∞ back into itself in a $\|\cdot\|_\infty$-continuous way. ∎

Remark 2.5. The above proposition (together with some by now fairly standard norm estimates) is equivalent to the Littlewood Subordination Principle and may thus be regarded as an operator theoretic formulation thereof (see for example the discussion on p. 231 of [17] for details). The appearance of similar behaviour in more general contexts may thus justifiably be regarded as a generalised Littlewood Subordination Principle.

3. THE DEFINITION OF COMPOSITION OPERATORS ON NON-COMMUTATIVE H^p-SPACES

Given a possibly non-commutative Hardy space $H^p(\mathcal{M})$ corresponding to some von Neumann algebra the main objective in this section is to identify a plausible definition of composition operators in this context. Whichever way one wishes to describe such objects, "plausibility" of such a description would at least demand that it yield exactly the standard theory in the special cases of the more classical Hardy spaces like $H^p(E)$ where E is one of $\mathbb{D}^n$ or $\mathbb{D}_n$. Part of this process of identifying a plausible description is therefore the encoding of much of the information in the previous section in Banach algebraic terms. In this programme the identification of linear maps which in some sense "preserve" irreducible representations is therefore essential.

Proposition 3.1. *Let $\mathcal{B}, \widetilde{\mathcal{B}}$ be unital commutative semisimple Banach algebras. Then $C : \mathcal{B} \to \widetilde{\mathcal{B}}$ is an identity preserving homomorphism if and only if for any irreducible representation π of $\widetilde{\mathcal{B}}$, $\pi \circ C$ is an irreducible representation of $\mathcal{B}$.*

Proof. Since here irreducible representations correspond to non-zero multiplicative functionals, the "only if" statement is obvious. Conversely, given $a, b \in \mathcal{B}$ note that if for each irreducible representation π of $\widetilde{\mathcal{B}}$ we have that $\pi \circ C(ab) = \pi \circ C(a) \cdot \pi \circ C(b) = \pi(C(a)C(b))$, then surely $\pi(C(ab) - C(a)C(b)) = 0$, i.e. $C(ab) - C(a)C(b) \in \operatorname{rad}(\mathcal{B}) = \{0\}$. Similarly, we may conclude that $C(\mathbb{I}_{\mathcal{B}}) = \mathbb{I}_{\widetilde{\mathcal{B}}}$ in this case. ∎

To pass from the commutative to the non-commutative a comparison of Propositions 2.3 and 3.1 suggests that at the level of $H^p(\mathcal{M})$ (as defined in [24]) we identify "composition operators" from $H^\infty(\mathcal{M})$ into $H^\infty(\mathcal{M})$ with bounded linear operators $C : H^\infty \to H^\infty$ whose adjoints not only preserve the set of all irreducible representations of $H^\infty(\mathcal{M})$ but also map the subset of all those irreducible representations which extend continuously to some (alternatively, to all) $H^p(\mathcal{M})$, $1 \leqslant p < \infty$, back into itself. The latter set then plays the role of a "non-commutative open disc" with the difference between the two sets playing the role of a non-commutative version of the Shilov-boundary of the disc-algebra (see [15], pp. 160, 173). To clarify these ideas, a non-commutative version of Proposition 2.1 needs to be investigated. This will be done in the next section. For the moment, we compile some classical results to present a non-commutative version of Proposition 3.1.

PROPOSITION 3.2. *Let $\mathcal{B}, \widetilde{\mathcal{B}}$ be unital Banach algebras and $C : \mathcal{B} \to \widetilde{\mathcal{B}}$ a linear operator.*

(i) *If for any irreducible representation π of $\widetilde{\mathcal{B}}$, $\pi \circ C$ is an irreducible (anti-)representation of $\mathcal{B}$, then $\pi_{\mathrm{rad}(\tilde{\mathcal{B}})} \circ C$ is an identity-preserving Jordan morphism. (Here $\pi_{\mathrm{rad}(\tilde{\mathcal{B}})}$ is the canonical quotient map $\widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}/(\mathrm{rad}(\widetilde{\mathcal{B}}))$.)*

(ii) *If $\pi : \widetilde{\mathcal{B}} \to B(X)$ is a continuous representation of $\widetilde{\mathcal{B}}$ and if C is a Jordan morphism for which $\pi(C(\mathcal{B}))$ is either dense* ([29], 6.3.7) *or else a subspace for which the extended enveloping Lie-ring is again a ring* ([16]), *then $\pi \circ C$ is either a homomorphism or an anti-morphism.*

Proof. (i) Given any irreducible representation π of $\widetilde{\mathcal{B}}$, $\pi \circ C$ is either a homomorphism or an anti-morphism. Hence, for each $a, b \in \mathcal{B}$, π necessarily annihilates $C(ab+ba) - C(a)C(b) - C(b)C(a)$. It follows that $C(ab+ba) - C(a)C(b) - C(b)C(a)$ belongs to $\mathrm{rad}(\widetilde{\mathcal{B}})$ for each $a, b \in \mathcal{B}$. For the sake of argument we may therefore suppose that $\mathrm{rad}(\widetilde{\mathcal{B}}) = \{0\}$. Since C is then a Jordan morphism we have that $C(a) = C(\mathbb{I})C(a)C(\mathbb{I})$ for all $a \in \mathcal{B}$ (see for example the identities on p. 212 of [6]). Thus $C(\mathbb{I})$ acts as a multiplicative identity on $C(\mathcal{B})$. Alongside the fact that for each irreducible representation π of $\widetilde{\mathcal{B}}$, $\pi \circ C(\mathcal{B})$ is an irreducible subalgebra of $B(X_\pi)$, this implies that $\pi(\mathbb{I}) = \pi(C(\mathbb{I}))$ for every such representation. Thus $\mathbb{I} - C(\mathbb{I}) \in \mathrm{rad}(\widetilde{\mathcal{B}}) = \{0\}$.

(ii) Let $\pi : \widetilde{\mathcal{B}} \to B(X_\pi)$ be an irreducible representation such that $\pi \circ C(\mathcal{B})$ contains the finite ranks and such that the (extended) enveloping Lie-ring of $\pi \circ C(\mathcal{B})$ is again a ring ([16], pp. 493, 495). This ring is then primitive with minimal ideals ([16], p. 489). But then $\pi \circ C$ is a Lie (anti-)morphism and hence an (anti-)morphism by Theorem 21 in [16]. To see this, observe that in the anti-morphism case

$$
\begin{aligned}
\pi \circ C(ab) &= \frac{1}{2}(\pi \circ C([a,b] + (ab+ba))) \\
&= \frac{1}{2}([\pi \circ C(b), \pi \circ C(a)] + (\pi \circ C(a)\pi \circ C(b) + \pi \circ C(b)\pi \circ C(a))) \\
&= \pi \circ C(b) \cdot \pi \circ C(a).
\end{aligned}
$$

The remaining case follows from Section 6.3.7 of [29]. ∎

The above proposition suggests a connection with identity-preserving Jordan-morphisms in the non-commutative setting. A problem we face in this setting is that the Banach algebra $H^\infty(\mathcal{M})$ need not be semi-simple. Consider for example the following:

EXAMPLE 3.3. Let $\mathcal{M} = M_2(\mathbb{C})$ and let $H^\infty(\mathcal{M}) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$. The only maximal ideal (and hence radical) of H^∞ would seem to be $\left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} : x, y \in \mathbb{C} \right\}$. The algebra $\mathcal{M} = M_2(\mathbb{C})$ already acts irreducibly on $\mathbb{C}^2$ whereas $H^\infty(\mathcal{M})$ does not. (It leaves the subspace $\left\{ \begin{bmatrix} w \\ 0 \end{bmatrix} : w \in \mathbb{C} \right\}$ invariant.) Thus the non-commutative version of Section II.4.1 from [12] fails in that not every irreducible representation of $L_\infty(\mathcal{M})$ engenders an irreducible representation of $H^\infty(\mathcal{M})$.

To circumvent any potential difficulties arising from the possible lack of semi-simplicity of $H^\infty(\mathcal{M})$ we take our cue from Proposition 3.2 and opt to rather try and describe composition operators on non-commutative H^p-spaces in terms of identity-preserving Jordan morphisms. In the commutative setting it is clear that if we are given a homomorphism $C : \mathcal{B} \to \widetilde{\mathcal{B}}$ and a non-zero multiplicative functional $\delta : \widetilde{\mathcal{B}} \to \mathbb{C}$ (irreducible representation), $\pi \circ C$ will *automatically* be an irreducible representation of $\mathcal{B}$ (non-zero multiplicative functional) provided C preserves the identity. By analogy with this setting we may therefore dispense with the requirement of irreducibility of $\pi \circ C$ in the non-commutative setting without compromising anything in the commutative. A plausible definition of a composition operator on $H^\infty(\mathcal{M})$ would then be an identity preserving contractive Jordan morphism C on $H^\infty(\mathcal{M})$ such that:

(i) for any irreducible representation π of $H^\infty(\mathcal{M})$, $\pi\circ C$ is either a morphism or an anti-morphism;

(ii) for some $1 \leqslant p < \infty$, $\pi \circ C$ extends continuously to $H^p(\mathcal{M})$ whenever π extends continuously to $H^p(\mathcal{M})$.

The subset of all such Jordan morphisms which extend continuously to continuous operators $H^p(\mathcal{M}) \to H^p(\mathcal{M})$ may then justifiably be regarded as composition operators from $H^p(\mathcal{M})$ to $H^p(\mathcal{M})$. Note that in this framework the expectation Φ associated with $H^\infty(\mathcal{M})$ also qualifies as a "composition operator" from $H^p(\mathcal{M})$ to $H^p(\mathcal{M})$ for each $1 \leqslant p \leqslant \infty$ ([24], 3.1, 3.2, 3.9).

Though not immediately obvious there is a measure of positivity encoded in the above definition of composition operators. If indeed our Jordan morphism C is contractive then for any state ω of $\mathcal{M}$, $\omega \circ C$ is the restriction of a state on $\mathcal{M}$. To see this observe that if C is identity-preserving and contractive then for any state ω of $\mathcal{M}$ we have $\|\omega \circ C\| \leqslant 1$ and $\omega \circ C(\mathbb{I}) = 1$. Any norm-preserving extension ν_ω of $\omega \circ C$ (à la Hahn-Banach) will therefore be a state ([6], 2.3.11). In particular, the restriction of C to $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^* = L_\infty(\mathcal{D})$ then proves to be order-preserving. Now, let $\mathcal{B}$ be a linear complement of $\mathcal{D}$ in $\mathcal{A} = H^\infty(\mathcal{M})$. With $\mathcal{B}^*$ denoting the set of all elements of $\mathcal{M}$ appearring as adjoints of elements of $\mathcal{B}$, it is an exercise to show that then $\mathcal{A}^* = \mathcal{D} \oplus \mathcal{B}^*$ and hence that $\mathcal{A} + \mathcal{A}^* = \mathcal{B} \oplus \mathcal{D} \oplus \mathcal{B}^*$. Given a contractive identity preserving Jordan morphism $C : H^\infty(\mathcal{M}) \to \mathcal{M}$, we may therefore extend C to a map $\widetilde{C}$ on all of $\mathcal{A} + \mathcal{A}^*$ by setting $\widetilde{C} = C$ on $\mathcal{A} = \mathcal{B} \oplus \mathcal{D}$ and defining the action on $\mathcal{B}^*$ by $\widetilde{C}(a) = C(a^*)^*$ for every $a \in \mathcal{B}^*$. Since C is order-preserving on $\mathcal{D}$, it follows that $\widetilde{C}$ as defined above preserves adjoints. In fact, $\widetilde{C}$ is even order-preserving and contractive on $\mathcal{A} + \mathcal{A}^*$. Each identity-preserving contractive Jordan-morphism C on $H^\infty(\mathcal{M}) = \mathcal{A}$ therefore appears as a restriction of a positive map on $\mathcal{A} + \mathcal{A}^*$. To see that $\widetilde{C}$ is in fact positive, let ω be a state of $\mathcal{M}$ and ν_ω the afore-mentioned state-extension of $\omega \circ C$ to $\mathcal{M}$. For each $a \in \mathcal{B}^*$, we then necessarily have

$$\nu_\omega(a) = \overline{\nu_\omega(a^*)} = \overline{\omega \circ C(a^*)} = \omega(C(a^*)^*) = \omega \circ \widetilde{C}(a)$$

(since $a^* \in \mathcal{B}$). Thus $\nu_\omega|_{\mathcal{A}+\mathcal{A}^*} = \omega \circ \widetilde{C}$. Since $\widetilde{C}$ maps states onto restrictions of states, it is order preserving and contractive. (To see that $\widetilde{C}$ is contractive, observe that if we are given $a \in \mathcal{A}+\mathcal{A}^*$ we may select ω so that $\omega(\widetilde{C}(a)) = \|\widetilde{C}(a)\|$ whence $\|\widetilde{C}(a)\| \leqslant \|\omega \circ \widetilde{C}\| \, \|a\| = \|a\|$.)

4. A NON-COMMUTATIVE LITTLEWOOD SUBORDINATION PRINCIPLE

Throughout this section we will assume $\mathcal{M}$ to be a von Neumann algebra with finite normalised faithful normal trace τ. Our primary task in this section is to obtain a non-commutative version of Proposition 2.4. As alluded to earlier, such a result may justifiably be regarded as a (noncommutative) operator theoretic version of the Littlewood Subordination Principle (see Remark 2.5). The "lemmas" that ultimately prove to be vital in affording such a result are the non-commutative factorisation results of Marsalli and West ([24], 4.2 and 4.3). To obtain the proposed theorem we will however require sharper versions of these results.

THEOREM 4.1. ([24]) *Let $1 \leqslant p \leqslant \infty$. For any $\varepsilon > 0$ and $z \in L^p(\mathcal{M})$ there exists $h_i \in H^p(\mathcal{M})$ and $v_i \in \mathcal{M}$ with $i = 1, 2$ such that:*

(i) $z = h_1 v_1 = v_2 h_2$;

(ii) $\|v_i\|_\infty \leqslant 1$;

(iii) $\|h_i\|_p < (1+\varepsilon)\|z\|_p$;

(iv) *h_i is invertible and $h_i^{-1} \in H^\infty$;*

(v) *if indeed $z \in L^\infty(\mathcal{M})$ as well, then $h_i \in H^\infty$.*

Proof. All but the final statement is contained in 4.2 of [24]. We therefore need only show how this may be obtained. First let $2 \leqslant p \leqslant \infty$. Observe that in the notation of the proof in [24] we have that $|y|^2 = y^*y = \delta^2\mathbb{I} + z^*z = \delta^2\mathbb{I} + |z|^2$ and hence that $|y|^2 \in L^\infty(\mathcal{M})$ if $z \in L^\infty(\mathcal{M}) = \mathcal{M}$. But then surely $|y| \in L^\infty(\mathcal{M})$ and therefore $y \in L^\infty(\mathcal{M})$. As noted in the proof of 4.2 in [24], $h_i = a_i^{-1}$ and y have the same singular function which by 2.5 (i) in [10] is sufficient to imply that $a_i^{-1} = h_i \in L^\infty(\mathcal{M})$ if $y \in L^\infty(\mathcal{M})$.

Now suppose $1 \leqslant p < 2$. Given the polar decomposition $z = v|z| = v|z|^{1/2}|z|^{1/2}$ we have $v|z|^{1/2}, |z|^{1/2} \in L^{2p}(\mathcal{M})$ since $z \in L^p(\mathcal{M})$. Thus since $2p \geqslant 2$ it follows from the first part of the proof that $|z|^{1/2} = w_2 g_2$ say with $w_2 \in \mathcal{M}_1$ (the unit ball of $\mathcal{M}$), $g_2 \in H^{2p}, g_2^{-1} \in H^\infty$ and $\|g_2\|_{2p} < (1+\varepsilon)^{1/2}\| \, |z|^{1/2}\|_{2p} = (1+\varepsilon)^{1/2}\|z\|_p^{1/2}$. Moreover, if $z \in L^\infty(\mathcal{M})$, then surely $|z|^{1/2} \in L^\infty(\mathcal{M})$ in which case g_2 may be selected so that $g_2 \in H^\infty$ as well.

Now since $v|z|^{1/2} \in L^{2p}(\mathcal{M})$ with $w_2 \in L^\infty(\mathcal{M})$, we have $v|z|^{1/2}w_2 \in L^{2p}(\mathcal{M})$ with $v|z|^{1/2}w_2 \in L^\infty(\mathcal{M})$ whenever $z = v|z| \in L^\infty(\mathcal{M})$. Thus, again by the first part, we may select w_1 and g_1 so that $v|z|^{1/2}w_2 = w_1 g_1$ with $\|w_1\|_\infty \leqslant 1$, $g_1 \in H^{2p}$, $g_1^{-1} \in H^\infty$, $\|g_1\|_{2p} < (1+\varepsilon)^{1/2}\|z\|_p^{1/2}$, and with $g_1 \in H^\infty$ if in addition $z \in L^\infty(\mathcal{M})$. Arguing as in 4.2 of [24], observe that if $z \in L^\infty(\mathcal{M}) \cap L^p(\mathcal{M})$ we then have by 4.1 of [24] and the above that $h_2 = g_1 g_2 \in H^p \cap H^\infty$. ∎

With the above modification of 4.2 of [24] at our disposal, we can now suitably modify the non-commutative Riesz Factorisation Theorem of [24]. In this regard, note that apart from the additional statement regarding H^∞ in the present version, the case $r = 1$ boils down to 4.3 of [24].

Theorem 4.2. (Riesz Factorisation Theorem) *Let* $1 \leqslant r, p, q \leqslant \infty$ *be given with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Given* $f \in H^r$ *and* $\varepsilon > 0$, *there exist* $g \in H^p$ *and* $h \in H^q$ *such that* $f = gh$ *and* $\|f\|_r \leqslant \|g\|_p \|h\|_q < (1+\varepsilon)\|f\|_r$. *If in addition* $f \in H^\infty$ *we may select* g *and* h *so that* $g, h \in H^\infty$. *If* $f \in H^1_0$ *we can arrange that* $g \in H^p_0$. *If* $f \in H^r_0$ *and* $1 < p \leqslant \infty$ *we can arrange that either* $g \in H^p_0$ *or* $h \in H^q_0$.

Proof. The proof of 4.3 in [24] generalises readily to the present context. To see the statement regarding H^∞, let $f \in H^r \cap H^\infty \subset L^r(\mathcal{M}) \cap L^\infty(\mathcal{M})$ and let $p \neq r$. (In the case $p = r$ we let $f = g$ and $h = \mathbb{I}$). If $f = v|f| = v|f|^{r/p}|f|^{r/q}$ is the polar decomposition of f, we then surely have $x = v|f|^{r/p} \in L^p(\mathcal{M}) \cap L^\infty(\mathcal{M})$ and $y = |f|^{r/q} \in L^q(\mathcal{M}) \cap L^\infty(\mathcal{M})$. By Theorem 4.1 we may then select $v \in \mathcal{M}$ and $h \in H^q \cap H^\infty$ such that $\|v\|_\infty \leqslant 1$, $h^{-1} \in H^\infty$ and $\|h\|_q \leqslant (1+\varepsilon)\|y\|_q$. Then $f = xy = (xv)h = gh$ with $g = xv \in L^\infty(\mathcal{M})$ if $f \in H^\infty$. The rest of the proof now follows as in 4.3 of [24]. ∎

As a first step towards a non-commutative subordination principle we present what is effectively a non-commutative version of Proposition 2.1.

Theorem 4.3. *Let* $\mathcal{B}$ *be a Banach algebra and* $\pi : H^\infty(\mathcal{M}) \to \mathcal{B}$ *a continuous homomorphism or anti-morphism. Then the following are equivalent:*

(i) *For some* $1 \leqslant p < \infty$, π *extends* $\|\cdot\|_p$*-continuously to a continuous map* $\pi : H^p(\mathcal{M}) \to \mathcal{B}$*;*

(ii) *For each* $1 \leqslant p < \infty$, π *extends* $\|\cdot\|_p$*-continuously to a continuous map* $\pi : H^p(\mathcal{M}) \to \mathcal{B}$.

The above equivalence holds in particular for (irreducible) (anti-)representations of $H^\infty(\mathcal{M})$.

Proof. For the sake of argument, let $\pi : H^\infty(\mathcal{M}) \to \mathcal{B}$ be a homomorphism. It suffices to show that for some arbitrarily given $1 < p < \infty$, π remains continuous when $H^\infty(\mathcal{M}) = \mathcal{A}$ is equipped with the norm $\|\cdot\|_p$ if and only if the same statement is valid for the norm $\|\cdot\|_1$.

First of all, let $1 < p < \infty$ be given and suppose that

$$\|\pi(a)\| \leqslant \|\pi\|_p \|a\|_p \quad \text{for all } a \in H^\infty(\mathcal{M}). \tag{4.1}$$

If in fact $p \geqslant 2$, then given any $f \in H^\infty(\mathcal{M})$ and any $\varepsilon > 0$ we may apply Theorem 4.2 to obtain $g, h \in H^\infty(\mathcal{M})$ so that $f = gh$ with $\|g\|_p\|h\|_p \leqslant (1+\varepsilon)\|f\|_{p/2}$. But then by (4.1)

$$\|\pi(f)\| = \|\pi(g)\pi(h)\| \leqslant \|\pi\|_p^2 \|g\|_p \|h\|_p \leqslant \|\pi\|_p^2 (1+\varepsilon)\|f\|_{p/2}.$$

Since both f and $\varepsilon > 0$ were arbitrary, it follows that if (4.1) holds and if $2 \leqslant p$, then

$$\|\pi(a)\| \leqslant \|\pi\|_p^2 \|a\|_{p/2} \quad \text{for all } a \in H^\infty(\mathcal{M}). \tag{4.2}$$

Thus, on applying this process inductively if necessary, it follows that we may then assume that (4.1) holds for some p with $1 \leqslant p < 2$. But since $\tau(\mathbb{I}) = 1$, Hölder's inequality then ensures that $L^2(\mathcal{M})$ injects continuously into $L^p(\mathcal{M})$ and hence that $H^2(\mathcal{M})$ injects continuously into $H^p(\mathcal{M})$. It follows that (4.1) holds for $p = 2$. But then by (4.2) we must have that

$$\|\pi(a)\| \leqslant \|\pi\|_2^2 \|a\|_1 \quad \text{for all } a \in H^\infty(\mathcal{M}) \tag{4.3}$$

as required. Conversely, note that by Hölders inequality each $H^p(\mathcal{M})$, $1 \leqslant p \leqslant \infty$, injects into $H^1(\mathcal{M})$. Thus, if (4.3) holds, then for each $1 \leqslant p < \infty$ we will have that

$$\|\pi(a)\| \leqslant \|\pi\|_1 \|a\|_p \quad \text{for all } a \in H^\infty(\mathcal{M}). \quad ∎$$

Remark 4.4. It is tempting to conjecture that if $\mathcal{B}$ is a C^*-algebra the above result will even hold for Jordan-morphisms. In this regard observe that if as in the proof we have $f = gh$ with π a Jordan-morphism, we would still be able to estimate $\|\pi(f)\| = \|\pi(gh)\|$ in terms of $\|\pi(g)\| \, \|\pi(h)\|$ if indeed $\pi(gh - hg)$ is normal. To see this observe that by Theorem 1 in [16] we would then have

$$\begin{aligned}
\|\pi(gh)\| &= \Big\| \frac{1}{2}(\pi([g,h]) + \pi(gh + hg)) \Big\| = \frac{1}{2}(\|\pi([g,h])\| + \|\pi(g)\pi(h) + \pi(h)\pi(g)\|) \\
&\leqslant \frac{1}{2}(\|\pi([g,h])^2\|^{1/2} + 2\|\pi(g)\| \, \|\pi(h)\| \\
&= \frac{1}{2}\|[\pi(g), \pi(h)]^2\|^{1/2} + \|\pi(g)\| \, \|\pi(h)\| \leqslant 2\|\pi(g)\| \, \|\pi(h)\|.
\end{aligned}$$

(Here normality is used to ensure that $\|\pi([g,h])\| = \|\pi([g,h])^2\|^{1/2}$.)

With Theorem 4.3 now at our disposal, it is clear that the suggested definition of a composition operator in Section 3 is unambiguous. We proceed to verify a series of lemmas with the eventual goal of obtaining the promised non-commutative version of Proposition 2.4.

Lemma 4.5. *Let $\varphi : \mathcal{B} \to \mathcal{C}$ be a Jordan-morphism, $\mathcal{B}$ a Banach algebra, and $\mathcal{C}$ a C^*-algebra in its reduced atomic representation* ([18], p. 740). *If φ has the property that for each irreducible representation π of $\mathcal{C}$, $\pi \circ \varphi$ is either a homomorphism or an anti-morphism, then there exists a projection $e \in \mathcal{C}' \cap \mathcal{C}''$ such that φ_e is a homomorphism and $\varphi_{\mathbb{I}-e}$ an anti-morphism. In addition, we then have that*

$$\Big| \varphi\Big(\prod_{n=1}^k h_n \Big) \Big|^p \leqslant \Big| \prod_{n=1}^k \varphi(h_n) \Big|^p + \Big| \prod_{n=1}^k \varphi(h_{k+1-n}) \Big|^p$$

for any $0 < p < \infty$ and any finite set of elements $h_1, h_2, \ldots, h_k$ in B.

Proof. Let $\mathcal{P}_0$ be a maximal set of pure states for which the associated irreducible representations are pairwise inequivalent ([18], 10.3.7). (These generate a reduced atomic representation of $\mathcal{C}$.) Corresponding to each $\omega \in \mathcal{P}_0$, we have a "support projection" $e_\omega \in \mathcal{C}' \cap \mathcal{C}''$ with $e_\omega \mathcal{C} e_\omega$ a $*$-isomorphic copy of $\pi_\omega(\mathcal{C})$, $e_\omega e_\rho = 0$ if $\omega \neq \rho$, and $\mathbb{I} = \sum_{\omega \in \mathcal{P}_0} e_\omega$. (For details see for example Proposition 6 in [22].) Thus, by the hypothesis, it follows that for each $\omega \in \mathcal{P}_0$, $e_\omega \varphi e_\omega$ is either a homomorphism or an anti-morphism. Hence, let e be the sum of all the e_ω's for which $e_\omega \varphi e_\omega$ is a homomorphism. Since $\varphi = \sum_{\omega \in \mathcal{P}_0} e_\omega \varphi e_\omega$, the first part follows. Now let $h_1, \ldots, h_k \in \mathcal{B}$ be given. By the above

$$\varphi\Big(\prod_{n=1}^k h_n \Big) = \Big(\prod_{n=1}^k \varphi(h_n) \Big) e + \Big(\prod_{n=1}^k \varphi(h_{k+1-n}) \Big)(\mathbb{I} - e). \tag{4.4}$$

For any element $a \in \mathcal{C}$, it is an exercise to show that $|a|^{2n} = |ae|^{2n} + |a(\mathbb{I}-e)|^{2n}$ for all $n \in \mathbb{N}$ whence $q(|a|^2) = q(|ae|^2)e + q(|a(\mathbb{I}-e)|^2)(\mathbb{I}-e)$ for all polynomials q. Selecting suitable polynomials it follows from the Stone-Weierstrass theorem and the functional calculus for positive elements that in fact $|a|^p = |ae|^p + |a(\mathbb{I}-e)|^p$ where p is an arbitrarily given positive real. Thus, it follows from (4.4) above that

$$\begin{aligned}\Big|\varphi\Big(\prod_{n=1}^{k} h_n\Big)\Big|^p &= \Big|\prod_{n=1}^{k}\varphi(h_n)\Big|^p e + \Big|\prod_{n=1}^{k}\varphi(h_{k+1-n})\Big|^p(\mathbb{I}-e)\\ &\leqslant \Big|\prod_{n=1}^{k}\varphi(h_n)\Big|^p + \Big|\prod_{n=1}^{k}\varphi(h_{k+1-n})\Big|^p\end{aligned}$$

as required. ∎

Lemma 4.6. *Let $1 \leqslant p < \infty$ be given and let $\varphi : H^\infty(\mathcal{M}) \to H^\infty(\mathcal{M})$ be a Jordan morphism such that $\pi \circ \varphi$ is either a morphism or an anti-morphism for each irreducible representation π of $L^\infty(\mathcal{M})$. If φ extends to a bounded map $\varphi : H^p(\mathcal{M}) \to H^p(\mathcal{M})$ with norm $\|\varphi\|_p$, then for any integer $0 < k \leqslant p$, φ extends to a map $\varphi : H^{p/k}(\mathcal{M}) \to H^{p/k}(\mathcal{M})$ with norm not exceeding $2(\|\varphi\|_p)^k$. If in fact φ is either a morphism or an anti-morphism, the norm of the induced map $\varphi : H^{p/k}(\mathcal{M}) \to H^{p/k}(\mathcal{M})$ does not exceed $(\|\varphi\|_p)^k$.*

Proof. We prove only the one statement. Since the statement regarding the moduli in Lemma 4.5 is independent of the representation we conclude that

$$\Big|\varphi\Big(\prod_{n=1}^{k} h_n\Big)\Big|^q \leqslant \Big|\prod_{n=1}^{k}\varphi(h_n)\Big|^q + \Big|\prod_{n=1}^{k}\varphi(h_{k+1-n})\Big|^q$$

for all $0 < q < \infty$ and all $h_1, \ldots, h_k \in H^\infty(\mathcal{M})$.

Now let $\varepsilon > 0$ and $f \in H^\infty(\mathcal{M})$ be given. On applying Theorem 4.2 inductively we may select $h_1, h_2, \ldots, h_k \in H^\infty(\mathcal{M})$ such that $f = h_1h_2\cdots h_k$ with $\prod_{n=1}^{k}\|h_n\|_p < (1+\varepsilon)^{k-1}\|f\|_{p/k}$. Thus, on applying the above inequality for the case $q = 1$, it follows from Lemma 4.6 and 4.1 of [24] that

$$\begin{aligned}\|\varphi(f)\|_{p/k} = \Big\|\varphi\Big(\prod_{n=1}^{k} h_n\Big)\Big\|_{p/k} &\leqslant \Big\|\Big|\prod_{n=1}^{k}\varphi(h_n)\Big| + \Big|\prod_{n=1}^{k}\varphi(h_{k+1-n})\Big|\Big\|_{p/k}\\ &\leqslant \Big\|\prod_{n=1}^{k}\varphi(h_n)\Big\|_{p/k} + \Big\|\prod_{n=1}^{k}\varphi(h_{k+1-n})\Big\|_{p/k} \leqslant 2\prod_{n=1}^{k}\|\varphi(h_n)\|_p\\ &\leqslant 2(\|\varphi\|_p)^k\prod_{h=1}^{k}\|h_n\|_p \leqslant 2(\|\varphi\|_p)^k(1+\varepsilon)^{k-1}\|f\|_{p/k}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the required estimate holds on the dense subspace $H^\infty(\mathcal{M})$ of $H^{p/k}(\mathcal{M})$. ∎

Remark 4.7. If as in the case of the spaces $H^p(\mathbb{D})$ each irreducible representation of $L^\infty(\mathcal{M})$ restricts to an irreducible representation of $H^\infty(\mathcal{M})$ ([12], II.4.1), the conditions imposed on the previous lemma would of course fall within the ambit of the proposed definition of "composition operators" in Section 3. However, as was pointed out earlier, this is not generally true in the non-commutative context (see Example 3.3).

A similar observation to that in Remark 4.4 suggests that possibly Lemma 4.6 may be extended to general Jordan morphisms $\varphi : H^\infty(\mathcal{M}) \to H^\infty(\mathcal{M})$.

In the following lemma the requirement of normality is an all too obvious restriction. In settings where the statement regarding the positive extension of contractive Jordan morphisms on $H^\infty(\mathcal{M})$ is automatically true, this alternative goes a long way to obviating this restriction. We will ultimately see that this extension property holds in particular for the spaces $H^p(\mathbb{D})$.

Lemma 4.8. *Let $\varphi : H^\infty(\mathcal{M}) \to H^\infty(\mathcal{M})$ be a continuous identity preserving Jordan morphism extending continuously to a map $\varphi : H^p(\mathcal{M}) \to H^p(\mathcal{M})$, $1 \leqslant p < \infty$, with norm $\|\varphi\|_p$. For any $k \in \mathbb{N}$ and any $a \in H^\infty(\mathcal{M})$ such that $\varphi(a)$ is normal we have that*

$$\|\varphi(a)\|_{pk} \leqslant (\|\varphi\|_p)^{1/k}\|a\|_{pk}. \tag{4.5}$$

If $1 < p < \infty$ and if in addition φ is contractive on $H^\infty(\mathcal{M})$, it extends to a unique bounded hermitian map $\widetilde{\varphi}$ on $L^p(\mathcal{M})$ which is positive on the dense subspace $\mathcal{A}+\mathcal{A}^* = H^\infty(\mathcal{M}) + H^\infty(\mathcal{M})^*$ of $L^p(\mathcal{M})$. If $\widetilde{\varphi}$ is positive on all of $L^p(\mathcal{M})$ we get

$$\|\varphi(a)\|_{p2^k} \leqslant \sqrt{2}(\|\widetilde{\varphi}\|_p)^{1/2^k}\|a\|_{p2^k} \tag{4.6}$$

for all $k \in \mathbb{N}$ and all $a \in H^\infty(\mathcal{M})$. If moreover $\widetilde{\varphi}$ satisfies the condition

$$\widetilde{\varphi}(|a^*|^{2^k} + |a|^{2^k}) \geqslant |\widetilde{\varphi}(a^*)|^{2^k} + |\widetilde{\varphi}(a)|^{2^k} \tag{P}$$

for all $k \in \mathbb{N}$ and all $a \in \mathcal{M}$, we obtain the estimate

$$\|\varphi(a)\|_{p2^k} \leqslant (2\|\widetilde{\varphi}\|_p)^{1/2^k}\|a\|_{p2^k}, \quad a \in H^\infty(\mathcal{M}),\ k \in \mathbb{N}.$$

(Note: All positive maps on C^*-algebras with $\widetilde{\varphi}(\mathbb{I}) \leqslant \mathbb{I}$ satisfy condition (P) for the case $k = 1$ (see 7.3 in [37]). The full condition (P) does however fail for general positive maps and seems in some way to be connected to the decomposability of the map in question. For an example of a positive map which fails condition (P), we refer to Choi's example of an indecomposable positive map on the algebra of 3×3 complex matrices ([7] and cf. [31], p. 294). This map fails the inequality for the case $k = 2$ and the matrix with $\alpha_{22} = 4$, $\alpha_{13} = 1$, and all other entries zero.)

Proof. Let $a \in H^\infty(\mathcal{M})$ be given such that $\varphi(a)$ is normal. Then for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\|\varphi(a)\|_{pk} &= (\| \, |\varphi(a)|^k \|_p)^{1/k} = (\| \, |\varphi(a)^k| \, \|_p)^{1/k} = (\|\varphi(a^k)\|_p)^{1/k} \\ &\leqslant (\|\varphi\|_p \|a^k\|_p)^{1/k} \leqslant (\|\varphi\|_p)^{1/k} \|a\|_{pk}.\end{aligned}$$

(The last inequality follows from a repeated application of Hölder's inequality.) Now let $1 < p < \infty$ and let φ be an identity-preserving contractive Jordan morphism on $H^\infty(\mathcal{M})$. The fact that φ uniquely extends to a positive map on $H^\infty(\mathcal{M}) + H^\infty(\mathcal{M})^*$ was discussed at the end of Section 3. In particular, φ then acts positively on the von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Now since $\mathcal{D}$ appears as a dense subspace of $L^p(\mathcal{D})$ and since $H^p(\mathcal{M}) = H^p_0(\mathcal{M}) \oplus L^p(\mathcal{D})$ ([24], 6.2), the continuous action of φ on $H^p(\mathcal{M})$ (and hence also on $L^p(\mathcal{M})$) ensures that the unique extension of φ to $H^p(\mathcal{M})$ acts positively on $L^p(\mathcal{D})$. The extension of φ to a hermitian map $\widetilde{\varphi}$ on $L^p(\mathcal{M})$ can now be done along the same lines as the construction in Section 3 by setting

$$\widetilde{\varphi}(a) = \varphi(a^*)^*, \quad a \in (H^p_0)^*$$

and applying 6.2 of [24]. Since this construction canonically contains the construction in Section 3, $\widetilde{\varphi}$ constructed as above will on restriction to $\mathcal{A} + \mathcal{A}^*$ yield precisely the map in Section 3. The $\| \cdot \|_p$-boundedness of $\widetilde{\varphi}$ follows from the $\| \cdot \|_p$-boundedness of φ, of the relevant projections in 6.2 of [24], and of the conjugate-linear map $a \to a^*$.

Given $1 \leqslant q < \infty$ recall that the symmetric q-modulus ([20]) of an element $a \in \mathcal{M}$ is given by

$$|a|_q = \Big(\frac{1}{2}(|a|^q + |a^*|^q)\Big)^{1/q}.$$

Observe that $|a|^q \leqslant 2|a|^q_q$ whence $|a| \leqslant 2^{1/q}|a|_q$. If indeed $\widetilde{\varphi}$ acts positively on all of $L^p(\mathcal{M})$ it will surely map $\| \cdot \|_\infty$-boundedly $\mathcal{M}$ into $\mathcal{M}$ ($0 \leqslant a \leqslant \|a\|_\infty \mathbb{I} \Rightarrow 0 \leqslant \widetilde{\varphi}(a) \leqslant \|a\|_\infty \mathbb{I}$). By Størmer's version of the generalized Schwarz inequality ([37], 7.3) we also have

$$\widetilde{\varphi}(a)^*\widetilde{\varphi}(a) + \widetilde{\varphi}(a)\widetilde{\varphi}(a)^* \leqslant \widetilde{\varphi}(a^*a + aa^*), \quad a \in \mathcal{M}.$$

An inductive application of this fact in the context of the symmetric 2-modulus now yields

$$\| \, |\widetilde{\varphi}(a)|_2 \|_{p2^k} \leqslant (\|\widetilde{\varphi}(|a|_2^{2^k})\|_p)^{1/2^k}, \quad a \in \mathcal{M}, \, k \in \mathbb{N}.$$

To see that the above inequality is generally valid observe that since the p-norm respects order and since $\widetilde{\varphi}(b)^2 \leqslant \widetilde{\varphi}(b^2)$ for any $b \in \mathcal{M}^+$, it follows for any fixed $k \in \mathbb{N}$ and any $b \in \mathcal{M}^+$ that $\|\widetilde{\varphi}(b)\|_{p2^k} = (\|\widetilde{\varphi}(b)^2\|_{p2^{k-1}})^{1/2} \leqslant (\|\widetilde{\varphi}(b^2)\|_{p2^{k-1}})^{1/2}$. Proceeding inductively we see that

$$\|\widetilde{\varphi}(b)\|_{p2^k} \leqslant (\|\widetilde{\varphi}(b^{2^k})\|_p)^{1/2^k}, \quad b \in \mathcal{M}^+, \, k \in \mathbb{N}.$$

Now since $|\widetilde{\varphi}(a)|_2^2 = \frac{1}{2}(|\widetilde{\varphi}(a)|^2 + |\widetilde{\varphi}(a^*)|^2) \leqslant \frac{1}{2}\widetilde{\varphi}(|a|^2 + |a^*|^2) = \widetilde{\varphi}(|a|_2^2)$ for all $a \in \mathcal{M}$, it is clear that then $\| \, |\widetilde{\varphi}(a)|_2 \|_{p2^k} = (\| \, |\widetilde{\varphi}(a)|_2^2 \|_{p2^{k-1}})^{1/2} \leqslant (\|\widetilde{\varphi}(|a|_2^2)\|_{p2^{k-1}})^{1/2}$. On taking $b = |a|_2^2$ in the previous inequality and combining it with the above, it now follows that

$$\| \, |\widetilde{\varphi}(a)|_2 \|_{p2^k} \leqslant \big((\|\widetilde{\varphi}\big((|a|_2^2)^{2^{k-1}}\big)\|_p)^{1/2^{k-1}}\big)^{1/2} = (\|\widetilde{\varphi}(|a|_2^{2^k})\|_p)^{1/2^k}, \quad a \in \mathcal{M}, \, k \in \mathbb{N}$$

as required.

Consequently, given $a \in H^\infty(\mathcal{M})$, positivity of $\widetilde{\varphi}$ and the fact that $|\varphi(a)| \leqslant \sqrt{2}|\widetilde{\varphi}(a)|_2$ yields

$$\begin{aligned}\|\varphi(a)\|_{p2^k} &= \|\,|\varphi(a)|\,\|_{p2^k} \leqslant \sqrt{2}\|\,|\widetilde{\varphi}(a)|_2\|_{p2^k}\\ &\leqslant \sqrt{2}(\|\,|\widetilde{\varphi}(a)|_2^{2^k}\|_p)^{1/2^k} \leqslant \sqrt{2}(\|\widetilde{\varphi}(|a|_2^{2^k})\|_p)^{1/2^k}\\ &\leqslant \sqrt{2}(\|\widetilde{\varphi}\|_p)^{1/2^k}\|\,|a|_2\|_{p2^k} \leqslant \sqrt{2}(\|\widetilde{\varphi}\|_p)^{1/2^k}\|a\|_{p2^k}.\end{aligned}$$

(The final inequality is a consequence of 4.10 from [20] and the fact that $p2^k \geqslant 2$.) Observing that $|a| \leqslant 2^{1/2^k}|a|_{2^k}$, a similar proof using $|a|_{2^k}$ instead of $|a|_2$ will suffice in the case that $\widetilde{\varphi}$ satisfies condition (P). ∎

The positivity requirement in the previous lemma could possibly be relaxed slightly. If the hermitian extension $\widetilde{\varphi}$ appears as the difference of two positive maps, similar estimates may be obtained by individually applying the lemma to each of the positive components of $\widetilde{\varphi}$. Now, if $\widetilde{\varphi}$ linearly maps $L^\infty(\mathcal{M})$ into $L^\infty(\mathcal{M})$, it is necessarily $\|\cdot\|_\infty$-continuous on $L^\infty(\mathcal{M})$ by the closed graph theorem and hence in this case the question regarding the achievability of such a decomposition reduces to the L^∞ case. (That the induced map has a closed graph may be seen from the continuity of $\widetilde{\varphi}$ on $L^p(\mathcal{M})$ and the fact that $L^\infty(\mathcal{M})$ continuously injects into $L^p(\mathcal{M})$). (See also Theorem 4.14.)

Lemma 4.9. *Let $1 < p \leqslant \infty$ be given and let (q_n) be a sequence of reals in $[1, p[$ increasing to p. Then $a \in L^p(\mathcal{M})$ if and only if $a \in \bigcap_{n=1}^{\infty} L^{q_n}(\mathcal{M})$ and $\sup_n \|a\|_{q_n} < \infty$. In this case $\lim_n \|a\|_{q_n} = \|a\|_p$.*

Proof. Since in the present context $\tau(\mathbb{I}) = 1$, Hölder's inequality ([10], 4.2) suffices to show that $L^p(\mathcal{M})$ contractively injects into each of the $L^{q_n}(\mathcal{M})$'s thereby establishing the one direction of the first statement. For the converse as well as the final statement it is evident from 2.5 (i), 2.6 and 2.8 of [10] that the general case follows from the $L^p[0,1]$ setting where it is probably fair to say that results of this nature are "standard". ∎

Lemma 4.10. *Let $1 \leqslant p < \infty$ be given and let (q_n) be a sequence of reals in $]p, \infty[$ decreasing to p. Given $a \in \bigcup_{n=1}^{\infty} L^{q_n}(\mathcal{M}) \subset L^p(\mathcal{M})$, we have that $\|a\|_p = \lim_n \|a\|_{q_n}$.*

Proof. Observe that $L^{q_n}(\mathcal{M}) \subset L^{q_{n+1}}(\mathcal{M}) \subset L^p(\mathcal{M})$ for each $n \in \mathbb{N}$ since $\tau(\mathbb{I}) = 1$. Once more an application of 2.6 and 2.8 of [10] reduces this to the case $L^p[0,1]$ where such results are "standard". ∎

LEMMA 4.11. *Let $p \in]0,\infty[$ be given. There exist sequences $(k_m), (n_m)$, $(\ell_m) \subset \mathbb{N}$ such that $\frac{2^m}{k_m}$ decreases to p and $\frac{2^{n_m}}{\ell_m}$ increases to p.*

Proof. Given $p > 0$ write $\frac{1}{p}$ in binary form, i.e.

$$\frac{1}{p} = \sum_{i \leqslant n} \varepsilon_i 2^i,$$

where n is the largest integer such that $\frac{1}{p} \geqslant 2^n$ and $\varepsilon_i \in \{0,1\}$ for each integer $i \leqslant n$. Given $m > \max\{-n, 0\}$, the terms $\sum\limits_{-m \leqslant i \leqslant n} \varepsilon_i 2^i$ clearly increase to $\frac{1}{p}$ as $m \to \infty$ and can moreover easily be seen to be of the form

$$\sum_{-m \leqslant i \leqslant n} \varepsilon_i 2^i = \frac{k_m}{2^m}$$

where $k_m \in \mathbb{N}$ by merely writing the sum over the common denominator 2^m. (More precisely $k_m = \sum\limits_{-m \leqslant i \leqslant n} \varepsilon_i 2^{i+m}$.)

Thus $\frac{2^m}{k_m}$ as constructed above decreases to p. To see the second statement let $(q_m) \subset]0,p[$ be any sequence of distinct reals increasing to p. On applying the first part we see that for any $m \geqslant 1$ we may find $n_m, \ell_m \in \mathbb{N}$ so that

$$q_m \leqslant \frac{2^{n_m}}{\ell_m} < q_{m+1}.$$

The result follows. ∎

With all the necessary lemmas now at our disposal we are finally in a position to verify the promised extension of Proposition 2.4. A comparison of the next two results clearly reveals that in the absence of commutativity a measure of positivity (of the map $\widetilde{\varphi}$) is required to afford a result analogous to Proposition 2.4. This implicit reliance on positivity is moreover quite natural in that it reflects the state of affairs in the classical case (see Theorem 4.14).

THEOREM 4.12. *Let $\varphi : H^\infty(\mathcal{M}) \to H^\infty(\mathcal{M})$ be a contractive identity preserving Jordan morphism such that:*

(α) $\pi \circ \varphi$ is either a homomorphism or anti-morphism for each irreducible representation of $L^\infty(\mathcal{M}) \supset H^\infty(\mathcal{M})$; and

(β) for some $1 < p < \infty$, φ extends to a bounded map $\varphi : H^p(\mathcal{M}) \to H^p(\mathcal{M})$.

If the canonical hermitian extension $\widetilde{\varphi}$ to all of $L^p(\mathcal{M})$ (see Lemma 4.8) is even positive, then for any other $1 \leqslant q < \infty$, φ will induce a unique closed operator $\overline{\varphi} : D(\overline{\varphi}) \subset H^q(\mathcal{M}) \to H^q(\mathcal{M})$ with the properties that:

(i) *$D(\overline{\varphi}) \supset \bigcup\limits_{r>q} H^r(\mathcal{M})$;*

(ii) *for any $r > q$, $\overline{\varphi}$ restricts to a bounded map $\overline{\varphi} : H^r(\mathcal{M}) \to H^q(\mathcal{M})$;*

(iii) *for any $q > s \geqslant 1$, $\overline{\varphi}$ extends uniquely to a bounded map $\overline{\varphi} : H^q(\mathcal{M}) \to H^s(\mathcal{M})$.*

If $\widetilde{\varphi} : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ satisfies the slightly stronger condition (P) in Lemma 4.8, then $\overline{\varphi} : H^q(\mathcal{M}) \to H^q(\mathcal{M})$ itself is a bounded everywhere defined operator.

Proof. Let $1 < q < \infty$ be given and suppose that $\widetilde{\varphi} : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is positive. (Apart from the fact that (iii) is vacuously satisfied if $q = 1$, the proof for this case is entirely analogous to the one presently under consideration.) Now for any $k, m \in \mathbb{N}$ such that $\frac{p2^m}{k} \geqslant 1$ it follows from Lemmas 4.6 and 4.8 that φ induces a unique bounded map

$$\varphi : H^{vp}(\mathcal{M}) \to H^{vp}(\mathcal{M}), \quad \text{where } v = \frac{2^m}{k}$$

with norm

$$\|\varphi\|_{vp} \leqslant 2(\|\varphi\|_{p2^m})^k \leqslant 2(\sqrt{2})^k(\|\widetilde{\varphi}\|_p)^{k/2^m}.$$

Therefore, it follows from Lemma 4.11 and for example [32], and cf. (4.1) in [24] that for a suitably selected sequence (v_m) of rationals decreasing to $\frac{q}{p}$ we have that φ extends to a bounded map $\varphi : H^{pv_m} \to H^{pv_m}$ for each $m \in \mathbb{N}$. But since $pv_m \geqslant q$, φ in fact induces a unique bounded map $\varphi : H^{pv_m} \to H^q$ for each $m \in \mathbb{N}$. Now recall that in this setting $H^u(\mathcal{M})$ injects into $H^w(\mathcal{M})$ whenever $u > w$. It therefore follows from what we have just verified that for each $r > q$, φ induces a unique bounded map from $H^r(\mathcal{M})$ into $H^q(\mathcal{M})$ and hence a linear map from $\bigcup_{r>q} H^r(\mathcal{M}) \subset H^q(\mathcal{M})$ into $H^q(\mathcal{M})$. (One need only note that for each $r > q$ it is possible to find some pv_m with $r \geqslant pv_m > q$ and argue from there.) If $\widetilde{\varphi}$ satisfied property (P) then for say $v_m = \frac{2^m}{k_m}$ we would have

$$\|\varphi\|_{pv_m} \leqslant 2(\|\varphi\|_{p2^m})^{k_m} \leqslant 2(2\|\widetilde{\varphi}\|_p)^{km/2^m} = 2(2\|\widetilde{\varphi}\|_p)^{1/v_m}.$$

Now since $\|\widetilde{\varphi}\|_p \geqslant 1$ (to see this note that $\widetilde{\varphi}(\mathbb{I}) = \mathbb{I}$) and $\frac{1}{v_m}$ increases to $\frac{p}{q}$, it follows that $(2\|\widetilde{\varphi}\|_p)^{1/v_m}$ increases to $(2\|\widetilde{\varphi}\|_p)^{p/q}$. Thus if $\widetilde{\varphi}$ satisfies property (P) then for each $pv_m > q$ and each $a \in H^{pv_m}$, the induced bounded map $\varphi : H^{pv_m} \to H^{pv_m}$ satisfies the inequality

$$\|\varphi(a)\|_q \leqslant \|\varphi(a)\|_{pv_m} \leqslant 2(2\|\widetilde{\varphi}\|_p)^{p/q}\|a\|_{pv_m}.$$

If now we pass to the dense subspace $\bigcup_{m=1}^{\infty} H^{pv_m} = \bigcup_{r>q} H^r$ of H^q and apply Lemma 4.10, we see that the aforementioned linear map from this subspace into H^q will in fact be $\|\cdot\|_q - \|\cdot\|_q$ continuous if $\widetilde{\varphi} : L^p \to L^p$ satisfies property (P).

It remains to show that in general positivity of $\widetilde{\varphi} : L^p \to L^p$ is sufficient to guarantee the validity of (iii) and the closability of the induced linear map from $\bigcup_{r>q} H^r \subset H^q$ into H^q. However, closability of this map proves to be a consequence of the fact that for any $q > s \geqslant 1$, φ extends uniquely to a bounded map $\varphi : H^q \to H^s$. To see this latter fact apply Lemma 4.11 to find a rational $v = \frac{2^m}{k}$ so that $q > pv > s$. By Lemmas 4.6 and 4.8, φ induces a unique bounded map from H^{pv} into H^{pv}. Since in addition H^q injects continuously into H^{pv} and H^{pv} continuously into H^s, this yields the required bounded map $\varphi : H^q \to H^s$. (Uniqueness follows from the density of H^∞ and the fact that this map is an extension of the Jordan morphism $\varphi : H^\infty \to H^\infty$.) ∎

THEOREM 4.13. *Let $\mathcal{M}$ be commutative and $\varphi : H^\infty \to H^\infty$ a contractive identity preserving homomorphism. If for some $1 \leqslant p < \infty$, φ extends to a bounded map $\varphi : H^p \to H^p$, it does so for each $1 \leqslant p < \infty$.*

Proof. In this setting all elements of H^∞ are normal. The proof is therefore essentially a simplification of the proof of Theorem 4.12 using the normality criteria in Lemma 4.8 rather than positivity. ∎

In hindsight the applicability of Theorem 4.12 is directly dependent on our ability to verify the positivity criteria. In this regard, it would therefore be useful to know if the map $\widetilde{\varphi}$ in Lemma 4.8 acts positively on all of $L^p(\mathcal{M})$ for $1 < p < \infty$ instead of just on $\mathcal{A} + \mathcal{A}^*$. Had $\mathcal{A} + \mathcal{A}^*$ been a subalgebra instead of a subspace an appeal to something like the Kaplansky density theorem would have been conceivable. As it is, we have to content ourselves with the following observation regarding the case $H^p(\mathbb{D})$ in support of the suspicion that $\widetilde{\varphi}$ may indeed be order-preserving on all of $L^p(\mathcal{M})$.

THEOREM 4.14. *Let $1 \leqslant p < \infty$ and let $C_T : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ be a composition operator induced by the analytic map $T : \mathbb{D} \to \mathbb{D}$. Then C_T extends uniquely to an order-preserving map $\widetilde{C_T}$ on $L^p(\mathbb{T})$.*

Proof. First assume that $T(0) = 0$ and that $|T(z)| < 1$ almost everywhere on $\mathbb{T}$. Then C_T extends to the integral operator K_T on $L^p(\mathbb{T})$ with kernel

$$k(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}t}) = \frac{1 - |T(\mathrm{e}^{\mathrm{i}s})|^2}{|\mathrm{e}^{\mathrm{i}t} - T(\mathrm{e}^{\mathrm{i}s})|^2}$$

(see [33]). By the assumption on T it is clear that $k(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}t}) > 0$ almost everywhere and hence that K_T is order-preserving on $L^p(\mathbb{T})$. By the uniqueness of the hermitian extension we must have $\widetilde{C_T} = K_T$.

More generally, if all we know is that $T(0) = 0$, it follows from Proposition 1 of [33] that $\widetilde{C_T}$ exists (even if $p = 1$) and is the strong limit of order-preserving integral operators and hence itself order-preserving. (To see this, note that for every $f \in L^p(\mathbb{T})$ and $g \in L^{p^*}(\mathbb{T})$ with $f, g \geqslant 0$ we have $(\widetilde{C_T}(f), g) = \lim\limits_{r \nearrow 1} (K_{rT}(f), g) \geqslant 0$.)

For general T we bring the Möbius transformation

$$S(z) = \frac{T(0) - z}{1 - \overline{T(0)}z}$$

into play. With S as above, $ST(0) = 0$ and $S^2T = T$. (For details see Prerequisites in [17].) Hence $C_T = C_{ST}C_S$. By the uniqueness of the hermitian extension we must have $\widetilde{C_T} = \widetilde{C_{ST}}\widetilde{C_S}$ provided $\widetilde{C_S}$ exists. Now, since by the first part of the proof $\widetilde{C_{ST}}$ exists and is order-preserving, we need only show that $\widetilde{C_S}$ exists and is order-preserving to conclude the proof. But since S maps $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ in an analytic way, S in fact induces a composition operator on $L^p(\mathbb{T})$ by means of the formula

$$\mathrm{e}^{\mathrm{i}t} \to f(S(\mathrm{e}^{\mathrm{i}t}))$$

by 2.1.2 in [35]. This operator can be shown to extend C_S in a natural way (consider radial limits; [15], pp. 38, 39, 136) and is order-preserving. (To see this, note that $f(S(\mathrm{e}^{\mathrm{i}t})) \geqslant 0$ for all t whenever $f(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ for all t since $S(\partial\mathbb{D}) = \partial\mathbb{D}$.) This operator is therefore nothing but the unique hermitian extension $\widetilde{C_S}$ of C_S. ∎

In concluding this section we present a generalisation of the fact that a bounded map $C : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is a composition operator if and only if it is multiplicative.

PROPOSITION 4.15. *Let $1 \leqslant p < \infty$ be given and let $\varphi : H^p(\mathcal{M}) \to H^p(\mathcal{M})$ be continuous and multiplicative in the sense that $\varphi(ab+ba) = \varphi(a)\varphi(b)+\varphi(b)\varphi(a)$ whenever a, b, ab, and ba all belong to $H^p(\mathcal{M})$. Given any $a \in H^\infty(\mathcal{M}) \subset H^p(\mathcal{M})$, we have that $\varphi(a) \in H^\infty(\mathcal{M})$ whenever $\varphi(a)$ is normal in which case $\|\varphi(a)\|_\infty \leqslant \|a\|_\infty$. In particular, if $\mathcal{M}$ is commutative φ restricts to a contractive, homomorphism from $H^\infty(\mathcal{M})$ into $H^\infty(\mathcal{M})$.*

Proof. By the hypothesis we have $\varphi(a^n) = \varphi(a)^n$ for all $n \in \mathbb{N}$ whenever $a \in H^\infty(\mathcal{M})$. If $\varphi(a)$ is normal, we may therefore argue as in the first part of the proof of Lemma 4.8 to see that then $\|\varphi(a)\|_{pn} \leqslant (\|\varphi\|_p)^{1/n}\|a\|_{pn}$. On letting $n \to \infty$ this yields $\|\varphi(a)\|_\infty \leqslant \|a\|_\infty$ by Lemma 4.9. The final statement follows from the fact that if $\mathcal{M}$ is commutative then the fact that $H^p(\mathcal{M})$ appears as a subspace of the commutative $*$-algebra $\widetilde{\mathcal{M}}$ of τ-measurable operators ensures that all elements of $H^p(\mathcal{M})$ are necessarily normal. Thus, by the above φ in this case then acts homomorphically on all of $H^\infty(\mathcal{M})$. ∎

5. APPLICATIONS TO ISOMETRIES ON H^p

In this final cycle of results we investigate the relationship between identity preserving linear isometries on non-commutative H^p-spaces and "composition operators". Though we do not quite manage to achieve a general non-commutative characterisation of such isometries, our results are sufficiently strong to imply results of this nature for isometries that have H^∞ as an invariant subspace. In fact, for identity preserving isometries this H^∞-invariance proves to be all but equivalent to "multiplicativity". Though much work remains to be done, this partial success strongly hints at the tenability of a general non-commutative characterisation. A serious obstacle to be overcomed in this regard is the absence of a satisfactory non-commutative analog of the elementary inner function $\imath : z \to z$, $z \in \mathbb{D}$. In fact, it is precisely the presence of this selfsame inner function that affords the by now classical characterisation of linear isometries on $H^p(\mathbb{D})$, $p \neq 2$, by guaranteeing a sufficient degree of H^∞-invariance for identity preserving isometries in this context. (See the first part of the proof of Theorem 1 in [11] for details.) Our main result amounts to a non-commutative extension of Proposition 2 from [11].

THEOREM 5.1. *Let $\mathcal{M}, \mathcal{W}$ be von Neumann algebras with finite normalised faithful (normal) traces τ and ν respectively. Let $\mathcal{A}$ be a subalgebra of $\mathcal{W}$ containing the identity and φ an identity-preserving transformation from $\mathcal{A}$ into $\mathcal{M}$ such that for some $1 \leqslant p < \infty$ with $p \neq 2$ we have*

$$\tau(|\varphi(a)|^p) = \nu(|a|^p) \quad \text{for all } a \in \mathcal{A}.$$

Then for each $a \in \mathcal{A}$ we have that

$$\tau(|\varphi(a)|^2) = \nu(|a|^2)$$

with $\varphi(a^2) - \varphi(a)^2$ belonging to the annihilator of $\varphi(\mathcal{A})$ in the sense that $\varphi(a^2) - \varphi(a)^2 \in \{x \in \mathcal{M} : \tau(\varphi(b)^* x) = 0,\, b \in \mathcal{A}\}$. Moreover $\varphi(a)^* = \varphi(a^*)$ if in addition $a^* \in \mathcal{A}$. If both a and $\varphi(a)$ are normal then $\varphi(a^2) = \varphi(a)^2$.

The above result is of course particularly applicable to the case of identity-preserving isometries on $H^p(\mathcal{M})$ with $1 \leqslant p < \infty$ and $p \neq 2$, which map $H^\infty(\mathcal{M}) = H^p(\mathcal{M}) \cap L^\infty(\mathcal{M})$ back into $H^\infty(\mathcal{M})$.

Proof. Let $a, b \in \mathcal{A}$ be given. Since by hypothesis $\varphi(\mathbb{I} + za) = \mathbb{I} + z\varphi(a)$ we have

$$\tau(|\mathbb{I} + z\varphi(a)|^p) = \nu(|\mathbb{I} + za|^p)$$

for all $z \in \mathbb{C}$. Now, for $r > 0$ small enough, we have that $\|za + \bar{z}a^* + |z|^2|a|^2\|_\infty < 1$ and $\|z\varphi(a) + \bar{z}\varphi(a)^* + |z|^2|\varphi(a)|^2\|_\infty < 1$ whenever $|z| \leqslant r$. Since the function $\lambda \to (1+\lambda)^{p/2}$ is analytic in the open unit disc it therefore follows from the analytic functional calculus that

$$|\mathbb{I} + za|^p = (\mathbb{I} + (za + \bar{z}a^* + |z|^2|a|^2))^{(p/2)} = \sum_{k=0}^{\infty} \binom{\frac{p}{2}}{k} (za + \bar{z}a^* + |z|^2|a|^2)^k$$

for all $|z| \leqslant r$ where the convergence of the series is absolute. We may now expand each term of the form $(za + \bar{z}a^* + |z|^2|a|^2)^k$ and suitably rearrange terms to get a series of the form $\sum_{i,j} z^i \bar{z}^j \alpha_{ij}$ where each α_{ij} is a suitable combination of permutations of $a^i(a^*)^j$. This rearrangement also converges absolutely. To see this observe that given α_{ij} with $i + j \geqslant m$ each permutation of $a^i(a^*)^j$ appearing in α_{ij} results from an expansion of some term in

$$\sum_{k \geqslant \lfloor m/2 \rfloor} \binom{\frac{p}{2}}{k} (za + \bar{z}a^* + |z|^2|a|^2)^k.$$

An application of the triangle inequality now yields

$$\sum_{i+j \geqslant m} |z^i \bar{z}^j| \, \|\alpha_{ij}\| \leqslant \sum_{k \geqslant \lfloor m/2 \rfloor} \binom{\frac{p}{2}}{k} (2|z| \, \|a\| + |z|^2 \|a\|^2)^k$$

with the right hand side converging to zero as $m \to \infty$ when $|z| \leqslant r$ and say $r \leqslant \frac{1}{3\|a\|}$. Recall that here $\nu(\mathbb{I}) = 1$. Thus $L^1(\mathcal{W}) \supseteq \mathcal{W}$ implying that ν extends to a continuous linear functional on $\mathcal{W}$. By continuity we then have that

$$\nu(|\mathbb{I} + za|^p) = \nu\Bigg(\sum_{i,j \geqslant 0} z^i \bar{z}^j \alpha_{ij}\Bigg) = \sum_{i,j \geqslant 0} z^i \bar{z}^j \nu(\alpha_{ij}) \tag{5.1}$$

for all $|z| \leqslant r$. A similar construction with respect to $\varphi(a)$ now yields an expression of the form

$$\tau(|\mathbb{I} + z\varphi(a)|^p) = \sum_{i,j \geqslant 0} z^i \bar{z}^j \tau(\beta_{ij}) \tag{5.2}$$

for all $|z| \leqslant r$ where each β_{ij} is obtained from combinations of permutations of $\varphi(a)^i(\varphi(a)^*)^j$ in exactly the same way that α_{ij} was obtained from combinations of permutations of $a^i(a^*)^j$. Since by hypothesis the two power series in (5.1) and

(5.2) agree on the disc of radius r, we must have $\tau(\beta_{ij}) = \nu(\alpha_{ij})$ for each $i, j \geqslant 0$. Now, for $0 \leqslant i \leqslant 2$, $0 \leqslant j \leqslant 1$, the trace property ensures that say ν applied to any permutation of $a^i(a^*)^j$ yields exactly $\nu((a^*)^j a^i)$ with a similar statement holding for τ. Consequently, since the fact that $p \neq 2$ ensures that $\binom{\frac{p}{2}}{k}$ is non-zero for at least $0 \leqslant k \leqslant 2$, the terms α_{ij} and β_{ij} will in fact be non-zero for $0 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 1$. By the aforementioned observation regarding the trace property, it therefore follows that

$$\nu((a^*)^j a^i) = \tau((\varphi(a)^*)^j \varphi(a)^i), \quad 0 \leqslant i \leqslant 2,\, 0 \leqslant j \leqslant 1,\, a \in \mathcal{A}. \tag{5.3}$$

Now, since $\nu(a^*a) = \tau(\varphi(a)^*\varphi(a))$ for all $a \in \mathcal{A}$, it follows from the polarisation identity for inner products that

$$\nu(b^*a) = \tau(\varphi(b)^*\varphi(a)), \quad a, b \in \mathcal{A} \tag{5.4}$$

and therefore by (5.3) and (5.4) that

$$\tau(\varphi(a)^*\varphi(a^2)) = \nu(a^*a^2) = \tau(\varphi(a)^*\varphi(a)^2), \quad a \in \mathcal{A}. \tag{5.5}$$

Now, replace a by $za + b$ in the equality above. A consideration of the coefficients of the z^2 term of the resultant polynomial yields the fact that

$$\tau(\varphi(b)^*\varphi(a^2)) = \tau(\varphi(b)^*\varphi(a)^2), \quad a, b \in \mathcal{A} \tag{5.6}$$

as required. To show that $\varphi(a^*) = \varphi(a)^*$ whenever $a, a^* \in \mathcal{A}$, it suffices to show that $\varphi(a) = \varphi(a)^*$ whenever $a^* = a \in \mathcal{A}$. To this end suppose $a = a^* \in \mathcal{A}$. By (5.3) with $j = 0$ and $i = 2$, and (5.4) with $b = a$ we then have

$$\begin{aligned}\tau(|\varphi(a)^* - \varphi(a)|^2) &= 2\tau(\varphi(a)^*\varphi(a)) - \tau((\varphi(a)^*)^2) - \tau(\varphi(a)^2)\\ &= 2\tau(\varphi(a)^*\varphi(a)) - \overline{\tau(\varphi(a)^2)} - \tau(\varphi(a)^2) = 2\nu(a^*a) - \overline{\nu(a^2)} - \nu(a^2) = 0.\end{aligned}$$

Since τ is faithful, we must have $|\varphi(a)^* - \varphi(a)|^2 = 0$ and hence $\varphi(a)^* - \varphi(a) = 0$ in this case. Finally, suppose that both $a \in \mathcal{A}$ and $\varphi(a)$ are normal. By what we have just verified

$$\tau(\varphi(a^2)^*\varphi(a)^2) = \tau(\varphi(a^2)^*\varphi(a^2))$$

and by duality

$$\tau((\varphi(a)^*)^2\varphi(a^2)) = \overline{\tau(\varphi(a^2)^*\varphi(a)^2)} = \tau(\varphi(a^2)^*\varphi(a^2)).$$

This in turn is sufficient to imply that $\tau(|\varphi(a)^2 - \varphi(a^2)|^2) = \tau(|\varphi(a)^2|^2) - \tau(|\varphi(a^2)|^2)$. By the normality of $\varphi(a)$ this yields $\tau(|\varphi(a)^2 - \varphi(a^2)|^2) = \tau(|\varphi(a)|^4) - \tau(|\varphi(a^2)|^2)$. On considering (5.3) we see that

$$\tau(|\varphi(a^2)|^2) = \nu(|a^2|^2) = \nu(|a|^4)$$

by the normality of a and the fact that $a^2 \in \mathcal{A}$. If indeed we can show that $\tau(|\varphi(a)|^4) = \nu(|a|^4)$, then surely $|\varphi(a)^2 - \varphi(a^2)|^2 = 0$ (and hence $\varphi(a)^2 - \varphi(a^2) = 0$) by the faithfulness of τ. It remains to show that $\tau(|\varphi(a)|^4 = \nu(|a|^4)$ if both a and $\varphi(a)$ are normal. To see this fact recall that $\binom{\frac{p}{2}}{2} \neq 0$ since $p \neq 2$. Thus, the trace property together with normality ensures that the coefficients of the $|z|^4 = \bar{z}^2 z^2$ terms in (5.1) and (5.2) are respectively non-zero multiples (by the same constant) of $\nu(|a|^4)$ and $\tau(|\varphi(a)|^4)$. Since the two series agree on a neighbourhood of 0, we must have $\nu(|a|^4) = \tau(|\varphi(a)|^4)$ as required. ∎

Proposition 5.2. *Let $1 \leqslant p < \infty$ where $p \neq 2$, and let φ be an identity preserving linear isometry on $H^p(\mathcal{M})$. If $\varphi(H^\infty) \subset H^\infty$, φ necessarily preserves both adjoints and the Jordan product (where defined) on $L^p(\mathcal{D}) = H^p(\mathcal{M}) \cap H^p(\mathcal{M})^*$, and furthermore induces a unique linear identity preserving isometry $\widetilde{\varphi}$ on $H^2(\mathcal{M})$ such that $\varphi|_{H^p \cap H^2} = \widetilde{\varphi}|_{H^p \cap H^2}$. If in addition φ is surjective, it preserves the Jordan product even on $H^\infty(\mathcal{M}) \subset H^p(\mathcal{M})$. If φ is surjective and $\varphi(H^\infty) = H^\infty$, it extends to an adjoint-preserving isometric linear bijection on all of $L^2(\mathcal{M})$. Conversely, if φ preserves the Jordan product on $H^\infty(\mathcal{M})$, then given any $a \in H^\infty(\mathcal{M})$, $\varphi(a) \in H^\infty(\mathcal{M})$ whenever $\varphi(a)$ is normal. In particular, if $\mathcal{M}$ is commutative, $H^\infty(\mathcal{M})$ is an invariant subspace of φ if and only if φ acts multiplicatively on $H^\infty(\mathcal{M})$.*

Proof. Let φ be given. In the case that φ is known to preserve the Jordan product on H^∞, the statements related to the necessity of the invariance of H^∞ under the action of φ are contained in Proposition 4.15. Hence suppose that $\varphi(H^\infty) \subset H^\infty$.

It is now a direct consequence of Theorem 5.1 that φ preserves adjoints on the dense subspace $\mathcal{D}$ of $L^p(\mathcal{D})$ as well as squares of elements $a \in H^\infty(\mathcal{M})$ for which both a and $\varphi(a)$ are normal. This is clearly sufficient to ensure the multiplicativity of φ on H^∞ in the commutative case ([29], 6.3.2). More generally, the fact that φ then certainly preserves squares on the selfadjoint portion of the selfadjoint subalgebra $\mathcal{D}$ is sufficient to ensure that φ preserves the Jordan product on all of $\mathcal{D}$. (The identity $(a+b)^2 - a^2 - b^2 = ab + ba$ may be used to verify the preservation of the Jordan product of selfadjoint elements. By linearity this fact then extends to all of $\mathcal{D}$.)

A further direct consequence of Theorem 5.1 is that $\varphi|_{H^\infty}$ extends uniquely to an identity preserving isometry $\widetilde{\varphi}$ on H^2. We therefore need only show that $\varphi|_{H^p \cap H^2} = \widetilde{\varphi}|_{H^p \cap H^2}$. To see this, suppose for the sake of argument that $2 < p < \infty$. Given any $a \in H^p \cap H^2$, select $\{a_n\} \subset H^\infty$ such that $\{a_n\}$ converges to a in H^p. Since in this case H^p injects continuously into H^2, the sequence $\{a_n\}$ converges to a in H^2 as well. But then, by continuity, $\{\varphi(a_n)\} = \{\widetilde{\varphi}(a_n)\}$ converges to $\varphi(a)$ in H^p and hence also in H^2. This clearly suffices to show that $\widetilde{\varphi}(a) = \varphi(a)$.

It remains to investigate the case where φ is surjective.

Now, given any $a \in H^\infty$, the fact that H^∞ is an invariant subspace of φ clearly suffices to verify that $\varphi(a^2) - \varphi(a)^2 \in H^\infty$. Thus by the non-commutative Hölder inequality ([10], 4.2) $x \to \tau((\varphi(a^2) - \varphi(a)^2)^* x)$ canonically defines a continuous linear functional ω on each H^q, $1 \leqslant q < \infty$. By 6.3 of [24] we need only show that $\omega = 0$ on some H^q with $1 < q < \infty$ in order to see that $\varphi(a^2) - \varphi(a)^2 = 0$. As a functional on H^p, ω must by Theorem 5.1 be in the annihilator of $(\varphi(H^\infty)) \subset H^p$. Since φ is surjective, it will map the dense subspace H^∞ of H^p onto the dense subspace $\varphi(H^\infty)$ of $\varphi(H^p) = H^p$. Thus $\varphi(H^\infty)$ will of necessity have a trivial annihilator implying that $\omega = 0$ on H^p. If $1 < p < \infty$ then on taking $p = q$ it follows that $\varphi(a^2) = \varphi(a)^2$. For the case $p = 1$ the same conclusion follows by taking $q = 2$ and observing that $H^2 \subset H^1$. Thus φ preserves the Jordan product on H^∞ ([29], 6.3.2).

Finally, suppose that φ is surjective and that $\varphi(H^\infty) = H^\infty$. On applying what we have just verified to both φ and φ^{-1}, we conclude that there exist isometries $\widetilde{\varphi}$ and $\widetilde{\varphi^{-1}}$ on H^2 which act as inverses to each other on the dense subspace $H^p \cap H^2$. By continuity, $\widetilde{\varphi}$ is therefore clearly invertible with $\widetilde{\varphi^{-1}} = \widetilde{\varphi}^{-1}$. By the first part of the theorem we know that both $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ will leave the self-adjoint portion of $H^2(\mathcal{M})$, viz $L^2(\mathcal{D})$, invariant. Clearly, this can only be the case if $\widetilde{\varphi}(L^2(\mathcal{D})) = L^2(\mathcal{D})$. It is known that $L^2(\mathcal{M})$ decomposes into the *orthogonal* direct sum $L^2(\mathcal{M}) = H_0^2 \oplus L^2(\mathcal{D}) \oplus (H_0^2)^*$ ([23]). Thus, since $\widetilde{\varphi}$ is a surjective isometry, that is a unitary operator, on the Hilbert space $H^2 = H_0^2 \oplus L^2(\mathcal{D})$, we also have that $\widetilde{\varphi}(H_0^2) = \widetilde{\varphi}(L^2(\mathcal{D})^\perp) = L^2(\mathcal{D})^\perp = H_0^2$. We may now linearly extend $\widetilde{\varphi}$ to all of $L^2(\mathcal{M})$ by defining the action on $(H_0^2)^*$ by $\widetilde{\varphi}(a) = \widetilde{\varphi}(a^*)^*$ for each $a \in (H_0^2)^*$. From what we have already verified, it now readily follows that this extension *isometrically* respectively maps each of the *orthogonal* subspaces H_0^2, $L^2(\mathcal{D})$, and $(H_0^2)^*$ of $L^2(\mathcal{M})$ back onto themselves. It is now an exercise to show this is sufficient to ensure that $\widetilde{\varphi}$ acts isometrically on all of $L^2(\mathcal{M})$. Since $\widetilde{\varphi}$ is known to preserve adjoints on $L^2(\mathcal{D})$, it also readily follows from the definition of the extension that in fact it preserves adjoints on all of $L^2(\mathcal{M})$. ∎

Remark 5.3. In the commutative setting it is possible to show that any identity preserving linear isometry on H^p, $p \neq 2$, induces an isometry on H^2 apart from any considerations regarding the invariance of H^∞ (see Proposition 1 in [11]). In fact, it is precisely this fact alongside the action of such maps on the function $\imath : z \to z$ that suffices to establish the invariance of H^∞ (and thereby multiplicativity of φ) in the setting of $H^p(\mathbb{D})$, $p \neq 2$. (See the first part of the proof of Theorem 1 in [11].) Given the apparent absence of a satisfactory analogue of $\imath$ in more general H^p spaces, any attempt at showing that even here φ automatically leaves H^∞ invariant would have to rely on more ingenious techniques (even in the case of a commutative von Neumann algebra $\mathcal{M}$).

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