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SHIFTS AS MODELS FOR SPECTRAL DECOMPOSABILITY ON HILBERT SPACE

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ABSTRACT. Let U be a bounded invertible linear mapping of the Hilbert space $\mathfrak{K}$ onto itself. Let $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$, and denote by $\ell^2(\mathcal{W})$ the corresponding weighted Hilbert space. Our main result shows that the right bilateral shift $\mathcal{R}$ on $\ell^2(\mathcal{W})$ serves as a model for spectral decomposability of U. Further aspects of this for multiplier transference are treated, and lead to an example wherein the discrete Hilbert kernel defines a bounded convolution operator on $\ell^2(\mathcal{W}^{(0)})$, but the analogues of the classical Marcinkiewicz Multiplier Theorem and the classical Littlewood-Paley Theorem fail to hold for $\ell^2(\mathcal{W}^{(0)})$.

KEYWORDS: *Operator, Hilbert space, spectral family, operator-valued weight sequence, shift.*

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1. INTRODUCTION AND NOTATION

Given a Banach space $\mathfrak{X}$, we shall denote by $\mathfrak{B}(\mathfrak{X})$ the algebra of all continuous linear mappings of $\mathfrak{X}$ into $\mathfrak{X}$. The identity operator on $\mathfrak{X}$ will be symbolized by I. Suppose that $V \in \mathfrak{B}(\mathfrak{X})$ is invertible, and that $q : \mathbb{T} \to \mathbb{C}$ is a trigonometric polynomial. Thus, $q(z) \equiv \sum_{j=-\infty}^{\infty} \widehat{q}(j) z^j$, where $\widehat{q}(j) = 0$ for all but finitely many values of j, and we define $q(V)$ by writing

$$q(V) = \sum_{j=-\infty}^{\infty} \widehat{q}(j) V^j.$$

The notion of *trigonometrically well-bounded operator*, introduced in [2], will play a central role in our considerations below because of the intimate connection between trigonometrically well-bounded operations and spectral decomposability, which we outline here, leaving precise details for the review of spectral theory included in Section 3. A continuous linear mapping $\mathcal{T}$ of the Banach space $\mathfrak{X}$ into itself is trigonometrically well-bounded if and only if $\mathcal{T}$ has a "unitary-like" spectral representation

$$\mathcal{T} = \int_{0^-}^{2\pi} \mathrm{e}^{\mathrm{i}t}\, \mathrm{d}E(t), \tag{1.1}$$

where $E(\,\cdot\,) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ is a unique idempotent-valued function (possessing various properties weaker than those associated with a countably additive Borel spectral measure), and the integral in (1.1) is a Riemann-Stieltjes integral existing in the strong operator topology. As shown in Corollary 2.17 of [2], when $\mathfrak{X}$ is reflexive an operator $V \in \mathfrak{B}(\mathfrak{X})$ is trigonometrically well-bounded if and only if V is invertible, and there is a constant γ such that for every trigonometric polynomial q,

$$\|q(V)\| \leqslant \gamma \|q\|_{BV(\mathbb{T})}, \tag{1.2}$$

where $\|q\|_{BV(\mathbb{T})}$ designates the norm of q in the Banach algebra $BV(\mathbb{T})$:

$$\|q\|_{BV(\mathbb{T})} = |q(1)| + \operatorname{var}(q, \mathbb{T}).$$

Trigonometrically well-bounded operators will also be useful below, because they are closely related to discrete Hilbert averages (see [3], Theorem (2.4)), and constitute a ready vehicle for the transference of Fourier multipliers (see, e.g., [1], [6], [7], and [8]).

Throughout all that follows, $\mathfrak{K}$ will be an arbitrary Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. An *operator-valued weight sequence* on $\mathfrak{K}$ will be a bilateral sequence $\mathfrak{W} = \{W_k\}_{k=-\infty}^{\infty} \subseteq \mathfrak{B}(\mathfrak{K})$ such that for each $k \in \mathbb{Z}$, W_k is a positive, invertible, self-adjoint operator. We associate with $\mathfrak{W}$ the weighted Hilbert space $\ell^2(\mathfrak{W})$ consisting of all sequences $x = \{x_k\}_{k=-\infty}^{\infty} \subseteq \mathfrak{K}$ such that

$$\sum_{k=-\infty}^{\infty} \langle W_k x_k, x_k \rangle < \infty,$$

and furnished with the inner product $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ specified by

$$\langle\!\langle x, y \rangle\!\rangle = \sum_{k=-\infty}^{\infty} \langle W_k x_k, y_k \rangle.$$

Thus, $\ell^2(\mathfrak{W})$ is a generalization to non-commutative analysis of the ℓ^2-spaces defined by scalar-valued weight sequences in the special case where $\mathfrak{K} = \mathbb{C}$. (For the continuous variable generalization from scalar-valued weights to operator-valued weights, see [19], [20].) We shall be especially concerned with avenues for interplay between $\ell^2(\mathfrak{W})$ and the discrete Hilbert kernel $h : \mathbb{Z} \to R$, specified by

$$h(k) = \begin{cases} k^{-1} & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$$

The formal operator of convolution by h on $\ell^2(\mathfrak{W})$ will be referred to as the *discrete Hilbert transform*, and will be symbolized by D. If h defines a bounded convolution operator mapping $\ell^2(\mathfrak{W})$ into $\ell^2(\mathfrak{W})$, we shall say that $\mathfrak{W}$ possesses the *Treil-Volberg Property* (see [9], [19], [20] for background facts-in particular, the reasoning in [19], [20] shows that if $\mathfrak{W}$ has the Treil-Volberg Property, then $\mathfrak{W}$ satisfies an operator-valued analogue of the A_2 weight condition).

Suppose now that $U \in \mathfrak{B}(\mathfrak{K})$ is an invertible operator, let $\mathcal{W}$ be the operator-valued weight sequence on $\mathfrak{K}$ given by $\mathcal{W} = \{(U^k)^* U^k\}_{k=-\infty}^{\infty}$, and let $\mathcal{R}$ be the right bilateral shift on $\ell^2(\mathcal{W})$. Our main result (Theorem 2.3) asserts that for every trigonometric polynomial q,

$$\|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \sup_{z\in\mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})}. \tag{1.3}$$

In view of the characterization in (1.2), it follows directly from (1.3) that the right shift $\mathcal{R}$ is trigonometrically well-bounded on $\ell^2(\mathcal{W})$ if and only if U is trigonometrically well-bounded on $\mathfrak{K}$ (Corollary 2.4). In this sense, $\mathcal{R}$ serves as a model for the spectral decomposability properties and the transference properties of U. In Section 4 we take up the multiplier theory associated with operator-valued weight sequences, including some transference consequences of the approach using $\mathcal{R}$ as a model (Theorems 4.8 and 4.11, Corollary 4.12). In Sections 5 and 6, we study the interactions of the classical Marcinkiewicz multiplier functions with $\ell^2(\mathcal{W})$, showing in Section 6 that there is an operator-valued weight sequence $\mathfrak{W}$ on Hilbert space such that $\mathfrak{W}$ enjoys the Treil-Volberg Property, but the analogues of the classical Littlewood-Paley Theorem and the classical Marcinkiewicz Multiplier Theorem fail to hold for $\ell^2(\mathfrak{W})$. This situation is in marked contrast to the well-known affirmative outcome for scalar-valued A_2 weights ([15], [16]).

2. SHIFTS AS MODELS FOR ESTIMATING OPERATOR NORMS

We shall denote by $\ell_0(\mathfrak{K})$ the space of all finitely supported $\mathfrak{K}$-valued functions defined on $\mathbb{Z}$. For the trivial operator-valued weight sequence $\mathfrak{W} = \{W_k\}_{k=-\infty}^{\infty}$ specified by $W_k = I$ for all $k \in \mathbb{Z}$, the corresponding space $\ell^2(\mathfrak{W})$ will be written as $\ell^2(\mathfrak{K})$. Henceforth normalized Haar measure on $\mathbb{T}$ will be denoted by λ, and the Fejér kernel for $\mathbb{T}$ will be symbolized by $\{\kappa_n\}_{n=0}^{\infty}$. Given an arbitrary operator-valued weight sequence $\mathfrak{W} = \{W_k\}_{k=-\infty}^{\infty}$ on $\mathfrak{K}$, for each $z \in \mathbb{T}$ we shall signify by $\mathcal{M}_z$ the linear isometry of $\ell^2(\mathfrak{W})$ onto $\ell^2(\mathfrak{W})$ defined by

$$\mathcal{M}_z x = \{z^k x_k\}_{k=-\infty}^{\infty}, \quad \text{for each } x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathfrak{W}). \tag{2.1}$$

We also associate with $\mathfrak{W}$ the operator-valued weight sequence $\widetilde{\mathfrak{W}}$ defined by putting $\widetilde{\mathfrak{W}} = \{W_{-k}\}_{k=-\infty}^{\infty}$. Notice that the mapping Δ which sends each $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathfrak{W})$ to the sequence $\Delta x = \{x_{-k}\}_{k=-\infty}^{\infty}$ is a linear isometry of $\ell^2(\mathfrak{W})$ onto $\ell^2(\widetilde{\mathfrak{W}})$. Since the discrete Hilbert kernel is an odd function, it is readily seen with the aid of Δ that $\mathfrak{W}$ has the Treil-Volberg property if and only if $\widetilde{\mathfrak{W}}$ has the Treil-Volberg property, and in this case

$$\|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} = \|D\|_{\mathfrak{B}(\ell^2(\tilde{\mathfrak{W}}))}. \tag{2.2}$$

Definition 2.1. If $U \in \mathfrak{B}(\mathfrak{K})$ is an invertible operator, with corresponding operator-valued weight sequence $\mathcal{W} = \{(U^k)^* U^k\}_{k=-\infty}^{\infty}$, let Γ be the *invertible linear isometry of* $\ell^2(\mathfrak{K})$ onto $\ell^2(\mathcal{W})$ specified by

$$\Gamma x = \{U^{-k} x_k\}_{k=-\infty}^{\infty} \quad \text{for all } x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathfrak{K}).$$

Before stating our main result, it will be convenient to dispose of the following technical item regarding the boundedness properties of shifts.

Lemma 2.2. *Let $U \in \mathfrak{B}(\mathfrak{K})$ be an invertible operator with corresponding operator-valued weight sequence $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$. Let $\mathcal{R}$ and $\mathcal{L}$ be, respectively, the right bilateral shift and the left bilateral shift defined for each $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathcal{W})$ by*

$$\mathcal{R}x = \{x_{j-1}\}_{j=-\infty}^{\infty}, \quad \mathcal{L}x = \{x_{j+1}\}_{j=-\infty}^{\infty}.$$

Then $\mathcal{R}$ and $\mathcal{L}$ are injective bounded linear mappings of $\ell^2(\mathcal{W})$ onto $\ell^2(\mathcal{W})$ such that $\|\mathcal{R}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \|U\|_{\mathfrak{B}(\mathfrak{K})}$, $\|\mathcal{L}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \|U^{-1}\|_{\mathfrak{B}(\mathfrak{K})}$, and $\mathcal{L} = \mathcal{R}^{-1}$.

Proof. For each $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathcal{W})$, we have:

$$\|\mathcal{R}x\|_{\ell^2(\mathcal{W})}^2 = \sum_{j=-\infty}^{\infty} \|U^j x_{j-1}\|_{\mathfrak{K}}^2 = \sum_{j=-\infty}^{\infty} \|UU^{j-1} x_{j-1}\|_{\mathfrak{K}}^2 \leqslant \|U\|^2 \sum_{j=-\infty}^{\infty} \|U^{j-1} x_{j-1}\|_{\mathfrak{K}}^2.$$

This shows that $\|\mathcal{R}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \|U\|_{\mathfrak{B}(\mathfrak{K})}$. To see that $\|U\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \|\mathcal{R}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}$, let $\alpha \in \mathfrak{K}$, and let $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$ be specified by putting $x_0 = \alpha$, and $x_j = 0$ for $j \in \mathbb{Z} \setminus \{0\}$. Thus,

$$\|U\alpha\|_{\mathfrak{K}}^2 = \|\mathcal{R}x\|_{\ell^2(\mathcal{W})}^2 \leqslant \|\mathcal{R}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}^2 \|\alpha\|_{\mathfrak{K}}^2.$$

Similar reasoning shows that $\|\mathcal{L}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \|U^{-1}\|_{\mathfrak{B}(\mathfrak{K})}$, after which the remaining conclusions of the lemma are evident. ∎

Our main result can now be formulated as follows.

Theorem 2.3. *Let $\mathfrak{K}$ be a Hilbert space, and let $U \in \mathfrak{B}(\mathfrak{K})$ be invertible. Let $\mathcal{W}$ be the operator-valued weight sequence on $\mathfrak{K}$ defined by $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$. Then the right bilateral shift $\mathcal{R}$ on $\ell^2(\mathcal{W})$ has the property that*

$$\|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \sup_{z \in \mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})}, \tag{2.3}$$

for every trigonometric polynomial q.

Proof. For each $y = \{y_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathfrak{K})$, let $\widehat{y} \in L^2(\mathbb{T}, \mathfrak{K})$ be the Fourier transform of y defined by putting

$$\widehat{y}(z) = \sum_{j=-\infty}^{\infty} z^{-j} y_j,$$

with series convergence in the norm topology of $L^2(\mathbb{T}, \mathfrak{K})$. Clearly the Parseval formula holds:

$$\int_{\mathbb{T}} \|\widehat{y}(z)\|_{\mathfrak{K}}^2 \, \mathrm{d}\lambda(z) = \|y\|_{\ell^2(\mathfrak{K})}^2. \tag{2.4}$$

Let $q(z) \equiv \sum_{k=-N}^{N} \widehat{q}(k) z^k$ be a trigonometric polynomial defined on $\mathbb{T}$. Trivial estimates using the triangle inequality show that $\sup_{z\in\mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})} < \infty$. Now let $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$. Observe that for each $j \in \mathbb{Z}$, the isometry Γ in Definition 2.1 satisfies

$$(\Gamma^{-1} q(\mathcal{R})\Gamma x)(j) = \sum_{k=-N}^{N} \widehat{q}(k) U^k x_{j-k}. \tag{2.5}$$

This shows, in particular, that $\Gamma^{-1} q(\mathcal{R})\Gamma x \in \ell_0(\mathfrak{K})$. Moreover, for each $z \in \mathbb{T}$, we have

$$\begin{aligned}(\Gamma^{-1} q(\mathcal{R})\Gamma x)^{\wedge}(z) &= \sum_{j=-\infty}^{\infty} z^{-j} \sum_{k=-N}^{N} \widehat{q}(k) U^k x_{j-k} \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-N}^{N} \widehat{q}(k) z^{-k} U^k (z^{k-j} x_{j-k}).\end{aligned} \tag{2.6}$$

We infer from (2.6) and the Parseval formula (2.4) that

$$\begin{aligned}\|\Gamma^{-1} q(\mathcal{R})\Gamma x\|^2_{\ell^2(\mathfrak{K})} &= \int_{\mathbb{T}} \Big\| \sum_{j=-\infty}^{\infty} \sum_{k=-N}^{N} \widehat{q}(k) z^{-k} U^k (z^{k-j} x_{j-k}) \Big\|^2_{\mathfrak{K}} \mathrm{d}\lambda(z) \\ &= \int_{\mathbb{T}} \Big\| \sum_{k=-N}^{N} \widehat{q}(k) z^{-k} U^k \Big(\sum_{j=-\infty}^{\infty} z^{k-j} x_{j-k} \Big) \Big\|^2_{\mathfrak{K}} \mathrm{d}\lambda(z).\end{aligned} \tag{2.7}$$

Since for each $z \in \mathbb{T}$ and each $k \in \mathbb{Z}$,

$$\sum_{j=-\infty}^{\infty} z^{k-j} x_{j-k} = \sum_{j=-\infty}^{\infty} z^{-j} x_j = \widehat{x}(z),$$

we can write (2.7) in the form

$$\begin{aligned}\|\Gamma^{-1} q(\mathcal{R})\Gamma x\|^2_{\ell^2(\mathfrak{K})} &= \int_{\mathbb{T}} \Big\| \sum_{k=-N}^{N} \widehat{q}(k) z^{-k} U^k (\widehat{x}(z)) \Big\|^2_{\mathfrak{K}} \mathrm{d}\lambda(z) \\ &\leqslant \big(\sup_{z\in\mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})} \big)^2 \int_{\mathbb{T}} \|\widehat{x}(z)\|^2_{\mathfrak{K}} \mathrm{d}\lambda(z) \\ &= \big(\sup_{z\in\mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})} \big)^2 \|x\|^2_{\ell^2(\mathfrak{K})}.\end{aligned}$$

This shows that

$$\|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \sup_{z\in\mathbb{T}} \|q(zU)\|_{\mathfrak{B}(\mathfrak{K})}. \tag{2.8}$$

For $\alpha \in \mathfrak{K}$, and $M \in \mathbb{N}$, define $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$ by writing for each $j \in \mathbb{Z}$,

$$x_j = \begin{cases} \alpha, & \text{if } |j| \leqslant M+N, \\ 0, & \text{if } |j| > M+N. \end{cases} \tag{2.9}$$

With the aid of (2.5) we see that

$$\begin{aligned}\sum_{j=-M}^{M}\left\|\sum_{k=-N}^{N}\widehat{q}(k)U^k x_{j-k}\right\|_{\mathfrak{K}}^2 &\leqslant \|\Gamma^{-1}q(\mathcal{R})\Gamma x\|_{\ell^2(\mathfrak{K})}^2 \\ &\leqslant \|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}^2\|x\|_{\ell^2(\mathfrak{K})}^2 \\ &= \|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}^2(2M+2N+1)\|\alpha\|_{\mathfrak{K}}^2.\end{aligned} \tag{2.10}$$

Employing in the left-hand member of (2.10) the definition in (2.9) for $x = \{x_j\}_{j=-\infty}^{\infty}$, we arrive at the inequality

$$(2M+1)\|q(U)\alpha\|_{\mathfrak{K}}^2 \leqslant \|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}^2(2M+2N+1)\|\alpha\|_{\mathfrak{K}}^2.$$

After dividing this estimate by $(2M+1)$ and letting $M \to \infty$, we infer that

$$\|q(U)\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}. \tag{2.11}$$

For each $w \in \mathbb{T}$, it is easily seen by direct calculation that the surjective isometry $\mathcal{M}_w$ of $\ell^2(\mathcal{W})$, defined in (2.1), satisfies

$$\mathcal{M}_w\mathcal{R}\mathcal{M}_w^{-1} = w\mathcal{R}. \tag{2.12}$$

Applying (2.11) to the trigonometric polynomial q_w in place of q, where $q_w(z) = q(wz)$ for all $z \in \mathbb{T}$, we find with the aid of (2.12) that for each $w \in \mathbb{T}$,

$$\|q(wU)\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \|q(w\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} = \|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))}.$$

This, together with (2.8), completes the proof of (2.3). ∎

COROLLARY 2.4. *Under the hypotheses and notation of Theorem* 2.3 *the following assertions are equivalent:*

(i) U *is trigonometrically well-bounded on* $\mathfrak{K}$*;*
(ii) $\mathcal{R}$ *is trigonometrically well-bounded on* $\ell^2(\mathcal{W})$*;*
(iii) $\mathcal{W}$ *has the Treil-Volberg Property.*

Proof. If U is trigonometrically well-bounded, then there is a constant δ such that for every trigonometric polynomial q, $\|q(U)\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \delta\|q\|_{BV(\mathbb{T})}$, and hence $\sup_{z\in\mathbb{T}}\|q(zU)\|_{\mathfrak{B}(\mathfrak{K})} \leqslant 2\delta\|q\|_{BV(\mathbb{T})}$. This, together with (2.3) shows that $\mathcal{R}$ is trigonometrically well-bounded. Conversely, if $\mathcal{R}$ is trigonometrically well-bounded, then there is a constant η such that for every trigonometric polynomial q, $\|q(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \eta\|q\|_{BV(\mathbb{T})}$, and it follows by (2.3) that U is trigonometrically well-bounded. The equivalence of conditions (ii) and (iii) above is a special case of the following result ([9], Theorem 4.12). ∎

THEOREM 2.5. *Suppose that* $\mathfrak{W}$ *is an operator-valued weight sequence on the arbitrary Hilbert space* $\mathfrak{K}$*. Then* $\mathfrak{W}$ *has the Treil-Volberg Property if and only if the right bilateral shift is a trigonometrically well-bounded operator on* $\ell^2(\mathfrak{W})$.

3. BACKGROUND ITEMS FROM SPECTRAL THEORY

In this section we recall the requisite features of trigonometrically well-bounded operators, and develop some useful items tailored to our purposes (Theorem 3.8 and Corollaries 3.9 and 3.10). In what follows, $\mathfrak{X}$ will denote an arbitrary Banach space. Our basic tool for expressing spectral decomposability will be the following notion of a spectral family in $\mathfrak{X}$, which will be used in conjunction with its associated theory of spectral integration.

DEFINITION 3.1. A *spectral family* in $\mathfrak{X}$ is an idempotent-valued function $E(\,\cdot\,) : \mathbb{R} \to \mathfrak{B}(X)$ with the following properties:

(i) $E(s)E(t) = E(t)E(s) = E(s)$ if $s \leqslant t$;

(ii) $\sup\{\|E(t)\| : t \in \mathbb{R}\} < \infty$;

(iii) with respect to the strong operator topology of $\mathfrak{B}(X)$, $E(\,\cdot\,)$ is right continuous and has a left-hand limit $E(t^-)$ at each point $t \in \mathbb{R}$;

(iv) $E(t) \to I$ as $t \to \infty$ and $E(t) \to 0$ as $t \to -\infty$, where each limit is with respect to the strong operator topology.

If, in addition, there exist $a, b \in \mathbb{R}$ with $a \leqslant b$ such that $E(t) = 0$ for $t < a$ and $E(t) = I$ for $t \geqslant b$, $E(\,\cdot\,)$ is said to be *concentrated on* $[a, b]$.

Given a spectral family $E(\,\cdot\,)$ in $\mathfrak{X}$ concentrated on a compact interval $J = [a, b]$, we can develop an associated theory of spectral integration as follows. For each bounded function $\varphi : J \to \mathbb{C}$ and each partition $\mathcal{P} = (t_0, t_1, \ldots, t_n)$ of J, where we take $t_0 = a$ and $t_n = b$, set

$$\mathcal{S}(\mathcal{P}; \varphi, E) = \sum_{k=1}^{n} \varphi(t_k)\{E(t_k) - E(t_{k-1})\}.$$

If the net $\{\mathcal{S}(\mathcal{P}; \varphi, E)\}$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ as $\mathcal{P}$ increases by refinement through the set of partitions of J, then the strong limit is called the *spectral integral of* φ with respect to $E(\cdot)$ and is denoted by $\int_J \varphi(t)\,\mathrm{d}E(t)$.

In this case, we define $\int_J^{\oplus} \varphi(t)\,\mathrm{d}E(t)$ by writing

$$\int_J^{\oplus} \varphi(t)\,\mathrm{d}E(t) \equiv \varphi(a)E(a) + \int_J \varphi(t)\,\mathrm{d}E(t).$$

Denote by $BV(J)$ the Banach algebra of functions $\varphi : J \to \mathbb{C}$ of bounded variation on J, with norm $\|\varphi\|_{BV(J)} = |\varphi(b)| + \operatorname{var}(\varphi, J)$. It can be shown (see [12], Chapter 17 or the simplified account in [4], Section 2) that the spectral integral $\int_J \varphi(t)\,\mathrm{d}E(t)$ exists for each $\varphi \in BV(J)$, and that the mapping $\varphi \to \int_J^{\oplus} \varphi(t)\,\mathrm{d}E(t)$ is an identity-preserving algebra homomorphism of $BV(J)$ into $\mathfrak{B}(\mathfrak{X})$ satisfying

$$\left\| \int_J^{\oplus} \varphi(t)\,\mathrm{d}E(t) \right\| \leqslant \|\varphi\|_{BV(J)} \sup\{\|E(t)\| : t \in \mathbb{R}\}. \tag{3.1}$$

As indicated in Section 1, we shall also be concerned with the Banach algebra $BV(\mathbb{T})$, which consists of all $F : \mathbb{T} \to \mathbb{C}$ such that the function $\widetilde{F}(t) \equiv F(\mathrm{e}^{\mathrm{i}t})$ belongs to $BV([0, 2\pi])$, and which is furnished with the norm

$$\|F\|_{BV(\mathbb{T})} = \|\widetilde{F}\|_{BV([0,2\pi])}.$$

DEFINITION 3.2. An operator $U \in \mathfrak{B}(\mathfrak{X})$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\,\cdot\,)$ in $\mathfrak{X}$ concentrated on $[0, 2\pi]$ such that

$$U = \int_{[0,2\pi]}^{\oplus} \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}E(t).$$

In this case, it is possible to arrange that $E((2\pi)^-) = I$, and with this additional property the spectral family $E(\,\cdot\,)$ is uniquely determined by U, and is called *the spectral decomposition* of U.

Notice that if $U \in \mathfrak{B}(\mathfrak{X})$ is a trigonometrically well-bounded operator with spectral decomposition $E(\,\cdot\,)$, then the multiplicativity property for spectral integration of functions having bounded variation shows that for $z \in \mathbb{C} \setminus \mathbb{T}$,

$$\int_{[0,2\pi]}^{\oplus} \frac{1}{z - \mathrm{e}^{\mathrm{i}t}}\,\mathrm{d}E(t) = (z - U)^{-1},$$

and hence the spectrum of U is a subset of $\mathbb{T}$. In particular, U is invertible, with $U^{-1} = \int_{[0,2\pi]}^{\oplus} \mathrm{e}^{-\mathrm{i}t}\,\mathrm{d}E(t)$, and for every trigonometric polynomial q,

$$q(U) = \int_{[0,2\pi]}^{\oplus} q(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t).$$

The class of trigonometrically well-bounded operators was introduced in [2], and its fundamental structural theory further developed in [3]. In particular, it follows from [2], Corollary 2.17, that an invertible operator on a Banach space is trigonometrically well-bounded if and only if its inverse is trigonometrically well-bounded. For examples of trigonometrically well-bounded operators and of integration with respect to their associated spectral families, see, e.g., [3], [4], [7], [8], and [11]. In particular, the powers of a trigonometrically well-bounded operator on $\mathfrak{X}$ need not be uniformly bounded (even when $\mathfrak{X}$ is a Hilbert space). As indicated by our earlier discussion of the condition in (1.2), when $\mathfrak{X}$ is reflexive trigonometrically well-bounded operators can be characterized by suitable boundedness conditions. The following proposition, which incorporates (1.2) in its statement, describes this state of affairs.

Proposition 3.3. *Suppose that $\mathfrak{X}$ is a reflexive Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ is invertible. The following assertions are equivalent:*

(i) *U is trigonometrically well-bounded;*

(ii) $\sup\{\|\psi(U)\| : \psi$ *is a trigonometric polynomial,* $\|\psi\|_{BV(\mathbb{T})} \leqslant 1\} < \infty$*;*

(iii) $\sup\Big\{\Big\| \sum\limits_{0<|k|\leqslant n} \Big(1 - \frac{|k|}{n+1}\Big) \frac{z^k U^k}{k} \Big\| : n \in \mathbb{N},\, z \in \mathbb{T}\Big\} < \infty.$

The characterization of the notion of trigonometrically well-bounded operator in the reflexive space setting expressed by condition (ii) (respectively, condition (iii)) of Proposition 3.3 follows from [2], Corollary 2.17 (respectively, [3], Theorem (2.4)), and indicates the "trigonometric" (respectively, ergodic Hilbert transform) aspects of the associated spectral decomposability. Notice that the sums appearing in Proposition 3.3 (iii) are the $(C,1)$ means of the ergodic Hilbert averages for the operators zU, $z \in \mathbb{T}$. The use of these $(C,1)$ means rather than the ergodic Hilbert averages *per se* for the operators zU, $z \in \mathbb{T}$, is essential for the generality asserted in Proposition 3.3 (see [3], Example (3.1)).

Remark 3.4. Without the hypothesis that $\mathfrak{X}$ is reflexive, each of the conditions in Proposition 3.3 (ii)–(iii) is necessary for U to be trigonometrically well-bounded (see [10], Theorems 2.1 and 5.2).

The following variant of Fejér's Theorem is valid for trigonometrically well-bounded operators (see [4], Theorem (3.10) (i)).

Proposition 3.5. *Suppose that U is a trigonometrically well-bounded operator on a Banach space $\mathfrak{X}$, and $E(\,\cdot\,)$ is the spectral decomposition of U. Let $\psi \in BV(\mathbb{T})$, and define $\psi^\ddagger \in BV([0,2\pi])$ by putting*

$$\psi^\ddagger(t) = \frac{1}{2}\big\{\lim_{s\to t^+} \psi(\mathrm{e}^{\mathrm{i}s}) + \lim_{s\to t^-} \psi(\mathrm{e}^{\mathrm{i}s})\big\}.$$

Then the formal series $\sum\limits_{k=-\infty}^{\infty} \widehat{\psi}(k) U^k$ is $(C,1)$ summable in the strong operator topology to (that is, the sequence $\Big\{\sum\limits_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big)\widehat{\psi}(k)U^k\Big\}_{n=1}^{\infty}$ converges in the strong operator topology to) $\int\limits_{[0,2\pi]}^{\oplus} \psi^\ddagger(t)\, \mathrm{d}E(t)$.

In the general Banach space setting, the spectral decomposition of a trigonometrically well-bounded operator U has the following description in terms of averages of U ([10], Theorem 5.3).

Theorem 3.6. *Suppose that U is a trigonometrically well-bounded operator on a Banach space $\mathfrak{X}$, and $E(\,\cdot\,)$ is the spectral decomposition of U. Then, with all convergence in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$, we have for each t such that $0 \leqslant t < 2\pi$,*

$$\begin{aligned} E(t) = (2\pi\mathrm{i})^{-1}\{t\mathrm{i}I - S_t + S_0\} &+ \lim_n \frac{1}{n}\sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)\mathrm{e}^{-\mathrm{i}kt}U^k \\ &+ \lim_n \frac{1}{n}\sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)U^k, \end{aligned} \tag{3.2}$$

where $S_t = \lim_n \sum_{0<|k|\leqslant n} \left(1 - \frac{|k|}{n+1}\right) \frac{\mathrm{e}^{-\mathrm{i}kt}}{k} U^k$ *for* $0 \leqslant t < 2\pi$.

The significance of the last two terms on the right of (3.2) is exhibited in the following theorem (see [4], Theorem (3.14) and [10], Corollary 4.4).

THEOREM 3.7. *Under the hypotheses of Theorem* 3.6, *for each* t *such that* $0 \leqslant t < 2\pi$, *we have, with convergence in the strong operator topology of* $\mathfrak{B}(\mathfrak{X})$,

$$\frac{2}{n}\sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)\mathrm{e}^{-\mathrm{i}kt}U^k \to E(t) - E(t^-)$$

and

$$\{E(t) - E(t^-)\}\mathfrak{X} = \{x \in \mathfrak{X} : Ux = \mathrm{e}^{\mathrm{i}t}x\}.$$

When the point spectrum of a trigonometrically well-bounded operator is empty, Theorem 3.7 can be used to eliminate the last two terms from the right-hand side of (3.3). For our later convenience, we illustrate this situation with the following example.

THEOREM 3.8. *Let* $\mathfrak{W} = \{W_k\}_{k=-\infty}^{\infty}$ *be an operator-valued weight sequence on the arbitrary Hilbert space* $\mathfrak{K}$, *and suppose that* $\mathfrak{W}$ *has the Treil-Volberg Property. Then the right bilateral shift* $\mathcal{R}$ *and the left bilateral shift* $\mathcal{L} = \mathcal{R}^{-1}$ *are trigonometrically well-bounded operators on* $\ell^2(\mathfrak{W})$, *each having empty point spectrum.*

Proof. By Theorem 2.5, $\mathcal{R}$ is trigonometrically well-bounded on $\ell^2(\mathfrak{W})$ (and hence so is $\mathcal{L} = \mathcal{R}^{-1}$). In order to complete the proof, it suffices to show that $\mathcal{R}$ has no eigenvalues. Suppose to the contrary that $\gamma \in \mathbb{C}$ is an eigenvalue for $\mathcal{R}$, and let $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathfrak{W}) \setminus \{0\}$ be an eigenvector associated with γ. Since $\mathcal{R}$ is trigonometrically well-bounded, $\sigma(\mathcal{R}) \subseteq \mathbb{T}$, and so $|\gamma| = 1$. From $\mathcal{R}x = \gamma x$ we infer that for all $k \in \mathbb{Z}$, $\mathcal{R}^{-k}x = \gamma^{-k}x$, whence $x_k = (\mathcal{R}^{-k}x)(0) = \gamma^{-k}x_0$. It follows, in particular, that $x_0 \neq 0$. We now have:

$$\infty > \|x\|_{\ell^2(\mathfrak{W})}^2 = \sum_{k=-\infty}^{\infty} \langle W_k x_k, x_k\rangle = \sum_{k=-\infty}^{\infty} \langle W_k x_0, x_0\rangle. \tag{3.3}$$

We claim that for each $\alpha \in \mathfrak{K} \setminus \{0\}$, the sequence $\{\langle W_k\alpha, \alpha\rangle\}_{k=-\infty}^{\infty}$ is a scalar-valued A_2 weight sequence. In order to see this, let $f : \mathbb{Z} \to \mathbb{C}$ be finitely supported, and put $y = \{f(k)\alpha\}_{k=-\infty}^{\infty} \in \ell^2(\mathfrak{W})$. Thus for each $k \in \mathbb{Z}$,

$$(Dy)(k) = \sum_{j=-\infty}^{\infty} h(k-j)f(j)\alpha = ((h * f)(k))\alpha.$$

Hence the inequality $\|Dy\|_{\ell^2(\mathfrak{W})}^2 \leqslant \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}^2 \|y\|_{\ell^2(\mathfrak{W})}^2$ can be written as

$$\sum_{k=-\infty}^{\infty} |(h*f)(k)|^2 \langle W_k\alpha, \alpha\rangle \leqslant \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}^2 \sum_{k=-\infty}^{\infty} |f(k)|^2 \langle W_k\alpha, \alpha\rangle.$$

Our claim follows directly from this by Theorem 10 of [15]. (We remark that an alternate proof of the claim can be obtained by careful adaptation of the ingredients used to establish Lemma 3.6 of [20].) Specializing α to be the vector $x_0 \in \mathfrak{K} \setminus \{0\}$ which occurs in (3.3), we now see that $\{\langle W_k x_0, x_0 \rangle\}_{k=-\infty}^{\infty}$ is a scalar-valued A_2 weight sequence, and hence by the Scholium in Section 4 of [5] we must have $\sum_{k=-\infty}^{\infty} \langle W_k x_0, x_0 \rangle = \infty$. This contradiction to (3.3) completes the proof of Theorem 3.8. ∎

COROLLARY 3.9. *Assume the hypotheses and notation of Theorem* 3.8*, and let* $\mathcal{E}(\,\cdot\,)$ *be the spectral decomposition of the trigonometrically well-bounded operator* $\mathcal{L}$ *on* $\ell^2(\mathfrak{W})$*. Then for each* t *such that* $0 \leqslant t < 2\pi$,

$$\mathcal{E}(t) = (2\pi \mathrm{i})^{-1} \{t\mathrm{i}I - S_t + S_0\}, \tag{3.4}$$

where, with convergence in the strong operator topology of $\mathfrak{B}(\ell^2(\mathfrak{W}))$,

$$S_t = \lim_n \sum_{0<|k|\leqslant n} \Big(1 - \frac{|k|}{n+1}\Big) \frac{\mathrm{e}^{-\mathrm{i}kt}}{k} \mathcal{L}^k \quad \text{for } 0 \leqslant t < 2\pi. \tag{3.5}$$

Moreover,

$$\sup_{t\in\mathbb{R}} \|\mathcal{E}(t)\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant 1 + \pi^{-1} \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}. \tag{3.6}$$

Proof. In view of Theorems 3.6, 3.7, and 3.8, it remains only to show that (3.6) holds. Suppose that $0 \leqslant t < 2\pi$. In terms of the notation introduced in conjunction with (2.2), it is easy to see from the $(C, 1)$ means on the right of (3.5) that we have for $x \in \ell^2(\widetilde{\mathfrak{W}})$ and $j \in \mathbb{Z}$,

$$(\Delta S_t \Delta^{-1} x)(j) = \mathrm{e}^{-\mathrm{i}jt} (Dy)(j),$$

where $y = \{y_k\}_{k=-\infty}^{\infty} \in \ell^2(\widetilde{\mathfrak{W}})$ is given by $y_k = \mathrm{e}^{\mathrm{i}kt} x_k$ for all $k \in \mathbb{Z}$. Consequently, we see with the aid of (2.2) that

$$\|S_t\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} = \|D\|_{\mathfrak{B}(\ell^2(\widetilde{\mathfrak{W}}))} = \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}. \tag{3.7}$$

Since $\mathcal{E}(\,\cdot\,)$ is concentrated on $[0, 2\pi]$, application of (3.7) to (3.4) suffices to establish (3.6). ∎

As a corollary of the method of proof for Theorem 3.8, we also have the following result.

COROLLARY 3.10. *Let* $\mathfrak{W} = \{W_k\}_{k=-\infty}^{\infty}$ *be an operator-valued weight sequence on the Hilbert space* $\mathfrak{K} \neq \{0\}$*, and suppose that* $\mathfrak{W}$ *has the Treil-Volberg Property. Then the discrete Hilbert transform* D *satisfies the estimate*

$$\pi \leqslant \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}.$$

Proof. Choose $\alpha \in \mathfrak{K} \setminus \{0\}$. The method of proof for Theorem 3.8 shows that $\mathfrak{w} = \{\langle W_k \alpha, \alpha \rangle\}_{k=-\infty}^{\infty}$ is a scalar-valued A_2 weight sequence such that the discrete Hilbert transform $D_{\mathfrak{w}} \in \mathfrak{B}(\ell^2(\mathfrak{w}))$ satisfies

$$\|D_{\mathfrak{w}}\|_{\mathfrak{B}(\ell^2(\mathfrak{w}))} \leqslant \|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}.$$

So we shall complete the proof by establishing that π does not exceed the spectral radius $r(D_{\mathfrak{w}})$ of $D_{\mathfrak{w}}$. Application of Theorem (4.2) and Corollary (4.8) from [5] shows that the right bilateral shift $\mathcal{R}_{\mathfrak{w}} : \ell^2(\mathfrak{w}) \to \ell^2(\mathfrak{w})$ is a trigonometrically well-bounded operator satisfying

$$-\mathcal{R}_{\mathfrak{w}} = \mathrm{e}^{-D_{\mathfrak{w}}}. \tag{3.8}$$

For each $z \in \mathbb{T}$, we find by easy direct calculations with the operator $\mathcal{M}_z \in \mathfrak{B}(\ell^2(\mathfrak{w}))$ in (2.1) that

$$\mathcal{M}_z \mathcal{R}_{\mathfrak{w}} \mathcal{M}_z^{-1} = z\mathcal{R}_{\mathfrak{w}}.$$

It follows that the spectrum $\Lambda(\mathcal{R}_{\mathfrak{w}})$ of $\mathcal{R}_{\mathfrak{w}}$ satisfies

$$\Lambda(\mathcal{R}_{\mathfrak{w}}) = z\Lambda(\mathcal{R}_{\mathfrak{w}}) \quad \text{for all } z \in \mathbb{T}. \tag{3.9}$$

Since $\mathcal{R}_{\mathfrak{w}}$ is trigonometrically well-bounded, we have $\Lambda(\mathcal{R}_{\mathfrak{w}}) \subseteq \mathbb{T}$, and this fact together with (3.9) shows that

$$\Lambda(\mathcal{R}_{\mathfrak{w}}) = \mathbb{T}. \tag{3.10}$$

Applying (3.10) to (3.8), we readily deduce by spectral mapping that $\Lambda(\mathcal{D}_{\mathfrak{w}})$ is pure-imaginary, and that $r(D_{\mathfrak{w}}) \geqslant \pi$. ∎

4. MULTIPLIERS FOR $\ell^2(\mathfrak{W})$ AND TRANSFERENCE

For a function ψ belonging to $L^1(\mathbb{T})$ (respectively, belonging to $L^2(\mathbb{T})$), we shall denote the Fourier transform (respectively, inverse Fourier transform) of ψ by $\widehat{\psi}$ (respectively, $\check{\psi}$). Thus, for $\psi \in L^2(\mathbb{T})$, $\check{\psi}(k) = \widehat{\psi}(-k)$, for all $k \in \mathbb{Z}$.

DEFINITION 4.1. Let $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ be an operator-valued weight sequence on the Hilbert space $\mathfrak{K}$. A bounded, Haar measurable function $\psi : \mathbb{T} \to C$ is called a *multiplier* for $\ell^2(\mathfrak{W})$ (in symbols, $\psi \in M_{\ell^2(\mathfrak{W})}$) provided that:

(i) for each $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathfrak{W})$, and each $k \in \mathbb{Z}$, the series

$$(\check{\psi} * x)(k) \equiv \sum_{j=-\infty}^{\infty} \check{\psi}(j) x_{k-j}$$

converges unconditionally in the norm topology of $\mathfrak{K}$, and

(ii) the operator $T_\psi : x \in \ell^2(\mathfrak{W}) \to \check{\psi} * x$ is a bounded linear mapping of $\ell^2(\mathfrak{W})$ into $\ell^2(\mathfrak{W})$. If this is the case, then the *multiplier norm* $\|\psi\|_{M_{\ell^2(\mathfrak{W})}}$ is defined by

$$\|\psi\|_{M_{\ell^2(\mathfrak{W})}} = \|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}.$$

The notation introduced in Definition 4.1 we remain in effect henceforth. The relationship between multiplier theory for $\ell^2(\mathfrak{W})$ and spectral decomposability is described in the next two theorems.

THEOREM 4.2. *Suppose that* $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ *is an operator-valued weight sequence on* $\mathfrak{K}$ *which has the Treil-Volberg Property, and let* $\mathcal{L}$ *be the left bilateral shift on* $\ell^2(\mathfrak{W})$. *For* $0 \leqslant t < 2\pi$, *let* ψ_t *be the characteristic function, defined on* $\mathbb{T}$, *of the arc*

$$\{e^{is} : 0 \leqslant s \leqslant t\}.$$

Then $\psi_t \in M_{\ell^2(\mathfrak{W})}$, *and the spectral decomposition* $\mathcal{E}(\cdot)$ *of the trigonometrically well-bounded operator* $\mathcal{L} \in \mathfrak{B}(\ell^2(\mathfrak{W}))$ *satisfies*

$$\mathcal{E}(t) = T_{\psi_t} \quad \text{for } 0 \leqslant t < 2\pi.$$

(In particular, $\mathcal{E}(0) = 0$.*)*

Proof. Suppose that $0 \leqslant t < 2\pi$, and $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$. Straightforward coordinatewise calculations proceeding from (3.4) and the $(C,1)$ means on the right of (3.5) show that

$$\mathcal{E}(t)x = (\check{\psi}_t) * x.$$

The desired conclusions follow from this by using the density of $\ell_0(\mathfrak{K})$ in $\ell^2(\mathfrak{W})$. ∎

THEOREM 4.3. *Assume the hypotheses and notation of Theorem* 4.2. *Suppose that* $\psi : \mathbb{T} \to C$ *is a bounded function such that* ψ *is continuous* λ-*a.e. on* $\mathbb{T}$, *and the spectral integral*

$$\int_{[0,2\pi]} \psi(e^{it})\, d\mathcal{E}(t) \tag{4.1}$$

exists. Then ψ *is a multiplier for* $\ell^2(\mathfrak{W})$, *and* T_ψ, *the multiplier transform of* ψ *on* $\ell^2(\mathfrak{W})$, *coincides with the spectral integral in* (4.1).

Proof. Let $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$ (say, $L \in \mathbb{N}$, and $x_j = 0$ for $|j| > L$), and let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,2\pi]$ with mesh-fineness tending to 0 such that the corresponding Riemann-Stieltjes approximating sums $\mathcal{S}(\mathcal{P}_n; \psi(e^{i(\cdot)}), \mathcal{E})x$ tend to $\int_{[0,2\pi]} \psi(e^{it})\, d\mathcal{E}(t)x$ in the norm topology of $\mathfrak{K}$. Let $\mathcal{P}_n = (s_0^{(n)}, s_1^{(n)}, \ldots, s_{N_n}^{(n)})$. Then in view of Theorem 4.2 we have for each $n \in \mathbb{N}$,

$$\mathcal{S}(\mathcal{P}_n; \psi(e^{i(\cdot)}), \mathcal{E})x = \sum_{k=1}^{N_n} \psi(e^{is_k^{(n)}})\, \check{\chi}_{J_k^{(n)}} * x, \tag{4.2}$$

where $\chi_{J_k^{(n)}}$ denotes the characteristic function, defined on $\mathbb{T}$ of the arc $J_k^{(n)}$ specified by

$$J_k^{(n)} = \{e^{it} : s_{k-1}^{(n)} < t \leqslant s_k^{(n)}\}.$$

Let $\psi_n \in BV(\mathbb{T})$ be defined by writing

$$\psi_n = \sum_{k=1}^{N_n} \psi(e^{is_k^{(n)}}) \chi_{J_k^{(n)}},$$

and notice that by our hypotheses, $\{\psi_n\}_{n=1}^\infty$ is a uniformly bounded sequence of functions on $\mathbb{T}$ which converges λ-a.e. on $\mathbb{T}$ to ψ. Consequently, by bounded convergence,

$$\lim_n \overset{\vee}{\psi}_n (k) = \overset{\vee}{\psi} (k) \quad \text{for all } k \in \mathbb{Z}.$$

By (4.2), we have for each $j \in \mathbb{Z}$,

$$(\mathcal{S}(\mathcal{P}_n; \psi(\mathrm{e}^{i(\cdot)}), \mathcal{E})x)(j) = (\overset{\vee}{\psi}_n * x)(j) = \sum_{k=-L}^{L} \overset{\vee}{\psi}_n (j-k)x_k.$$

It obviously follows from this by letting $n \to \infty$ that for $x = \{x_j\}_{j=-\infty}^\infty \in \ell_0(\mathfrak{K})$,

$$\int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}\mathcal{E}(t)x = \overset{\vee}{\psi} * x.$$

The desired conclusions follow from this by using the density of $\ell_0(\mathfrak{K})$ in $\ell^2(\mathfrak{W})$. ∎

As noted in (3.1) functions of bounded variation can be integrated with respect to a spectral family. In view of this fact and the estimates in (3.1), (3.6), and Corollary 3.10, we see that Theorem 4.3 has the following corollary, which transfers the classical Fourier multiplier theorem of Stečkin ([12], Theorem 20.7) to $\ell^2(\mathfrak{W})$.

COROLLARY 4.4. *Suppose that* $\mathfrak{W} = \{W_j\}_{j=-\infty}^\infty$ *is an operator-valued weight sequence on* $\mathfrak{K}$ *which has the Treil-Volberg Property. Then each* $\psi \in BV(\mathbb{T})$ *is a multiplier for* $\ell^2(\mathfrak{W})$ *whose corresponding multiplier transform* T_ψ *satisfies the estimate*

$$\|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant (1 + \pi^{-1})\|D\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}\, \|\psi\|_{BV(\mathbb{T})}. \tag{4.3}$$

We now pass to transference estimates for Rademacher averages. Rademacher averages will be expressed in terms of Haar integration over the group $\mathfrak{D}^N$, which is defined as follows. Let $\mathfrak{D}$ denote the multiplicative group consisting of the real numbers 1 and -1. For each $N \in \mathbb{N}$, we denote by $\mathfrak{D}^N$ the direct product of N copies of $\mathfrak{D}$. The generic element ε of $\mathfrak{D}^N$ will be written in the form $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, and normalized Haar measure on $\mathfrak{D}^N$ will be denoted by $\mathrm{d}\varepsilon$. It is easily seen from Khinchin's Inequality and the Khinchin-Kahane Inequality (see [17], Theorem 2.b.3, and [18], Theorem 1.e.13, respectively) that L^p norms of square functions can be generalized from the Lebesgue space framework to the setting of an arbitrary Banach space $\mathfrak{X}$ by replacing the square function norms with expressions of the form

$$\left(\int_{\mathfrak{D}^N} \left\|\sum_{n=1}^N \varepsilon_n \alpha_n\right\|_{\mathfrak{X}}^2 \mathrm{d}\varepsilon\right)^{1/2} \qquad \text{for } N \in \mathbb{N},\ \{\alpha_n\}_{n=1}^N \subseteq \mathfrak{X}.$$

In view of this fact, the following notion is a generalization to the Banach space setting of the classical M. Riesz Property for square functions defined by Riesz projections (see, e.g., Theorem 6.5.2, [13] for the classical case).

DEFINITION 4.5. ([6], Definition (2.4)). A family $\mathcal{F}$ of bounded linear operators mapping a Banach space $\mathfrak{X}$ into itself is said to have the *R-property* provided there is a constant $K_{\mathcal{F}}$ such that for all $N \in \mathbb{N}$, all $\{T_n\}_{n=1}^N \subseteq \mathcal{F}$, and all $\{\alpha_n\}_{n=1}^N \subseteq \mathfrak{X}$, we have

$$\Big(\int\limits_{\mathfrak{D}^N} \Big\| \sum_{n=1}^N \varepsilon_n T_n \alpha_n \Big\|_{\mathfrak{X}}^2 \mathrm{d}\varepsilon\Big)^{1/2} \leqslant K_{\mathcal{F}} \Big(\int\limits_{\mathfrak{D}^N} \Big\| \sum_{n=1}^N \varepsilon_n \alpha_n \Big\|_{\mathfrak{X}}^2 \mathrm{d}\varepsilon\Big)^{1/2}.$$

While the R-property plays an important role in abstract versions of the Marcinkiewicz Multiplier Theorem (see [6], Section 4, which includes Proposition 5.1 below), and will accordingly be useful to us in subsequent discussions, we note for our purposes that the R-property has the following particularly simple characterization in the Hilbert space setting.

PROPOSITION 4.6. *A family $\mathcal{F}$ of bounded linear operators mapping a Hilbert space $\mathfrak{K}$ into itself has the R-property if and only if $\mathcal{F}$ is uniformly bounded.*

Proof. We can assume without loss of generality that $\mathfrak{K} = L^2(\mu)$, for some measure μ. Since the Rademacher functions form an orthonormal sequence in $L^2([0,1])$, it is clear that

$$\int\limits_{\mathfrak{D}^N} \Big\| \sum_{n=1}^N \varepsilon_n f_n \Big\|_{L^2(\mu)}^2 \mathrm{d}\varepsilon = \sum_{n=1}^N \|f_n\|_{L^2(\mu)}^2.$$

This permits us to recast the R-property in terms of square function estimates as follows. $\mathcal{F}$ has the R-property if and only if there is a constant $C_{\mathcal{F}}$ such that for all $N \in \mathbb{N}$, all $\{T_n\}_{n=1}^N \subseteq \mathcal{F}$, and all $\{f_n\}_{n=1}^N \subseteq L^2(\mu)$, we have

$$\sum_{n=1}^N \int\limits_{\Omega} |T_n f_n|^2 \,\mathrm{d}\mu \leqslant C_{\mathcal{F}} \sum_{n=1}^N \int\limits_{\Omega} |f_n|^2 \,\mathrm{d}\mu.$$

The desired conclusion is evident from this. ∎

COROLLARY 4.7. *Suppose that U is a trigonometrically well-bounded operator on the arbitrary Hilbert space $\mathfrak{K}$, and let $E(\cdot)$ be the spectral decomposition of U. Then $\{E(t) : t \in \mathbb{R}\}$ has the R-property.*

With obvious modifications, the methods used to establish Theorem 2.3 readily furnish it with the following generalization to Rademacher averages.

THEOREM 4.8. *Suppose that $\mathfrak{K}$ is a Hilbert space, and $U \in \mathfrak{B}(\mathfrak{K})$ is an invertible operator. Denote by $\mathcal{W}$ the operator-valued weight sequence on $\mathfrak{K}$ defined by putting $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$, and denote by $\mathcal{R}$ the right bilateral shift on $\ell^2(\mathcal{W})$. Let $N \in \mathbb{N}$ and let trigonometric polynomials $q_n, 1 \leqslant n \leqslant N$, be given. Let ξ_1 be the smallest constant such that*

$$\Big(\int\limits_{\mathfrak{D}^N} \Big\| \sum_{n=1}^N \varepsilon_n q_n(zU)\alpha \Big\|_{\mathfrak{K}}^2 \mathrm{d}\varepsilon\Big)^{1/2} \leqslant \xi_1 \|\alpha\|_{\mathfrak{K}}, \quad \textit{for all } z \in \mathbb{T}, \textit{ and all } \alpha \in \mathfrak{K}.$$

Let ξ_2 be the smallest constant such that

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n q_n(\mathcal{R})x \right\|_{\ell^2(\mathcal{W})}^2 \mathrm{d}\varepsilon\right)^{1/2} \leqslant \xi_2 \|x\|_{\ell^2(\mathcal{W})}, \quad \textit{for all } x \in \ell^2(\mathcal{W}).$$

Then $\xi_1 = \xi_2$.

After setting the stage with the next two propositions regarding convolutions, we shall use Theorem 4.8 as a vehicle for transference of multiplier estimates from $\ell^2(\widetilde{\mathcal{W}})$ to the Hilbert space of reference $\mathfrak{K}$ in the context of Theorem 2.3 and Corollary 2.4, with $U \in \mathfrak{B}(\mathfrak{K})$ trigonometrically well-bounded (see Theorem 4.11 and Corollary 4.12 below).

PROPOSITION 4.9. *Let $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ be an operator-valued weight sequence on $\mathfrak{K}$. Suppose that $\mathfrak{k} \in L^1(\mathbb{T})$, and ψ is a multiplier for $\ell^2(\mathfrak{W})$. Then the convolution $\mathfrak{k} * \psi$ is a multiplier for $\ell^2(\mathfrak{W})$ whose corresponding multiplier transform $T_{\mathfrak{k}*\psi}$ on $\ell^2(\mathfrak{W})$ has the following description in terms of $\ell^2(\mathfrak{W})$-valued Bochner integration:*

$$T_{\mathfrak{k}*\psi}x = \int_{\mathbb{T}} \mathfrak{k}(z)\mathcal{M}_z T_\psi \mathcal{M}_{\overline{z}} x \mathrm{d}\lambda(z), \ \textit{for each } x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathfrak{W}), \tag{4.4}$$

where $\mathcal{M}_z$ is the isometry of $\ell^2(\mathfrak{W})$ given by (2.1). *Moreover,*

$$\|T_{\mathfrak{k}*\psi}\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant \|\mathfrak{k}\|_{L^1(\lambda)} \|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}. \tag{4.5}$$

Proof. For $x \in \ell^2(\mathfrak{W})$, let Ωx denote the Bochner integral on the right in (4.4). Clearly $\Omega \in \mathfrak{B}(\ell^2(\mathfrak{W}))$, with

$$\|\Omega\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant \|\mathfrak{k}\|_{L^1(\lambda)} \|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))}. \tag{4.6}$$

Since projection onto each coordinate is continuous on $\ell^2(\mathfrak{W})$, we see that for each $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$, and each $j \in \mathbb{Z}$,

$$\begin{aligned} (\Omega x)(j) &= \int_{\mathbb{T}} \mathfrak{k}(z) z^j \sum_{m=-\infty}^{\infty} \overset{\vee}{\psi}(j-m) z^{-m} x_m \, \mathrm{d}\lambda(z) \\ &= \sum_{m=-\infty}^{\infty} \int_{\mathbb{T}} \mathfrak{k}(z) z^j \overset{\vee}{\psi}(j-m) z^{-m} x_m \, \mathrm{d}\lambda(z) \\ &= \sum_{m=-\infty}^{\infty} \overset{\vee}{\mathfrak{k}}(j-m) \overset{\vee}{\psi}(j-m) x_m. \end{aligned} \tag{4.7}$$

For each $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathfrak{W})$, $\|x - x^{(n)}\|_{\ell^2(\mathfrak{W})} \to 0$, where, for each $n \in \mathbb{N}$, the vector $x^{(n)} = \{x_j^{(n)}\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$ is obtained by taking $x_j^{(n)} = x_j$ when $|j| \leqslant n$, and $x_j^{(n)} = 0$ otherwise. In view of this, the desired conclusions are now apparent from (4.6) and (4.7). ∎

PROPOSITION 4.10. *Let $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ be an operator-valued weight sequence on $\mathfrak{K}$. Suppose that $\mathfrak{k} \in L^1(\mathbb{T})$, $N \in \mathbb{N}$, and ψ_n, $1 \leqslant n \leqslant N$ are multipliers for $\ell^2(\mathfrak{W})$. Let ρ be any constant such that*

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\psi_n} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|x\|_{\ell^2(\mathfrak{W})}, \tag{4.8}$$

for all $x \in \ell^2(\mathfrak{W})$. Then

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\mathfrak{k}*\psi_n} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|\mathfrak{k}\|_{L^1(\lambda)} \|x\|_{\ell^2(\mathfrak{W})},$$

for all $x \in \ell^2(\mathfrak{W})$.

Proof. We start by observing with the aid of (4.4) that for $x \in \ell^2(\mathfrak{W})$,

$$\begin{aligned} &\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\mathfrak{k}*\psi_n} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \\ &\quad = \left(\int_{\mathfrak{D}^N} \left\| \int_{\mathbb{T}} \mathfrak{k}(z) \sum_{n=1}^{N} \varepsilon_n \mathcal{M}_z T_{\psi_n} \mathcal{M}_{\overline{z}} x \,\mathrm{d}\lambda(z) \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \\ &\quad \leqslant \left(\int_{\mathfrak{D}^N} \left(\int_{\mathbb{T}} |\mathfrak{k}(z)| \left\| \sum_{n=1}^{N} \varepsilon_n \mathcal{M}_z T_{\psi_n} \mathcal{M}_{\overline{z}} x \right\|_{\ell^2(\mathfrak{W})} \mathrm{d}\lambda(z) \right)^2 \mathrm{d}\varepsilon \right)^{1/2}. \end{aligned}$$

Applying Minkowski's Inequality for integrals to the right member, we find that

$$\begin{aligned} &\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\mathfrak{k}*\psi_n} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \\ &\quad \leqslant \int_{\mathbb{T}} |\mathfrak{k}(z)| \left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n \mathcal{M}_z T_{\psi_n} \mathcal{M}_{\overline{z}} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \mathrm{d}\lambda(z) \\ &\quad = \int_{\mathbb{T}} |\mathfrak{k}(z)| \left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\psi_n} \mathcal{M}_{\overline{z}} x \right\|_{\ell^2(\mathfrak{W})}^2 \mathrm{d}\varepsilon \right)^{1/2} \mathrm{d}\lambda(z). \end{aligned} \tag{4.9}$$

Using (4.8) in the right member of (4.9) completes the proof. ∎

THEOREM 4.11. *Suppose that $U \in \mathfrak{B}(\mathfrak{K})$ is trigonometrically well-bounded with spectral decomposition $E(\cdot)$, and let $\mathcal{W} = \{W_j\}_{j=-\infty}^{\infty}$ be the operator-valued weight sequence on $\mathfrak{K}$ defined by putting $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$. Let*

$$\widetilde{\mathcal{W}} = \{(U^{-j})^* U^{-j}\}_{j=-\infty}^{\infty}$$

be the operator-valued weight sequence corresponding to the trigonometrically well-bounded operator U^{-1}. Suppose that we are given $N \in \mathbb{N}$, and functions ψ_n, $1 \leqslant$

$n \leqslant N$, belonging to $BV(\mathbb{T})$. If ρ is any constant such that

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n T_{\psi_n} x \right\|^2_{\ell^2(\widetilde{\mathcal{W}})} \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|x\|_{\ell^2(\widetilde{\mathcal{W}})}, \tag{4.10}$$

for all $x \in \ell^2(\widetilde{\mathfrak{W}})$, then, in the notation of Proposition 3.5, we have for all $\alpha \in \mathfrak{K}$,

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n \int_{[0,2\pi]}^{\oplus} \psi_n^{\ddagger}(t)\, \mathrm{d}E(t)\alpha \right\|^2_{\mathfrak{K}} \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|\alpha\|_{\mathfrak{K}}.$$

Proof. Let $m \in \mathbb{N}$, and let κ_m be the Fejér kernel of order m for $\mathbb{T}$. For $1 \leqslant n \leqslant N$, let $q_{mn} : \mathbb{T} \to C$ be specified by putting

$$q_{mn} = \kappa_m * \psi_n.$$

It is easy to see for each of these trigonometric polynomials q_{mn}, that, in terms of the right bilateral shift $\mathcal{R} \in \mathfrak{B}(\ell^2(\widetilde{\mathcal{W}}))$, the operator $T_{q_{mn}} \in \mathfrak{B}(\ell^2(\widetilde{\mathcal{W}}))$ coincides with $q_{mn}(\mathcal{R}^{-1})$. Hence we can apply Proposition 4.10 (in the setting of $\widetilde{\mathcal{W}}$) to κ_m and (4.10) to infer that

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n q_{mn}(\mathcal{R}^{-1}) x \right\|^2_{\ell^2(\widetilde{\mathcal{W}})} \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|x\|_{\ell^2(\widetilde{\mathcal{W}})},$$

for all $x \in \ell^2(\widetilde{\mathcal{W}})$. Using the isometry Δ described at the outset of Section 2, we can rewrite this in the form

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n q_{mn}(\mathcal{R}) x \right\|^2_{\ell^2(\mathcal{W})} \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|x\|_{\ell^2(\mathcal{W})},$$

for all $x \in \ell^2(\mathcal{W})$. We now invoke Theorem 4.8 to infer that for $m \in \mathbb{N}$,

$$\left(\int_{\mathfrak{D}^N} \left\| \sum_{n=1}^{N} \varepsilon_n q_{mn}(U) \alpha \right\|^2_{\mathfrak{K}} \mathrm{d}\varepsilon \right)^{1/2} \leqslant \rho \|\alpha\|_{\mathfrak{K}}, \tag{4.11}$$

for all $\alpha \in \mathfrak{K}$. To complete the proof of Theorem 4.11, let $m \to \infty$ in (4.11), while applying Proposition 3.5 to ψ_n and $q_{mn} = \kappa_m * \psi_n$, for $1 \leqslant n \leqslant N$. ∎

Taking $N = 1$ in Theorem 4.11, we arrive at the following result on transference of single multiplier estimates.

COROLLARY 4.12. *Suppose that $U \in \mathfrak{B}(\mathfrak{K})$ is trigonometrically well-bounded with spectral decomposition $E(\cdot)$, and let*

$$\widetilde{\mathcal{W}} = \{(U^{-j})^* U^{-j}\}_{j=-\infty}^{\infty}$$

be the operator-valued weight sequence on $\mathfrak{K}$ corresponding to the trigonometrically well-bounded operator U^{-1}. Suppose that $\psi \in BV(\mathbb{T})$. Then

$$\left\| \int_{[0,2\pi]}^{\oplus} \psi^{\ddagger}(t)\, \mathrm{d}E(t) \right\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \|T_\psi\|_{\mathfrak{B}(\ell^2(\widetilde{\mathcal{W}}))}.$$

5. THE MARCINKIEWICZ FUNCTION CLASS

Let $\{t_k\}_{k=-\infty}^{\infty}$ be the bilateral sequence of dyadic points in the interval $(0, 2\pi)$:

$$t_k = \begin{cases} 2^{k-1}\pi & \text{if } k \leqslant 0, \\ 2\pi - 2^{-k}\pi, & \text{if } k > 0. \end{cases} \tag{5.1}$$

A function $\psi : \mathbb{T} \to C$ is said to be a *Marcinkiewicz function* (in symbols, $\psi \in \mathfrak{M}(\mathbb{T})$) provided

$$\|\psi\|_{\mathfrak{M}(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\psi(z)| + \sup_{k\in\mathbb{Z}} \operatorname{var}(\psi(\mathrm{e}^{\mathrm{i}(\,\cdot\,)}), [t_k, t_{k+1}]) < \infty.$$

As is well-known, $\mathfrak{M}(\mathbb{T})$ is a unital Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathfrak{M}(\mathbb{T})}$. For the role accorded the Marcinkiewicz functions as Fourier multipliers by the classical Marcinkiewicz Multiplier Theorem (see Theorem 8.4.2, [13]).

In the Banach space setting the following result ([6], Theorem (4.4)) provides a link between the function class $\mathfrak{M}(\mathbb{T})$ and spectral integration.

PROPOSITION 5.1. *Let $E(\,\cdot\,)$ be a spectral family in a Banach space $\mathfrak{X}$. Suppose that: $E(\,\cdot\,)$ is concentrated on $[0, 2\pi]$; $\{E(s) : 0 \leqslant s \leqslant 2\pi\}$ has the R-property; and*

$$\sup\left\{\left\|\sum_{n=N}^{M} \varepsilon_n\{E(t_{n+1}) - E(t_n)\}\right\|_{\mathfrak{B}(\mathfrak{X})}\right\} < \infty, \tag{5.2}$$

where $\{t_k\}_{k=-\infty}^{\infty}$ is the sequence of dyadic points in $(0, 2\pi)$, and the supremum is extended over all $N \in \mathbb{Z}$, $M \in \mathbb{Z}$ such that $N \leqslant M$, and all choices of $\varepsilon_n = \pm 1$ for $N \leqslant n \leqslant M$. Then there is a constant ϑ such that for each $\psi \in \mathfrak{M}(\mathbb{T})$ the spectral integral

$$\int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}s})\,\mathrm{d}E(s)$$

exists, and

$$\left\|\ \overset{\oplus}{\int_{[0,2\pi]}} \psi(\mathrm{e}^{\mathrm{i}s})\,\mathrm{d}E(s)\right\| \leqslant \vartheta\|\psi\|_{\mathfrak{M}(\mathbb{T})}.$$

In the setting of $\ell^2(\mathfrak{W})$ we introduce the following definition, which formulates the natural analogue of the classical Marcinkiewicz Multiplier Theorem.

DEFINITION 5.2. Suppose that $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ is an operator-valued weight sequence on the Hilbert space $\mathfrak{K}$. We say that $\mathfrak{W}$ *has the Marcinkiewicz Multiplier Property* provided that there is a constant ζ such that each $\psi \in \mathfrak{M}(\mathbb{T})$ is a multiplier for $\ell^2(\mathfrak{W})$ having multiplier transform T_ψ satisfying

$$\|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant \zeta\|\psi\|_{\mathfrak{M}(\mathbb{T})}. \tag{5.3}$$

The following theorem lists some pleasant consequences which arise when $\mathfrak{W}$ enjoys the Marcinkiewicz Multiplier Property.

THEOREM 5.3. *Suppose that* $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ *is an operator-valued weight sequence on* $\mathfrak{K}$ *such that* $\mathfrak{W}$ *possesses the Marcinkiewicz Multiplier Property. Then* $\mathfrak{W}$ *has the Treil-Volberg Property, and the left bilateral shift* $\mathcal{L}$ *is a trigonometrically well-bounded operator on* $\ell^2(\mathfrak{W})$*. Let* $\mathcal{E}(\,\cdot\,)$ *denote the spectral decomposition of* $\mathcal{L}$*. Then if* $N \in \mathbb{Z}$*,* $M \in \mathbb{Z}$*,* $N \leqslant M$*, and* $\varepsilon_n = \pm 1$ *for* $N \leqslant n \leqslant M$*, we have in the notation* (5.1) *for the dyadic points of* $(0, 2\pi)$*:*

$$\left\| \sum_{n=N}^{M} \varepsilon_n \{\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n)\} \right\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant 3\zeta, \tag{5.4}$$

where ζ *is the constant appearing in* (5.3)*. For each* $\psi \in \mathfrak{M}(\mathbb{T})$*, the spectral integral*

$$\int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}\mathcal{E}(t)$$

exists and coincides with the corresponding multiplier transform $T_\psi \in \mathfrak{B}(\ell^2(\mathfrak{W}))$*.*

Proof. Clearly $BV(\mathbb{T}) \subseteq \mathfrak{M}(\mathbb{T})$, with each $\phi \in BV(\mathbb{T})$ satisfying

$$\|\phi\|_{\mathfrak{M}(\mathbb{T})} \leqslant 2\|\phi\|_{BV(\mathbb{T})}. \tag{5.5}$$

Specializing ϕ to be the trigonometric polynomial $q_1(z) \equiv z$ (respectively, $q_2(z) \equiv z^{-1}$), we see from the Marcinkiewicz Multiplier Property that $\mathcal{L} = T_{q_1} \in \mathfrak{B}(\ell^2(\mathfrak{W}))$ (respectively, $\mathcal{R} = T_{q_2} \in \mathfrak{B}(\ell^2(\mathfrak{W}))$). For each trigonometric polynomial q, it is now clear that $q(\mathcal{L}) = T_q$, and so by (5.3) and (5.5),

$$\|q(\mathcal{L})\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant 2\zeta \|q\|_{BV(\mathbb{T})}.$$

In view of Proposition 3.3, this shows that $\mathcal{L}$ is a trigonometrically well-bounded operator on $\ell^2(\mathfrak{W})$, and hence so is $\mathcal{R} = \mathcal{L}^{-1}$. An application of Theorem 2.5 now furnishes the Treil-Volberg Property for $\mathfrak{W}$.

Suppose next that $N \in \mathbb{Z}$, $M \in \mathbb{Z}$, $N \leqslant M$, and $\varepsilon_n = \pm 1$ for $N \leqslant n \leqslant M$. Using the bilateral sequence $\{t_k\}_{k=-\infty}^{\infty}$ of dyadic points in the interval $(0, 2\pi)$, let χ_n, $N \leqslant n \leqslant M$, be the characteristic function, defined on $\mathbb{T}$, of the arc

$$\{\mathrm{e}^{\mathrm{i}t} : t_n < t < t_{n+1}\},$$

and define $\psi \in BV(\mathbb{T}) \subseteq \mathfrak{M}(\mathbb{T})$ to be the function given by

$$\psi = \sum_{n=N}^{M} \varepsilon_n \chi_n.$$

Clearly $\|\psi\|_{\mathfrak{M}(\mathbb{T})} = 3$, and hence application of (5.3) shows that

$$\|T_\psi\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant 3\zeta. \tag{5.6}$$

In view of the description for $\mathcal{E}(\,\cdot\,)$ stated in Theorem 4.2, we can obviously rewrite (5.6) as (5.4). Moreover, we know from Corollary 4.7 (applied to $\ell^2(\mathfrak{W})$ in place of $\mathfrak{K}$) that

$$\{\mathcal{E}(t) : t \in \mathbb{R}\}$$

has the R-property. It now follows from Proposition 5.1 that for each $\Phi \in \mathfrak{M}(\mathbb{T})$, the spectral integral

$$\int_{[0,2\pi]} \Phi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}\mathcal{E}(t)$$

exists. The proof of Theorem 5.3 can now be completed by an appeal to Theorem 4.3. ∎

REMARK 5.4. In view of the characterization of $\mathcal{E}(\cdot)$ by multiplier transforms (Theorem 4.2) and the standard definition of the dyadic points in (5.1), the conclusion in Theorem 5.3 asserting the existence of a finite supremum for

$$\left\{ \left\| \sum_{n=N}^{M} \varepsilon_n \{\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n)\} \right\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} : N \in \mathbb{Z}, M \in \mathbb{Z}, N \leqslant M, \text{ and } \varepsilon_n = \pm 1 \right\}$$

can be regarded as a direct analogue of the classical Littlewood-Paley Theorem (see, e.g., Theorem 7.2.1, [13], for the classical setting). Hence we can view (5.4) as asserting a Littlewood-Paley Property for $\mathcal{E}(\cdot)$. More generally, if $E(\cdot)$ is a spectral family in a Banach space $\mathfrak{X}$ and is concentrated on $[0, 2\pi]$, we shall say that $E(\cdot)$ has the *Littlewood-Paley Property* provided that (5.2) holds. Such a state of affairs arises in the following corollary of Theorem 5.3.

COROLLARY 5.5. *Suppose that $\mathfrak{K}$ is a Hilbert space, and $U \in \mathfrak{B}(\mathfrak{K})$ is an invertible operator. Let $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$ be the corresponding operator-valued weight sequence on $\mathfrak{K}$. Suppose that $\mathcal{W}$ has the Marcinkiewicz Multiplier Property, with constant ζ as in* (5.3). *Then U is trigonometrically well-bounded. Let $E(\cdot)$ denote the spectral decomposition of U. If $N \in \mathbb{Z}$, $M \in \mathbb{Z}$, $N \leqslant M$, and $\varepsilon_n = \pm 1$ for $N \leqslant n \leqslant M$, then, in the notation of* (5.1), *we have:*

$$\left\| \sum_{n=N}^{M} \varepsilon_n \{E(t_{n+1}) - E(t_n)\} \right\|_{\mathfrak{B}(\mathfrak{K})} \leqslant 9\zeta. \tag{5.7}$$

There is a constant ϑ such that for each $\psi \in \mathfrak{M}(\mathbb{T})$, the spectral integral $\int_{[0,2\pi]} \psi(e^{it})\, dE(t)$ exists, and satisfies

$$\left\| \int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, dE(t) \right\|_{\mathfrak{B}(\mathfrak{K})} \leqslant \vartheta \|\psi\|_{\mathfrak{M}(\mathbb{T})}. \tag{5.8}$$

Moreover, the mapping

$$\psi \in \mathfrak{M}(\mathbb{T}) \longmapsto \int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, dE(t)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{T})$ into $\mathfrak{B}(\mathfrak{K})$.

Proof. Application of Theorem 5.3 to $\mathcal{W}$ shows that $\mathcal{W}$ has the Treil-Volberg Property, and so U is trigonometrically well-bounded by Corollary 2.4. Suppose $N \in \mathbb{Z}$, $M \in \mathbb{Z}$, $N \leqslant M$, and $\varepsilon_n = \pm 1$ for $N \leqslant n \leqslant M$. Let $\delta > 0$ be small enough so that for $N \leqslant n \leqslant M$, $t_n + \delta < t_{n+1}$, and $t_{n+1} + \delta < t_{n+2}$. Denote by $\chi_n \in BV(\mathbb{T})$ the characteristic function of the arc

$$\{e^{it} : t_n + \delta \leqslant t \leqslant t_{n+1} + \delta\}. \tag{5.9}$$

Define $\psi_\delta \in BV(\mathbb{T})$ by writing

$$\psi_\delta = \sum_{n=N}^{M} \varepsilon_n \chi_n, \tag{5.10}$$

and notice that

$$\|\psi_\delta\|_{\mathfrak{M}(\mathbb{T})} \leqslant 9. \tag{5.11}$$

We next define the auxiliary function $\widetilde{\psi}_\delta \in BV(\mathbb{T})$ by putting

$$\widetilde{\psi}_\delta(z) = \psi_\delta(\overline{z}) \quad \text{for all } z \in \mathbb{T}. \tag{5.12}$$

Clearly we have

$$\|\widetilde{\psi}_\delta\|_{\mathfrak{M}(\mathbb{T})} = \|\psi_\delta\|_{\mathfrak{M}(\mathbb{T})} \leqslant 9. \tag{5.13}$$

Since $\mathcal{W}$ has the Treil-Volberg Property, so does $\widetilde{\mathcal{W}} = \{(U^{-j})^* U^{-j}\}_{j=-\infty}^{\infty}$, and it follows by Corollary 4.4 that each of ψ_δ and $\widetilde{\psi}_\delta$ is a multiplier for both $\ell^2(\mathcal{W})$ and $\ell^2(\widetilde{\mathcal{W}})$. In terms of the surjective isometry $\Delta : \ell^2(\mathcal{W}) \to \ell^2(\widetilde{\mathcal{W}})$ described in conjunction with (2.2), it is readily seen that $T_{\psi_\delta} \in \mathfrak{B}(\ell^2(\widetilde{\mathcal{W}}))$ and $T_{\widetilde{\psi}_\delta} \in \mathfrak{B}(\ell^2(\mathcal{W}))$ satisfy the relation

$$T_{\widetilde{\psi}_\delta} = \Delta^{-1} T_{\psi_\delta} \Delta. \tag{5.14}$$

From (5.14), together with the Marcinkiewicz Multiplier Property for $\mathcal{W}$ and (5.13), we see that

$$\|T_{\psi_\delta}\|_{\mathfrak{B}(\ell^2(\widetilde{\mathcal{W}}))} = \|T_{\widetilde{\psi}_\delta}\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \zeta \|\widetilde{\psi}_\delta\|_{\mathfrak{M}(\mathbb{T})} \leqslant 9\zeta. \tag{5.15}$$

In view of (5.15), we can invoke Corollary 4.12 to infer that

$$\Big\| \int_{[0,2\pi]}^{\oplus} \psi_\delta^{\ddagger}(t)\,\mathrm{d}E(t) \Big\|_{\mathfrak{B}(\mathfrak{K})} \leqslant 9\zeta. \tag{5.16}$$

Simple direct calculations show that

$$\begin{aligned} \int_{[0,2\pi]}^{\oplus} \psi_\delta^{\ddagger}(t)\,\mathrm{d}E(t) &= \sum_{n=N}^{M} \varepsilon_n \int_{[0,2\pi]} \chi_n^{\ddagger}(t)\,\mathrm{d}E(t) \\ &= \sum_{n=N}^{M} \frac{\varepsilon_n}{2} \{E(t_{n+1}+\delta) + E((t_{n+1}+\delta)^-) - E(t_n+\delta) - E((t_n+\delta)^-)\}. \end{aligned}$$

Hence as $\delta \to 0^+$,

$$\int_{[0,2\pi]}^{\oplus} \psi_\delta^{\ddagger}(t)\,\mathrm{d}E(t) \to \sum_{n=N}^{M} \varepsilon_n \{E(t_{n+1}) - E(t_n)\},$$

in the strong operator topology of $\mathfrak{B}(\mathfrak{K})$. Using this in (5.16) we arrive at (5.7).

By virtue of Proposition 5.1, the conclusion in (5.8) follows from (5.7) and Corollary 4.7. All the remaining conclusions are evident except for the multiplicativity of spectral integrals in the present setting. Suppose then that $\psi_1 \in \mathfrak{M}(\mathbb{T})$, and $\psi_2 \in \mathfrak{M}(\mathbb{T})$. If

$$0 < a < b < 2\pi,$$

then it is easy to see that for $j = 1, 2$,

$$\int\limits_{[a,b]} \psi_j(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t) = \int\limits_{[0,2\pi]} \psi_j(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\{E(b) - E(a)\}.$$

Since $\psi_j(\mathrm{e}^{\mathrm{i}(\,\cdot\,)}) \in BV([a,b])$ for $j = 1, 2$, we also have

$$\Big(\int\limits_{[a,b]} \psi_1(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)\Big(\int\limits_{[a,b]} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big) = \int\limits_{[a,b]} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t),$$

and so

$$\begin{aligned}\Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)&\Big(\int\limits_{[0,2\pi]} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)(E(b) - E(a))\\ &= \Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)(E(b) - E(a)).\end{aligned}$$

Letting $b \to (2\pi)^-$ and $a \to 0^+$, we obtain from this the relation

$$\begin{aligned}\Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)&\Big(\int\limits_{[0,2\pi]} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)(I - E(0))\\ &= \Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)(I - E(0)).\end{aligned}$$

But

$$\Big(\int\limits_{[0,2\pi]} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)E(0) = \Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)E(0) = 0,$$

and so

$$\Big(\int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)\Big(\int\limits_{[0,2\pi]} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big) = \int\limits_{[0,2\pi]} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t).$$

Hence

$$\Big(\int\limits_{[0,2\pi]}^{\oplus} \psi_1(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big)\Big(\int\limits_{[0,2\pi]}^{\oplus} \psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\Big) = \int\limits_{[0,2\pi]}^{\oplus} \psi_1(\mathrm{e}^{\mathrm{i}t})\psi_2(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t). \qquad \blacksquare$$

We now take up some examples where the hypotheses of Theorem 5.3 are fulfilled. One straightforward example occurs when $\mathfrak{W} = \{W_j\}_{j=-\infty}^{\infty}$ is an operator-valued weight sequence on $\mathfrak{K}$ such that the right bilateral shift $\mathcal{R}$ is an invertible power-bounded operator on $\ell^2(\mathfrak{W})$— that is,

$$c \equiv \sup_{n\in\mathbb{Z}} \|\mathcal{R}^n\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} < \infty. \tag{5.17}$$

Since $\ell^2(\mathfrak{W})$, being a Hilbert space, is automatically a UMD space, it follows from Theorem (4.5), [11], that $\mathcal{R}$ is trigonometrically well-bounded (alternatively, by Sz.-Nagy's Theorem for bounded abelian groups of Hilbert space operators ([12], Theorem 8.1), $\mathcal{R}$ is similar to a unitary operator, and so the condition in Proposition 3.3 (ii) holds). By Theorem 2.5 $\mathfrak{W}$ has the Treil-Volberg Property. Using the spectral decomposition $\mathcal{E}(\,\cdot\,)$ of the power-bounded, trigonometrically well-bounded operator $\mathcal{L} = \mathcal{R}^{-1}$, we can apply Theorem (1.1) (ii), [6], to get a constant η such that for each $\psi \in \mathfrak{M}(\mathbb{T})$,

$$\int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}\mathcal{E}(t)$$

exists, with

$$\left\| \int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}\mathcal{E}(t) \right\|_{\mathfrak{B}(\ell^2(\mathfrak{W}))} \leqslant \eta \|\psi\|_{\mathfrak{M}(\mathbb{T})}.$$

Reference to Theorem 4.3 now shows that each $\psi \in \mathfrak{M}(\mathbb{T})$ is a multiplier for $\ell^2(\mathfrak{W})$, with

$$T_\psi = \int_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}\mathcal{E}(t).$$

So it is clear that in the present set-up $\mathfrak{W}$ has the Marcinkiewicz Multiplier Property. However, the following reasoning makes it clear that the context of this example is rather special. Let $\alpha \in \mathfrak{K}$, and define $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell_0(\mathfrak{K})$ by taking $x_0 = \alpha$, and $x_j = 0$ for $j \neq 0$. Then for each $k \in \mathbb{Z}$,

$$\|\mathcal{R}^k x\|^2_{\ell^2(\mathfrak{W})} = \langle W_k \alpha, \alpha \rangle.$$

Hence for $N \in \mathbb{Z}$, $j \in \mathbb{Z}$, $M \in \mathbb{Z}$, with $N \leqslant j \leqslant M$, we have, with the aid of the power-boundedness of $\mathcal{R}$ assumed in (5.17),

$$\begin{aligned} \frac{1}{M-N+1} \sum_{k=N}^{M} \langle W_k \alpha, \alpha \rangle &= \frac{1}{M-N+1} \sum_{k=N}^{M} \|\mathcal{R}^k x\|^2_{\ell^2(\mathfrak{W})} \leqslant c^2 \|x\|^2_{\ell^2(\mathfrak{W})} \\ &\leqslant c^4 \|\mathcal{R}^j x\|^2_{\ell^2(\mathfrak{W})} = c^4 \langle W_j \alpha, \alpha \rangle. \end{aligned}$$

In the terminology of [9], this shows that $\mathfrak{W} \in A_1(\mathfrak{K})$. It follows by Corollary 2.31, [9], together with another application of (5.17), that

$$\sup_{k \in \mathbb{Z}} \|W_k\|_{\mathfrak{B}(\mathfrak{K})} + \sup_{k \in \mathbb{Z}} \|W_k^{-1}\|_{\mathfrak{B}(\mathfrak{K})} < \infty.$$

Consequently when (5.17) holds, the vector spaces $\ell^2(\mathfrak{W})$ and $\ell^2(\mathfrak{K})$ coincide, and the norms $\|\cdot\|_{\ell^2(\mathfrak{W})}, \|\cdot\|_{\ell^2(\mathfrak{K})}$ are equivalent. Hence in this framework the Marcinkiewicz Multiplier Property for $\mathfrak{W}$ merely states the abstract Marcinkiewicz Multiplier Theorem for $\ell^2(\mathfrak{K})$ (which holds by virtue of Theorem (4.5), [6], applied to the special case of the UMD space $\mathfrak{K}$).

We now take up a more delicate class of examples where the hypotheses of Corollary 5.5, and thereby the hypotheses of Theorem 5.3, are fulfilled. We refer the reader to [8] for the relevant terminology and background facts, which, for the present purposes, are specialized from the case $1 < p < \infty$ to the Hilbert space setting of $p = 2$. The transference result in Theorem 2.3 will also play a key role.

Theorem 5.6. *Suppose that (X, μ) is a sigma-finite measure space. Let $\mathfrak{K} = L^2(\mu)$ and let $U \in \mathfrak{B}(L^2(\mu))$ be an invertible, separation-preserving operator whose linear modulus $|U|$ satisfies the condition*

$$\sup_{n \geqslant 0} \left\| \frac{1}{2n+1} \sum_{j=-n}^{n} |U|^j \right\|_{\mathfrak{B}(L^2(\mu))} < \infty.$$

Let $\mathcal{W} = \{(U^j)^* U^j\}_{j=-\infty}^{\infty}$ be the operator-valued weight sequence on $\mathfrak{K}$ associated with U. Then $\mathcal{W}$ has the Marcinkiewicz Multiplier Property.

Proof. We begin by observing that by Theorem (4.2), [8], U is trigonometrically well-bounded, and so by Corollary 2.4 above, the right bilateral shift $\mathcal{R}$ and the left bilateral shift $\mathcal{L}$ are trigonometrically well-bounded operators on $\ell^2(\mathcal{W})$, and $\mathcal{W}$ has the Treil-Volberg Property. Let $\{h_j\}_{j=-\infty}^{\infty}$, $\{\Phi_j\}_{j=-\infty}^{\infty}$, and $\{J_j\}_{j=-\infty}^{\infty}$ be associated with $\{U_j\}_{j=-\infty}^{\infty}$ as in Section 2, [8]. With suitable modifications, we now follow the pattern of reasoning used in the proof of Lemma (5.5), [8]. Let $N \in \mathbb{Z}$, $M \in \mathbb{Z}$, $N \leqslant M$, and $\varepsilon_n = \pm 1$ for $N \leqslant n \leqslant M$. Let $\{t_k\}_{k=-\infty}^{\infty}$ be the sequence of dyadic points in $(0, 2\pi)$ given by (5.1). Choose $\delta > 0$ as was done at the outset of the proof for Corollary 5.5, and let $\psi_\delta \in BV(\mathbb{T})$, $\widetilde{\psi}_\delta \in BV(\mathbb{T})$, be the functions defined by (5.9), (5.10), and (5.12). In particular, ψ_δ, $\widetilde{\psi}_\delta$ satisfy the estimate in (5.13). Fix $w \in \mathbb{T}$, and for each $m \in \mathbb{N}$, let κ_m be the Fejér kernel of order m for $\mathbb{T}$, and define the trigonometric polynomial τ_m by writing

$$\tau_m(z) = \kappa_m(wz), \quad \text{for all } z \in \mathbb{T}.$$

Let $f \in \mathfrak{K} = L^2(\mu)$, and temporarily fix $m \in \mathbb{N}$. For each $L \in \mathbb{N}$, we have with the aid of (2.14), [8]:

$$\begin{aligned}
&\int_X \left| \sum_{j=-m}^{m} \widehat{\tau}_m(j) \big(\widetilde{\psi}_\delta\big)^\wedge(j) U^j f \right|^2 \mathrm{d}\mu \\
&= \frac{1}{2L+1} \int_X \sum_{v=-L}^{L} \left| \Phi_v \bigg(\sum_{j=-m}^{m} \overset{\vee}{\tau}_m(j) \big(\widetilde{\psi}_\delta\big)^\vee(j) U^{-j} f \bigg) \right|^2 J_v \,\mathrm{d}\mu \\
&= \frac{1}{2L+1} \int_X \sum_{v=-L}^{L} \left| \Phi_v \bigg(\sum_{j=-m}^{m} \overset{\vee}{\tau}_m(j) \big(\widetilde{\psi}_\delta\big)^\vee(j) h_{-j} \Phi_{-j}(f) \bigg) \right|^2 J_v \,\mathrm{d}\mu \qquad (5.18) \\
&= \frac{1}{2L+1} \int_X \sum_{v=-L}^{L} \left| \sum_{j=-m}^{m} \overset{\vee}{\tau}_m(j) \big(\widetilde{\psi}_\delta\big)^\vee(j) \Phi_v(h_{-j}) \Phi_{v-j}(f) \right|^2 J_v \,\mathrm{d}\mu \\
&= \frac{1}{2L+1} \int_X \sum_{v=-L}^{L} \left| \sum_{j=-m}^{m} \overset{\vee}{\tau}_m(j) \big(\widetilde{\psi}_\delta\big)^\vee(j) h_{v-j} \Phi_{v-j}(f) \right|^2 |h_\nu|^{-2} J_v \,\mathrm{d}\mu.
\end{aligned}$$

We now examine the behavior of the integrand in the last member of (5.18). Denote by $\chi_{L,m}$ the characteristic function, defined on $\mathbb{Z}$, of $\{k \in \mathbb{Z} : |k| \leqslant L + m\}$. We have μ-a.e. on X:

$$\begin{aligned}\sum_{v=-L}^{L}\left|\sum_{j=-m}^{m}\overset{\vee}{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\vee}(j)h_{v-j}\Phi_{v-j}(f)\right|^2|h_\nu|^{-2}J_v \\ = \sum_{v=-L}^{L}\left|\sum_{j=-\infty}^{\infty}\overset{\vee}{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\vee}(j)\chi_{L,m}(v-j)h_{v-j}\Phi_{v-j}(f)\right|^2|h_\nu|^{-2}J_v.\end{aligned} \tag{5.19}$$

In the right member of (5.19), we now apply the Marcinkiewicz Multiplier Theorem for the Lebesgue (sequence) spaces associated with scalar-valued A_p weight sequences, $1 < p < \infty$ (see Theorem (5.1), [8], which is essentially due to Kurtz ([16])), combined with the estimate in (5.13) above and Theorems (3.2) (iii) and (5.2), [8]. This procedure shows that there is a constant θ, depending only on U, such that μ-a.e. on X ,

$$\sum_{v=-L}^{L}\left|\sum_{j=-m}^{m}\overset{\vee}{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\vee}(j)h_{v-j}\Phi_{v-j}(f)\right|^2|h_\nu|^{-2}J_v \leqslant \theta\sum_{j=-L-m}^{L+m}|\Phi_j(f)|^2J_j.$$

Using this estimate on the right of (5.18), we find that

$$\begin{aligned}\int_X\left|\sum_{j=-m}^{m}\widehat{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\wedge}(j)U^jf\right|^2 \mathrm{d}\mu &\leqslant \frac{\theta}{2L+1}\sum_{j=-L-m}^{L+m}\int_X|\Phi_j(f)|^2J_j\,\mathrm{d}\mu \\ &= \frac{\theta}{2L+1}\sum_{j=-L-m}^{L+m}\int_X\Phi_j(|f|^2)J_j\,\mathrm{d}\mu,\end{aligned}$$

and so, by (2.11), [8], applied to the majorant in this estimate, we have

$$\int_X\left|\sum_{j=-m}^{m}\widehat{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\wedge}(j)U^jf\right|^2 \mathrm{d}\mu \leqslant \theta\left(\frac{2L+2m+1}{2L+1}\right)\|f\|^2_{L^2(\mu)}. \tag{5.20}$$

We now let $L \to \infty$ on the right of (5.20) to obtain for $m \in \mathbb{N}$, and each $f \in L^2(\mu)$,

$$\left\|\sum_{j=-m}^{m}\widehat{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\wedge}(j)U^jf\right\|_{L^2(\mu)} \leqslant \theta^{1/2}\|f\|_{L^2(\mu)},$$

and so for all $m \in \mathbb{N}$,

$$\left\|\sum_{j=-m}^{m}\widehat{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\wedge}(j)U^j\right\|_{\mathfrak{B}(L^2(\mu))} \leqslant \theta^{1/2}. \tag{5.21}$$

From the definition of τ_m, it is clear that

$$\sum_{j=-m}^{m}\widehat{\tau}_m(j)\big(\widetilde{\psi}_\delta\big)^{\wedge}(j)U^j = (\kappa_m * \widetilde{\psi}_\delta)(wU).$$

Substitution of this in (5.21) yields

$$\sup_{w\in\mathbb{T}}\|(\kappa_m * \widetilde{\psi}_\delta)(wU)\|_{\mathfrak{B}(L^2(\mu))} \leqslant \theta^{1/2}. \tag{5.22}$$

Using (5.22), we now apply (2.3) to the trigonometric polynomial $(\kappa_m * \widetilde{\psi}_\delta)$, and thereby infer that

$$\|(\kappa_m * \widetilde{\psi}_\delta)(\mathcal{R})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \theta^{1/2}.$$

From the relationship (5.12) between ψ_δ and $\widetilde{\psi}_\delta$ it is immediate that

$$(\kappa_m * \widetilde{\psi}_\delta)(\mathcal{R}) = (\kappa_m * \psi_\delta)(\mathcal{L}),$$

and so we have shown that for all sufficiently small $\delta > 0$, we have

$$\|(\kappa_m * \psi_\delta)(\mathcal{L})\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \theta^{1/2} \quad \text{for all } m \in \mathbb{N}. \tag{5.23}$$

Letting $m \to \infty$ in (5.23) while applying Proposition 3.5 to the trigonometrically well-bounded operator $\mathcal{L}$ and its spectral decomposition $\mathcal{E}(\,\cdot\,)$, we find that for all sufficiently small $\delta > 0$,

$$\Big\| \int_{[0,2\pi]}^{\oplus} \psi_\delta^{\ddagger}(t)\, \mathrm{d}\mathcal{E}(t) \Big\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \theta^{1/2}. \tag{5.24}$$

By letting $\delta \to 0^+$ in (5.24) we can now use reasoning analogous to that occurring right after (5.16) in the proof of (5.7), and thereby deduce that

$$\Big\| \sum_{n=N}^{M} \varepsilon_n \{\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n)\} \Big\|_{\mathfrak{B}(\ell^2(\mathcal{W}))} \leqslant \theta^{1/2}. \tag{5.25}$$

The proof of Theorem 5.6 can now be completed by using (5.25) and Corollary 4.7 (applied to $\mathcal{L}$ and $\mathcal{E}(\cdot)$) in conjunction with Proposition 5.1 and Theorem 4.3. ∎

6. AN EXAMPLE: FAILURE OF THE MARCINKIEWICZ MULTIPLIER PROPERTY AND THE LITTLEWOOD-PALEY PROPERTY

In classical single-variable Fourier analysis, as well as in its generalizations to weighted norm inequalities and to Lebesgue spaces of vector-valued functions, the boundedness of the relevant Hilbert transform insures that the natural analogues of the Marcinkiewicz Multiplier Theorem and the Littlewood-Paley Theorem hold ([13], [15], [16], [6]). In marked contrast to this state of affairs, we shall show in this section that there is an operator-valued weight sequence $\mathcal{W}^{(0)}$ on a Hilbert space $\mathfrak{K}_0$ such that $\mathcal{W}^{(0)}$ possesses the Treil-Volberg Property, but $\mathcal{W}^{(0)}$ does not possess the Marcinkiewicz Multiplier Property, and the natural analogue of the Littlewood-Paley Theorem does not hold for $\ell^2(\mathcal{W}^{(0)})$. Broadly speaking, such an example can arise because the general framework of operator-valued weight sequences is highly non-commutative in nature, and lacks the scope for suitable analogues of classical maximal functions such as the Hardy-Littlewood maximal operator.

Let $\mathfrak{K}_0$ be the usual separable Hilbert space $\ell^2(\mathbb{N})$ consisting of all unilateral sequences of complex numbers $x = \{x_k\}_{k=1}^{\infty}$ such that

$$\|x\|_{\ell^2(\mathbb{N})}^2 = \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

The example we shall present, which continues the reasoning in (5.36), [6], has its origins in the interplay (treated in [14]) between spectral theory and a suitable conditional basis for $\mathfrak{K}_0$. From the construction of this conditional basis it was shown in [14] that there is a sequence of rank-one idempotent operators $\{P_n\}_{n=1}^{\infty} \subseteq \mathfrak{B}(\mathfrak{K}_0)$ such that:

(i) $P_n P_m = 0$ for $n \neq m$;

(ii) $\sum\limits_{n=1}^{\infty} P_n$ converges to the identity operator in the strong operator topology of $\mathfrak{B}(\mathfrak{K}_0)$;

(iii) $\Big\| \sum\limits_{j=1}^{n} P_{2j} \Big\|_{\mathfrak{B}(\mathfrak{K}_0)} \to \infty$ as $n \to \infty$.

We define the strictly decreasing sequence $\{\xi_n\}_{n=1}^{\infty} \subseteq (0, 2\pi)$ by putting $\xi_n = 2^{-n-1}\pi$ for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, ξ_n is the dyadic point t_{-n} in (5.1). For $t \in (0, \xi_1]$, let N be the smallest $N \in \mathbb{N}$ such that $\xi_N \leqslant t$, and define $E_0(t)$ to be the strongly convergent series $\sum\limits_{n=N}^{\infty} P_n$. We also put $E_0(t) = 0$ for $t \leqslant 0$, and $E_0(t) = I$ for $t > \xi_1$. It is readily verified that $E_0(\cdot)$ is a spectral family of projections in $\mathfrak{K}_0$ concentrated on $[0, \xi_1]$ $=[0, \pi/4]$. Now let $U_0 \in \mathfrak{B}(\mathfrak{K}_0)$ be defined by writing $U_0 = \int\limits_{[0,2\pi]}^{\oplus} \mathrm{e}^{\mathrm{i}t} \, \mathrm{d}E_0(t)$. Thus U_0 is a trigonometrically well-bounded operator on $\mathfrak{K}_0$ having $E_0(\,\cdot\,)$ as its spectral decomposition. (We remark in passing that, as shown in (5.10), [4], $\sup\{\|U_0^j\|_{\mathfrak{B}(\mathfrak{K}_0)} : j \in \mathbb{Z}\} = \infty$.) It is easy to see directly from the definition of $E_0(\,\cdot\,)$ that if $\psi : \mathbb{T} \to \mathbb{C}$ is a bounded function such that $\int\limits_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}E_0(t)$ exists, then we have (with series convergence in the strong operator topology),

$$\text{(6.1)} \qquad \int\limits_{[0,2\pi]}^{\oplus} \psi(\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}E_0(t) = \int\limits_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}E_0(t) = \sum_{n=1}^{\infty} \psi(\mathrm{e}^{\mathrm{i}\xi_n}) P_n.$$

Since U_0 is trigonometrically well-bounded, its associated operator-valued weight sequence

$$\text{(6.2)} \qquad \mathcal{W}^{(0)} = \{(U_0^j)^* U_0^j\}_{j=-\infty}^{\infty}$$

has the Treil-Volberg Property, by Corollary 2.4. Nevertheless, we have the following result.

THEOREM 6.1. *The operator-valued weight sequence* $\mathcal{W}^{(0)}$ *given by* (6.2) *does not possess the Marcinkiewicz Multiplier Property, and the spectral decomposition* $\mathcal{E}(\,\cdot\,)$ *of the left bilateral shift* $\mathcal{L} \in \mathfrak{B}(\ell^2(\mathcal{W}^{(0)}))$ *does not possess the Littlewood-Paley Property defined in Remark* 5.4.

Proof. If $\mathcal{W}^{(0)}$possessed the Marcinkiewicz Multiplier Property, then by Corollary 5.5, for each $\psi \in \mathfrak{M}(\mathbb{T})$, the spectral integral

$$\int\limits_{[0,2\pi]} \psi(\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}E_0(t)$$

would exist. However, this conclusion is false, as was shown in (5.36), [6]. The failure of this conclusion can also be readily seen as follows. For each $n \in \mathbb{Z}$, let χ_n be the characteristic function, defined on $\mathbb{T}$, of the dyadic arc $\{\mathrm{e}^{\mathrm{i}t} : t_{n-1} < t \leqslant t_n\}$. Clearly $\mathfrak{M}(\mathbb{T})$ contains the function ϕ specified by writing $\phi = \sum_{n=-\infty}^{\infty} (-1)^n \chi_n$. If $\int_{[0,2\pi]} \phi(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}E_0(t)$ existed, then by (6.1) the series $\sum_{n=1}^{\infty} (-1)^n P_n$ would converge in the strong operator topology. Taken in combination with Property (ii) listed above for the sequence $\{P_n\}_{n=1}^{\infty}$, this would imply the strong convergence of the series $\sum_{n=1}^{\infty} P_{2n}$, in contradiction to Property (iii) of $\{P_n\}_{n=1}^{\infty}$.

Since $\mathcal{E}(\,\cdot\,)$ has the R-property by Corollary 4.7 (applied to $\mathcal{L}$ and $\ell^2(\mathcal{W}^{(0)})$), the failure of the Marcinkiewicz Multiplier Property for $\mathcal{W}^{(0)}$ implies by Proposition 5.1 and Theorem 4.3 that $\mathcal{E}(\,\cdot\,)$ does not have the Littlewood-Paley Property. ∎

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REFERENCES

1. E. Berkson, J. Bourgain, T.A. Gillespie, On the almost everywhere convergence of ergodic averages for power-bounded operators on L^p-subspaces, *Integral Equations Operator Theory* **14**(1991), 678–715.
2. E. Berkson, T.A. Gillespie, *AC* functions on the circle and spectral families, *J. Operator Theory* **13**(1985), 33–47.
3. E. Berkson, T.A. Gillespie, Fourier series criteria for operator decomposability, *Integral Equations Operator Theory* **9**(1986), 767–789.
4. E. Berkson, T.A. Gillespie, Stečkin's theorem, transference, and spectral decompositions, *J. Funct. Anal.* **70**(1987), 140–170.
5. E. Berkson, T.A. Gillespie, The spectral decomposition of weighted shifts and the A_p condition, *Colloq. Math.* (special issue dedicated to A. Zygmund), **60**–**61**(1990), 507–518.
6. E. Berkson, T.A. Gillespie, Spectral decompositions and harmonic analysis on UMD spaces, *Studia Math.* **112**(1994), 13–49.
7. E. Berkson, T.A. Gillespie, The q-variation of functions and spectral integration of Fourier multipliers, *Duke Math. J.* **88**(1997), 103–132.
8. E. Berkson, T.A. Gillespie, Mean-boundedness and Littlewood-Paley for separation-preserving operators, *Trans. Amer. Math. Soc.* **349**(1997), 1169–1189.
9. E. Berkson, T.A. Gillespie, Mean$_2$-bounded operators on Hilbert space and weight sequences of positive operators, *Positivity* **3**(1999), 101–133.
10. E. Berkson, T.A. Gillespie, Spectral decompositions, ergodic averages, and the Hilbert transform, *Studia Math.* **144**(2001), 39–61.
11. E. Berkson, T.A. Gillespie, P.S. Muhly, Abstract spectral decompositions guaranteed by the Hilbert transform, *Proc. London Math. Soc. (3)* **53**(1986), 489–517.
12. H.R. Dowson, *Spectral Theory of Linear Operators*, London Math. Soc. Monogr., vol. 12, Academic Press, New York 1978.

13. R.E. Edwards, G.I. Gaudry, *Littlewood-Paley and Multiplier Theory*, Ergeb. Math. Grenzgeb., vol. 90, Springer, Berlin 1977.
14. T.A. Gillespie, Commuting well-bounded operators on Hilbert spaces, *Proc. Edinburgh Math. Soc. (2)* **20**(1976), 167–172.
15. R. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.* **176**(1973), 227–251.
16. D.S. Kurtz, Littlewood-Paley and multiplier theorems on weighted L^p spaces, *Trans. Amer. Math. Soc.* **259**(1980), 235–254.
17. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces: Sequence Spaces.* I, Ergeb. Math. Grenzgeb., vol. 92, Springer, Berlin 1977.
18. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces: Function Spaces.* II, Ergeb. Math. Grenzgeb., vol. 97, Springer, Berlin 1979.
19. S. Treil, A. Volberg, Wavelets and the angle between past and future, *Math. Gottingensis* **7**(1995), 1–33.
20. S. Treil, A. Volberg, Wavelets and the angle between past and future, *J. Funct. Anal.* **143**(1997), 269–308.

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