J. OPERATOR THEORY
50(2003), 131–156

OPERATOR SPACE TENSOR PRODUCTS AND HOPF CONVOLUTION ALGEBRAS

EDWARD G. EFFROS and ZHONG-JIN RUAN

Communicated by William B. Arveson

Abstract. It is shown how one may use operator space tensor product to define Hopf algebraic operations on the preduals of Hopf von Neumann algebras. A careful discussion of the extended Haagerup tensor product is presented which includes a useful technique for handling computations with products of infinite matrices.

Keywords: *Operator spaces, tensor products, Hopf algebras, quantum groups.*

MSC (2000): Primary 46L07, 47L25, 20G42; Secondary 46L52.

1. INTRODUCTION

The convolution algebra $L^1(G)$ of a locally compact group G provides a Banach algebraic generalization of the classical group algebra $\mathbb{C}[G]$ of a discrete group G. In particular, the uniformly bounded Banach space representations of G are in one-to-one correspondence with the bounded Banach space representations of $L^1(G)$ (see [30]). On the other hand, in contrast to group algebras, convolution algebras are generally not provided with a comultiplication, i.e., a "Hopf structure" (see Section 2).

One may define natural analogues of the convolution algebra for quantum groups. Perhaps the simplest example is provided by the Fourier algebra of a non-commutative group G, which may be thought of as the convolution algebra of the "dual quantum group" $\widehat{G}$ (see [13]). In this more general context, the lack of a Hopf structure is a serious flaw, since a natural comultiplication enables one to define the tensor product of representations. As a result, functional analysts have instead used various alternative "dual" constructions, such as Hopf C^*-algebras, Hopf von Neumann algebras, and multiplicative unitaries (see [25], [23], [22], [31], [2], [42], [43], [44], [20] and [27]).

Although considerable progress has been made in the functional analytic theory of quantum groups, there are still good reasons for considering a convolution algebraic approach. Perhaps the most important of these is that various group theoretic notions, such as amenability, are most conveniently described in terms of the convolution algebras (see, e.g., [36]). It can also be argued that convolution algebras would enable one to avoid some of the technicalities associated with the corepresentations and coactions of the existing theory (see the discussion in [28]).

It is evident from the existing theory that in order to define a Hopf convolution algebra for quantum groups, one must first replace the classical L^1-Banach spaces by their "non-commutative L^1-space" analogues, the preduals of von Neumann algebras. If one wishes to consider algebraic operations on such spaces it is also necessary to use their underlying *operator space* structure (for a general survey of this subject see [21]).

Approximately ten years ago the authors showed that one can define Hopf algebraic operations on non-commutative L^1-spaces by using the operator space projective and extended Haagerup tensor products. These results were circulated in an unpublished manuscript ([19]). In the intervening years the tensor product theory has become more familiar to specialists (for a recent example see [29]). In addition the Hopf algebra techniques have proved to be quite useful in formulating the notion of amenability for Kac algebras ([36]).

Since an increasing number of authors have referred to the manuscript, we believe that it would be useful to make this material available to a wider audience. We have modified the paper in several ways. We have substantially improved the discussion of the extended Haagerup tensor product by using a more precise limiting technique. This has enabled us to give a simple proof of the multivariable version of an important embedding result of Blecher and Smith ([7]; see (5.20)). We have shortened the discussion of the operator nuclear, projective and Haagerup tensor products, since many of the details can now be found elsewhere (see, e.g., [21]). We have also postponed much of the discussion of Fourier-Stieltjes algebras to a subsequent paper.

We begin in Section 2 by considering how operator space tensor products naturally arise in the theory of Hopf algebras. In Section 3 we briefly discuss some infinite matrix manipulations. The relevant tensor products are described in Section 4 and Section 5, and an important "shuffle theorem" is proved in Section 6. In Section 7 we conclude the discussion in Section 2, and in particular we indicate how one can construct tensor products of representations of Hopf convolution algebras.

Given a Hilbert space H, we use the expression "weak* topology" for the usual σ-weak operator topology on $B(H)$.

Unless otherwise indicated, we assume that all operator spaces are norm complete.

2. HOPF ALGEBRAS

Analysts use the Hopf algebraic terminology in a more inclusive and less precise sense than that found in the algebraic literature (for the elementary algebraic theory see [3] and [40]). In general, a Hopf algebra (A, m, δ) consists of a linear space A with norms or matrix norms, an associative bilinear multiplication $m = m_A : A \times A \to A$, and a coassociative comultiplication $\delta = \delta_A : A \to A \mathbin{\widetilde{\otimes}} A$, where $\widetilde{\otimes}$ is a suitable tensor product, and δ is an algebraic homomorphism (this links the two operations). The maps are assumed to be bounded in some appropriate sense.

A *Hopf von Neumann algebra* (R, m, δ) is a von Neumann algebra R together with its multiplication operation m, and a weak* continuous $*$-isomorphic unital coassociative injection

$$\delta : R \to R \mathbin{\overline{\otimes}} R,$$

where $R \mathbin{\overline{\otimes}} R$ is the usual von Neumann algebraic tensor product. We may associate two Hopf von Neumann algebras with a locally compact group G. Let us fix a left invariant Haar measure on G. We have that $(L^\infty(G), m, \delta)$ is a Hopf von Neumann algebra, where m is the point-wise multiplication and $\delta f(x, y) = f(xy)$. On the other hand if $R(G)$ is the von Neumann algebra generated by the left regular representation $\lambda : G \to L^2(G)$, and the normal homomoprhism $\delta : R \to R \mathbin{\overline{\otimes}} R$ is determined by the map $\lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$ (see [31], Section 2 for the details).

From the finite-dimensional theory one might expect that the predual R_* of a Hopf von Neumann algebra R is again a Hopf algebra. Using the fact that

$$(R \mathbin{\overline{\otimes}} R)_* = R_* \mathbin{\widehat{\otimes}} R_*,$$

where $\widehat{\otimes}$ is the operator space projective tensor product (see Section 4), the preadjoint of $\delta = \delta_R$ is a natural associative multiplication

$$m = m_{R_*} : R_* \mathbin{\widehat{\otimes}} R_* \to R_*.$$

In particular, if $R = R(G)$, this is the usual multiplication of the *Fourier algebra* $A(G) = R_*$. If G is abelian, the Banach algebra $A(G)$ may be identified with the usual *convolution algebra* $L^1(\widehat{G})$ of the dual group $\widehat{G}$, whereas for non-commutative groups and more generally quantum groups, it is thought of as the "convolution algebra of the dual quantum group".

In order to complete the duality, and most importantly, to define the tensor product for representations of the algebra R_*, it is also necessary to define a comultiplication on R_* which is dual to the multiplication map $m = m_R : R \times R \to R$. Our first task is to linearize m by using a suitable tensor product. Even if R is commutative, m does not extend to a contractive linear map $R \mathbin{\overline{\otimes}} R \to R$. Fortunately there is a natural operator space tensor product, the *normal Haagerup tensor product* $R \overset{\sigma\mathrm{h}}{\otimes} R$ (introduced in [12]), which is ideally suited for linearizing bilinear functions of this type on R. The map m extends uniquely to a weak* continuous completely contractive map $m : R \overset{\sigma\mathrm{h}}{\otimes} R \to R$.

The space $R \overset{\sigma\mathrm{h}}{\otimes} R$ has a natural predual, the *extended Haagerup tensor product* $R_* \overset{\mathrm{eh}}{\otimes} R_*$. We define the comultiplication $\delta : R_* \to R_* \overset{\mathrm{eh}}{\otimes} R_*$ to be the preadjoint of $m : R \overset{\sigma\mathrm{h}}{\otimes} R \to R$. We call a Hopf algebra $(A = R_*, m, \delta)$ which arises in this manner a *Hopf convolution algebra.* We may summarize these constructions with the diagram

$$\begin{array}{ccc} \text{Hopf von Neumann algebra } R & & \text{Hopf convolution algebra } R_* \\ \delta = \delta_R : R \to R \mathbin{\overline{\otimes}} R & & m = (\delta_R)_* : R_* \widehat{\otimes} R_* \to R_* \\ & \longleftrightarrow & \\ m = m_R : R \overset{\sigma\mathrm{h}}{\otimes} R \to R & & \delta = (m_R)_* : R_* \to R_* \overset{\mathrm{eh}}{\otimes} R_* \end{array} . \tag{2.1}$$

In order to verify that R_* is a Hopf algebra, we must also prove the non-trivial fact that

$$\delta : R_* \to R_* \overset{\mathrm{eh}}{\otimes} R_*$$

is an algebraic homomorphism. To make sense of this we must first prove that the "shuffle" linear map of algebraic tensor products

$$\mathcal{S} : (R_* \otimes R_*) \otimes (R_* \otimes R_*) \to (R_* \otimes R_*) \otimes (R_* \otimes R_*)$$

defined by

$$(x \otimes y) \otimes (u \otimes v) \mapsto (x \otimes u) \otimes (y \otimes v)$$

has a natural extension to a complete contraction

$$\mathcal{S}_{\mathrm{e}} : (R_* \overset{\mathrm{eh}}{\otimes} R_*) \mathbin{\widehat{\otimes}} (R_* \overset{\mathrm{eh}}{\otimes} R_*) \to (R_* \mathbin{\widehat{\otimes}} R_*) \overset{\mathrm{eh}}{\otimes} (R_* \mathbin{\widehat{\otimes}} R_*). \tag{2.2}$$

This result was proved in [15]. In Section 6 we show that it follows from a shuffle result for arbitrary operator spaces. For this purpose it is necessary to use the *nuclear* tensor product $V \overset{\mathrm{nuc}}{\otimes} W$, which is a natural complete quotient of the projective tensor product $V \widehat{\otimes} W$ (see Section 4). For von Neumann algebras R and S we have that $R_* \overset{\mathrm{nuc}}{\otimes} S_* = R_* \mathbin{\widehat{\otimes}} S_*$. Given operator spaces V_1, V_2, W_1 and W_2, we show in Theorem 6.1 that $\mathcal{S}$ extends to a complete contraction

$$\mathcal{S}_{\mathrm{e}} : (V_1 \overset{\mathrm{eh}}{\otimes} W_1) \overset{\mathrm{nuc}}{\otimes} (V_2 \overset{\mathrm{eh}}{\otimes} W_2) \to (V_1 \overset{\mathrm{nuc}}{\otimes} V_2) \overset{\mathrm{eh}}{\otimes} (W_1 \overset{\mathrm{nuc}}{\otimes} W_2). \tag{2.3}$$

Since we have a natural complete quotient map

$$(R_* \overset{\mathrm{eh}}{\otimes} R_*) \mathbin{\widehat{\otimes}} (R_* \overset{\mathrm{eh}}{\otimes} R_*) \to (R_* \overset{\mathrm{eh}}{\otimes} R_*) \overset{\mathrm{nuc}}{\otimes} (R_* \overset{\mathrm{eh}}{\otimes} R_*),$$

(2.2) is an immediate consequence of (2.3).

3. INFINITE MATRICES

Given an operator space V and index sets I and J, we let $M_{I,J}(V)$ denote the vector space of matrices $[v_{i,j}]$, $i \in I$, $j \in J$, for which the finite submatrices are uniformly bounded in norm (this is again an operator space; [14], [16]), and as usual we let $M_J(V) = M_{J,J}(V)$ and $M_J = M_J(\mathbb{C}) = B(\ell^2(J))$. We also use the notation $T_{I,J}(V) = T_{I,J} \widehat{\otimes} V$, where $T_{I,J}$ is the predual of $M_{I,J}$.

Products of bounded infinite scalar matrices must be handled with some care. Given $a \in M_{I,K}$, $b \in M_{K,J}$, we have that the series $\sum_k a_{ik} b_{kj}$ converges absolutely, and thus unconditionally since

$$\sum_k |a_{ik} b_{kj}| \leqslant \Big(\sum_k |a_{ik}|^2\Big)^{1/2} \Big(\sum_k |b_{kj}|^2\Big)^{1/2} = \|a^*(e_i)\| \, \|b(e_j)\| < \infty.$$

The series involved in products of more than two matrices need not converge absolutely. As a result one must justify changes in the order of summation of series of this type. Fortunately there is a modified form of unconditionality that is valid. To illustrate this, let us suppose that we are given $a \in M_{I,K}$, $b \in M_{K,L}$, and $c \in M_{L,J}$ for index sets I, J, K, L. Given a subset $S \subseteq K$, we let $P(S)$ be the corresponding projecton on $\ell^2(K)$. This determines a projection valued measure on K. Similarly we let $Q(T)$ be the projection on $\ell^2(L)$ determined by a subset $T \subseteq L$. If we restrict to finite sets $F \subseteq K$ and $G \subseteq L$, we may regard $F \to P(F)$ and $G \to Q(G)$ as nets of projections, each of which converges to the identity in the strong operator topology. Since multiplication is jointly continuous in the strong operator topology on bounded sets of operators, we have that

$$abc = \lim_{F,G} aP(F)bQ(G)c$$

and thus we get a limit of finite sums

$$(abc)_{i,j} = \lim_{F,G} \sum_{k \in F, l \in G} a_{i,k} b_{k,l} c_{l,j}.$$

Similarly we have

$$abc = \lim_F aP(F)bc$$

and therefore

$$(abc)_{i,j} = \lim_F \sum_{k \in F} a_{i,k} (bc)_{k,j}.$$

We conclude this section with a review of certain operator space conventions and results. Given operator spaces V and W, we let $CB(V,W)$ denote the operator space of completely bounded maps $\varphi : V \to W$. If V and W are the duals of operator spaces, then we let $CB^\sigma(V,W)$ be the weak* continuous maps in $CB(V,W)$.

If H and K are Hilbert spaces with bases $(e_j)_{j \in J}$ and $(f_i)_{i \in I}$ indexed by sets J and I, we may identify $B(H,K)$ with $M_{I,J}$.

We have a natural complete isometry

$$CB(V, M_{I,J}) \cong M_{I,J}(V^*), \tag{3.1}$$

where given a matrix $f = [f_{ij}] \in M_{I,J}(V^*)$, the corresponding map $\varphi_f : V \to M_{I,J}$ is defined by $\varphi_f(v) = [f_{ij}(v)]$. This is immediate from the identifications

$$M_{I,J}(V^*) = (T_{I,J} \widehat{\otimes} V)^* = CB(V, M_{I,J})$$

(see [21], (10.1.8)).

On the other hand, we have the natural complete isometry

$$CB^\sigma(V^*, M_{I,J}) \cong M_{I,J}(V). \tag{3.2}$$

This is proved as follows. Let us suppose that $\varphi \in CB^\sigma(V^*, M_{I,J})$. Then from (3.1) there is a matrix $F = [F_{i,j}] \in M_{I,J}(V^{**})$ such that $\varphi(f) = [F_{ij}(f)]$. By hypothesis, $f \mapsto [F_{ij}(f)]$ is continuous in the weak* topologies. It follows that each function F_{ij} is weak* continuous, and thus has the form $F_{ij}(f) = f(v_{ij})$ for some element $v_{ij} \in V$, and thus $F = [v_{i,j}]$. Conversely, if $F = [v_{i,j}]$ where $v_{i,j} \in V$, then we claim that $\varphi_F \in CB^\sigma(V^*, M_{I,J})$. To prove this it suffices to show that the restriction of $f \mapsto F(f)$ to the unit ball B of V^* is continuous in those topologies. Since $F(B)$ is bounded, the weak* topology coincides with the weak operator topology on $F(B)$. In turn, it suffices to show that $f \mapsto F'(f)$ is weak* continuous for finite submatrices F' of F, and this is immediate from the weak* continuity of the entries $v_{i,j}$.

Given operator spaces V_k, $k = 1, \dots, p$, index sets I_k and J_k and rectangular matrices $v_k = [v^{(k)}_{i_k,j_k}] \in M_{I_k,J_k}(V_k)$, we define the *Kronecker product* by

$$v_1 \otimes \cdots \otimes v_p = [v^{(1)}_{i_1,j_1} \otimes \cdots \otimes v^{(p)}_{i_p,j_p}] \in M_{I,J}(V_1 \otimes \cdots \otimes V_p),$$

where $I = I_1 \times \cdots \times I_p$ and $J = J_1 \times \cdots \times J_p$. In particular given $v_k = [v^{(k)}_{i_k,j_k}] \in M_{m_k,n_k}(V_k)$, we have

$$v_1 \otimes \cdots \otimes v_p \in M_{m,n}(V_1 \otimes \cdots \otimes V_p), \tag{3.3}$$

where $m = m_1 \cdots m_p$ and $n = n_1 \cdots n_p$.

Given $v \in M_{J_1,J_2}(V)$ and $f \in M_{I_1,I_2}(V^*)$, we shall use often the "pairing" notation

$$\langle f, v\rangle = f_{J_1J_2}(v) = [f_{i_1,i_2}(v_{j_1j_2})] \in M_{I_1\times J_1, I_2 \times J_2}. \tag{3.4}$$

This formalism is particularly useful for considering dual operator spaces. Given an operator space V and matrices $v \in M_n(V^*)$, we have

$$\|f\| = \sup\{\|\langle f, v\rangle\| : \|v\| \leqslant 1, v \in M_n(V)\} \tag{3.5}$$

and

$$\|v\| = \sup\{\|\langle f, v\rangle\| : \|f\| \leqslant 1, f \in M_n(V^*)\}. \tag{3.6}$$

4. THE PROJECTIVE AND NUCLEAR TENSOR PRODUCTS

We use the tensor product terminology in the usual functorial sense. Thus given operator spaces $V_1, \ldots, V_p$, a tensor product $\widetilde{\otimes}$ determines a corresponding operator space $V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_p$, and given completely contractive maps $\varphi_k : V_k \to W_k$ we have a corresponding complete contraction

$$\varphi = \varphi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \varphi_p : V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_p \to W_1 \widetilde{\otimes} \cdots \widetilde{\otimes} W_p.$$

We say that $\widetilde{\otimes}$ is *injective* if completely isometric injections φ_k determine a completely isometric injection φ, and that $\widetilde{\otimes}$ is *projective* if complete quotient maps φ_k determine a complete quotient map φ.

We will make only peripheral use of the operator space injective tensor product $\overset{\vee}{\otimes}$ for operator spaces (see [21]). We begin by reviewing the notion of complete boundedness for multilinear maps and their linearization via the operator space projective tensor product $\widehat{\otimes}$.

Given another operator space W and a multilinear map

$$\varphi : V_1 \times \cdots \times V_p \to W, \tag{4.1}$$

we also write φ for its linear extension

$$\varphi : V_1 \otimes \cdots \otimes V_p \to W,$$

as well as the the multilinear and linear maps

$$\varphi : M_{I_1,J_1}(V_1) \times \cdots \times M_{I_p,J_p}(V_p) \to M_{I,J}(W)$$

and

$$\varphi : M_{I_1,J_1}(V_1) \otimes \cdots \otimes M_{I_p,J_p}(V_p) \to M_{I,J}(W)$$

determined by

$$\varphi(v_1, \ldots, v_p) = \varphi(v_1 \otimes \cdots \otimes v_p) = [\varphi(v^{(1)}_{i_1,j_1} \otimes \cdots \otimes v^{(p)}_{i_p,j_p})].$$

φ is said to be *completely bounded* (in the sense of Choi [8]) if there is a constant K such that

$$\|\varphi(v_1 \otimes \cdots \otimes v_p)\| = \left\|[\varphi(v^{(1)}_{i_1,j_1} \otimes \cdots \otimes v^{(p)}_{i_p,j_p})]\right\| \leqslant K\|v_1\| \cdots \|v_p\|$$

for all $v_k \in M_{m_k,n_k}(V)$, where m_k and n_k are arbitrary integers. If φ is completely bounded, we define its *completely bounded norm* $\|\varphi\|_{\mathrm{cb}}$ to be the least such constant K, i.e.,

$$\|\varphi\|_{\mathrm{cb}} = \sup\left\{\|\varphi(v_1 \otimes \cdots \otimes v_p)\| : \|v_1\| \cdots \|v_p\| \leqslant 1\right\}.$$

Given operator spaces V_k, $k = 1, \ldots, p$ and a matrix $u \in M_m(V_1 \otimes \cdots \otimes V_p)$, we define *operator space projective tensor norm* $\|u\|_\wedge$ by

$$\|u\|_\wedge = \inf\{\|\alpha\| \, \|v_1\| \cdots \|v_n\| \, \|\beta\| : u = \alpha(v_1 \otimes \cdots \otimes v_n)\beta\},$$

where $v_k \in M_{n_k}(V_k)$, $\alpha \in M_{m,n}$, and $\beta \in M_{n,m}$ with $n = n_1 \cdots n_p$. We let $V_1 \otimes_\wedge \cdots \otimes_\wedge V_p$ denote the corresponding (incomplete) operator space, and we define the *operator space projective tensor product* $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$ to be its completion.

We may also represent a matrix in $M_m(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)$ by using infinite matrices (see [21]). Given $u \in M_m(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)$ and $\varepsilon > 0$, there exist index sets

J_k and matrices $v_k \in M_{J_k}(V_k)$ with $k = 1, \ldots, p$, $\alpha \in M_{m,J}$, and $\beta \in M_{J,m}$, where $J = J_1 \times \cdots \times J_p$, such that

$$u = \alpha(v_1 \otimes \cdots \otimes v_p)\beta \quad \text{and} \quad \|u\|_\wedge \leqslant \|\alpha\| \, \|v_1\| \cdots \|v_p\| \, \|\beta\| \leqslant \|u\|_\wedge + \varepsilon. \tag{4.2}$$

If we let F_k range over the finite subsets of J_k, and we let $F = F_1 \times \cdots \times F_p$, and we define $v_k^{F_k}$, α^F and β^F to be the corresponding finite truncated matrices in $M_{F_k}(V_k)$, $M_{m,F}$, and $M_{F,m}$, respectively, then u is the norm limit in $M_m(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)$ of the net

$$F \mapsto \alpha^F(v^{F_1} \otimes \cdots \otimes v^{F_p})\beta^F.$$

Furthermore an element $u \in M_m(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)$ may always be written in the form (4.2) with $v_k \in K_\infty(V_k)$ and $\alpha \in K_{m,\infty^p}$ and $\beta \in K_{\infty^p,m}$, where $K_\infty(V_k)$ (respectively, K_{m,∞^p} and $K_{\infty^p,m}$) consists of the norm limits of finitely non-zero matrices in $M_\infty(V_k)$ (respectively, M_{m,∞^p} and $M_{\infty^p,m}$).

Any completely contractive multilinear map

$$\varphi : V_1 \times \cdots \times V_p \to W$$

determines a completey contractive linear map

$$\widetilde{\varphi} : V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \to W$$

with $\|\widetilde{\varphi}\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$, and this in turn provides us with a natural identification

$$CB(V_1 \times \cdots \times V_p, W) \cong CB(V \widehat{\otimes} \cdots \widehat{\otimes} V_p, W). \tag{4.3}$$

Given an element $u = \alpha(v_1 \otimes \cdots \otimes v_p)\beta \in V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$, it is easily verified that

$$\widetilde{\varphi}(u) = \alpha\varphi(v_1, \ldots, v_p)\beta.$$

If we are given complete contractions $\varphi_k : V_k \to W_k$, then we let

$$\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_p : V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \to W_1 \widehat{\otimes} \cdots \widehat{\otimes} W_p \tag{4.4}$$

be the linear map determined by the completely contractive multilinear map

$$(v_1, \ldots, v_p) \mapsto \varphi_1(v_1) \otimes \cdots \otimes \varphi_p(v_p) \in W_1 \widehat{\otimes} \cdots \widehat{\otimes} W_p.$$

In particular, if $f_k \in V_k^*$, and $u = \alpha(v_1 \otimes \cdots \otimes v_p)\beta \in V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$, then the linear functional

$$f_1 \otimes \cdots \otimes f_p : V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \to \mathbb{C}$$

satisfies

$$\langle f_1 \otimes \cdots \otimes f_p, u\rangle = \alpha(\langle f_1, v_1\rangle \otimes \cdots \otimes \langle f_p, v_p\rangle)\beta, \tag{4.5}$$

where $\langle f_k, v_k\rangle = [f_k(v_{(i,j)}^{(k)})] \in M_{J_k}$.

If V^* is a dual operator space, it has a *weak* faithful representation*, i.e., there is a Hilbert space H and a weak* homeomorphic complete isometry of V^* onto a weak* closed subspace of $B(H)$ ([17], Proposition 5.1). Given weak* closed subspaces $V_k^* \subseteq B(H_k)$, $k = 1, \ldots, p$, we define the *normal spatial tensor product* $V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^*$ to be the weak* closure of $V_1^* \otimes \cdots \otimes V_p^*$ in $B(H_1 \otimes \cdots \otimes H_p)$. We define the *Fubini tensor product* $V_1^* \overline{\otimes}_F \cdots \overline{\otimes}_F V_p^*$ to be the space of all operators $b \in B(H_1 \otimes \cdots \otimes H_p)$ such that for each k with $1 \leqslant k \leqslant p$ and functionals $\omega_j \in B(H_j)_*$, $j \neq k$ the "slice"

$$\langle b, \omega_1 \otimes \cdots \otimes \omega_{k-1} \otimes \mathrm{id}_k \otimes \cdots \otimes \omega_p\rangle$$

lies in V_k^*. From the following result we see that neither of these tensor products depends upon the given weak* faithful representations $V_k^* \subseteq B(H_k)$.

Theorem 4.1. *For any operator spaces $V_1, \ldots, V_p$ and arbitrary weak* closed representations $V_k^* \subseteq B(H_k)$, we have a completely isometric weak* homeomorphism*

$$(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)^* = V_1^* \overline{\otimes}_F \cdots \overline{\otimes}_F V_p^*. \tag{4.6}$$

$V_1^* \otimes \cdots \otimes V_p^*$ is dense in $V_1^* \overline{\otimes}_F \cdots \overline{\otimes}_F V_p^*$ in the $V_1 \otimes \cdots \otimes V_p$ topology.

Proof. This may be found in [13] and [21]. ∎

It follows from Theorem 4.1 that the identification of the Fubini tensor product with the dual of $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$ carries the normal spatial tensor product $V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^*$ onto the closure of $V_1^* \otimes \cdots \otimes V_p^*$ in the topology determined by the completion $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$. We conclude that the normal spatial tensor product does not depend on the embeddings $V_k^* \subseteq B(H_k)$. However, there is a more explicit way of seeing this.

We define the *nuclear tensor product* $V_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} V_p$ of operator spaces $V_1, \ldots, V_p$ by

$$V_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} V_p = (V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p)/\ker \Psi,$$

where Ψ is the canonical complete contraction

$$\Psi : V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \to V_1 \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} V_p. \tag{4.7}$$

Theorem 4.2. *For any dual operator spaces $V_1^*, \ldots, V_p^*$ we have a completely isometric weak* homeomorphism*

$$(V_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} V_p)^* \cong V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^*. \tag{4.8}$$

Proof. The inclusion

$$V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^* \hookrightarrow V_1^* \overline{\otimes}_F \cdots \overline{\otimes}_F V_p^*$$

determines a complete quotient map

$$V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \to (V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^*)_*,$$

and thus

$$(V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^*)_* \cong V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p / N \tag{4.9}$$

where

$$N = [\, V_1^* \overline{\otimes} \cdots \overline{\otimes} V_p^* \,]_\perp = [\, V_1^* \otimes \cdots \otimes V_p^* \,]_\perp. \tag{4.10}$$

On the other hand, since the natural map

$$V_1 \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} V_p \to (V_1^* \widehat{\otimes} \cdots \widehat{\otimes} V_p^*)^*$$

is completely isometric ([6]), we have that

$$\ker \Psi = [\, V_1^* \otimes \cdots \otimes V_p^* \,]_\perp,$$

and our result follows from (4.9) and (4.10). ∎

PROPOSITION 4.3. *Given dual operator spaces* V_k^* *and* W_k^* *and weak* continuous completely contractive maps* $\varphi_k : V_k^* \to W_k^*$, $1 \leqslant k \leqslant p$, *the algebraic tensor product* $\varphi_1 \otimes \cdots \otimes \varphi_p$ *extends uniquely to a complete contraction*

$$\varphi_1 \otimes \cdots \otimes \varphi_p : V_1^* \,\overline{\otimes}_F \cdots \overline{\otimes}_F V_p^* \to W_1^* \,\overline{\otimes}_F \cdots \overline{\otimes}_F W_p^*, \tag{4.11}$$

which is continuous in the $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$, $W_1 \widehat{\otimes} \cdots \widehat{\otimes} W_p$ *topologies. Similarly, there is a unique extension*

$$\varphi_1 \otimes \cdots \otimes \varphi_p : V_1^* \,\overline{\otimes} \cdots \overline{\otimes}\, V_p^* \to W_1^* \,\overline{\otimes} \cdots \overline{\otimes}\, W_p^*, \tag{4.12}$$

which is continuous in the $V_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} V_p$, $W_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} W_p$ *topologies.*

Proof. For each k we have that $\varphi_k = (\varphi_{k*})^*$ for some complete contraction $\varphi_{k*} : W_k \to V_k$. The corresponding map

$$\varphi_\wedge = \varphi_{1*} \otimes \cdots \varphi_{p*} : W_1 \widehat{\otimes} \cdots \widehat{\otimes} W_p \to V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p$$

is a complete contraction for which the adjoint is (4.11), and which is obviously continuous in the stated topology. On the other hand, the maps φ_{k_*} determine a commutative diagram

$$\begin{tikzcd}
W_1 \widehat{\otimes} \cdots \widehat{\otimes} W_p \arrow[r, "\varphi_\wedge = \varphi_{1*} \otimes \cdots \otimes \varphi_{p*}"] \arrow[d, "\Psi_W"] & V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_p \arrow[d, "\Psi_V"] \\
W_1 \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} W_p \arrow[r, "\varphi_\vee = \varphi_{1*} \otimes \cdots \otimes \varphi_{p*}"] & V_1 \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} V_p
\end{tikzcd}$$

and in particular, we have that $\varphi_\wedge(\ker \Psi_W) \subseteq \ker \Psi_V$. It follows that $\varphi_\wedge$ induces a completely contractive map

$$\varphi_{\text{nuc}} : W_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} W_p \to V_1 \overset{\text{nuc}}{\otimes} \cdots \overset{\text{nuc}}{\otimes} V_p.$$

We obtain from this the desired map

$$\varphi_1 \otimes \cdots \otimes \varphi_p = \varphi_{\text{nuc}}^* : V_1^* \,\overline{\otimes} \cdots \overline{\otimes}\, V_p^* \to W_1^* \,\overline{\otimes} \cdots \overline{\otimes}\, W_p^*,$$

which is again continuous in the weak* topologies.

It is immediate that (4.10) and (4.11) extend the algebraic tensor product, and they are unique because the algebraic tensor products are dense in the corresponding topologies. ∎

5. THE EXTENDED AND NORMAL HAAGERUP TENSOR PRODUCTS

The Haagerup tensor product was first considered in unpublished notes of Haagerup (see [24]). An early discussion of Haagerup's theory appeared in [10]. We begin by reviewing this material.

Given matrices $v_k = \left[v^{(k)}_{j_{k-1},j_k}\right] \in M_{n_{k-1},n_k}(V_k)$, $k = 1, \ldots, p$, we define the *multiplicative product*

$$v_1 \odot \cdots \odot v_p \in M_{n_0,n_p}(V_1 \otimes \cdots \otimes V_p)$$

by "matrix multiplication", i.e.,

$$(v_1 \odot \cdots \odot v_p)_{j_0,j_p} = \sum_{j_1,\ldots,j_{p-1}} v^{(1)}_{j_0,j_1} \otimes \cdots \otimes v^{(p)}_{j_{p-1}j_p}. \tag{5.1}$$

In particular, if $j_0 = j_p = n$, then $v_1 \odot \cdots \odot v_p \in M_n(V_1 \otimes \cdots \otimes V_p)$.

Given operator spaces $V_1, \ldots, V_p$ and W and a multilinear map

$$\varphi : V_1 \times \cdots \times V_p \to W,$$

or equivalently a linear map

$$\varphi : V_1 \otimes \cdots \otimes V_p \to W,$$

we say that φ is *multiplicatively bounded* if there is a constant K such that for all $n \in \mathbb{N}$

$$\|\varphi_n(v_1 \odot \cdots \odot v_p)\| = \left\| \Big[\sum_{j_1 \cdots j_{p-1}} \varphi(v^{(1)}_{j_0,j_1} \otimes \cdots \otimes v^{(p)}_{j_{p-1}j_p}) \Big] \right\| \leqslant K\|v_1\| \cdots \|v_p\|$$

for all $v_k \in M_{n_{k-1},n_k}(V)$, where $n_0 = n_p = n$, and $n_1, \ldots, n_{p-1}$ are arbitrary. If φ is multiplicatively bounded, we define its *multiplicative norm* $\|\varphi\|_{mb}$ to be the least such constant K, i.e.,

$$\|\varphi\|_{\mathrm{mb}} = \sup\left\{\|\varphi_n(v_1 \odot \cdots \odot v_p)\| : \|v_1\| \cdots \|v_p\| \leqslant 1\right\}.$$

These matrix norms determine an operator space structure on the linear space $CB_{\mathrm{m}}(V_1 \times \cdots \times V_p, W)$ of all such maps. If the V_k and W are dual operator spaces, we again say that φ is *normal* if it is weak* continuous in each variable, and we let $CB^{\sigma}_{\mathrm{m}}(V_1 \times \cdots \times V_p, W)$ be the operator subspace of normal maps. These notions were introduced by Christensen and Sinclair ([9]).

Theorem 5.1. *A multilinear map*

$$\varphi : V_1 \times \cdots \times V_p \to B(H_p, H_0),$$

is multiplicatively contractive if and only if there exist Hilbert spaces $H_1, \ldots, H_{p-1}$ and complete contractions $\varphi_k : V_k \to B(H_k, H_{k-1})$ such that

$$\varphi(v_1, \ldots, v_p) = \varphi_1(v_1) \cdots \varphi_p(v_p). \tag{5.2}$$

If each V_k is a dual space and φ is normal, then we may assume that each φ_k is weak continuous.*

Proof. The representation (5.2) is just a restatement of the Christensen-Sinclair theorem [9] as generalized to operator spaces by Paulsen and Smith (see

[32] or [21]). The theorem for normal maps is well-known to specialists. We have included a simple proof for the convenience of the reader.

Let us assume that for fixed $v_i \in V_i$, $i \neq k$, and $\xi_p \in H_p$, $\eta_0 \in H_0$

$$v_k \mapsto \langle s_1(v_1) \cdots s_k(v_k) \cdots s_p(v_p)\xi_p \mid \eta_0 \rangle \tag{5.3}$$

is weak* continuous. We begin by noting that

$$\begin{aligned}&\langle s_1(v_1)s_2(v_2) \cdots s_r(v_p)\xi_p \mid \eta_0 \rangle \\ &\qquad = \langle s_k(v_k)s_{k+l}(v_{k+l}) \cdots s_p(v_p)\xi_p \mid s_{k-1}(v_{k-1})^* \cdots s_1(v_1)^*\eta_0 \rangle.\end{aligned}$$

We may assume that

$$H'_k = \{s_{k+1}(v_{k+1}) \cdots s_p(v_p)\xi_p : v_i \in V_i (i \geqslant k+1),\, \xi_p \in H_p\}$$

is dense in H_k, since otherwise we may replace H_k by the norm closure $\mathrm{cl}(H'_k)$ and s_k by $s_k\,|_{\mathrm{cl}(H'_k)}$ without affecting the equality in (3.2) or the continuity that might be assumed in any of the variables. Similarly we may assume that

$$H''_{k-1} = \{s_{k-1}(v_{k-1})^* \cdots s_1(v_1)^*\eta_0 : v_i \in V_i \ (i \leqslant k-1), \eta_0 \in H_0\}$$

is dense in H_{k-1}.

By hypothesis we have that $v_k \mapsto \langle s_k(v_k)\xi \mid \eta \rangle$ is weak* continuous for $\xi \in H'_k$ and $\eta \in H''_{k-1}$. If we let $\xi_n \in H'_k$ and $\eta_n \in H''_{k-1}$ be sequences converging to vectors $\xi \in H_k$ and $\eta \in H_{k-1}$, then the functions $v_k \mapsto \langle s_k(v_k)\xi_n | \eta_n \rangle$ converge uniformly on the closed unit ball of V_k to the function $v_k \mapsto \langle s_k(v_k)\xi \mid \eta \rangle$. It follows that the latter function is weak* continuous on that ball and thus on all of V_k. ∎

Given operator spaces V_k, $k = 1, \ldots, p$, and a matrix $u \in M_n(V_1 \otimes \cdots \otimes V_p)$, we define the *Haagerup norm* of u by

$$\|u\|_{\mathrm{h}} = \inf \left\{ \|v_1\| \cdots \|v_p\| : u = v_1 \odot \cdots \odot v_p, v_k \in M_{n_{k-1},n_k}(V_k) \right\}, \tag{5.4}$$

where $n_0 = n$, $n_p = n$, and n_k is arbitary for $1 \leqslant k \leqslant p-1$. These matrix norms determine an operator space structure on $V_1 \otimes \cdots \otimes V_p$, and we call its completion $V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p$ the *Haagerup tensor product.*

A multilinear map $\varphi :\ V_1 \times \cdots \times V_p \to W$ is multiplicatively contractive if and only if there is a completely contractive map $\widetilde{\varphi} : V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p \to W$ with

$$\varphi(v_1, \ldots, v_p) = \widetilde{\varphi}(v_1 \otimes \cdots \otimes v_p).$$

In this manner we obtain the completely isometric identification

$$CB_{\mathrm{m}}(V_1 \times \cdots \times V_p, W) \cong CB(V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p, W). \tag{5.5}$$

Given complete contractions $\varphi_k : V_k \to W_k$, we have that

$$\varphi_1 \otimes \cdots \otimes \varphi_p : V_1 \times \cdots \times V_p \longrightarrow W_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_p$$

is multiplicatively contractive, and thus determines a complete contraction

$$\varphi_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} \varphi_p : V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p \longrightarrow W_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_p$$

(see, e.g., [21], Proposition 9.2.5). The Haagerup tensor product is both injective and projective. Furthermore, it is associative, but it is generally not commutative.

We define the *extended Haagerup tensor product* $V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p$ to be the space of all normal multiplicatively bounded maps $u : V_1^* \times \cdots \times V_p^* \to \mathbb{C}$, i.e.,

$$V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p = (V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p)^*_\sigma = CB^\sigma_{\mathrm{m}}(V_1^* \times \cdots \times V_p^*, \mathbb{C}) \tag{5.6}$$

(see [11], [12], and [15]) and we let $\| \cdot \|_{\mathrm{eh}}$ denote the relative matrix norms inherited from the operator space $CB_{\mathrm{m}}(V_1^* \times \cdots \times V_p^*, \mathbb{C})$. Equivalently, the matrix norms are determined by the identification

$$M_n(CB^\sigma_{\mathrm{m}}(V_1^* \times \cdots \times V_p^*, \mathbb{C})) = CB^\sigma_{\mathrm{m}}(V_1^* \times \cdots \times V_p^*, M_n).$$

We may use Theorem 5.1 to write the elements $u \in M_n(V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p)$ in terms of infinite matrices over the V_k. It follows from the normal mapping result in Theorem 5.1 and (3.2) that if $\|u\|_{\mathrm{eh}} \leqslant 1$, then there exist contractive matrices $v_k \in M_{J_{k-1},J_k}(V_k)$, where $J_0 = J_p = \{1, \ldots, n\}$, for which

$$u(f_1, \ldots, f_p) = \langle f_1, v_1 \rangle \cdots \langle f_p, v_p \rangle. \tag{5.7}$$

If that is the case, we use the notation

$$u = v_1 \odot \cdots \odot v_p = v_1 \odot_{J_1} \cdots \odot_{J_{p-1}} v_p. \tag{5.8}$$

Changing to matrix notation, we have from the discussion in Section 3 that

$$\begin{aligned}\langle f_1 \otimes \cdots \otimes f_k, u \rangle &= \langle f_1, v_1 \rangle \cdots \langle f_p, v_p \rangle \\ &= \lim_{F_1 \cdots F_{p-1}} \Big[\sum_{i_1 \in F_1, \ldots, i_{p-1} \in F_{p-1}} f_1(v^{(1)}_{i_0,i_1}) \cdots f_p(v^{(p)}_{i_{p-1},i_p}) \Big]\end{aligned} \tag{5.9}$$

where the limit is taken over finite subsets $F_k \subseteq J_k$, $1 \leqslant k \leqslant p-1$. If we let $F = F_1 \times \cdots \times F_p$ and $v_k^F \in M_{F_{k-1},F_k}(V_k)$ be the obvious truncation of v_k, we see that the net

$$F \to u_F = v_1^F \odot \cdots \odot v_p^F \in V_1 \otimes \cdots \otimes V_p$$

converges to u in the topology determined by $V_1^* \otimes \cdots \otimes V_p^*$. Since it is evident that

$$\|u_F\|_{\mathrm{eh}} \leqslant \|u_F\|_{\mathrm{h}} \leqslant \|v_1\| \cdots \|v_p\|$$

it also converges in the topology determined by $V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^*$.

It is clear from our discussion above that

$$\|u\|_{\mathrm{eh}} = \inf\{\|v_1\| \cdots \|v_p\|\}, \tag{5.10}$$

where the infimum extends over all representations (5.8).

Given completely bounded maps $\varphi_k : V_k \to W_k$, the corresponding map

$$\overline{\varphi} = (\varphi_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} \varphi_n^*)^* : (V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_n^*)^* \to (W_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_n^*)^*$$

satisfies

$$\overline{\varphi}((V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^*)^*_\sigma) \subseteq (W_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_p^*)^*_\sigma$$

since each map $\varphi_j^* : W_j^* \to V_j^*$ is weak* continuous. We let

$$\varphi_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} \varphi_p : V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p \to W_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} W_p \tag{5.11}$$

be the restriction of $\overline{\varphi}$ to $V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p$. We note that if the φ_k are complete contractions, then the same is true for $\overline{\varphi}$ and thus for $\varphi_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} \varphi_p$.

LEMMA 5.2. *Suppose that* V_j, W_j, $j = 1, \ldots, p$, *are operator spaces, and that* $\varphi_j : V_j \to W_j$ *are completely bounded. Then given index sets* J_j *with* $J_0 = J_p = \{1\}$ *and* $v_j \in M_{J_{j-1} J_j}(V_j)$ *we have*

$$(\varphi_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} \varphi_p)(v_1 \odot \cdots \odot v_p) = \varphi_1(v_1) \odot \cdots \odot \varphi_p(v_p).$$

Proof. If we let $\varphi = \varphi_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} \varphi_p$, then for $g_j \in W_j^*$ we have

$$\begin{aligned}
\langle \varphi(v_1 \odot \cdots \odot v_p), g_1 \otimes \cdots \otimes g_p \rangle &= \langle v_1 \odot \cdots \odot v_p, \varphi_1^*(g_1) \otimes \cdots \otimes \varphi_p^*(g_p) \rangle \\
&= \langle v_1, \varphi_1^*(g_1) \rangle \cdots \langle v_p, \varphi_p^*(g_p) \rangle \\
&= \langle (\varphi)_{J_1 J_2}(v_1), g_1 \rangle \cdots \langle (\varphi_p)_{J_{p-1} J_p}(v_p), g_p \rangle \\
&= \langle \varphi_1(v_1) \odot \cdots \odot \varphi_p(v_p), g_1 \otimes \cdots \otimes g_p \rangle.
\end{aligned}$$

Since elements of $W_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} W_n$ are determined by the values they assume on elements of $W_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_n^*$, or equivalently, on elements of $W_1^* \otimes \cdots \otimes W_n^*$, we conclude that

$$\varphi(v_1 \odot \cdots \odot v_n) = \varphi_1(v_1) \odot \cdots \odot \varphi_n(v_n). \quad ∎$$

We conclude that $\overset{\mathrm{eh}}{\otimes}$ is a tensor product in the sense described in the previous section. We will prove that it is associative below.

In [7] Blecher and Smith characterized the dual of the Haagerup tensor product in terms of what they called the *weak* Haagerup tensor product*. In the following we see that this coincides with the extended Haagerup tensor product of dual operator spaces. It should be noted that Stephen Allen has studied the weak* Haagerup tensor product for operator spaces that are not necessarily dual spaces ([1]).

THEOREM 5.3. *Suppose that* $V_1, \ldots, V_p$ *are operator spaces. Then we have the complete isometry*

$$(V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p)^* \cong V_1^* \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p^*.$$

Proof. From Theorem 5.1 and (5.7), elements f of both

$$M_n((V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p)^*) = CB(V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p, M_n)$$

and of

$$M_n(V_1^* \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p^*) = CB^\sigma_{\mathrm{m}}(V_1^{**} \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^{**}, M_n)$$

have representations of the form

$$f(v_1, \ldots, v_p) = \langle f_1, v_1 \rangle \cdots \langle f_p, v_p \rangle, \tag{5.12}$$

where in the first case $v_k \in V_k$ and $f_k : V_k \to B(H_k, H_{k-1})$ is completely bounded, and in the second case $v_k \in V_k^{**}$ and $f_k : V_k^{**} \to B(H_k, H_{k-1})$ is weak* continuous and completely bounded. Thus it suffices to show that we have a natural identification

$$CB^\sigma(V^{**}, B(H,K)) \cong CB(V, B(H,K)).$$

Changing to matrix notation, this is evident from (3.2) and (3.1) since we have

$$CB^\sigma(V^{**}, M_{I,J}) = M_{I,J}(V^*) = CB(V, M_{I,J}). \quad ∎$$

We note that if we are given $f = f_1 \odot \cdots \odot f_p \in M_n(V_1^* \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p^*)$ and $v_k \in V_k$, then from (5.12) we have the matrix product

$$\begin{aligned} \langle f, v_1 \otimes \cdots \otimes v_p \rangle &= \langle f_1, v_1 \rangle \cdots \langle f_p, v_p \rangle \\ &= \lim_{G_1 \cdots G_{p-1}} \Big[\sum_{j_1 \in G_1, \ldots, j_{p-1} \in G_{p-1}} f^{(1)}_{j_0 j_1}(v_1) \cdots f^{(p)}_{j_{p-1} j_p}(v_p) \Big], \end{aligned} \tag{5.13}$$

where the limit is taken over finite subsets $G_k \subseteq J_k$, $1 \leqslant k \leqslant p-1$.

Lemma 5.4. *Suppose that V_k, W_k, $k = 1, \ldots, p$, are operator spaces, and that for each k, $\varphi_k : V_k \to W_k$ is completely isometric. Then* (5.11) *is completely isometric.*

Proof. Let us suppose that the φ_k are completely isometric. Then the maps $\varphi_k : W_k^* \to V_k^*$ are complete quotient maps, and since the Haagerup tensor product is projective, the same is true for the map

$$\varphi_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} \varphi_p^* : W_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_p^* \to V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^*.$$

It follows that the bottom row of the following diagram is a completely isometric injection, and thus the same is true for the top row:

$$\begin{tikzcd} V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p \arrow[r, "\varphi_1 \otimes \cdots \otimes \varphi_p"] \arrow[d, phantom, sloped, "\subseteq"] & W_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} W_p \arrow[d, phantom, sloped, "\subseteq"] \\ (V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^*)^* \arrow[r, "(\varphi_1^* \otimes \cdots \otimes \varphi_p^*)^*"] & (W_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} W_p^*)^* \end{tikzcd} . \quad ∎$$

We conclude that the extended Haagerup tensor product is injective. In contrast to the Haagerup tensor product, the extended Haagerup tensor product is not projective, i.e., if one is given an operator space X and a complete quotient map $Y \to Y_1$, then the induced map $X \overset{\mathrm{eh}}{\otimes} Y \to X \overset{\mathrm{eh}}{\otimes} Y_1$ need not be a quotient map. We are indebted to David Blecher for the following argument.

Proposition 5.5. *The extended Haagerup tensor product is not projective.*

Proof. We recall from [4] that an operator space X is said to be *projective* if given operator spaces V and W and a complete quotient map $\pi : V \to W$, then any map $\varphi : X \to W$ with $\|\varphi\|_{\mathrm{cb}} < 1$ can be lifted to a map $\widetilde{\varphi} : X \to V$ with $\|\widetilde{\varphi}\|_{\mathrm{cb}} < 1$. Equivalently, the induced map $CB(X, V) \to CB(X, W)$ is a Banach space quotient map. If X is projective, then the latter is in fact a complete quotient map since we may identify $\pi_n : M_n(CB(X, M_n(X))) \to M_n(CB(X, M_n(W)))$ with the complete quotient map $CB(X, M_n(V)) \to CB(X, M_n(W))$.

Taking adjoint maps, it is evident that an operator space X is projective if and only if for any weak* homeomorphic completely isometric injection $\psi : W^* \to V^*$, the corresponding map

$$CB^\sigma(W^*, X^*) \to CB^\sigma(V^*, X^*)$$

is isometric (or completely isometric). In other words, any weak* continuous complete contraction $\psi : W^* \to X^*$ has a weak* continuous completely contractive extension $\psi : V^* \to X^*$. This is the case if $X = T_{m,n} = (M_{m,n})_*$. On the other hand this weak* version of injectivity was shown to be false for M_∞ in [14], and thus $T_\infty = (M_\infty)_*$ is not projective.

If X and Y are projective operator spaces, then the same is true for $X \mathbin{\widehat{\otimes}} Y$. To see this we note that if we are given a complete quotient map, then the induced map $CB(Y, V) \to CB(Y, W)$ is a complete quotient map, and therefore

$$CB(X \mathbin{\widehat{\otimes}} Y, V) = CB(X, CB(Y, V)) \to CB(X, CB(Y, W)) = CB(X \mathbin{\widehat{\otimes}} Y, W)$$

is a complete quotient map. It follows that the column Hilbert space $M_{\infty,1}$ is not projective, since if it were, then its conjugate operator space $M_{1,\infty}$ would also be projective, and therefore $T_\infty = M_{\infty,1} \mathbin{\widehat{\otimes}} M_{1,\infty}$ would also be projective, a contradiction (see [21] for a discussion of the conjugate operator space).

Changing notation, we have that $M_{\infty,1} = H_{\mathrm{c}}$, where $H = \ell^2$. For any operator space V, we have the complete isometries

$$M_{\infty,1} \overset{\mathrm{eh}}{\otimes} V = ((H_{\mathrm{c}})^* \overset{\mathrm{h}}{\otimes} V^*)^*_\sigma \cong ((H_{\mathrm{c}})^* \mathbin{\widehat{\otimes}} V^*)^*_\sigma \cong CB^\sigma(V^*, H_{\mathrm{c}}) \cong CB((H_{\mathrm{c}})^*, V)$$

(see [5], [6], [16], [18], and [21], (9.3.5)), where the identification on the right is the inverse of the adjoint map $\varphi \mapsto \varphi^*$.

Let us suppose that $\overset{\mathrm{eh}}{\otimes}$ is projective in the second variable. It follows from the above relation that for any complete quotient map $V \to W$, the corresponding map

$$CB((H_{\mathrm{c}})^*, V) \to CB((H_{\mathrm{c}})^*, W) \tag{5.14}$$

is a complete quotient map, i.e., $(H_{\mathrm{c}})^* = M_{1,\infty}$ and therefore its conjugate operator space $M_{\infty,1}$ is projective, a contradiction. ∎

It is evident that the identity map $V_1 \otimes \cdots \otimes V_p \to V_1 \otimes \cdots \otimes V_p$ is completely contractive with respect to the Haagerup and extended Haagerup tensor products since the extended product norm uses more decompositions. In fact the map

$$V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p \to V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p \tag{5.15}$$

is a completely isometric injection. This is apparent from the diagram

$$\begin{tikzcd}
V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p \arrow[d] \arrow[dr] & \\
V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p \arrow[r] & (V_1^* \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p^*)^*
\end{tikzcd},$$

where the diagonal map is a completely isometric injection owing to the "self-duality" of the Haagerup tensor product ([5], [18]) and the bottom map is completely isometric by definition.

We turn next to a surprising result of Blecher and Smith (see [7] for the case $p = 2$). If $f \in V_1^* \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p^* = (V_1 \overset{\mathrm{h}}{\otimes} \cdots \overset{\mathrm{h}}{\otimes} V_p)^*$, then we may extend it to elements $u \in V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p$. In fact, we may define a pairing of $n \times n$ matrices f and u over these two spaces as follows. If $f = f_1 \odot \cdots \odot f_p$, and $u = v_1 \odot \cdots \odot v_p$, where $v_k \in M_{J_{k-1},J_k}(V_k)$ and $f_k \in M_{I_{k-1},I_k}(V_k^*)$, and $I_0 = I_p = J_0 = J_p = \{1, \ldots, n\}$, we wish to define

$$\langle f, u \rangle = \langle f_1, v_1 \rangle \cdots \langle f_p, v_p \rangle. \tag{5.16}$$

The right hand side makes sense because it is the product of the bounded scalar matrices

$$\langle f_k, v_k \rangle \in M_{I_{k-1} \times J_{k-1}, I_k \times J_k}. \tag{5.17}$$

Proposition 5.6. *The pairing* (5.16) *does not depend upon the decompositions $f = f_1 \odot \cdots \odot f_p$ and $u = v_1 \odot \cdots \odot v_p$.*

Proof. If we let F_k range over the finite sets in I_k, $1 \leqslant k \leqslant p-1$, then the projections $P(F_k \times J_k)$ converge to the identity operator in the strong operator topology. It follows from (5.9) that

$$\begin{aligned}\langle f, u \rangle &= \lim_{F_1 \cdots F_{p-1}} \langle f_1, v_1 \rangle P(F_1 \times J_1) \cdots P(F_{p-1} \times J_{p-1}) \langle f_p, v_p \rangle \\ &= \lim_{F_1 \cdots F_{p-1}} \left[\sum_{i_k \in F_k} \sum_{j_k \in J_k} f^{(1)}_{i_0 i_1}(v^{(1)}_{j_0 j_1}) \cdots f^{(p)}_{i_{p-1} i_p}(v^{(p)}_{j_{p-1} j_p}) \right] \\ &= \lim_{F_1 \cdots F_{p-1}} \sum_{i_k \in F_k} \langle f^{(1)}_{i_0 i_1} \otimes \cdots \otimes f^{(p)}_{i_{p-1} i_p}, v_1 \odot \cdots \odot v_p \rangle \\ &= \lim_{F_1 \cdots F_{p-1}} \sum_{i_k \in F_k} \langle f^{(1)}_{i_0 i_1} \otimes \cdots \otimes f^{(p)}_{i_{p-1} i_p}, u \rangle\end{aligned} \tag{5.18}$$

(this is a norm limit of matrices in $M_{I_0 \times J_0, I_p \times J_p} \cong M_{n^2}$) and thus (5.16) does not depend upon the decomposition $u = v_1 \odot \cdots \odot v_p$.

On the other hand if we let G_k range over the finite sets in J_k, then the projections $P(I_k \times G_k)$ converge to the identity operator in the strong operator topology. Thus from (5.13),

$$\begin{aligned}\langle f, u \rangle &= \lim_{G_1 \cdots G_{p-1}} \langle f_1, v_1 \rangle P(I_1 \times G_1) \cdots P(I_{p-1} \times G_{p-1}) \langle f_p, v_p \rangle \\ &= \lim_{G_1 \cdots G_{p-1}} \left[\sum_{j_k \in G_k} \sum_{i_k \in I_k} f^{(1)}_{i_0 i_1}(v^{(1)}_{j_0 j_1}) \cdots f^{(p)}_{i_{p-1,1}}(v^{(p)}_{j_{p-1}, j_p}) \right] \\ &= \lim_{G_1 \cdots G_{p-1}} \sum_{j_k \in G_k} \langle f_1 \odot \cdots \odot f_p, v^{(1)}_{i_0, j_1} \otimes \cdots \otimes v^{(p)}_{j_{p-1}, j_p} \rangle \\ &= \lim_{G_1 \cdots G_{p-1}} \sum_{j_k \in G_k} \langle f, v^{(1)}_{j_0, j_1} \otimes \cdots \otimes v^{(p)}_{j_{p-1}, j_p} \rangle\end{aligned} \tag{5.19}$$

and (5.16) does not depend upon the decomposition $f = f_1 \odot \cdots \odot f_p$. ∎

We conclude from these considerations the following result.

THEOREM 5.7. (see [7] for the case $p = 2$) *For any operator spaces* $V_1, \ldots, V_p$, *the pairing* (5.16) *determines a completely isometric inclusion*

$$V_1^* \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p^* \hookrightarrow (V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p)^*. \tag{5.20}$$

Proof. Given $f \in M_n(V_1^* \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p^*)$ and $u \in M_n(V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p)$, we have from (5.19) that

$$\langle f, u \rangle = \lim_{G_1 \cdots G_{p-1}} \langle f, v_G^{(1)} \odot \cdots \odot v_G^{(p)} \rangle,$$

where we let $G = G_1 \times \cdots \times G_p$, and $v_G^{(k)} \in M_{G_{k-1}, G_k}(V_k)$ be the obvious truncation of $v^{(k)}$. If $\|u\|_{\text{eh}} \leqslant 1$, then we may assume that $\|v^{(k)}\| \leqslant 1$, and thus $\|v_{G_k}^{(k)}\| \leqslant 1$. If we let $u_G = v_G^{(1)} \odot \cdots \odot v_G^{(p)}$, then from Theorem 5.3,

$$\|\langle f, u_G \rangle\| \leqslant \|f\|_{\text{eh}} \|u_G\|_{\text{h}} \leqslant \|f\|_{\text{eh}} \|v_G^{(1)}\| \cdots \|v_G^{(p)}\| \leqslant \|f\|_{\text{eh}}.$$

It follows that $\|\langle f, u \rangle\| \leqslant \|f\|_{\text{eh}}$ and thus from (3.5), (5.20) is completely contractive. It is immediate from (5.15) that this mapping is a complete isometry. ∎

Given dual operator spaces $V_1^*, \ldots, V_r^*$ the *normal Haagerup tensor product* (see [12]) is defined by

$$V_1^* \overset{\sigma\text{h}}{\otimes} \cdots \overset{\sigma\text{h}}{\otimes} V_p^* = (V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p)^*.$$

LEMMA 5.8. *If* $V_1, \ldots, V_p$ *are operator spaces, then* $V_1^* \otimes \cdots \otimes V_p^*$ *is dense in* $V_1^* \overset{\sigma\text{h}}{\otimes} \cdots \overset{\sigma\text{h}}{\otimes} V_p^*$ *in the weak* topology defined by* $V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p$.

Proof. If $u \in V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p$ satisfies

$$\langle f_1 \otimes \cdots \otimes f_p, u \rangle = 0$$

for all $f_k \in V_k^*$ then from (5.6) it is evident that $u = 0$, and thus from the bipolar theorem, we have the density result. ∎

PROPOSITION 5.9. *Given a normal multiplicatively bounded multilinear map* $\varphi : V_1^* \times \cdots \times V_p^* \to W^*$ *there is a unique weak* continuous completely bounded map* $\varphi_{\sigma\text{h}} : V_1^* \overset{\sigma\text{h}}{\otimes} \cdots \overset{\sigma\text{h}}{\otimes} V_p^* \to W^*$ *such that*

$$\varphi(f_1, \ldots, f_n) = \varphi_{\sigma\text{h}}(f_1 \otimes \cdots \otimes f_n).$$

Proof. If $w \in W$, then $w \circ \varphi : V_1^* \times \cdots \times V_p^* \to \mathbb{C}$ is normal and multiplicatively contractive, and thus an element of $V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p$. This determines a complete contraction map

$$\varphi_* : W \to V_1 \overset{\text{eh}}{\otimes} \cdots \overset{\text{eh}}{\otimes} V_p, \quad w \mapsto w \circ \varphi,$$

and we may let $\varphi_{\sigma\text{h}} = (\varphi_*)^*$. ∎

In particular, if H, K and L are Hilbert spaces, the multiplication map

$$B(K,L) \times B(H,K) \to B(H,L)$$

is both normal and multiplicatively contractive, and thus determines a weak* continuous complete contraction

$$B(K,L) \overset{\sigma\mathrm{h}}{\otimes} B(H,K) \to B(H,L). \tag{5.21}$$

Using a simple elaboration of the proof of Proposition 5.1, we obtain the natural identification

$$CB^{\sigma}_{\mathrm{m}}(V_1^* \times \cdots \times V_n^*, W^*) = CB^{\sigma}(V_1^* \overset{\sigma\mathrm{h}}{\otimes} \cdots \overset{\sigma\mathrm{h}}{\otimes} V_n^*, W^*). \tag{5.22}$$

In particular, weak* continuous complete contractions $\varphi_k : V_k^* \to W_k^*$ determines a weak* continuous complete contraction

$$\varphi_1 \overset{\sigma\mathrm{h}}{\otimes} \cdots \overset{\sigma\mathrm{h}}{\otimes} \varphi_p = (\varphi_{1*} \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} \varphi_{p*})^* : V_1^* \overset{\sigma\mathrm{h}}{\otimes} \cdots \overset{\sigma\mathrm{h}}{\otimes} V_p^* \to W_1^* \overset{\sigma\mathrm{h}}{\otimes} \cdots \overset{\sigma\mathrm{h}}{\otimes} W_p^*.$$

Owing to the injectivity of the extended Haagerup tensor product, the normal tensor product is projective for weak* closed subspaces. On the other hand, since extended Haagerup tensor product is not projective, the normal Haagerup tensor product is not injective.

We may use the normal tensor product to prove that the extended Haagerup tensor product is associative. It suffices to consider the case $p = 3$. Given operator spaces V, W, X, the tensor product $(V \overset{\mathrm{eh}}{\otimes} W) \overset{\mathrm{eh}}{\otimes} X$ by definition consists of the normal multiplicatively bounded maps

$$u : (V \overset{\mathrm{eh}}{\otimes} W)^* \times X^* \to \mathbb{C}.$$

But any such map has the form $u = u_1 \odot_J x$ where

$$u_1 : V^* \overset{\sigma\mathrm{h}}{\otimes} W^* = (V \overset{\mathrm{eh}}{\otimes} W)^* \to M_{I,J}$$

is weak* continuous and completely bounded. It follows that u_1 corresponds to a normal multiplicatively bounded map of $V^* \times W^*$ into $\mathbb{C}$, and from Theorem 5.1 there are elements $v \in M_{I,J}(V)$ and $w \in M_{I,J}(W)$ for which

$$u_1(f,g) = \langle v, f\rangle \, \langle w, g\rangle$$

(this is a matrix product). It follows that u uniquely determines a unique element

$$\widetilde{u} = v \odot_I w \odot_J x \in V \overset{\mathrm{eh}}{\otimes} W \overset{\mathrm{eh}}{\otimes} X.$$

Since the reverse argument is also clear, we obtain a canonical identification of $(V \overset{\mathrm{eh}}{\otimes} W) \overset{\mathrm{eh}}{\otimes} X$ with $V \overset{\mathrm{eh}}{\otimes} W \overset{\mathrm{eh}}{\otimes} X$, and a similar argument applies to $V \overset{\mathrm{eh}}{\otimes} (W \overset{\mathrm{eh}}{\otimes} X)$.

Finally, we note that (5.20) provides us with a natural inclusion

$$V_1^* \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p^* \subseteq V_1^* \overset{\sigma\mathrm{h}}{\otimes} \cdots \overset{\sigma\mathrm{h}}{\otimes} V_p^*, \tag{5.23}$$

and the adjoint of (5.15) determines a natural weak* continuous projection of the second space onto the first.

6. THE SHUFFLE THEOREM

The following theorem was proved for von Neumann algebras in [15], and a variation of this result was proved for operator spaces in [13].

THEOREM 6.1. *Suppose that V_k, W_k, $k = 1, 2$, are operator spaces. Then the shuffle map*

$$\mathcal{S} : (V_1^* \otimes V_2^*) \otimes (W_1^* \otimes W_2^*) \to (V_1^* \otimes W_1^*) \otimes (V_2^* \otimes W_2^*) \tag{6.1}$$

extends uniquely to a weak continuous complete contraction*

$$\mathcal{S}_\sigma : (V_1^* \overline{\otimes} V_2^*) \overset{\sigma\mathrm{h}}{\otimes} (W_1^* \overline{\otimes} W_2^*) \to (V_1^* \overset{\sigma\mathrm{h}}{\otimes} W_1^*) \overline{\otimes} (V_2^* \overset{\sigma\mathrm{h}}{\otimes} W_2^*). \tag{6.2}$$

On the other hand, the shuffle map

$$\mathcal{S} : (V_1 \otimes W_1) \otimes (V_2 \otimes W_2) \to (V_1 \otimes V_2) \otimes (W_1 \otimes W_2) \tag{6.3}$$

may be extended to a complete contraction

$$\mathcal{S}_\mathrm{e} : (V_1 \overset{\mathrm{eh}}{\otimes} W_1) \overset{\mathrm{nuc}}{\otimes} (V_2 \overset{\mathrm{eh}}{\otimes} W_2) \to (V_1 \overset{\mathrm{nuc}}{\otimes} V_2) \overset{\mathrm{eh}}{\otimes} (W_1 \overset{\mathrm{nuc}}{\otimes} W_2). \tag{6.4}$$

Proof. We may fix faithful weak* representations

$$\Phi : V_1^* \overset{\sigma\mathrm{h}}{\otimes} W_1^* \hookrightarrow B(H_1) \quad \text{and} \quad \Psi : V_2^* \overset{\sigma\mathrm{h}}{\otimes} W_2^* \hookrightarrow B(H_2).$$

Since the normal spatial tensor product of dual operator spaces is independent of the choice of Hilbert spaces, we may identify $(V_1^* \overset{\sigma\mathrm{h}}{\otimes} V_2^*) \overline{\otimes} (W_1^* \overset{\sigma\mathrm{h}}{\otimes} W_2^*)$ with the weak* closure

$$\mathrm{cl}_{\mathrm{w}^*}\{\Phi(V_1^* \overset{\sigma\mathrm{h}}{\otimes} W_1^*) \otimes \Psi(V_2^* \overset{\sigma\mathrm{h}}{\otimes} W_2^*)\} \subseteq B(H_1 \otimes H_2).$$

From Theorem 5.1 there exist Hilbert spaces $H_k^{'}$ and weak* continuous complete contractions $s_k : V_k^* \to B(H_k^{'}, H_k)$ and $t_k : W_k^* \to B(H_k, H_k^{'})$, $k = 1, 2$, for which

$$\Phi(f_1 \otimes g_1) = s_1(f_1)t_1(g_1) \quad \text{and} \quad \Psi(f_2 \otimes g_2) = s_2(f_2)t_2(g_2). \tag{6.5}$$

These induce weak* continuous maps

$$s = s_1 \otimes s_2 : V_1^* \overline{\otimes} V_2^* \to B(H_1' \otimes H_2', H_1 \otimes H_2)$$

and

$$t = t_1 \otimes t_2 : W_1^* \overline{\otimes} W_2^* \to B(H_1 \otimes H_2, H_1' \otimes H_2')$$

and thus a weak* continuous complete contraction

$$\mathcal{S}_\sigma = st : (V_1^* \overline{\otimes} V_2^*) \overset{\sigma\mathrm{h}}{\otimes} (W_1^* \overline{\otimes} W_2^*) \to B(H_1 \otimes H_2).$$

We claim that $\mathcal{S}_\sigma$ extends $\mathcal{S}$. We may use Φ and Ψ to identify $f_i \otimes g_i$ with their images $s_i(f_i)t_i(g_i)$, $i = 1, 2$. It follows that

$$\begin{aligned}\mathcal{S}_\sigma((f_1 \otimes f_2) \otimes (g_1 \otimes g_2)) &= s(f_1 \otimes f_2)t(g_1 \otimes g_2) \\ &= (s_1(f_1) \otimes s_2(f_2))(t_1(g_1) \otimes t_2(g_2)) = s_1(f_1)t_1(g_1) \otimes s_2(f_2)t_2(g_2) \\ &= (f_1 \otimes g_1) \otimes (f_2 \otimes g_2) = \mathcal{S}((f_1 \otimes f_2) \otimes (g_1 \otimes g_2)).\end{aligned}$$

It is obvious that $\mathcal{S}$ has range in $(V_1^* \overset{\sigma\mathrm{h}}{\otimes} W_1^*) \overline{\otimes} (V_2^* \overset{\sigma\mathrm{h}}{\otimes} W_2^*)$, and thus to show that the same is true for $\mathcal{S}_\sigma = st$ it suffices to prove that

$$X_0 = V_1^* \otimes V_2^* \otimes W_1^* \otimes W_2^*$$

is weak* dense in

$$X = (V_1^* \overline{\otimes} V_2^*) \overset{\sigma\mathrm{h}}{\otimes} (W_1^* \overline{\otimes} W_2^*).$$

We have that $V_1^* \otimes V_2^*$ and $W_1^* \otimes W_2^*$ are weak* dense in $V_1^* \overline{\otimes} V_2^*$ and $W_1^* \overline{\otimes} W_2^*$, respectively. Since the bilinear map

$$(V_1^* \overline{\otimes} V_2^*) \times (W_1^* \overline{\otimes} W_2^*) \to (V_1^* \overline{\otimes} V_2^*) \overset{\sigma\mathrm{h}}{\otimes} (W_1^* \overline{\otimes} W_2^*)$$

is weak* continuous in each variable, it follows that X_0 is weak* dense in

$$(V_1^* \overline{\otimes} V_2^*) \otimes (W_1^* \overline{\otimes} W_2^*)$$

and thus from Lemma 5.8 it is dense in X.

Since $\mathcal{S}_\sigma$ is weak* continuous, we have that $\mathcal{S}_\sigma = T^*$ for some complete contraction

$$T : (V_1 \overset{\mathrm{eh}}{\otimes} W_1) \overset{\mathrm{nuc}}{\otimes} (V_2 \overset{\mathrm{eh}}{\otimes} W_2) \to (V_1 \overset{\mathrm{nuc}}{\otimes} V_2) \overset{\mathrm{eh}}{\otimes} (W_1 \overset{\mathrm{nuc}}{\otimes} W_2).$$

To check that this extends (6.3), we note that for $v_k \in V_k$ and $w_k \in W_k$ we have

$$\begin{aligned}
&\langle T((v_1 \otimes w_1) \otimes (v_2 \otimes w_2)), (f_1 \otimes f_2) \otimes (g_1 \otimes g_2)\rangle \\
&\quad = \langle (v_1 \otimes w_1) \otimes (v_2 \otimes w_2)), \mathcal{S}_\sigma((f_1 \otimes f_2) \otimes (g_1 \otimes g_2))\rangle \\
&\quad = \langle (v_1 \otimes w_1) \otimes (v_2 \otimes w_2), (f_1 \otimes g_1) \otimes (f_2 \otimes g_2)\rangle \\
&\quad = \langle \mathcal{S}((v_1 \otimes w_1) \otimes (v_2 \otimes w_2)), (f_1 \otimes f_2) \otimes (g_1 \otimes g_2)\rangle .
\end{aligned}$$

Since we have already seen that $V_1^* \otimes V_2^* \otimes W_1^* \otimes W_2^*$ is weak* dense in

$$(V_1^* \overline{\otimes} V_2^*) \overset{\sigma\mathrm{h}}{\otimes} (W_1^* \overline{\otimes} W_2^*),$$

we obtain (6.4). ∎

We note that a simple induction may be used to show that the *multiple shuffle* map $\mathcal{S} : (V_1 \otimes \cdots \otimes V_p) \otimes (W_1 \otimes \cdots \otimes W_p) \to (V_1 \otimes W_1) \otimes \cdots \otimes (V_p \otimes W_p)$ determined by

$$\mathcal{S}((v_1 \otimes \cdots \otimes v_p) \otimes (w_1 \otimes \cdots \otimes w_p)) = (v_1 \otimes w_1) \otimes \cdots \otimes (v_p \otimes w_p) \tag{6.6}$$

extends to a completely contractive map

$$\mathcal{S}_\sigma : (V_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} V_p) \overset{\mathrm{nuc}}{\otimes} (W_1 \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} W_p) \to (V_1 \overset{\mathrm{nuc}}{\otimes} W_1) \overset{\mathrm{eh}}{\otimes} \cdots \overset{\mathrm{eh}}{\otimes} (V_p \overset{\mathrm{nuc}}{\otimes} W_p).$$

7. HOPF CONVOLUTION ALGEBRAS

Finally we return to the discussion of Hopf convolution algebras begun in Section 2. Given a Hopf von Neumann algebra (R, m, δ), we have a corresponding triple $(A = R_*, m, \delta)$ defined by the diagram (2.1). We note that if $A = R_*$ and $B = S_*$ are two such convolution algebras, then $A \overset{\mathrm{eh}}{\otimes} B$ is a completely contractive Banach algebra, i.e., the multiplication map

$$m : (A \overset{\mathrm{eh}}{\otimes} B) \widehat{\otimes} (A \overset{\mathrm{eh}}{\otimes} B) \to (A \overset{\mathrm{eh}}{\otimes} B)$$

is completely contractive, since m is the composition of the complete contractions

$$(A \overset{\mathrm{eh}}{\otimes} B) \mathbin{\widehat{\otimes}} (A \overset{\mathrm{eh}}{\otimes} B) \to (A \overset{\mathrm{eh}}{\otimes} B) \overset{\mathrm{nuc}}{\otimes} (A \overset{\mathrm{eh}}{\otimes} B) \xrightarrow{\mathcal{S}_{\mathrm{e}}} (A \overset{\mathrm{nuc}}{\otimes} A) \overset{\mathrm{eh}}{\otimes} (B \overset{\mathrm{nuc}}{\otimes} B)$$

and

$$(A \overset{\mathrm{nuc}}{\otimes} A) \overset{\mathrm{eh}}{\otimes} (B \overset{\mathrm{nuc}}{\otimes} B) = (A \mathbin{\widehat{\otimes}} A) \overset{\mathrm{eh}}{\otimes} (B \mathbin{\widehat{\otimes}} B) \xrightarrow{m_A \otimes m_B} A \overset{\mathrm{eh}}{\otimes} B.$$

THEOREM 7.1. *If $(A = R_*, m, \delta)$ is a Hopf convolution algebra, then δ is a completely contractive homomorphism.*

Proof. The hypothesis that δ_R is an algebraic homomorphism is encoded in the commutative diagram

$$\begin{tikzcd}
R \overset{\sigma\mathrm{h}}{\otimes} R \arrow[r, "\delta_R \otimes \delta_R"] \arrow[d, "m_R"] & (R \mathbin{\overline{\otimes}} R) \overset{\sigma\mathrm{h}}{\otimes} (R \mathbin{\overline{\otimes}} R) \arrow[r, "\mathcal{S}_\sigma"] & (R \overset{\sigma\mathrm{h}}{\otimes} R) \mathbin{\overline{\otimes}} (R \overset{\sigma\mathrm{h}}{\otimes} R) \arrow[d, "m_R \otimes m_R"] \\
R \arrow[rr, "\delta_R"] & & R \mathbin{\overline{\otimes}} R
\end{tikzcd} \tag{7.1}$$

.

Taking the preadjoint of (7.1), we find that δ_A is again a homomorphism:

$$\begin{tikzcd}
A \overset{\mathrm{eh}}{\otimes} A & (A \overset{\mathrm{nuc}}{\otimes} A) \overset{\mathrm{eh}}{\otimes} (A \overset{\mathrm{nuc}}{\otimes} A) \arrow[l, "m_A \otimes m_A"'] & (A \overset{\mathrm{eh}}{\otimes} A) \overset{\mathrm{nuc}}{\otimes} (A \overset{\mathrm{eh}}{\otimes} A) \arrow[l, "\mathcal{S}_{\mathrm{e}}"'] \\
A \arrow[u, "\delta_A"] & & A \mathbin{\widehat{\otimes}} A = A \overset{\mathrm{nuc}}{\otimes} A \arrow[ll, "m_A"'] \arrow[u, "\delta_A \otimes \delta_A"]
\end{tikzcd} \tag{7.2}$$

. ∎

We define a *representation* $\pi : A \to B(H)$ of a Hopf convolution algebra A on a Hilbert space H to be a completely bounded homomorphism $\pi : A \to B(H)$. The comultiplication $\delta = \delta_A$ may be used to define the *tensor product* of completely bounded representations of A. Given representations $\pi_1 : A \to B(H)$ and $\pi_2 : A \to B(K)$ we define $\pi_1 \times \pi_2 : A \to B(H \otimes K)$ to be the composition

$$A \xrightarrow{\delta} A \overset{\mathrm{eh}}{\otimes} A \xrightarrow{\pi_1 \otimes \pi_2} B(H) \overset{\mathrm{eh}}{\otimes} B(K) \subseteq B(H) \overset{\sigma\mathrm{h}}{\otimes} B(K) \xrightarrow{\theta} B(H \otimes K), \tag{7.3}$$

where θ is determined by taking the product of the maps

$$B(H) \to B(H \otimes K) : T \mapsto T \otimes I_K \quad \text{and} \quad B(K) \to B(H \otimes K) : T \mapsto I_{\mathrm{h}} \otimes T$$

(see (5.23) and (5.21)).

Turning to some examples, if G is a locally compact group with Haar measure μ, then both $L^\infty(G) = L^\infty(G, \mu)$ and the left regular von Neumann algebra $L(G)$

are Hopf von Neumann algebras (see [31] and [39]). The corresponding Hopf convolution algebras are the usual convolution algebra $L^1(G) = L^1(G, \mu)$ and the Fourier algebra $A(G)$. We note that since $L^\infty(G)$ is a commutative von Neumann algebra, the operator space structure on $L^1(G)$ is just the *maximal* operator space structure associated with the underlying Banach space (see [6]). Thus we have that any bounded map φ from $L^1(G)$ to an operator space X is automatically completely bounded with $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$.

Given a Hopf convolution algebra A, a completely bounded representation $\pi : A \longrightarrow B(H_\pi)$, and vectors $\xi, \eta \in H_\pi$, we say that the functional $c(a) = \langle \pi(a)\xi \,|\, \eta \rangle \in R = A^*$ is a *coefficient operator* of π, and if π is completely contractive, we say that c is a *Fourier-Stieltjes coefficient operator*. Letting $\mathcal{C}(A)$ (respectively, $\mathcal{B}(A)$) be all coefficient operators (respectively, all Fourier-Stieltjes coefficient operators), we have

$$\mathcal{B}(A) \subseteq \mathcal{C}(A) \subseteq R = A^*.$$

If $A = L^1(G)$ for G a locally compact group, any contractive representation π of A is automatically completely contractive. Given a Fourier-Stieltjes coefficient operator

$$b(a) = \langle \pi(a)\xi \,|\, \eta \rangle,$$

we may assume that π is non-degenerate, i.e., that $\pi(A)H_\pi$ is dense in H_π. To see this, let H_0 be the closure of $\pi(A)H_\pi$, and let π_0 be the corresponding subrepresentation of π. π_0 is non-degenerate since $L^1(G)$ has a contractive approximate identity u_γ. Letting $\xi_0 \in H_0$ be a weak limit point of the net $\pi(u_\gamma)\xi$ and η_0 be the orthogonal projection of η onto H_0, it is evident that

$$b(a) = \langle \pi_0(a)\xi_0 \,|\, \eta_0 \rangle.$$

The usual argument (see [30], Section 32) shows that π uniquely determines a contractive unital representation π_0 of G. Given $s \in G$, we have that both $\pi_0(s)$ and $\pi_0(s^{-l})$ are contractive and thus unitary. It follows that $\mathcal{B}(A)$ coincides with $B(G)$, the usual Fourier-Stieltjes algebra of the group G.

If $A = A(G)$ is the Fourier algebra of a non-commutative locally compact group G, then a completely contractive representation $\pi : A(G) \to B(H)$ corresponds to a contraction $W \in L(G) \mathbin{\overline{\otimes}} B(H)$ since we have the natural isomorphism

$$CB(A(G), B(H)) \cong (A(G) \mathbin{\widehat{\otimes}} B(H)_*)^* \cong L(G) \mathbin{\overline{\otimes}} B(H).$$

It is easy to see that W satisfies the Nakagami-Takesaki "associativity" condition (A.2) in the Appendix of their monograph [31]. If we could prove that W is unitary, we would have that π determines a "corepresentation" of G on H (see [28]). Of course if G is abelian, this is true since we then have that $A(G) = L^1(\widehat{G})$.

Returning to the general theory we have

THEOREM 7.2. *If A is a Hopf convolution algebra, then $\mathcal{C}(A)$ is a subalgebra of $R = A^*$.*

Proof. Given coefficient functions

$$c_k(a) = \langle \pi_k(a)\xi_k \,|\, \eta_k \rangle \in R, \quad k = 1, 2$$

and $\lambda \in \mathbb{C}$, we have that

$$\begin{aligned} (c_1 + c_2)(a) &= \langle \pi_1 \otimes \pi_2(a)(\xi_1 \oplus \xi_2) \,|\, (\eta_1 \oplus \eta_2) \rangle \\ (\lambda c_1)(a) &= \langle \pi_1(a)(\lambda \xi_1) \,|\, \eta_1 \rangle. \end{aligned}$$

Turning to multiplication, the functionals $\omega_k \in B(H_k)_*$ defined by $\omega_k(b_k) = \langle b_k \xi_k \,|\, \eta_k \rangle$, determine linear functionals $\omega_1 \otimes \omega_2$ in the commutative diagram

$$\begin{tikzcd}
B(H) \overset{\mathrm{eh}}{\otimes} B(K) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\omega_1\otimes\omega_2"] & B(H) \overset{\sigma\mathrm{h}}{\otimes} B(K) \arrow[r, "\theta"] \arrow[d, "\omega_1\otimes\omega_2"'] & B(H \otimes K) \arrow[dl, "\omega_1\otimes\omega_2"'] \\
& \mathbb{C} &
\end{tikzcd} .$$

Thus since the multiplication operation on R is the adjoint of the coffiultlplication $\delta : A \to A \overset{\mathrm{eh}}{\otimes} A$, and $c_k = \omega_k \circ \pi_k$, we have $c_1 \overset{\mathrm{eh}}{\otimes} c_2 = (\omega_1 \otimes \omega_2) \circ (\pi_1 \overset{\mathrm{eh}}{\otimes} \pi_2)$ and

$$\begin{aligned} (c_1 c_2)(a) &= c_1 \overset{\mathrm{eh}}{\otimes} c_2(\delta(a)) = \langle (\pi_1 \overset{\mathrm{eh}}{\otimes} \pi_2)(\delta(a)), \omega_1 \otimes \omega_2 \rangle \\ &= \langle \pi_1 \otimes \pi_2(\delta(a))(\xi_1 \otimes \xi_2) | \, \eta_1 \otimes \eta_2 \rangle = \langle (\pi_1 \times \pi_2)(a)(\xi_1 \otimes \xi_2) \,|\, \eta_1 \otimes \eta_2 \rangle, \end{aligned}$$

where $\pi_1 \times \pi_2$ is again a completely bounded representation of A. ∎

If $\pi_k : A \to B(H_k)$ are completely contractive representations, then it is evident that the same is true for $\pi_1 \times \pi_2$. It follows that $\mathcal{B}(A)$ is a subalgebra of $\mathcal{C}(A)$, and we shall refer to it as the *Fourier-Stieltjes algebra* of A.

In order to go further, it is necessary to introduce more structure. In particular, if one wishes to obtain a satisfactory duality theory, one must introduce $*$-algebraic structure, and ultimately a discussion of antipodes. Since this would take us far afield from the present discussion, we shall consider this theory in a subsequent paper.

Acknowledgements. We wish to thank Stephen Allen, David Blecher, and Roger Smith, who provided us the preprints [1], [7] and [38], in which they pointed out that the dual of the Haagerup tensor product should be regarded as another type of tensor product. We are indebted to David Blecher for discussions on those papers.

This research was partially supported by the NSF.

REFERENCES

1. S. ALLEN, Ideals of completely bounded operators, Ph. D. Dissertation, Edinburgh University, Edinburgh 1991.
2. S. BAAJ, G. SKANDALIS, Unitaires mutiplicatifs et dualité pour les produits croisés de C^*-algebres, *Ann. Sci. École Norm. Sup. (4)* **26**(1993), 425–488.

3. R. Blattner, S. Montegomery, A duality theory for Hopf module algebras, *J. Algebra* **95**(1985), 153–172.
4. D. Blecher, The standard dual of an operator space, *Pacific J. Math.* **153**(1992), 15–30.
5. D. Blecher, Tensor products of operator spaces. II, *Canad. J. Math.* **44**(1992), 75–90.
6. D. Blecher, V. Paulsen, Tensor products of operator spaces, *J. Funct. Anal.* **99**(1991), 262–292.
7. D. Blecher, R. Smith, The dual of the Haagerup tensor product, *J. London Math. Soc.* **45**(1992), 126–144.
8. M. Choi, unpublished note.
9. E. Christensen, A. Sinclair, Representations of completely bounded multilinear operators, *J. Funct. Anal.* **72**(1987), 151–181.
10. E. Effros, On multilinear completely bounded module maps, *Contemp. Math.*, vol. 62, Amer. Math. Soc., Providence, RI, 1987, pp. 479–501.
11. E. Effros, R. Exel, On multilinear double commutant theorems, in *Operator Algebras and Applications*, vol. 1 (Eds. Evans and Takesaki), London Math. Soc. Lecture Notes Ser., vol. 135, Cambridge Univ. Press, Cambridge, 1988, pp. 81–94.
12. E. Effros, A. Kishimoto, Module maps and Hochschild-Johnson cohomology, *Indiana Univ. Math. J.* **36**(1987), 257–276.
13. E. Effros, J. Kraus, Z-J. Ruan, On two quantum tensor products, in *Operator Algebras, Mathematical Physics, and Low Dimensional Topology, (Istanbul 1991)*, Res. Math. Notes, vol. 5, A K Peters, Wellesley, MA, 1993, pp. 125–145.
14. E. Effros, Z-J. Ruan, Representations of operator bimodules and their applications, *J. Operator Theory* **19**(1988), 137–157.
15. E. Effros, Z-J. Ruan, Multivariable multipliers for groups and their operator algebras, *Proc. Sympos. Pure Math.*, vol. 51, Part I, Amer. Math. Soc., Providence, RI, 1990, pp. 197–218.
16. E. Effros, Z-J. Ruan, On approximation properties for operator spaces, *Internat. J. Math.* **1**(1990), 163–187.
17. E. Effros, Z-J. Ruan, A new approach to operator spaces, *Canad Math. Bull.* **34**(1991), 329–337.
18. E. Effros, Z-J. Ruan, Self-duality for the Haagerup tensor product, *J. Funct. Anal.* **100**(1991), 257–284.
19. E. Effros, Z-J. Ruan, Operator convolution algebras: an approach to quantum groups, unpublished manuscript, 1991.
20. E. Effros, Z-J. Ruan, Discrete quantum groups. I. The Haar measure, *Internat. J. Math.* **5**(1994), 681–723.
21. E. Effros, Z-J. Ruan, *Operator Spaces*, London Math. Soc. Monogr. (N.S.), vol. 23, Oxford Univ. Press, New York 2000.
22. M. Enock, J.-M. Schwartz, *Kac Algebras and Duality of Locally Compact Groups*, Springer-Verlag, Berlin 1992.
23. J. Ernest, *Hopf-von Neumann Algebras*, Functional Analysis (Proc. Conf., Irvine, Calif., 1966), Academic Press, London; Thompson Book Co., Washington, D.C., pp. 195–215
24. U. Haagerup, Decomposition of completely bounded maps on operator algebras, unpublished manuscript, 1980.
25. G. Kac, A generalization of the principle of duality for groups [Russian], *Dokl. Akad. Nauk SSSR* **138**(1961), 275–278.
26. E. Kirchberg, Representations of coinvolutive Hopf-W^*-algebras and non-Abelian duality, *Bull. Acad. Polon. Sci. Sèr. Sci. Math. Astronom. Phys.* **25**(1977), no. 2, 117–122.

27. J. KUSTERMANS, Locally compact quantum groups in the universal setting, *Internat. J. Math.* **12**(2001), 289–338.
28. M. LANDSTAD, J. PHILLIPS, I. RAEBURN, C.E. SUTHERLAND, Representations of crossed products by coactions and principal bundles, *Trans. Amer. Math. Soc.* **299**(1987), 747–784.
29. C. LE MERDY, B. MAGAJNA, A factorization problem for normal completely bounded mappings, *J. Funct. Anal.* **181**(2001), 313–345.
30. L. LOOMIS, *An Introduction to Abstract Harmonic Analysis*, D. van Nostrand, New York 1953.
31. Y. NAKAGAMI, M. TAKESAKI, *Duality for Crossed Products of von Neumann Algebras*, Lecture Notes in Math., vol. 731, Springer Verlag, Berlin 1979.
32. V. PAULSEN, R. SMITH, Multilinear maps and tensor norms on operator systems, *J. Funct. Anal.* **92**(1994), 225–288.
33. G. PISIER, The operator Hilbert space OH, complex interpolation and tensor norms, *Mem. Amer. Math. Soc.* **122**(1996), No. 585.
34. G. PISIER, Non-commutative vector valued L_p-spaces and completely p-summing maps, *Astérisque* **247**(1998).
35. Z-J. RUAN, Subspaces of C^*-algebras, *J. Funct. Anal.* **76**(1988), 217–230.
36. Z-J. RUAN, Amenability of Hopf von Neumann algebras and Kac algebras, *J. Funct. Anal.* **139**(1996), 466–499.
37. A. SINCLAIR, R. SMITH, *Hochschild Cohomology of von Neumann Algebras*, London Math. Soc. Lecture Note Ser., vol. 203,Cambridge Univ. Press, Cambridge 1995.
38. R. SMITH, Completely bounded module maps and the Haagerup tensor product, *J. Funct. Anal.* **102**(1991), 156–175.
39. Ş. STRĂTILĂ, *Modular Theory in Operator Algebras*, Abacus Press, Turnbridge Wells, 1981.
40. M. SWEEDLER, *Hopf Algebras*, Math. Lecture Note Ser., Benjamin, New York 1969.
41. M. WALTER, On a new method for defining the norm of a Fourier-Stieltjes Algebra, *Pacific J. Math.* **137**(1989), 209–223.
42. S.L. WORONOWICZ, Twisted SU(2) group. An example of a noncommutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23**(1987), 117–181.
43. S.L. WORONOWICZ, Compact matrix pseudogroups, *Comm. Math. Phys.* **111**(1987), 613–665.
44. S.L. WORONOWICZ, Compact quantum groups, *Symétries quantiques (Les Houches, 1995)*, North-Holland, Amsterdam 1998, pp. 845–884.

EDWARD G. EFFROS
Mathematical Department
UCLA
Los Angeles, CA 90095–1555
USA
E-mail: ege@math.ucla.edu

ZHONG-JIN RUAN
Mathematical Department
Univerity of Illinois
Urbana, IL 61801
USA
E-mail: ruan@math.uiuc.edu

Received April 20, 2001.