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HIGHER-DIMENSIONAL NUMERICAL RANGES OF QUADRATIC OPERATORS

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ABSTRACT. We show that for every positive integer k, the k-numerical range of a square-zero operator on a (separable) Hilbert space is an (open or closed) circular disc centered at the origin. The radius and the closedness of the disc can be completely determined in terms of the "singular numbers" of the operator. The k-numerical range of idempotent operators is more difficult to describe since its boundary is in general not any familiar curve. What we do is to give enough information, again in terms of the singular numbers of the idempotent operator under consideration, so as to have a general idea of its shape and location.

KEYWORDS: *k-numerical range, square-zero operator, idempotent operator, quadric operator.*

MSC (2000): 47A12, 15A60.

1. INTRODUCTION

Let T be a bounded linear operator on a Hilbert space H. For every positive integer k, the *k-numerical range* of T, in symbols $W_k(T)$, is the set

$$\left\{ \sum_{j=1}^{k} \langle Tx_j, x_j \rangle : x_1, \ldots, x_k \text{ are } k \text{ orthonormal vectors in } H \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. When $k = 1$, this coincides with the (classical) numerical range $W(T)$. An operator T is *quadratic* if it satisfies $T^2 + aT + bI = 0$ for some scalars a and b. The purpose of this paper is to give descriptions, as precise as possible, of k-numerical ranges of quadratic operators.

Historically, the k-numerical range is one of the earliest generalizations of the numerical range. Its convexity was first asked by Halmos and proved by Berger ([2]). Considered as an intermediate object between the classical numerical range and the more general C-numerical range, it has proved its worth in the past by shedding information on many properties of the operator. Most of the early works on the k-numerical range involve operators on finite-dimensional spaces or finite matrices. Thus, for example, it is known that, for an n-by-n matrix $T, W_k(T)$ is a polygonal region for some k, $(n/2)-1 \leqslant k \leqslant (n/2)+1$, if and only if T is normal (cf. [9]). The symmetry of $W_k(T)$ with respect to the x-axis is known to imply, under some extra conditions, the unitary equivalence of T to a real normal matrix (cf. [10]). The equality of some of the sets $(1/k)W_k(T)$ also yields information on T and is related to Kippenhahn's conjecture on Hermitian pencils (cf. [8]). In general, the information encoded in the k-numerical ranges is more precise than in the classical one. On the other hand, although some of the above results can be generalized to the context of C-numerical ranges, the ones involving k-numerical ranges are more economical in terms of the data required. The present paper launches a study of the k-numerical ranges of some special-type operators on an infinite-dimensional space. Hopefully, the information obtained may shed more light on their general behavior.

As is well-known, to determine the numerical range, let alone the k-numerical ranges, of a certain operator is in general a difficult task. In the first paper on this subject ([12]), Toeplitz did this for operators on a two-dimensional space: if T is such an operator, then $W(T)$ is a closed circular or elliptic disc (or its degenerate form) depending on whether the eigenvalues of T are equal or otherwise. This was generalized in [13] to numerical ranges of the more general quadratic operators. In this paper, we will show that, up to a certain degree, analogous results hold for k-numerical ranges of quadratic operators. Since every quadratic operator is a linear function of a *square-zero operator* ($T^2 = 0$) or an *idempotent* ($T^2 = T$), we need only consider for these two subclasses. For the former, the k-numerical ranges are circular discs. Their radii as well as the condition for their closedness can be expressed in terms of the "singular numbers" of the operator under consideration. For the latter, the k-numerical ranges are, unfortunately, not elliptic discs in general. We will derive enough information of them, which is again based on the singular numbers of the idempotent under consideration, so as to give a rough description of their shape and location.

In Section 2 below, we derive the part of the theory which we need on singular numbers of general operators including the extremal property of sums of singular numbers originally due to von Neumann and Fan. This will be used in Sections 3 and 4 for the descriptions of k-numerical ranges of square-zero operators and idempotents, respectively.

Before starting, we list some basic properties of k-numerical ranges. As with the (classical) numerical range, the k-numerical range $W_k(T)$ of any operator T is a nonempty bounded convex subset of the complex plane (cf. [6], Problem 211). If the dimension of the underlying space is finite, say, n, then $W_k(T)$ is even compact and satisfies $W_k(T) = \operatorname{tr} T - W_{n-k}(T)$, $1 \leqslant k \leqslant n-1$, and $W_n(T) = \{\operatorname{tr} T\}$. The *$k$-numerical radius* of T, in symbols $w_k(T)$, is by definition the quantity $\sup\{|z| : z \in W_k(T)\}$.

For ease of exposition, in this paper we only consider operators on separable Hilbert spaces.

2. SINGULAR NUMBERS

The theory of singular numbers for finite matrices and compact operators on an infinite-dimensional space is well-established in the literature (cf. [7], p. 205 and [5], Chapter II). For our purposes, we need an extension of (a part of) the theory to the general operators, some of which can be found in Chapter II of Section 7 in [5].

For an operator T from H_1 to H_2, let $E_T(\,\cdot\,)$ denote the spectral measure of $|T| \equiv (T^*T)^{1/2}$ on H_1, and s_T the number $\inf\{t \geqslant 0 : \operatorname{rank} E_T((t,\infty)) < \infty\}$. The *singular numbers* of T are the eigenvalues (counting multiplicity and in decreasing order) $s_1(T) \geqslant s_2(T) \geqslant \cdots$ of $|T|$ in the open interval (s_T, ∞) with the provision that if $|T|$ has only finitely many eigenvalues in (s_T, ∞) then let the remaining $s_n(T)$'s be all equal to s_T. Such a definition is consistent with the one we usually encounter for finite matrices or compact operators. Let n_T $(\leqslant \infty)$ denote the sum of multiplicities of all the eigenvalues of $|T|$ in $[s_T, \infty)$. In particular, we have $s_1(T) = \|T\|$, $s_n(T) = s_n(T^*)$ for all $n \geqslant 1$, and, in the finite-dimensional case, $s_T = 0$, $n_T = \dim H_1$ and $s_n(T) = 0$ for all $n > \dim H_1$. The key result which we need for our later developments is the following extremal property for sums of singular numbers. It (partially) generalizes the corresponding result due to von Neumann and Fan (cf. [11], p. 514).

THEOREM 2.1. *If T is an operator from H_1 to H_2 and $p_1 \geqslant p_2 \geqslant \cdots \geqslant 0$ is a decreasing sequence of nonnegative numbers, then for any positive integer $n \leqslant \min\{\dim H_1, \dim H_2\}$, we have*

$$\sup \operatorname{Re} \sum_{j=1}^{n} p_j \langle Tx_j, y_j \rangle = \sup \left| \sum_{j=1}^{n} p_j \langle Tx_j, y_j \rangle \right| = \sum_{j=1}^{n} p_j s_j(T),$$

where both suprema are taken over orthonormal sets $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ in H_1 and H_2, respectively. Moreover, either of the suprema is attained if and only if the number of nonzero products $p_j s_j(T)$, $j = 1, \ldots, n$, is at most n_T.

This theorem will be proved after some preparatory work.

LEMMA 2.2. *If T is a positive semidefinite operator on the infinite-dimensional space H, then, for any $\varepsilon > 0$, T has a matrix representation*

$$\left[\begin{array}{ccc|c} t_1 & & 0 & \\ & t_2 & & * \\ 0 & & \ddots & \\ \hline & * & & * \end{array}\right],$$

where the t_n's are nonnegative real numbers such that $t_n = s_n(T)$ if $n \leqslant n_T$ and $|t_n - s_n(T)| < \varepsilon$ if otherwise.

Proof. If T has infinitely many eigenvalues in $[s_T, \infty)$, then obviously T has the representation

$$\left[\begin{array}{ccc|c} s_1(T) & & 0 & \\ & s_2(T) & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & * \end{array}\right].$$

On the other hand, if T has only finitely many eigenvalues, say, $s_1(T), \ldots, s_m(T)$ in $[s_T, \infty)$, then, letting T_1 be the restriction of T to the subspace $H_1 \equiv E_T([0, s_T])H$, we have $s_n(T) = s_T = \|T_1\| \in \sigma_e(T_1) \subseteq W_e(T_1)$ for all $n > m$, where $\sigma_e(T_1)$ and $W_e(T_1)$ are the essential spectrum and essential numerical range of T_1, respectively (cf. [4]). Hence T_1 can be represented as

$$\left[\begin{array}{ccc|c} t_{m+1} & & 0 & \\ & t_{m+2} & & * \\ 0 & & \ddots & \\ \hline & * & & * \end{array}\right],$$

where t_n's converge to $\|T_1\|$ (cf. [1], Lemma 2). Thus T has the representation

$$\begin{bmatrix} s_1(T) & & 0 \\ & \ddots & \\ 0 & & s_m(T) \end{bmatrix} \bigoplus \left[\begin{array}{ccc|c} t_{m+1} & & 0 & \\ & t_{m+2} & & * \\ 0 & & \ddots & \\ \hline & * & & * \end{array}\right],$$

which satisfies the required properties. ∎

The next lemma is a (partial) generalization of the usual variational characterization for eigenvalues of Hermitian matrices. It appeared in pp. 59–60 of [5] without proof.

LEMMA 2.3. *If T is a positive semidefinite operator on H, then*

$$\begin{aligned} s_n(T) &= \sup\{\inf\{\langle Tx, x\rangle : x \in K \text{ and } \|x\| = 1\} : \dim K = n\} \\ &= \inf\{\sup\{\langle Tx, x\rangle : x \in K \text{ and } \|x\| = 1\} : \dim H \ominus K = n-1\} \end{aligned}$$

for any $n \geqslant 1$.

Proof. We only prove the first equality for the case when T (on an infinite-dimensional H) has finitely many eigenvalues, say, $s_1(T), \ldots, s_m(T)$ in $[s_T, \infty)$ and leave the remaining cases to the reader. Let M denote the right-hand side of this equality.

Assume first that $n \leqslant m$. Let K_n be the subspace of H spanned by the eigenvectors of T corresponding to $s_1(T), \ldots, s_n(T)$. Since $W(T|K_n)$ equals the closed interval $[s_n(T), s_1(T)]$ (cf. [6], Problem 216), we have

$$s_n(T) = \min W(T|K_n) = \min\{\langle Tx, x\rangle : x \in K_n \text{ and } \|x\| = 1\},$$

which proves $s_n(T) \leqslant M$. For the other direction, we need check that $s_n(T) \geqslant \min W(P_K T|K)$ for any n-dimensional subspace K, where P_K denotes the (orthogonal) projection from H onto K. Let K_n be as above, $L = K \vee K_n$ and $T_1 = P_L T|L$. Then T_1 is of the form

$$T_1 = \left[\begin{array}{ccc|c} s_1(T) & & 0 & \\ & \ddots & & 0 \\ 0 & & s_n(T) & \\ \hline & 0 & & * \end{array}\right] \quad \text{on } L = K_n \oplus (L \ominus K_n).$$

By the validity of the asserted equality in the finite-dimensional case, we have $s_n(T_1) \geqslant \min W(P_K T_1|K)$. Since $s_n(T_1) = s_n(T)$ and $P_K T_1|K = P_K T|K$, we infer that

$$s_n(T) \geqslant \min W(P_K T|K) = \min\{\langle Tx, x\rangle : x \in K \text{ and } \|x\| = 1\},$$

and hence $s_n(T) \geqslant M$.

Next we assume that $n > m$. For any $\varepsilon > 0$, Lemma 2.2 implies that T has a matrix representation

$$\left[\begin{array}{ccc|c} t_1 & & 0 & \\ & \ddots & & * \\ 0 & & t_n & \\ \hline & * & & * \end{array}\right] \quad \text{on } H = K \oplus K^{\perp},$$

where the t_j's are nonnegative numbers satisfying $|t_j - s_j(T)| < \varepsilon$. Hence we have

$$\begin{aligned} s_n(T) - \varepsilon = s_T - \varepsilon < \min t_j &= \min W(P_k T|K) \\ &= \min\{\langle Tx, x\rangle : x \in K \text{ and } \|x\| = 1\}. \end{aligned}$$

It follows that $s_n(T) \leqslant M$. On the other hand, for any n-dimensional subspace K, let $L = K \ominus K_m$, where K_m is the m-dimensional subspace spanned by the eigenvectors corresponding to $s_1(T), \ldots, s_m(T)$. Then L is a nonzero subspace of $N = E_T([0, s_T])H$. Hence we have $w(P_L T|L) \leqslant \|T|N\| = s_T = s_n(T)$. But the containment $W(P_L T|L) \subseteq W(P_K T|K)$ implies that min $W(P_K T|K) \leqslant$ min $W(P_L T|L)$. Therefore,

$$s_n(T) \geqslant \min W(P_K T|K) = \min\{\langle Tx, x\rangle : x \in K \text{ and } \|x\| = 1\},$$

and hence $s_n(T) \geqslant M$ concluding the proof. ∎

COROLLARY 2.4. *If* $A = \begin{bmatrix} B & * \\ * & * \end{bmatrix}$ *is positive semidefinite, then* $s_n(A) \geqslant s_n(B)$ *for all* $n \geqslant 1$.

Proof. This follows easily from the variational characterization in Lemma 2.3. ∎

Note that with some more work we can even prove Corollary 2.4 without the positive-semidefiniteness assumption. However, for the present purpose the restricted form already suffices.

Proof of Theorem 2.1. It suffices to prove the assertions for a positive semidefinite operator T on a Hilbert space. Indeed, if $\dim\ker T \geqslant \dim\ker T^*$, then the polar decomposition of T yields $T = V|T|$ for some coisometry V from H_1 to H_2 (cf. [6], Problem 135). Hence for any orthonormal sets $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ in H_1 and H_2, respectively, we have

$$\sum_j p_j\langle Tx_j, y_j\rangle = \sum_j p_j\langle V|T|x_j, y_j\rangle = \sum_j p_j\langle |T|x_j, V^* y_j\rangle$$

with $\{V^* y_j\}_{j=1}^n$ an orthonormal set in H_1. On the other hand, any n-element orthonormal set in H_1 is of the form $\{V^* y_j\}_{j=1}^n$ as above. This shows that the suprema in question remain unaffected when T is replaced by $|T|$. Similar arguments apply in case $\dim\ker T \leqslant \dim\ker T^*$. Hence in the following we assume that T is positive semidefinite on H, $n \leqslant \dim H$ and $p_1,\ldots,p_n$ are all nonzero.

We first check that $\sup \operatorname{Re} \sum_{j=1}^{n} p_j\langle Tx_j, y_j\rangle \geqslant \sum_{j=1}^{n} p_j s_j(T)$. By Lemma 2.2, for any $\varepsilon > 0$, T has a representation

$$\left[\begin{array}{ccc|c} t_1 & & 0 & \\ & \ddots & & * \\ 0 & & t_n & \\ \hline & * & & * \end{array}\right] \quad \text{on } H = H_1 \oplus H_2,$$

where the t_j's satisfy $|t_j - s_j(T)| < \varepsilon$. If $\{x_j\}_{j=1}^n$ denotes the orthonormal basis of H_1 for which $\langle Tx_j, x_j\rangle = t_j$, $j = 1,\ldots,n$, then

$$\operatorname{Re} \sum_j p_j\langle Tx_j, x_j\rangle = \sum_j p_j t_j \geqslant \sum_j p_j s_j(T) - \sum_j p_j\varepsilon.$$

The asserted inequality follows immediately.

If $n \leqslant n_T$, then T has the representation

$$\left[\begin{array}{ccc|c} s_1(T) & & 0 & \\ & \ddots & & 0 \\ 0 & & s_n(T) & \\ \hline & 0 & & * \end{array}\right].$$

With $\{x_j\}_{j=1}^n$ as above, we have $\operatorname{Re} \sum_j p_j\langle Tx_j, x_j\rangle = \sum_j p_j s_j(T)$. This shows the attainment of the supremum when $n \leqslant n_T$.

To prove the inequality $\sup \left|\sum_{j=1}^{n} p_j\langle Tx_j, y_j\rangle\right| \leqslant \sum_{j=1}^{n} p_j s_j(T)$, let $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ be two orthonormal sets in H. If K denotes the subspace of H spanned by all these x's and y's and $T_1 = P_K T|K$, then

$$\left|\sum_j p_j\langle Tx_j, y_j\rangle\right| = \left|\sum_j p_j\langle T_1x_j, y_j\rangle\right| \leqslant \sum_j p_j s_j(T_1)$$

by the corresponding inequality in the finite-dimensional case (cf. [11], p. 514). Since $s_j(T_1) \leqslant s_j(T)$ for all j by Corollary 2.4, the asserted inequality holds. If $n \leqslant n_T$, the attainment of the supremum follows as in the last paragraph.

If the x_j's and y_j's are such that $\operatorname{Re} \sum_j p_j\langle Tx_j, y_j\rangle$ or $\left|\sum_j p_j\langle Tx_j, y_j\rangle\right|$ is equal to $\sum_j p_j s_j(T)$, then, by considering $\exp(\mathrm{e}^{\mathrm{i}\theta_j})x_j$ instead of x_j for some suitable real θ_j, we may assume that $\sum_j p_j\langle Tx_j, y_j\rangle = \sum_j p_j s_j(T)$. From the arguments in last paragraph, we obtain $\sum_j p_j s_j(T_1) = \sum_j p_j s_j(T)$. This implies, by Corollary 2.4, that $s_j(T_1) = s_j(T)$ for all j. Hence

$$T = \left[\begin{array}{ccc|c} s_1(T) & & 0 & \\ & \ddots & & * \\ 0 & & s_n(T) & \\ \hline & * & & * \end{array}\right].$$

Since

$$s_1(T)I - T = \left[\begin{array}{cccc|c} 0 & & & 0 & \\ & s_1(T)-s_2(T) & & & \\ & & \ddots & & * \\ 0 & & & s_1(T)-s_n(T) & \\ \hline & * & & & * \end{array}\right]$$

is positive semidefinite, we infer that the first row and first column of the above matrix are both zero. Successively considering the positive semidefinite $s_j(T)I - T_{(j)}$, where $T_{(j)}$ is the matrix obtained from that of T by deleting its first $j-1$ rows and columns, we conclude that T is of the form

$$\left[\begin{array}{ccc|c} s_1(T) & & 0 & \\ & \ddots & & 0 \\ 0 & & s_n(T) & \\ \hline & 0 & & * \end{array}\right].$$

Hence $s_1(T), \ldots, s_n(T)$ are eigenvalues of T and $n \leqslant n_T$ as asserted. This completes the proof. ∎

3. SQUARE-ZERO OPERATOR

The main result of this section is the following description of k-numerical ranges of a square-zero operator. For any real number t, $[t]$ denotes the largest integer which is less than or equal to t.

THEOREM 3.1. *Let T be a square-zero operator on H and k a positive integer.*

(i) *If $\dim H = n < \infty$, then $W_k(T)$ is the closed circular disc with center the origin and radius $(1/2)\sum\limits_{j=1}^{k} s_j(T)$ (respectively, $(1/2)\sum\limits_{j=1}^{n-k} s_j(T)$) for $1 \leqslant k \leqslant [n/2]$ (respectively, $[n/2] < k < n-1$).*

(ii) *If $\dim H = \infty$, then $W_k(T)$ is the (open or closed) circular disc with center the origin and radius $(1/2)\sum\limits_{j=1}^{k} s_j(T)$. $W_k(T)$ is open if and only if $k > n_T$.*

In particular, for $k = 1$ we have

COROLLARY 3.2. *If T is a square-zero operator, then $W(T)$ is the (open or closed) circular disc with center the origin and radius $\|T\|/2$. Moreover, $W(T)$ is closed if and only if T attains its norm, that is, $\|T\| = \|Tx\|$ for some unit vector x.*

This is a special case of Theorem 2.1 from [13].

COROLLARY 3.3. *If T is a square-zero operator on an infinite-dimensional space, then $W_k(T) \subseteq W_{k+1}(T)$ for all $k \geqslant 1$. Moreover, if $W_k(T)$ is open for some k, then so is every $W_l(T)$, $l > k$.*

Note that every square-zero operator T can be represented as

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \quad \text{on } H = H_1 \oplus H_2, \tag{3.1}$$

where H_1 and H_2 are either both of infinite dimension or both of finite dimension with $\dim H_1 = [(1/2)\dim H]$. Indeed, with respect to the decomposition $H = \ker T \oplus (\ker T)^\perp$, T has the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. Since $\overline{\operatorname{ran} T} \subseteq \ker T$, we have $\dim \ker T \geqslant \dim \overline{\operatorname{ran} T} = \dim \overline{\operatorname{ran} T^*} = \dim(\ker T)^\perp$ and hence we may cut down the size of $\ker T$ in the above representation to obtain (3.1).

We now begin the preparations for the proof of Theorem 3.1. The next lemma says that k-numerical ranges of a square-zero operator must be circular discs centered at the origin.

LEMMA 3.4. *Let T be a square-zero operator and k a positive integer. Then*

(i) *$W_k(T)$ is the (open or closed) circular disc centered at the origin with radius $w_k(T)$, and*

(ii) *$W_k(T)$ is closed if and only if $w_k(T)$ is attained, that is, $w_k(T) = |z|$ for some z in $W_k(T)$.*

Proof. It is easily seen from the representation (3.1) that T is unitarily equivalent to $e^{i\theta}T$ and hence $W_k(T) = e^{i\theta}W_k(T)$ for any real θ. Since $W_k(T)$ is convex, this implies the assertions in (i) and (ii). ∎

LEMMA 3.5. *If K is a finite-dimensional subspace of $H = H_1 \oplus H_2$, then there is an orthonormal basis $\{u_j\}$ of K with $u_j = v_j \oplus w_j$, where $v_j \in H_1$ and $w_j \in H_2$, such that $\{v_j\}$ and $\{w_j\}$ are orthogonal sets in H_1 and H_2, respectively.*

Proof. Let P be the (orthogonal) projection from H onto H_1. If

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \quad \text{on } H = K \oplus K^\perp,$$

then, since P_1 is Hermitian, there exists an orthonormal basis $\{u_j\}$ of K consisting of eigenvectors of P_1. Let $u_j = v_j \oplus w_j$, where $v_j \in H_1$ and $w_j \in H_2$. Then

$$\langle v_i, v_j \rangle = \langle v_i, u_j \rangle = \langle P u_i, u_j \rangle = \langle P_1 u_i, u_j \rangle = 0$$

for $i \neq j$, and also

$$\langle w_i, w_j \rangle = \langle v_i \oplus w_i, v_j \oplus w_j \rangle = \langle u_i, u_j \rangle = 0$$

for $i \neq j$. ∎

The next lemma gives, via Theorem 2.1, values of $w_k(T)$'s for a square-zero operator T.

LEMMA 3.6. *If* $T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ *on* $H = H_1 \oplus H_2$ *and* k *is a positive integer* $\leqslant \min\{\dim H_1, \dim H_2\}$, *then*

$$w_k(T) = \frac{1}{2} \sup \operatorname{Re} \sum_{j=1}^{k} \langle Ay_j, x_j \rangle,$$

where the supremum is taken over all orthonormal sets $\{x_j\}_{j=1}^k$ *and* $\{y_j\}_{j=1}^k$ *in* H_1 *and* H_2, *respectively. In this case,* $w_k(T)$ *is attained if and only if the above supremum is attained.*

Proof. For any λ in $W_k(T)$, $\lambda = \sum_{j=1}^{k} \langle Tz_j, z_j \rangle$ for some orthonormal vectors $z_1, \ldots, z_k$. Let K be the subspace spanned by the z_j's. Lemma 3.5 implies that there exists an orthonormal basis $\{u_j\}_{j=1}^k$ of K with $u_j = v_j \oplus w_j$, where $v_j \in H_1$, and $w_j \in H_2$, such that $\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0$ for $i \neq j$. Let $x_j = v_j / \|v_j\|$ for any nonzero v_j and enlarge such x_j's to an orthonormal subset $\{x_1, \ldots, x_k\}$ of H_1. Similarly, we obtain an orthonormal subset $\{y_1, \ldots, y_k\}$ of H_2. Replacing each x_j by a suitable $\exp(\mathrm{i}\theta_j)x_j$, we may assume that $\langle Ay_j, x_j \rangle \geqslant 0$ for all j. Hence

$$\begin{aligned} |\lambda| &= \Big| \sum_j \langle Tz_j, z_j \rangle \Big| = \Big| \sum_j \langle Tu_j, u_j \rangle \Big| = \Big| \sum_j \langle Aw_j, v_j \rangle \Big| \\ &\leqslant \sum_j |\langle Aw_j, v_j \rangle| = \sum_j \|w_j\| \cdot \|v_j\| \langle Ay_j, x_j \rangle \leqslant \frac{1}{2} \sum_j \langle Ay_j, x_j \rangle, \end{aligned}$$

where the last inequality follows from the fact that $\|v_j\|^2 + \|w_j\|^2 = \|u_j\|^2 = 1$. This shows that $w_k(T) \leqslant (1/2) \sup \operatorname{Re} \sum_j \langle Ay_j, x_j \rangle$.

To prove the reverse inequality, let $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$ be orthonormal sets in H_1 and H_2, respectively. Then the vectors $z_j = (1/\sqrt{2})(x_j \oplus y_j)$, $j = 1, \ldots, k$, are orthonormal in $H_1 \oplus H_2$ and hence

$$\frac{1}{2} \operatorname{Re} \sum_j \langle Ay_j, x_j \rangle = \operatorname{Re} \sum_j \langle Tz_j, z_j \rangle \in \operatorname{Re} W_k(T) \subseteq W_k(T)$$

by Lemma 3.4 (i). This shows that $(1/2) \sup \operatorname{Re} \sum_j \langle Ay_j, x_j \rangle \leqslant w_k(T)$.

The assertion on the attainment follows easily from the above proof. ∎

Proof of Theorem 3.1. Since it is easily seen that $s_k(T) = s_k(A)$ for all k, where A is the operator in the representation (3.1) of T, the assertions in the theorem follow from Lemmas 3.4, 3.6, Theorem 2.1 and, in the n-dimensional case, from the relation $W_k(T) = \operatorname{tr} T - W_{n-k}(T)$, $1 \leqslant k \leqslant n-1$. ∎

4. IDEMPOTENT OPERATOR

Although the numerical range of an idempotent operator is an elliptic disc (or its degenerate form), its k-numerical ranges are, as we will see later on, in general they are not. In this section, we will give a general description of their shape and location. A precise one does not seem possible.

Note that every idempotent T has the representation

$$\begin{bmatrix} 1 & A \\ 0 & 0 \end{bmatrix} \quad \text{on } H = H_1 \oplus H_2, \tag{4.1}$$

where $H_1 = \operatorname{ran} T$ and $H_2 = \ker T^*$.

LEMMA 4.1. *If T is an idempotent operator represented as in* (4.1)*, then*

$$s_n(T) = \begin{cases} (1 + s_n(A)^2)^{1/2} & \textit{if } 1 \leqslant n \leqslant p, \\ 1 & \textit{if } p < n \leqslant \operatorname{rank} T, \\ 0 & \textit{otherwise,} \end{cases}$$

where $p = \dim(\operatorname{ran} T \ominus (\operatorname{ran} T \cap \operatorname{ran} T^*))$.

Proof. Since $\ker T = \{-Ay \oplus y : y \in H_2\}$ and $\ker T^* = \{0 \oplus y : y \in H_2\}$, we have $\dim \ker T = \dim H_2 = \dim \ker T^*$. It follows easily from the polar decomposition that T^*T and TT^* are unitarily equivalent. Hence $|T|$ is unitarily equivalent to $(TT^*)^{1/2} = (1 + AA^*)^{1/2} \oplus 0$. Our assertion follows immediately. ∎

The next result gives an expression for the k-numerical range of idempotents as a union of (open or closed) circular discs. For this, we need some notations. For a complex number z_0, let $B(z_0, r) = \{z \in \mathcal{C} : |z - z_0| < r\}$ if $r > 0$ and $B(z_0, 0) = \{z_0\}$. For $a_1, \ldots, a_k \geqslant 0$, let

$$r_{(a_1,\ldots,a_k)}(t) = \max\left\{ \sum_{j=1}^{k} a_j (t_j(1 - t_j))^{1/2} : 0 \leqslant t_j \leqslant 1 \text{ for all } j \text{ and } \sum_{j=1}^{k} t_j = t \right\}$$

for $0 \leqslant t \leqslant k$. It is obvious that $r_{(a_1,\ldots,a_k)}$ is symmetric with respect to $k/2$.

PROPOSITION 4.2. *Let T be an idempotent operator with $\operatorname{rank} T = m$ and $\dim \ker T^* = n$ where $0 \leqslant m, n \leqslant \infty$. Then:*

$$W_k(T) = \begin{cases} \bigcup\limits_{0 \leqslant t \leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t))^- & \textit{if } 1 \leqslant k \leqslant m, n, n_T, \\ \bigcup\limits_{0 \leqslant t \leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t)) & \textit{if } n_T < k \leqslant m, n, \\ (k - n) + W_n(T) & \textit{if } n < k \leqslant m, \\ W_m(T) & \textit{if } m < k \leqslant n, \\ \operatorname{tr} T - W_{m+n-k}(T) & \textit{if } m, n \leqslant k \leqslant m + n, \end{cases}$$

where $a_j = (s_j(T)^2 - 1)^{1/2}$ for $j = 1, \ldots, k$ and $W_0(T)$ is understood to be $\{0\}$.

To prove this proposition, we need the following elementary lemma.

LEMMA 4.3. *Assume that* $a_1,\dots,a_k \geqslant 0$ *with* $a_{j_0} > 0$, $1 \leqslant j_0 \leqslant k$. *If for some* t, $0 < t < k$, *the maximum value* $r_{(a_1,\dots,a_k)}(t)$ *is assumed at* $t_1,\dots,t_k (0 \leqslant t_j \leqslant 1$ *for all* j *and* $\sum_j t_j = t)$, *then* $0 < t_{j_0} < 1$.

Proof. We may assume that $a_1 > 0, t_1 = 0$ and $t_2 > 0$. Let $f(t) = (t(1-t))^{1/2}$ for $0 \leqslant t \leqslant 1$. It suffices to check that for some small $a > 0$, the numbers

$$s_j = \begin{cases} a & \text{if } j = 1,\\ t_2 - a & \text{if } j = 2,\\ t_j, & \text{if } j = 3,\dots,k \end{cases}$$

satisfy $\sum_j a_j f(s_j) > \sum_j a_j f(t_j)$. Indeed, with the unspecified a we have

$$\begin{aligned}\sum_j a_j f(s_j) - \sum_j a_j f(t_j) &= a_1(f(a) - f(0)) + a_2(f(t_2 - a) - f(t_2))\\ &= a_1 a f'(b) + a_2(-a) f'(c)\end{aligned}$$

for some b and c satisfying $0 < b < a$ and $t_2 - a < c < t_2$ by the mean-value theorem. Since $f'' < 0$ on the open interval $(0,1)$, f' is strictly decreasing on $(0,1)$. Therefore,

$$a_1 a f'(b) - a_2 a f'(c) > a(a_1 f'(a) - a_2 f'(t_2 - a)).$$

Since $\lim_{a\to 0+}(a_1 f'(a) - a_2 f'(t_2 - a)) = +\infty$, we may choose a small $a > 0$ to obtain the asserted inequality $\sum_j a_j f(s_j) > \sum_j a_j f(t_j)$. Hence $t_1 > 0$. By symmetry, we also have $t_1 < 1$. ∎

Proof of Proposition 4.2. Let T be represented as in (4.1).

(a) Assume that $1 \leqslant k \leqslant m, n, n_T$. For any λ in $W_k(T), \lambda = \sum_{j=1}^{k} \langle T z_j, z_j\rangle$ for some orthonormal vectors $z_1,\dots,z_k$. We proceed as in the proof of Lemma 3.6. Let K be the subspace spanned by the z_j's. By Lemma 3.5, there exists an orthonormal basis $\{u_j\}_{j=1}^k$ of K with $u_j = v_j \oplus w_j$ ($v_j \in H_1$ and $w_j \in H_2$) such that $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle = 0$ for $i \neq j$. For nonzero v_j and w_j, let $x_j = v_j/\|v_j\|$ and $y_j = w_j/\|w_j\|$. We enlarge such vectors to orthonormal sets $\{x_j\}_{j=1}^k$ in H_1 and $\{y_j\}_{j=1}^k$ in H_2, and may assume that $\langle A y_j, x_j\rangle \geqslant 0$ for all j. Let $t = \sum_j \|v_j\|^2$. Then $0 \leqslant t \leqslant k$ and

$$\lambda = \sum_j \langle T u_j, u_j\rangle = \sum_j \langle v_j + A w_j, v_j\rangle = t + \sum_j \langle A w_j, v_j\rangle.$$

We may assume that the values of the product $\|v_j\| \cdot \|w_j\|$ are decreasing. Hence

$$\begin{aligned}|\lambda - t| &\leqslant \sum_j |\langle A w_j, v_j\rangle| = \sum_j \|v_j\|\cdot\|w_j\|\langle A y_j, x_j\rangle\\ &\leqslant \sum_j \|v_j\|\cdot\|w_j\| s_j(A) = \sum_j \|v_j\|(1 - \|v_j\|^2)^{1/2} a_j \qquad (4.2)\\ &\leqslant r_{(a_1,\dots,a_k)}(t)\end{aligned}$$

by Theorem 2.1 and Lemma 4.1. The inclusion $W_k(T) \subseteq \bigcup_{0\leqslant t\leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t))^-$ is proved.

For the converse, let $0 \leqslant t \leqslant k$ and $t_1, \ldots, t_k$ where $0 \leqslant t_j \leqslant 1$ for all j and $\sum_j t_j = t$, be such that $\sum_j a_j(t_j(1-t_j))^{1/2} = r_{(a_1,\ldots a_n)}(t)$. Our assumption $k \leqslant m, n, n_T$ implies that $k \leqslant n_A$. Hence, by the polar decomposition, there are orthonormal sets $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$ in H_1 and H_2, respectively, such that $\langle Ay_j, x_j\rangle = s_j(A)$ for all j. For any real θ, let $u_j = t_j^{1/2}x_j \oplus (\mathrm{e}^{\mathrm{i}\theta}(1-t_j)^{1/2}y_j)$. Then $\{u_j\}_{j=1}^k$ is an orthonormal set and

$$\begin{aligned}\sum_j \langle Tu_j, u_j\rangle &= \sum_j \langle t_j^{1/2}x_j + \mathrm{e}^{\mathrm{i}\theta}(1-t_j)^{1/2}Ay_j, t_j^{1/2}x_j\rangle\\ &= \sum_j t_j + \mathrm{e}^{\mathrm{i}\theta}\sum_j (t_j(1-t_j))^{1/2}s_j(A)\\ &= t + \mathrm{e}^{\mathrm{i}\theta} r_{(a_1,\ldots,a_k)}(t).\end{aligned}$$

This shows that the circle centered at t with radius $r_{(a_1,\ldots,a_k)}(t)$ is contained in $W_k(T)$. The convexity of the latter implies that $B(t, r_{(a_1,\ldots,a_k)}(t))^- \leqslant W_k(T)$. Hence in this case $W_k(T)$ is the union of such closed discs.

(b) Assume that $n_T < k \leqslant m, n$. Since in this case $n_T < \infty$, we can easily deduce that $m = n = \infty$, $n_T = n_A$ and $a_j = s_j(A) > 0$ for $j = 1, \ldots, k$. If $\lambda \in W_k(T)$, then, as in the first paragraph of (a), we construct u_j, v_j, w_j, x_j and y_j where $1 \leqslant j \leqslant k$, and let $t = \sum_j \|v_j\|^2$. If $\lambda = 0$ (respectively, $\lambda = k$), then λ is in $B(0,0)$ (respectively, $B(k,0)$). Hence we may assume that $\lambda \neq 0, k$. This will imply that $t \neq 0, k$. Indeed, if $t = 0$ (respectively, $t = k$), then $v_j = 0$ (respectively, $w_j = 0$) for all j and an easy computation yields that $\lambda = \sum_j \langle Tu_j, u_j\rangle = 0$ (respectively, $\lambda = k$), a contradiction. Thus, in particular, $r_{(a_1,\ldots,a_k)}(t) > 0$. Assume that $|\lambda - t| = r_{(a_1,\ldots,a_k)}(t)$. Then from (4.2) we obtain

$$\sum_j \|v_j\| \cdot \|w_j\| \langle Ay_j, x_j\rangle = \sum_j \|v_j\| \cdot \|w_j\| s_j(A) = r_{(a_1,\ldots,a_k)}(t).$$

By Theorem 2.1, this implies that the number of nonzero values of $\|v_j\|\cdot\|w_j\|s_j(A)$ is at most n_A. Since $k > n_T = n_A$, we have $\|v_k\|\cdot\|w_k\|s_k(A) = 0$ and thus $v_k = 0$ or $w_k = 0$. This means that the maximum value $r_{(a_1,\ldots,a_k)}(t)$ is assumed at $t_j = \|v_j\|^2$ where $1 \leqslant j \leqslant k$, with $t_k = 0$ or 1, which contradicts Lemma 4.3. Hence we have $|\lambda - t| < r_{(a_1,\ldots,a_k)}(t)$ and therefore $W_k(T) \subseteq \bigcup_{0\leqslant t\leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t))$.

To prove the converse inclusion, let $0 \leqslant t \leqslant k$ and $t_1, \ldots, t_k$ be as in the second paragraph of (a). Since $k > n_T = n_A$, by Lemma 2.2 and the polar decomposition there exist, for any $\varepsilon > 0$, orthonormal sets $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$ in H_1 and H_2, respectively, such that $\langle Ay_j, x_j\rangle = s_j(A)$ for $1 \leqslant j \leqslant n_A$ and $\langle Ay_j, x_j\rangle > s_j(A) - \varepsilon$ for $n_A < j \leqslant k$. For any real θ, let $u_j = t_j^{1/2}x_j \oplus (\mathrm{e}^{\mathrm{i}\theta}(1-$

$t_j)^{1/2}y_j)$. Then as in (a) we have

$$\sum_j \langle Tu_j, u_j\rangle = t + \mathrm{e}^{\mathrm{i}\theta}\sum_j (t_j(1-t_j))^{1/2}\langle Ay_j, x_j\rangle,$$

where

$$\sum_j (t_j(1-t_j))^{1/2}\langle Ay_j, x_j\rangle > r_{(a_1,\ldots,a_k)}(t) - \varepsilon \sum_{j=n_A+1}^{k} (t_j(1-t_j))^{1/2},$$

and deduce that $B(t, r_{(a_1,\ldots,a_k)}(t)) \subseteq W_k(T)$.

(c) Assume that $n < k \leqslant m$. For $\lambda \in W_k(T)$, let $u_j = v_j \oplus w_j$, $j = 1, \ldots, k$, be as in the first paragraph of (a). Since $k > n$, we may assume that $w_j = 0$ for $n < j \leqslant k$. Hence

$$\lambda = \sum_{j=1}^{k}\langle Tu_j, u_j\rangle = \sum_{j=1}^{n}\langle Tu_j, u_j\rangle + \sum_{j=n+1}^{k}\langle v_j, v_j\rangle = \sum_{j=1}^{n}\langle Tu_j, u_j\rangle + k - n.$$

This shows that $W_k(T) \subseteq (k-n) + W_n(T)$.

For the converse, let $\lambda \in W_n(T)$ and let $u_j = v_j \oplus w_j$, $j = 1, \ldots, n$, be as above. Choose unit vectors $v_{n+1}, \ldots, v_k$ so that $v_1, \ldots, v_k$ are mutually orthogonal. Let $w_{n+1} = \cdots = w_k = 0$. Then $u_j = v_j \oplus w_j$, $j = 1, \ldots, k$, form an orthonormal set and

$$\lambda = \sum_{j=1}^{n}\langle Tu_j, u_j\rangle = \sum_{j=1}^{k}\langle Tu_j, u_j\rangle - \sum_{j=n+1}^{k}\langle v_j, v_j\rangle = \sum_{j=1}^{k}\langle Tu_j, u_j\rangle - (k-n).$$

This shows that $W_n(T) \subseteq -(k-n) + W_k(T)$.

The assertion for $m < k \leqslant n$ can be proved analogously as in (c); that for $m, n \leqslant k \leqslant m+n$ follows from the symmetry property for k-numerical ranges of operators on a finite-dimensional space. ∎

Now we use Proposition 4.2 to determine when the k-numerical range of an idempotent operator is open (or closed). It turns out that the condition is exactly the same as that for square-zero operators. For the proof, we need the following lemma.

LEMMA 4.4. *If $a_1 > 0$ and $a_2, \ldots, a_k \geqslant 0$, then $r_{(a_1,\ldots,a_k)}(t) > t$ for small $t > 0$ and $r_{(a_1,\ldots,a_k)}(t) > k - t$ for large $t < k$.*

Proof. Let $f(t) = (t(1-t))^{1/2}$ for $0 \leqslant t \leqslant 1$. Since $\lim_{t\to 0+} f(t)/t = +\infty$, we have $a_1 f(t) > t$ for small $t > 0$ and hence $r_{(a_1,\ldots,a_k)}(t) > t$ for such t. The other assertion can be proved analogously. ∎

PROPOSITION 4.5. *The k-numerical range of an idempotent T is open if $k > n_T$ and closed if $k \leqslant n_T$.*

Proof. Let T be represented as in (4.1) and assume that $k > n_T$. Then, as noted in part (b) of the proof of Proposition 4.2, $n_T < \infty$ implies that $m = n = \infty$ and $A \neq 0$. Thus $a_1 = s_1(A) > 0$ and Lemma 4.4 implies that 0 (respectively, k) is in $B(t, r_{(a_1,\ldots,a_k)}(t))$ for small $t > 0$ (respectively, large $t < k$). By Proposition 4.2, we have $W_k(T) = \bigcup_{0<t<k} B(t, r_{(a_1,\ldots,a_k)}(t))$, which shows that $W_k(T)$ is open.

Now assume that $k \leqslant n_T$. If $k \leqslant m, n$, then $W_k(T) = \bigcup_{0\leqslant t\leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t))^-$ by Proposition 4.2. Let T' be an idempotent operator on a $2k$-dimensional space with singular numbers $s_1(T), \ldots, s_k(T), 0, \ldots, 0$. Note that if $s_k(T) = 0$, then $k \leqslant n_T$ implies that $s_T = 0$ and hence $m < \infty$ from which, since $k \leqslant m$, we deduce $s_k(T) \geqslant 1$, a contradiction. Hence $s_j(T) > 0$ for $j = 1, \ldots, k$. Then $k = \operatorname{rank} T' = \dim\ker T'^* $ and $k < n_{T'} = 2k$, and therefore $W_k(T') = \bigcup_{0\leqslant t\leqslant k} B(t, r_{(a_1,\ldots,a_k)}(t))^- = W_k(T)$. This shows the closedness of $W_k(T)$. On the other hand, if $k > m$ (respectively, $k > n$), then, assuming $\dim H = \infty$, we have $m < \infty$ and $n = n_T = \infty$ (respectively, $n < \infty$ and $m = n_T = \infty$). Hence $W_m(T)$ (respectively, $W_n(T)$) is closed by what we just proved. Thus $W_k(T) = W_m(T)$ (respectively, $W_k(T) = (k - n) + W_n(T)$) is also closed. ∎

We now try to gain more information concerning the shape and location of k-numerical ranges of idempotents through Proposition 4.2. This is achieved via the following several lemmas.

LEMMA 4.6. (i) *If $a_1, \ldots, a_k > 0$ and $a_{k+1} = \cdots = a_l = 0$, then*

$$r_{(a_1,\ldots,a_l)}(t) = \begin{cases} r_{(a_1,\ldots,a_k)}(t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}k, \\ \frac{1}{2}\sum_{j=1}^{k} a_j & \text{for } \frac{1}{2}k \leqslant t \leqslant l - \frac{1}{2}k, \\ r_{(a_1,\ldots,a_k)}(l - t) & \text{for } l - \frac{1}{2}k \leqslant t \leqslant l. \end{cases}$$

(ii) *If $a_1, \ldots, a_k > 0$, then $r_{(a_1,\ldots,a_k)}$ is a positive continuous function on $[0, k]$, which is strictly positive on $(0, k)$, strictly increasing on $[0, k/2]$ and symmetric with respect to $k/2$.*

Proof. (i) If $0 \leqslant t \leqslant k/2$, then we need check that $r_{(a_1,\ldots,a_l)}(t) \leqslant r_{(a_1,\ldots,a_k)}(t)$. For this purpose, let $t_1, \ldots, t_l$ be such that $0 \leqslant t_j \leqslant 1$ and $\sum_{j=1}^{l} t_j = t$. We may assume that there is some p, $0 \leqslant p \leqslant k$, such that $0 \leqslant t_j \leqslant 1/2$ for $1 \leqslant j \leqslant p$ and $1/2 < t_j \leqslant 1$ for $p < j \leqslant k$. Since

$$\sum_{j=k+1}^{l} t_j = t - \sum_{j=1}^{k} t_j \leqslant \frac{1}{2}k - \sum_{j=1}^{k} t_j = \sum_{j=1}^{k}\Big(\frac{1}{2} - t_j\Big) \leqslant \sum_{j=1}^{p}\Big(\frac{1}{2} - t_j\Big),$$

we can choose $s_1, \ldots, s_p$ with the property that $t_j \leqslant s_j \leqslant 1/2$ for each j and $\sum_{j=1}^{p}(s_j - t_j) = \sum_{j=k+1}^{l} t_j$. Let

$$s_j = \begin{cases} t_j & \text{if } p < j \leqslant k, \\ 0 & \text{if } k < j \leqslant l. \end{cases}$$

Then $0 \leqslant s_j \leqslant 1$ for all j, $\sum_{j=1}^{l} s_j = \sum_{j=1}^{l} t_j = t$ and

$$\sum_{j=1}^{k} a_j (s_j(1-s_j))^{1/2} \geqslant \sum_{j=1}^{l} a_j (t_j(1-t_j))^{1/2}.$$

This shows that $r_{(a_1,\dots,a_l)}(t) \leqslant r_{(a_1,\dots,a_k)}(t)$ as asserted.

We obviously have $r_{(a_1,\dots,a_l)}(t) \leqslant (1/2) \sum_{j=1}^{k} a_j$ for all t. To prove the reverse inequality for $k/2 \leqslant t \leqslant l-(k/2)$, let $t_1, \dots, t_l$ be such that $t_j = 1/2$ for $1 \leqslant j \leqslant k$, $0 \leqslant t_j \leqslant 1$ for $k < j \leqslant l$ and $\sum_{j=1}^{l} t_j = t$. Then

$$\sum_{j=1}^{l} a_j (t_j(1-t_j))^{1/2} = \frac{1}{2} \sum_{j=1}^{k} a_j$$

and our assertion follows.

If $l-(k/2) \leqslant t \leqslant l$, then

$$r_{(a_1,\dots,a_l)}(t) = r_{(a_1,\dots,a_l)}(l-t) = r_{(a_1,\dots,a_k)}(l-t)$$

by what we proved above. This completes the proof of (i).

(ii) To show the strict increase of $r_{(a_1,\dots,a_k)}$ on $[0, k/2]$, we argue as in (i). Let $t < s$ be in $[0, k/2]$ and $t_1, \dots, t_k$ be such that $0 \leqslant t_j \leqslant 1$ for all j, $\sum_j t_j = t$ and $\sum_j a_j (t_j(1-t_j))^{1/2} = r_{(a_1,\dots,a_k)}(t)$. We may assume that there is some p, $0 \leqslant p \leqslant k$, such that $0 \leqslant t_j \leqslant 1/2$ for $1 \leqslant j \leqslant p$ and $1/2 < t_j \leqslant 1$ for $p < j \leqslant k$. Since

$$s - t = s - \sum_{j=1}^{k} t_j \leqslant \frac{1}{2}k - \sum_{j=1}^{k} t_j = \sum_{j=1}^{k} \left(\frac{1}{2} - t_j\right) \leqslant \sum_{j=1}^{p} \left(\frac{1}{2} - t_j\right),$$

we can choose $s_1, \dots, s_p$ with the property that $t_j \leqslant s_j \leqslant 1/2$ for each j and $\sum_{j=1}^{p} (s_j - t_j) = s - t$. Note that $t < s$ implies that $t_j < s_j$ for some j. Let $s_j = t_j$ for $p < j \leqslant k$. Then $0 \leqslant s_j \leqslant 1$ for all j, $\sum_j s_j = s$ and

$$\sum_j a_j (s_j(1-s_j))^{1/2} > \sum_j a_j (t_j(1-t_j))^{1/2}.$$

It follows that $r_{(a_1,\dots,a_k)}(t) < r_{(a_1,\dots,a_k)}(s)$.

Other assertions of $r_{(a_1,\dots,a_k)}$ are easy to verify. ∎

Lemma 4.7. *If Λ is a closed convex subset of the complex plane which is the union of closed (nondegenerate) circular discs, then the boundary of Λ is a differentiable curve.*

Proof. Let p be a point on the boundary of Λ, and let L be any straight line passing p with Λ in one side of L. If $B \subseteq \Lambda$ is any closed circular disc containing p, then p is on the boundary of B and L is tangent to B at p. This shows the uniqueness of the line L and hence the differentiability of the boundary of Λ at p. ∎

Together with Proposition 4.2, this lemma implies that the boundary of the k-numerical range of idempotents must be differentiable. The next lemma improves on this; it shows, among other things, that the boundary is even continuously differentiable.

LEMMA 4.8. *For $s > 0$, let $\Lambda = \bigcup_{0 \leqslant t \leqslant s} B(t, r(t))^-$ be a compact convex subset of the complex plane, where r is a continuous function on $[0, s]$, increasing (respectively, strictly increasing) on $[0, s/2]$, strictly positive on $(0, s/2]$ with $r(0) = 0$ and $r(t) > t$ for small $t > 0$, and symmetric with respect to $s/2$. Then the boundary of Λ is a continuously differentiable curve. It defines a positive function f on $[t_1, t_2]$, where $t_1 = \inf(\Lambda \cap \mathbb{R})$ and $t_2 = \sup(\Lambda \cap \mathbb{R})$, which is increasing (respectively, strictly increasing) on $[t_1, s/2]$ strictly positive on $[t_1, s/2]$ with $f(t_1) = 0$ and symmetric with respect to $s/2$.*

Proof. Let f be the function on $[t_1, t_2]$ whose graph is the boundary of Λ in the upper-half plane. Since Λ is symmetric with respect to the x-axis and the vertical line $x = s/2$, we only need check, in case r is strictly increasing on $[0, s/2]$, that f is strictly increasing and continuously differentiable on $[t_1, s/2]$. The proof for the case of r increasing is similar.

By Lemma 4.7 f is differentiable on $[t_1, t_2]$. Hence for any t in (t_1, t_2), the graph of f has a unique tangent line passing through $t + \mathrm{i}f(t)$. Let $\alpha(t)$ denote the intersection of the x-axis and the normal line through $t + \mathrm{i}f(t)$. Note that $B(\alpha(t), r(\alpha(t)))^-$ is the unique disc containing the point $t + \mathrm{i}f(t)$.

We first show that α is increasing on $(t_1, s/2]$. Let $t_3 < t_4$ be in $(t_1, s/2]$. Since $t_3 + \mathrm{i}f(t_3)$ is on the boundary of Λ, it cannot be in the open disc $B(\alpha(t_4), r(\alpha(t_4)))$. Hence

$$|t_3 + \mathrm{i}f(t_3) - \alpha(t_4)| \geqslant r(\alpha(t_4)) = |t_4 + \mathrm{i}f(t_4) - \alpha(t_4)|$$

or, equivalently,

$$(t_3 - \alpha(t_4))^2 + f(t_3)^2 \geqslant (t_4 - \alpha(t_4))^2 + f(t_4)^2.$$

From this, we have

$$f(t_4)^2 \leqslant (t_3 - \alpha(t_4))^2 + f(t_3)^2 - (t_4 - \alpha(t_4))^2,$$

and hence

$$\begin{aligned} r(\alpha(t_3))^2 - |t_4 + \mathrm{i}f(t_4) - \alpha(t_3)|^2 &= |t_3 + \mathrm{i}f(t_3) - \alpha(t_3)|^2 - (t_4 - \alpha(t_3))^2 - f(t_4)^2 \\ &\geqslant (t_3 - \alpha(t_3))^2 + f(t_3)^2 - (t_4 - \alpha(t_3))^2 - (t_3 - \alpha(t_4))^2 - f(t_3)^2 + (t_4 - \alpha(t_4))^2 \\ &= 2(t_4 - t_3)(\alpha(t_3) - \alpha(t_4)). \end{aligned}$$

Therefore, if $\alpha(t_3) > \alpha(t_4)$, then $r(\alpha(t_3)) > |t_4 + \mathrm{i}f(t_4) - \alpha(t_3)|$, which will imply that $t_4 + \mathrm{i}f(t_4)$ is in $B(\alpha(t_3), r(\alpha(t_3)))$ and hence $t_4 + \mathrm{i}f(t_4)$ is in the interior of Λ, a contradiction. Thus we must have $\alpha(t_3) \leqslant \alpha(t_4)$ as asserted. Define $\alpha(t_1) = \lim_{t \to t_1+} \alpha(t)$. Then α is increasing on $[t_1, s/2]$.

To show that f is strictly increasing on $[t_1, s/2]$, let $t_3 < t_4$ in $[t_1, s/2]$ and $t_0 = t_4 - t_3 + \alpha(t_3)$. We have

$$|(t_4 + \mathrm{i}f(t_3)) - t_0| = |(t_3 + \mathrm{i}f(t_3)) - \alpha(t_3)| = r(\alpha(t_3)) < r(t_0)$$

if $t_0 \leqslant s/2$. Hence $t_4 + \mathrm{i}f(t_3)$ is in $B(t_0, r(t_0))$. Since the latter is contained in the interior of Λ, this implies that $f(t_3) < f(t_4)$. On the other hand, if $t_0 > s/2$, then

$$\left|(t_4 + \mathrm{i}f(t_3)) - \frac{1}{2}s\right| < |(t_4 + \mathrm{i}f(t_3)) - t_0| = r(\alpha(t_3)) \leqslant r\Big(\frac{1}{2}s\Big)$$

since $\alpha(t_3) \leqslant \alpha(s/2) = s/2$ and r is strictly increasing. Hence $t_4 + \mathrm{i}f(t_3)$ is in $B(s/2, r(s/2))$. As before, this implies that $f(t_3) < f(t_4)$.

Since $f'(t) = (\alpha(t) - t)/f(t)$ for t in $[t_1, s/2]$, the continuous differentiability of f' will follow from the continuity of α. To prove the latter, let t_3 in $[t_1, s/2]$ and $a = \lim\limits_{t\to t_3+} \alpha(t)$. We have

$$|t_3 + \mathrm{i}f(t_3) - a| = \lim_{t\to t_3+} |t + \mathrm{i}f(t) - \alpha(t)| = \lim_{t\to t_j+} r(\alpha(t)) = r(a),$$

which shows that $t_3+\mathrm{i}f(t_3)$ is on the boundary of $B(a, r(a))^-$. By the uniqueness of such a disc, we infer that $\alpha(t_3) = a$. Similarly, we can also prove $\lim\limits_{t\to t_3^-} \alpha(t) = \alpha(t_3)$. This shows the continuity of α and hence the continuous differentiability of f. ∎

The next lemma gives the smallest rectangle, with sides parallel to the x- and y-axis, which contains the k-numerical range of any operator. The rectangle is described in terms of quantities associated with Hermitian operators just as the singular numbers with positive operators. Indeed, for a Hermitian T, let $r_T = \inf\{t \in \sigma(T) : \operatorname{rank} E_T((t,\infty)) < \infty\}$ and $r_1^+(T) \geqslant r_2^+(T) \geqslant \cdots$ be the eigenvalues (counting multiplicity and in decreasing order) of T in (r_T, ∞). Here, as before, if T has only finitely many eigenvalues in (r_T, ∞), then let the remaining $r_n^+(T)$'s be all equal to r_T. Let $r_n^-(T) = -r_n^+(-T)$ for $n \geqslant 1$.

Lemma 4.9. *For any operator T and $k \geqslant 1$, $W_k(T)$ is contained in the rectangle formed by the lines $x = \sum\limits_{j=1}^{k} r_j^+(\operatorname{Re} T)$, $x = \sum\limits_{j=1}^{k} r_j^-(\operatorname{Re} T)$, $y = \sum\limits_{j=1}^{k} r_j^+(\operatorname{Im} T)$ and $y = \sum\limits_{j=1}^{k} r_j^-(\operatorname{Im} T)$, where $\operatorname{Re} T = (T + T^*)/2$ and $\operatorname{Im} T = (T - T^*)/(2\mathrm{i})$.*

Proof. From the definition, we easily derive that $\operatorname{Re} W_k(T) = W_k(\operatorname{Re} T)$. Since the quantity $\sup W_k(\operatorname{Re} T)$ can be shown (as in Theorem 2.1) to be $\sum\limits_{j=1}^{k} r_k^+(\operatorname{Re} T)$, $W_k(T)$ is contained in the left-half plane determined by the line $x = \sum\limits_{j=1}^{k} r_j^+(\operatorname{Re} T)$. Other sides of the rectangle can be obtained analogously. ∎

If T is an idempotent on H, then a simple computation yields that

$$r_n^+(\operatorname{Re} T) = \begin{cases} \frac{1}{2}(1 + s_n(T)) & \text{if } 1 \leqslant n \leqslant p, \\ 1 & \text{if } p < n \leqslant m, \\ 0 & \text{if } m < n \leqslant l - p, \\ \frac{1}{2}(1 - s_{l-n+1}(T)) & \text{if } l - p < n \leqslant l, \end{cases}$$

where $l = \dim H$, $m = \operatorname{rank} T$ and $p = \dim(\operatorname{ran} T \ominus (\operatorname{ran} T \cap \operatorname{ran} T^*))$. There are similar expressions relating $r_n^-(\operatorname{Re} T), r_n^+(\operatorname{Im} T)$ and $r_n^-(\operatorname{Im} T)$ to $s_n(T)$.

Now we are ready for the main result of this section; it is a consequence of the previous lemmas and propositions.

THEOREM 4.10. *Let T be an idempotent operator, and let $m = \operatorname{rank} T$, $n = \dim\ker T^*$ and $p = \dim(\operatorname{ran} T \ominus (\operatorname{ran} T \cap \operatorname{ran} T^*))$, $0 \leqslant m, n, p \leqslant \infty$.*

(i) *For $k \geqslant 1$, $W_k(T)$ is an (open or closed) convex region in the complex plane with continuously differentiable boundary. It is open if $k > n_T$ and closed if $k \leqslant n_T$.*

(ii) *If $1 \leqslant k \leqslant m, n$, then $W_k(T)$ is contained in the rectangular region $[t_1, t_2] \times [-s, s]$, where $t_1 = \Big(k - \sum_{j=1}^{k} s_j(T)\Big)/2$, $t_2 = \Big(k + \sum_{j=1}^{k} s_j(T)\Big)/2$ and $s = (1/2)\sum_{j=1}^{q} (s_j(T)^2 - 1)^{1/2}$, $q = \min\{k, p\}$, and is symmetric with respect to the x-axis and the line $x = k/2$.*

(iii) *If $1 \leqslant k \leqslant p$, then the part of the boundary of $W_k(T)$ in the upper-half plane is strictly increasing on $[t_1, k/2]$ and strictly decreasing on $[k/2, t_2]$. If $p < k \leqslant m, n$, then it is strictly increasing on $[t_1, p/2]$, constant with value s on $[p/2, k - (p/2)]$ and strictly decressing on $[k - (p/2), t_2]$.*

(iv) *If $n < k \leqslant m$ (respectively, $m < k \leqslant n$), then $W_k(T) = (k - n) + W_n(T)$ (respectively, $W_k(T) = W_m(T)$). If $m, n < k \leqslant m + n$, then $W_k(T) = \operatorname{tr} T - W_{m+n-k}(T)$. ($W_0(T)$ is understood to be $\{0\}$.)*

COROLLARY 4.11. *Let T be an idempotent operator and $k \geqslant 1$. Then the boundary of $W_k(T)$ contains a horizontal (nondegenerate) line segment if and only if $p < k < \dim H - p$ and $m, n > p$ in case $\dim H < \infty$, and $k, m, n > p$ in case $\dim H = \infty$, where m, n and p are as in Theorem 4.10.*

We remark that in the preceding corollary even if $W_k(T)$ contains no line segment, it may still not be an elliptic disc. In fact, in a recent work ([3]) the first author was able to characterize those idempotents T (on a finite-dimensional space) for which $W_k(T)$, $1 \leqslant k \leqslant p$, is: they are exactly those T with the first k singular numbers all equal.

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