AUTOMORPHISMS OF FREE PRODUCT-TYPE
AND THEIR CROSSED-PRODUCTS

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ABSTRACT. A continuous family of non-outer conjugate aperiodic automorphisms whose crossed-products are all isomorphic is given on every interpolated free group factor. An explicit "duality" relationship between compact co-commutative Kac algebra (minimal) free product actions and free shift actions is also discussed.

KEYWORDS: *Free product, crossed-product, aperiodic automorphism, amalgamated free product, free shift, interpolated free group factor, free Araki-Woods factor.*

MSC (2000): 46L54, 46L40.

1. INTRODUCTION

In this notes, we will study, as a natural continuation of [31], actions of the following type: Let P, Q and N be von Neumann algebras and α be an action on N of a group-like object G such as groups or Kac algebras. Our objective here is to analyze the natural extended action $\widetilde{\alpha}$ of G on the free product von Neumann algebra $M := (P \otimes N) * Q$ defined by $\widetilde{\alpha} = (\mathrm{Id}_P \otimes \alpha) * \mathrm{Id}_Q$.

As a simple application, we would like to point out that every interpolated free group factor $L(\mathbb{F}_r)$ (with arbitrary $r \in (1, \infty]$) has continuously many non-outer conjugate aperiodic automorphisms whose crossed-products are all isomorphic. This is a simple supplementary remark to a famous result of J. Phillips ([16]; also see [17]). The same is true for a large class of free Araki-Woods factors (including all the unique type III_λ cases with $\lambda \neq 0, 1$) too. We also discuss an explicit relationship between the (minimal) action of compact co-commutative Kac algebra $\mathbb{K}_G$ (coming from a discrete group G) considered in [31] and a certain G-free shift action. This simple but interesting observation by the first-named author was the starting point of this joint work.

2. APERIODIC AUTOMORPHISMS OF INTERPOLATED FREE GROUP FACTORS

Let us assume that (Q, τ_Q) is one of the following:

(1) an interpolated free group factor $L(\mathbb{F}_r)$, $1 < r \leqslant \infty$, with the unique tracial state;

(2) the group von Neumann algebra $L(\mathbb{Z})$ with the canonical tracial state;

(3) a finite direct sum of the trivial algebra $\mathbb{C}$ with a faithful (tracial) state, whose free dimension (see [7], [9]) is in $(0, 1)$.

Then we will consider the free product (see [36])

$$(2.1) \qquad (M, \tau_M) := (R \otimes L^\infty(\mathbb{T}), \tau_R \otimes \mu) * (Q, \tau_Q),$$

where R is the hyperfinite type II_1 factor and μ is the expectation by the Haar probability measure on the 1-dimensional torus $\mathbb{T}$. A result of K. Dykema (see [7]) says that every interpolated free group factor (see [8] and [22]) can be realized in this way. Note that the algebra $L^\infty(\mathbb{T})$ has the canonical generator $u(z) = z$, $z \in \mathbb{T}$, being a Haar unitary, i.e., $\mu(u^n) = 0$ as long as $n \neq 0$.

Then the free product von Neumann algebra M has the following natural action of the 1-dimensional torus $\mathbb{T}$:

$$\alpha_z|R \cup Q := \text{the trivial action}, \quad \alpha_z(u) := zu \text{ (the multiplication of } z)$$

for $z \in \mathbb{T}$. This action is nothing less than a free product action considered in [31] and [33] (also see [28]) for the other purpose.

LEMMA 2.1. (i) *The action α is continuous in the p-topology (or equivalently, in the u-topology).*

(ii) *The action α is outer, that is, $\alpha_z \notin \mathrm{Int}(M)$ for any $z \neq 1$.*

Proof. (i) Straightforward.

(ii) It suffices to show that $(M^{(\alpha, \mathbb{T})})' \cap M = \mathbb{C}1$, where $M^{(\alpha, \mathbb{T})}$ denotes the fixed-point algebra of the action α of $\mathbb{T}$. By Corollary 2 of [34] (based on the idea of Lemma 2.5 in [18], or the use of Theorem 4.1 of [19]) we have $(R \otimes \mathbb{C}1)' \cap M = \mathbb{C}1_R \otimes L^\infty(\mathbb{T})$. Since both the free components do never commute with and since $R \otimes \mathbb{C}$ and Q sit in $M^{(\alpha, \mathbb{T})}$, the relative commutant in question must be trivial. ∎

REMARK 2.2. (The free group factor version of Blattner's result in [1]) Let G be a separable locally compact group G. If the torus action on $L^\infty(\mathbb{T})$ was replaced, in the above construction, by a faithful action of G on a hyperfinite von Neumann algebra, then one would get an outer (continuous) action of G on an interpolated free group factor $L(\mathbb{F}_r)$ with arbitrary $r \in (1, \infty]$. Note here that R.-J. Blattner ([1]; also see [30], p. 47) showed that any separable locally compact group can outerly act on the hyperfinite type II_1 factor R so that the construction does really work. Hence, Blattner's result mentioned just above remains still valid for any interpolated free group factor. This observation was the initial motivation of the work ([31]).

LEMMA 2.3. (cf. [16]) *Let $z_1, z_2 \in \mathbb{T}$ be irrationals, i.e., $z_k = e^{2\pi i \theta_k}$ with $\theta_k \notin \mathbb{Q}$, $k = 1, 2$. If neither $z_1 = z_2$ nor $z_1 = \bar{z}_2$ holds, then the corresponding aperiodic automorphisms α_{z_1} and α_{z_2} never become outer conjugate to each other.*

Proof. We make use of the idea given in [16]. Namely the τ-invariant (an invariant for outer conjugacy introduced in [3] for modular actions) will be used to distinguish. The τ-invariant $\tau(M, \alpha_{z_k})$, $k = 1, 2$, is the weakest topology on $\mathbb{Z}$ making the mapping $n \mapsto \alpha_{z_k}^n \in \text{Out}(M)$ continuous, where the quotient $\text{Out}(M) = \text{Aut}(M)/\text{Int}(M)$ has the perfect topological sense since M is known to be a full factor. It is straightforward to see that $\tau(M, \alpha_{z_k})$ is captured as the weak topology on $\mathbb{Z}$ induced by the mapping $n \mapsto z_k^n \in \mathbb{T}$ thanks to Lemma 2.1 (i). Hence Lemma 1.3 in [16] does work. Here, note that $z_1 \neq z_2, \bar{z}_2$ means $\{z_1^n : n \in \mathbb{Z}\} \neq \{z_2^n : n \in \mathbb{Z}\}$. ∎

Here we should mention that the τ-invariant has been examined in Section 8 of [27] by D. Shlyakhtenko for some automorphisms on free products of von Neumann algebras. The result there may be useful for further investigations.

LEMMA 2.4. (cf. Proposition 1, [34]) *If z is irrational, i.e., $z = e^{2\pi i \theta}$ with $\theta \notin \mathbb{Q}$, then the crossed-product $M \rtimes_{\alpha_z} \mathbb{Z}$ is isomorphic to the amalgamated free product (see [19], [32], [36])*

$$(2.2) \qquad (R \otimes (L^\infty(\mathbb{T}) \rtimes_{\alpha_z} \mathbb{Z})) *_{\mathbb{C} \rtimes \mathbb{Z}} (Q \otimes L(\mathbb{Z})),$$

and its isomorphism class does not depend on the choice of z (or say θ). Here, the amalgamated free product in (2.2) is taken with respect to the conditional expectations $\tau_R \otimes E_\theta$ and $\tau_Q \otimes \text{Id}_{L(\mathbb{Z})}$ with the canonical conditional expectation $E_\theta : L^\infty(\mathbb{T}) \rtimes_{\alpha_z} \mathbb{Z} \to \mathbb{C} \rtimes \mathbb{Z}$.

Proof. By Proposition 1 of [34], it suffices to show that the amalgamated free product in (2.2) does not depend on the choice of z. We first note that

$$L^\infty(\mathbb{T}) \rtimes_{\alpha_z} \mathbb{Z} = \{u, v\}'' \cong A_\theta'',$$

where v denotes the generator of $\mathbb{Z}$ in the crossed-product so that $vu = zuv = e^{2\pi i \theta} uv$, and A_θ'' means the weak-closure of the irrational rotation algebra A_θ via the GNS-representation associated with the unique tracial state. Both the abelian subalgebras $\{u\}''$ and $\{v\}''$ are known to be Cartan subalgebras in A_θ'', and hence the inclusion

$$L^\infty(\mathbb{T}) \rtimes_{\alpha_z} \mathbb{Z} \supseteq \mathbb{C} \rtimes \mathbb{Z}$$

forms a pair of the hyperfinite type II_1 factor and a Cartan subalgebra. A. Connes, J. Feldman and B. Weiss' result ([5]) says that all the Cartan subalgebras in any fixed hyperfinite factor are conjugate, which implies the assertion. ∎

Summing up the above three lemmas, we get the following theorem.

Theorem 2.5. *Every interpolated free group factor has continuously many non-outer conjugate aperiodic automorphisms whose crossed-products are all isomorphic type* II_1 *factors.*

We further investigate the crossed-product of M by an aperiodic automorphism of the form α_z with irrational $z \in \mathbb{T}$. Thanks to Lemma 2.4 (and the proof), we may investigate the following amalgamated free product:

$$N := (R \otimes R_0) *_D (Q \otimes D),$$

where $R_0 \supseteq D$ is a (unique) pair of the hyperfinite type II_1 factor and a Cartan subalgebra.

Theorem 2.6. (cf. Theorem 8, [35]) *Let* ω *be a free ultrafilter. Then the crossed-product* $N = M \rtimes_{\alpha_z} \mathbb{Z}$ *with irrational* z *satisfies*

$$(2.3) \qquad\qquad N_\omega = N' \cap N^\omega = R_0' \cap D^\omega,$$

which implies that N *has the property* Γ *(i.e., not full) but not McDuff* ([13]).

Proof. We make use of the same idea as in [35]. Since R is a type II_1 factor, we can choose a unitary u in R ($= R \otimes \mathbb{C}1 \subseteq M$) with $\tau_R(u^n) = 0$ as long as $n \neq 0$. Towards making use of the above-mentioned idea, we have to choose an invertible element y in Q with $\tau_Q(y) = 0$. If (Q, τ_Q) is as in the case (1) or (2), then one can choose a Haar unitary so that there is no problem. When (Q, τ_Q) is as in the case (3), we write

$$Q = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} \quad (n \text{ times}),$$

and the (faithful) tracial state comes from a vector of weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$. The above invertible y should be a vector $\mu = (\mu_1, \mu_2, \ldots \ldots, \mu_n)$ in $\mathbb{C}^n \cong Q$ such that all the μ_i's are non-zero and μ is orthogonal to λ with respect to the usual inner product on $\mathbb{C}^n$. For choosing such a vector μ, it suffices to show that the complement $(\mathbb{C}\lambda)^\perp$ is not contained in any proper subspace generated by a part of the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$. If so was, then there would be a subset $\{e_{i_1}, \ldots, e_{i_j}\}$ such that $[e_{i_1}, \ldots, e_{i_j}] \subseteq ((\mathbb{C}\lambda)^\perp)^\perp = \mathbb{C}\lambda$, which is a contradiction to the condition that all the $\lambda_i \neq 0$. Hence we have shown the existence of an invertible y in Q with $\tau_Q(y) = 0$.

As in Proposition 5 of [35] we have, for each $x \in \{u\}' \cap N^\omega$,

$$\|y(x - (E_D^N)^\omega(x))\|_{\tau_N^\omega} \leqslant \|yx - xy\|_{\tau_N^\omega},$$

and hence if x furthermore commutes with y, then $x = (E_D^N)^\omega(x) \in D^\omega$ since y is invertible. Therefore, we have obtained the formula (2.3).

The relative commutant $R_0' \cap D^\omega$ can be identified (abstractly) with the von Neumann algebra generated by equivalence classes of ω-centralizing sequences (under the action of an ergodic finite-measure preserving transformation) consisting of Borel subsets in a non-atomic Lebesgue space X with $D = L^\infty(X)$. In this case, the set of those ω-centralizing sequences is known to be very large (see Theorems 3.1 and 3.3, [24]), and thus the latter assertion follows. ∎

The pair $R_0 \supseteq D$ is known to be constructed from the type II_1 amenable discrete equivalence relation $\mathcal{R}$ over a non-atomic Lebesgue space X, that is,

$$R_0 = W^*(\mathcal{R}) \supseteq D = L^\infty(X).$$

(See [5] and [11].) The Galois group $\mathrm{Gal}(R_0 \supseteq D) := \{\alpha \in \mathrm{Aut}(R_0) : \alpha|D = \mathrm{Id}\}$ is identified with the group $Z^1(\mathcal{R}; \mathbb{T})$ of 1-cocycles on $\mathcal{R}$ via the mapping $c \in Z^1(\mathcal{R}; \mathbb{T}) \mapsto M_c \in \mathrm{Gal}(R_0 \supseteq D)$, where M_c is the multiplier defined on $R_0 = W^*(\mathcal{R})$ (see [11]). We define the homomorphism $\Phi : Z^1(\mathcal{R}; \mathbb{T}) \to \mathrm{Aut}(N)$ (or more precisely into $\mathrm{Gal}(N \supseteq D)$) by

$$\Phi(c) := (\mathrm{Id}_R \otimes M_c) *_D (\mathrm{Id}_Q \otimes \mathrm{Id}_D), \quad c \in Z^1(\mathcal{R}; \mathbb{T}).$$

We first point out that $\Phi(B^1(\mathcal{R}; \mathbb{T})) = \mathrm{Int}(N, D)$ $(:= \{\mathrm{Ad}\, u \in \mathrm{Aut}(N) : u \in \mathcal{U}(D)\})$, and that if β is an approximate inner automorphism on N, then it must be of the form $\alpha = \mathrm{Ad}\, X \circ \beta_0$ with $X \in \mathcal{U}(N)$, $\beta_0 \in \overline{\mathrm{Int}(N, D)}$ thanks to Theorem 2.6 and Connes' method (see Subsection 2.3, [35], for summary). Then β_0 should act on both R and Q trivially and satisfy that its restriction to R_0 is in $\overline{\mathrm{Int}(R_0, D)}$, and hence $\beta_0 = (\mathrm{Id}_R \otimes M_c) *_D (\mathrm{Id}_Q \otimes \mathrm{Id}_D) = \Phi(c)$ for some $c \in \overline{B^1(\mathcal{R}; \mathbb{T})}$. What we explained here is exactly the same as in Section 4 of [35], and we get the following proposition in this way (see [35], Section 4) for details.

PROPOSITION 2.7. (cf. Theorem 14, [35]) *The crossed-product $N = M \rtimes_{\alpha_z} \mathbb{Z}$ with irrational z satisfies*

$$\chi(N) = \overline{\mathrm{Int}(N)}/\mathrm{Int}(N) \cong H^1(\mathcal{R}; \mathbb{T}),$$

and the isomorphism is induced by Φ. Here, $\chi(N)$ is the χ-group ([4]) and $H^1(\mathcal{R}; \mathbb{T})$ is the first cohomology group of the type II_1 ergodic amenable discrete equivalence relation (see [24]).

The free Araki-Woods factor $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ (equipped with the so-called free quasi-free state φ_U) associated with a (non-trivial) one-parameter group of orthogonal transformations on a real Hilbert space $\mathcal{H}_\mathbb{R}$ was introduced by D. Shlyakhtenko ([25]). It was shown in [25] and [26] that $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is a factor of type III_λ, $0 < \lambda < 1$, if U_t is periodic with period $2\pi/\log\lambda$ and a factor of type III_1 if U_t is non-periodic. Replacing in (2.1) (Q, τ_Q) by $(\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U)$, we consider

$$(M, \varphi) := (R \otimes L^\infty(\mathbb{T}), \tau_R \otimes \mu) * (\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U)$$

and aperiodic automorphisms α_z for irrational $z \in \mathbb{T}$ defined as above. If U_t has an eigenvalue not equal to 1, then we notice, by Corollary 5.5, [25] and [7], the following state-preserving isomorphisms:

$$\begin{aligned}
(M, \varphi) &\cong (R \otimes L^\infty(\mathbb{T}), \tau_R \otimes \mu) * (L(\mathbb{F}_\infty), \tau) * (\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U) \\
&\cong (L(\mathbb{F}_\infty), \tau) * (\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U) \\
&\cong (\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U),
\end{aligned}$$

and all the arguments above can work in this setting as well. With $R_0 \supseteq D$ as above, we hence have

$$M \rtimes_{\alpha_z} \mathbb{Z} \cong (R \otimes R_0) *_D (\Gamma(\mathcal{H}_\mathbb{R}, U_t)'' \otimes D),$$

which is a factor of the same III_λ-type, $0 < \lambda \leqslant 1$, as $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ (see e.g. Theorem 2.6, Corollary 4.5, [32]). The following theorem is obtained in this way.

Theorem 2.8. *The free Araki-Woods factor $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ with U_t having an eigenvalue not equal to 1 has continuously many non-outer conjugate aperiodic automorphisms, all of whose crossed-products are isomorphic to a non-full factor of the same III_λ-type as $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$, not McDuff and having the same χ-group as in Proposition 2.7.*

3. DUALITY BETWEEN FREE PRODUCT ACTIONS AND FREE SHIFT ACTIONS

We are now in turn going to deal with actions whose crossed-products still stay in the "category" consisting of interpolated free group factors. It is probably well-known that free shift actions are such typical examples.

We would like here to point out an explicit relationship between the action of a compact (co-commutative) Kac algebra $\mathbb{K}_G$ considered in [31] and a certain free shift action (or free permutation action associated with G).

Let P, Q be von Neumann algebras with specific faithful normal states φ_P, φ_Q, respectively. Let G be a discrete (countable) group, and λ_g, $g \in G$, means its left regular representation. Let $L(G)$ be the group von Neumann algebra with the canonical trace τ_G. The compact co-commutative Kac algebra $\mathbb{K}_G = (L(G), \delta_G, \kappa_G, \tau_G)$ can act on the free product

$$(M, \varphi) := (P \otimes L(G), \varphi_P \otimes \tau_G) * (Q, \varphi_Q)$$

by the free product (co-)action

$$\Gamma^G = (\mathrm{Id}_P \otimes \delta_G) * (\mathrm{Id}_Q \otimes 1_{L(G)})$$

in the sense of [31]. This action is nothing less than that considered in Section 4 of [31].

Consider P, Q and $L(G)$ as subalgebras of M naturally, and set

$$N := \Big(P \cup \bigcup_{g \in G} \lambda_g Q \lambda_g^* \Big)''.$$

Since $\mathrm{Ad}\,\lambda_g(x) = x$, $x \in P$, and $\mathrm{Ad}\,\lambda_g(\lambda_h y \lambda_h^*) = \lambda_{gh} y \lambda_{gh}^*$, $y \in Q$, $h \in G$, we can define the action α of G on N by $\alpha_g := \mathrm{Ad}\,\lambda_g|N$, $g \in G$.

It is rather easy to show that $\{P, \{\lambda_g Q \lambda_g^*\}_{g \in G}\}$ forms a free family with respect to φ and that each λ_g, $g \neq e$, is orthogonal to $L^2(N, \varphi)$ in $L^2(M, \varphi)$. (See [2] for example, where a slightly generalized situation was treated in the C^*-algebraic setting.) Note here that $\sigma_t^\varphi(\lambda_g) = \lambda_g$ and $\sigma_t^\varphi(N) = N$ (globally invariant) so that there exists the φ-conditional expectation $E : M \to N$ satisfying $E(\lambda_g) = 0$ as long as $g \neq e$. Therefore, we see that

$$(N, \varphi|N) \cong (P, \varphi_P) * \big(*_{g \in G}(Q, \varphi_Q)_g \big), \quad M \cong N \rtimes_\alpha G,$$

and that α can be naturally identified with the free product action $\mathrm{Id}_P * \gamma^G$ under the above identification, where γ^G denotes the G-free shift action on $*_{g \in G}(Q, \varphi_Q)_g$. What we want to point out here is the following "duality" between the G-free shift action γ^G and the free product action Γ^G:

THEOREM 3.1. *In the current setting, the following assertions hold:*

(i) $(M, \Gamma^G) \cong (N \rtimes_\alpha G, \widehat{\alpha})$ *with the dual action* $\widehat{\alpha}$ $(= \widehat{\mathrm{Id}_P * \gamma^G})$ *(see* [15]*);*

(ii) $N = M^{\Gamma^G}$;

(iii) $M \rtimes_{\Gamma^G} \mathbb{K}_G \cong N \otimes B(\ell^2(G))$.

Proof. (i) Under the identification $M = N \rtimes_\alpha G$ explained above, the dual action $\widehat{\alpha}$ acts on M in the following manner:

$$\widehat{\alpha}(y) = y \otimes 1_{L(G)}, \qquad y \in N,$$
$$\widehat{\alpha}(\lambda_g) = \lambda_g \otimes \lambda_g, \qquad g \in G.$$

On the other hand, the definition of δ_G gives

$$\Gamma^G(\lambda_g) = \lambda_g \otimes \lambda_g, \qquad\qquad g \in G,$$
$$\Gamma^G(x) = x \otimes 1_{L(G)}, \qquad\qquad x \in P,$$
$$\Gamma^G(\lambda_g y \lambda_g^*) = (\lambda_g y \lambda_g^*) \otimes 1_{L(G)}, \qquad y \in Q, \ g \in G,$$

and hence $\Gamma^G = \widehat{\alpha}$ follows.

(ii) comes from (i) and the well-known formula $N = (N \rtimes_\alpha G)^{\widehat{\alpha}}$ (see [15]).

(iii) comes from (i), (ii) and the Takesaki duality (see [15]). ∎

Furthermore, similarly to Lemma 2.1 (ii), one can see that if the centralizer P_{φ_P} is diffuse and $Q \neq \mathbb{C}$, then the action α is outer.

REMARKS 3.2. (i) According to Proposition 1 of [33] (or the proof of Proposition 3.2, [28]) one has

$$M \rtimes_{\Gamma^G} \mathbb{K}_G \cong \big((P \otimes L(G)) \rtimes_{\mathrm{Id}_P \otimes \delta_G} \mathbb{K}_G\big) *_{\ell^\infty(G)} \big(Q \rtimes_{\mathrm{Id}_Q \otimes 1_{L(G)}} \mathbb{K}_G\big)$$
$$\cong \big(P \otimes B(\ell^2(G))\big) *_{\ell^\infty(G)} \big(Q \otimes \ell^\infty(G)\big).$$

In the above, we omitted the mention of the specific conditional expectations since they are clear from the context. (And, in what follows, we will do omit it as long as when no confusion is possible.) The above computation says the following (probably known) simple fact: If A is a (unique) atomic MASA in $B(\mathcal{H})$ with arbitrary $\dim \mathcal{H}$, then

$$\big(P \otimes B(\mathcal{H})\big) *_A \big(Q \otimes A\big) \cong \big(P * \big(Q^{*(\dim \mathcal{H})}\big)\big) \otimes B(\mathcal{H}).$$

(Compare with Theorem 3.3 of [21].) Note here that this can be directly shown by simple algebraic method (without any random matrix type technique). What we explained here was one of the motivations of the works [33] and [28].

(ii) Theorem 3.1 suggests us what "free shift actions" associated with the duals of compact quantum groups (e.g., $\widehat{\mathrm{SU}_q(N)}$) should be. In fact, the theorem says that the minimal free product action of $\mathrm{SU}_q(2)$ given in [33] ($P = \mathbb{C}$ in that case) can be regarded as the dual (co-)action of the "free $\widehat{\mathrm{SU}_q(2)}$-shift" action. We will return in the future to this point of view.

As mentioned before, the action α is the free product action $\mathrm{Id}_P * \gamma^G$, and one has

$$M = (P \otimes L(G)) * Q \cong (P * (*_{g \in G} Q_g)) \rtimes_\alpha G$$
$$\cong (P \otimes L(G)) *_{L(G)} \big((*_{g \in G} Q_g) \rtimes_{\gamma^G} G\big)$$

with $Q_g := Q$, $g \in G$. This, in particular, says that $(*_{g \in G} Q_g) \rtimes_{\gamma^G} G \cong L(G) * Q$, and hence we get

$$(3.1) \qquad (P \otimes L(G)) * Q \cong (P \otimes L(G)) *_{L(G)} (L(G) * Q).$$

In fact, we have the following slightly more general fact.

FACT 3.3. (A special case of Proposition 2.7 in [14]) *Let P, Q and A be von Neumann algebras with faithful normal states φ_P, φ_Q and φ_A, respectively. Then*

$$(P \otimes A, \varphi_P \otimes \varphi_A) *_A (A * Q, \varphi_A * \varphi_Q) \cong ((P \otimes A) * Q, (\varphi_P \otimes \varphi_A) * \varphi_Q).$$

This can be shown by checking a suitable freeness; the proof is straightforward. Quite recently, A. Nica, D. Shlyakhtenko and R. Speicher ([14]) showed a more general fact as a corollary of what they developed on the operator-valued R-transform.

For each amenable group G and each $r \in (1, \infty]$, let us specify (P, τ_P) and (Q, τ_Q) for which we have

$$M \cong L(\mathbb{F}_r), \quad N \cong L(\mathbb{F}_{(r-1)|G|+1}).$$

To do so, it suffices, for example, to assume that P is the hyperfinite type II_1 factor R, (Q, τ_Q) is one of those given at the beginning of Section 2 and the free dimension of (Q, τ_Q) is $r - 1$. Then the above realization of M, N can be easily seen by simple computations of free dimensions based on results of K. Dykema ([7]). In this way, we have obtained the following result.

PROPOSITION 3.4. *For each amenable (countable) group G and each $r \in (1, \infty]$, there exists an outer action α of G on $L(\mathbb{F}_{(r-1)|G|+1})$ such that*

$$L(\mathbb{F}_{(r-1)|G|+1}) \rtimes_\alpha G \cong L(\mathbb{F}_r).$$

In particular, for each $r \in (1, \infty]$, there exists an aperiodic automorphism α on $L(\mathbb{F}_\infty)$ such that

$$L(\mathbb{F}_\infty) \rtimes_\alpha \mathbb{Z} \cong L(\mathbb{F}_r).$$

REMARK 3.5. When G is an amenable group and P, Q are specified as above, the fact (3.1) says that the formula of free dimension for a certain class of amalgamated free products (including the multi-matrix algebra situation) given in [9] is valid even in a case where the amalgamation subalgebra is non-atomic.

Finally, we would like to give small comments on the free analogs of Bernoulli shifts, introduced by replacing the tensor product "$\otimes$" by the free product "$*$".

The "free Bernoulli shift" consisting of the free group factor $L(\mathbb{F}_\infty)$ and its ergodic, aperiodic automorphism σ_p associated with a (non-degenerate) probability vector $p = (p_1, \ldots, p_n)$ is the free analog of Connes-Størmer's Bernoulli shift ([6]), and it is constructed in the following manner (see e.g. [20]): Let

$$(3.2) \qquad (N, \psi) := *_{k \in \mathbb{Z}} (M_n(\mathbb{C}), \varphi_p)_k$$

be the free product, where the state φ_p has the diagonal density matrix $\mathrm{diag}(p_1, \ldots \ldots, p_n)$. Then the $\mathbb{Z}$-free shift α preserves the free product state ψ so that one can consider its restriction $\sigma_p := \alpha|N_\psi$ to the centralizer N_ψ. (Note that, in the case of equal probabilities, i.e., $p_1 = \cdots = p_n = 1/n$, the state ψ itself is a trace

and $N \cong L(\mathbb{F}_\infty)$ due to [7] so that we simply set $\sigma_p := \alpha$.) According to [10], the centralizer N_ψ is isomorphic to $L(\mathbb{F}_\infty)$ (when ψ is not a trace). The discussion given before Theorem 3.1 tells us that the crossed-product $N \rtimes_\alpha \mathbb{Z}$ is isomorphic to the free product

$$(M, \varphi) := (L(\mathbb{Z}), \tau_\mathbb{Z}) * (M_n(\mathbb{C}), \varphi_p),$$

which is known to be isomorphic to $L(\mathbb{F}_{2-1/n^2})$ in the case of equal probabilities (see [7]), or otherwise to the free Araki-Woods factor whose Connes' Sd invariant is the multiplicative group generated by p_i/p_j, $1 \leqslant i, j \leqslant n$, (see [25], p. 365). In fact, N, ψ and α are realized as

$$N = \Big(\bigcup_{k \in \mathbb{Z}} \lambda^k M_n(\mathbb{C}) \lambda^{*k} \Big)'' \, (\subseteq M), \quad \psi = \varphi|N, \quad \alpha = \mathrm{Ad}\,\lambda,$$

where λ is the generating unitary of $L(\mathbb{Z})$. Moreover, note that $\varphi = \psi \circ E$ and $\sigma_t^\varphi|N = \sigma_t^\psi$, where $E : M\,(\cong N \rtimes_\alpha \mathbb{Z}) \to N$ is the canonical conditional expectation. Now, let E_φ be the φ-conditional expectation from M onto the centralizer M_φ, and E_ψ the ψ-conditional expectation from N onto N_ψ. Then it is easy to see that $E_\varphi(\lambda) = \lambda$ (thanks to $\lambda \in M_\varphi$) and $E_\varphi|N = E_\psi$ (by the commuting square property thanks to $E(M_\varphi) \subseteq N_\psi$). Therefore, we notice

$$M_\varphi = (N_\psi \cup \{\lambda\})'' \cong N_\psi \rtimes_{\sigma_p} \mathbb{Z}.$$

Since $M_\varphi \cong L(\mathbb{F}_\infty)$ (see [10], [23], [25], [26]), we have obtained the following proposition.

PROPOSITION 3.6. *Let* $(L(\mathbb{F}_\infty), \sigma_p)$ *be the free Bernoulli shift associated with a probability vector* $p = (p_1, \ldots, p_n)$ *defined above. Then*

$$L(\mathbb{F}_\infty) \rtimes_{\sigma_p} \mathbb{Z} \cong \begin{cases} L(\mathbb{F}_{2-1/n^2}) & \text{if } p_1 = \cdots = p_n = 1/n, \\ L(\mathbb{F}_\infty) & \text{otherwise.} \end{cases}$$

One can also define a more direct free analog of the classical Bernoulli shift, which may be called the free "commutative" Bernoulli shift associated with a probability vector $p = (p_1, \ldots, p_n)$. It consists of $L(\mathbb{F}_\infty)$ and its ergodic, aperiodic automorphism γ_p again, provided by replacing the full matrix algebra $M_n(\mathbb{C})$ in (3.2) by its diagonal subalgebra $\mathbb{C}^n$. Since no type III situation appears in this case, we simply define γ_p by the $\mathbb{Z}$-free shift α. Similarly to the above consideration, we see that the crossed-product $L(\mathbb{F}_\infty) \rtimes_{\gamma_p} \mathbb{Z}$ is isomorphic to the free product

$$(L(\mathbb{Z}), \tau_\mathbb{Z}) * (\mathbb{C}^n, p),$$

which is isomorphic to $L(\mathbb{F}_{2-\|p\|_2^2})$ due to [7]. Here, $\|p\|_2$ denotes the ℓ^2-norm of the probability vector p. Therefore, the following proposition is obtained.

PROPOSITION 3.7. *Let* $(L(\mathbb{F}_\infty), \gamma_p)$ *be the above free Bernoulli shift associated with a probability vector* $p = (p_1, \ldots, p_n)$. *Then*

$$L(\mathbb{F}_\infty) \rtimes_{\gamma_p} \mathbb{Z} \cong L(\mathbb{F}_{2-\|p\|_2^2}),$$

where $\|p\|_2$ *is the* ℓ^2*-norm of* p.

Remarks 3.8. (i) It is interesting to note that aperiodic automorphisms on $L(\mathbb{F}_\infty)$ given in this section have the characteristic very different from that of automorphisms given in Section 2. More precisely, if α is a $\mathbb{Z}$-free shift on $L(\mathbb{F}_\infty)$ as in Propositions 3.4, 3.6 and 3.7, then the crossed-product $L(\mathbb{F}_\infty) \rtimes_\alpha \mathbb{Z}$ is a certain interpolated free group factor and hence full. This means (see [12], [16], [17]) that $\{\alpha^n : n \in \mathbb{Z}\}$ forms a discrete subgroup of $\mathrm{Out}(L(\mathbb{F}_\infty))$. On the other hand, aperiodic automorphisms in Theorem 2.5 have non-full crossed-products as stated in Theorem 2.6.

(ii) As proved in [8] and [22] independently, the interpolated free group factors are either all isomorphic or all non-isomorphic (i.e., $L(\mathbb{F}_r) \not\cong L(\mathbb{F}_{r'})$ for $r \neq r'$). If the latter case were true, then one would have a continuous family of non-cocycle conjugate (or equivalently non-outer conjugate) $\mathbb{Z}$-free shifts on $L(\mathbb{F}_\infty)$ by Proposition 3.4 as well as some cocycle conjugacy classification results for free Bernoulli shifts σ_p and γ_p given in Propositions 3.6 and 3.7. In this connection, it may be pointed out that the classification up to conjugacy by means of Connes-Størmer's dynamical entropy ([6]) is meaningless for these free Bernoulli shifts since all such free shifts have zero entropy (see e.g. [29]).

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